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Research Article

Solution and Stability of Quartic Functional Equations in Modular Spaces by Using Fatou Property

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We propose a novel generalized quartic functional equation and investigate its Hyers–Ulam stability in modular spaces using a fixed point technique and the Fatou property in this paper.

1. Introduction

The idea of functional equation stability occurs when a functional equation is replaced by an inequality that acts as a perturbation on the equation. Take into account that the topic of functional equation stability was prompted by a query posed by Ulam in 1940 [1], and Hyers response was published in [2]. By considering an unbounded Cauchy difference, Aoki [3] and Rassias [4] extended Hyers' theorem for additive mappings and linear mappings, respectively.

In [5–8], the authors examined the Hyers–Ulam-Rassias stability findings for functional equations involving many variables. They discussed the approximate solution of the septic functional equation in [9]. They established that this equation is stable in quasi-β-Banach spaces and in (β, p)-Banach spaces. Additionally, they established the instability of the preceding radical functional equation in a pertinent example. Khamsi investigated the notion of quasicontraction mappings in modular function spaces in the absence of the Δ_2-condition, establishing the presence of fixed points and thoroughly analysing their uniqueness in [10].

Kim and Tamilvanan [11] introduced a novel class of quartic functional equations and examined their Hyers–Ulam stability in fuzzy normed spaces using both the direct and fixed point methods. They demonstrated how sums and products of powers of norms may be used to manage the stability of this quartic functional equation. Wongkum et al. [12] studied the extended Ulam–Hyers–Rassias stability of quadratic functional equations extensively using fixed point theory. Their conclusions are achieved in the context of modular spaces whose modulars are lower semicontinuous but do not meet any Δ_2-related requirements.

Nakano established the theory of modulars on linear spaces and the accompanying theory of modular linear spaces, which was further refined by Koshi and Shimogaki [13] and Yamamuro [14] and others. Luxemburg [15], Musielak [16], and Turpin [17] and their collaborators contributed the most comprehensive elaboration of these ideas. At the moment, modulars and modular spaces theory is widely employed, most notably in the study of different Orlicz spaces [18] and interpolation theory [19, 20], both of which have several applications [16]. The significance for applications stems from the complex structure of modular function spaces, which, in addition to being Banach spaces (or F-spaces in a more generic setting), contain modular equivalents of norm or metric concepts.

Notably, Nakano pioneered modular spaces with relation to order spaces in 1950 [21]. The spaces were produced using the Orlicz spaces theory, which substitutes a particular, integrated nonlinear functionality for the abstract functional that regulates the expansion of space members. They established several features of modular space in [22] and shown that every uniformly limited composition operator mapping this space onto itself must satisfy the so-called Matkowski's requirements. In 2014, Sadeghi [23] explored the stability of functional equations in modular space using a fixed point technique.

In [24], the existence and Hyers-Ulam stability of the nearly periodic solution to the fractional differential equation with impulse and fractional Brownian motion were studied under nonlocal circumstances. The research was conducted using the semigroups of operators and MÃ¶nch fixed point technique, as well as the basic theory of Hyers-Ulam stability. In [25], the authors examined at the existence and Hyers-Ulam stability of random impulsive stochastic functional differential equations with finite delays. They demonstrated that moderate solutions to the equations exist using Krasnoselskii's fixed point approach. They then examined the Hyers-Ulam stability results under the Lipschitz condition on a bounded and closed interval. Finally, they demonstrated their findings with an example. The reader is recommended to see [26–28] and the references therein for further information on fixed point theory in modular spaces.

Definition 1 (see [29]). Let V be a vector space over $\mathbb{K}$ ($\mathbb{C}$ or $\mathbb{R}$). A generalized functional $\rho : V \longrightarrow [0,\infty]$ is called a modular if for arbitrary $u, v \in V$, ρ satisfies

(a) $\rho(u) = 0$ if and only if $u = 0$

(b) $\rho(\beta u) = \rho(u)$ for every scalar β with $|\beta| = 1$

(c) $\rho(\beta u + \gamma v) \le \rho(u) + \rho(v)$, whenever $\beta, \gamma \ge 0$ and $\beta + \gamma = 1$

If we replace (c) by.

(c') $\rho(\beta u + \gamma v) \le \beta\rho(u) + \gamma\rho(v)$, whenever $\beta, \gamma \ge 0$ and $\beta + \gamma = 1$, then, the modular ρ is called convex. A modular ρ defines a corresponding modular space, i.e., the vector space V_ρ given by:

$$V_\rho = \{u \in V \mid \rho(cu) \longrightarrow 0 \, as \, c \longrightarrow 0\}. \tag{1}$$

Definition 2 (see [29]). If V_ρ is a modular space and the sequence $\{v_n\}$ in V_ρ, then

(i) $v_n \xrightarrow{\rho} v$ if $\rho(v_n - v) \longrightarrow 0$ as $n \longrightarrow \infty$

(ii) $\{v_n\}$ is known as ρ-Cauchy if $\rho(v_l - v_n) \longrightarrow 0$ as $l, n \longrightarrow \infty$

(iii) A subset $A \subseteq V_\rho$ is known as ρ-complete iff every ρ-Cauchy sequence is ρ-convergent in A

Definition 3 (see [29]). Let V_ρ be a modular space and a non-empty subset $A \subseteq V_\rho$. The mapping $J : A \longrightarrow A$ is referred to as a quasicontraction, if there is $k < 1$ satisfies

$$\rho(Jl - Jm) \le k \max\{\rho(l-m), \rho(l-Jm), \rho(m-Jl), \rho(l-Jl), \rho(m-Jm)\}, \tag{2}$$

for any $l, m \in A$.

Definition 4 (see [29]). Let V_ρ be a modular space, a non-empty subset $A \subseteq V_\rho$, and a function $J : A \longrightarrow A$, the J orbit around a point v is

$$O(J) \coloneqq \{u, Ju, J^2u, \cdots\}, \tag{3}$$

the quantity

$$Y_\rho(J) \coloneqq \sup\{\rho(p-q) \mid p, q \in O(J)\}, \tag{4}$$

is then related to J and is referred to as the orbital diameter of J at v. If $Y_\rho(J) < \infty$, in particular, one says that J has an orbit of v that is limited to v.

Fatou property: the ρ-modular will have the Fatou property iff $\rho(v) \le \lim_{m\longrightarrow\infty} \inf \rho(v_m)$ whenever $\{v_m\} \xrightarrow{\rho} v$. A modular function is stated to fulfil the conditions Δ_3 if there is $k > 0$ which satisfies $\rho(3v) \le k\rho(v)$, for every $v \in V_\rho$.

In this work, we introduce new generalized quartic functional equation

$$\begin{aligned}\sum_{1\le i\le n} \phi\left(-v_i + \sum_{j=1;i\ne j}^{n} v_j\right) &= (n-8) \sum_{1\le i<j<k<l\le n} \phi(v_i + v_j + v_k + v_l) \\ &\quad - \left(n^2 - 12n + 28\right) \sum_{1\le i<j<k\le n} \phi(v_i + v_j + v_k) \\ &\quad + \left(\frac{n^3 - 15n^2 + 60n - 68}{2}\right) \sum_{1\le i<j\le n} \phi(v_i + v_j) \\ &\quad + 2 \sum_{1\le i<j\le n} \phi(v_i - v_j) + \sum_{i=1}^{n} \phi(3v_i) \\ &\quad - \left(\frac{n^4 - 17n^3 + 86n^2 - 148n + 558}{6}\right) \sum_{i=1}^{n} \phi(v_i),\end{aligned} \tag{5}$$

where $n \ge 5$, and investigate Hyers-Ulam stability of this quartic functional equation in modular space by using the fixed point method with the help of Fatou property.

2. Solution

We denote V and W as two real vector spaces.

Theorem 5. *If an even mapping $\phi : V \longrightarrow W$ satisfies the functional equation (5) for all $v_1, v_2, \cdots, v_n \in V$, then, the function ϕ is quartic.*

Proof. In the view of evenness, we obtain $\phi(-v) = \phi(v)$ for all v in V. Now, setting $v_1 = v_2 = \cdots = v_n = 0$ in equation (5), we have $\phi(0) = 0$. Replacing $(v_1, v_2, \cdots, v_n)$ by $(v, 0, \cdots, 0)$ in equation (5), we get

$$\phi(3v) = 3^4\phi(v), \tag{6}$$

for all $v \in V$. Replacing v by $3v$ in equation (6), we have

$$\phi\left(3^2 v\right) = 3^{4(2)}\phi(v), \tag{7}$$

for all $v \in V$. Again, replacing v by $3v$ in equation (7), we obtain

$$\phi\left(3^3 v\right) = 3^{4(3)}\phi(v), \tag{8}$$

for all $v \in V$. For any nonnegative integer $n \geq 0$, we can generalize the result that

$$\phi\left(3^n v\right) = 3^{4(n)}\phi(v), \tag{9}$$

for all $v \in V$. Similarly, we have

$$\phi\left(3^{-n} v\right) = 3^{-4n}\phi(v), \tag{10}$$

for all $v \in V$. Next, replacing $(v_1, v_2, \cdots, v_n)$ by $(v_1, v_1, v_2, 0, \cdots, 0)$, we obtain

$$\phi(2v_1 + v_2) + \phi(2v_1 - v_2) = 4\phi(v_1 - v_2) + 4\phi(v_1 + v_2) + 24\phi(v_1) - 6\phi(v_2), \tag{11}$$

for all $v_1, v_2 \in V$. Hence, the function ϕ is quartic. □

3. Hyers-Ulam Stability

We assume that ρ is a convex modular on ρ-complete modular spaces W_ρ with the Fatou property that meets the Δ_3 -condition with $0 < k \leq 3$ in this section. Let V be a linear space as well. For convenience, we call a function $\phi : V \longrightarrow W_\rho$ by the shorthand $\phi : V \longrightarrow W_\rho$:

$$\begin{aligned} D\phi(v_1, v_2, \cdots, v_n) = &\sum_{1\leq i\leq n} \phi\left(-v_i + \sum_{j=1; i\neq j}^{n} v_j\right) \\ &- (n-8) \sum_{1\leq i<j<k<l\leq n} \phi\left(v_i + v_j + v_k + v_l\right) \\ &+ \left(n^2 - 12n + 28\right) \sum_{1\leq i<j<k\leq n} \phi\left(v_i + v_j + v_k\right) \\ &- \left(\frac{n^3 - 15n^2 + 60n - 68}{2}\right), \end{aligned} \tag{12}$$

$$\begin{aligned} &\sum_{1\leq i<j\leq n} \phi\left(v_i + v_j\right) - 2\sum_{1\leq i<j\leq n} \phi\left(v_i - v_j\right) - \sum_{i=1}^{n} \phi(3v_i) \\ &+ \left(\frac{n^4 - 17n^3 + 86n^2 - 148n + 558}{6}\right) \sum_{i=1}^{n} \phi(v_i), \end{aligned} \tag{13}$$

for all $v_1, v_2, \cdots, v_n \in V$.

Theorem 6. *Let $\psi : V^n \longrightarrow [0, +\infty)$ be a function such that*

$$\lim_{m\longrightarrow\infty} \frac{1}{3^{4m}} \psi(3^m v_1, 3^m v_2, \cdots, 3^m v_n) = 0, \tag{14}$$

$$\psi(3v_1, 3v_2, \cdots, 3v_n) \leq 3^4 L\psi(v_1, v_2, \cdots, v_n), \tag{15}$$

for all $v_1, v_2, \cdots, v_n \in V$, with $L < 1$. If an even mapping $\phi : V \longrightarrow W_\rho$ with $\phi(0) = 0$ and such that

$$\rho(D\phi(v_1, v_2, \cdots, v_n)) \leq \psi(v_1, v_2, \cdots, v_n), \tag{16}$$

for all $v_1, v_2, \cdots, v_n \in V$, then, there exists a unique quartic mapping $Q_4 : V \longrightarrow W_\rho$ satisfying

$$\rho(Q_4(v) - \phi(v)) \leq \frac{1}{3^4(1-L)} \psi(v, 0, \cdots, 0), \tag{17}$$

for all $v \in V$.

Proof. We consider the set

$$\Lambda = \left\{p : V \longrightarrow W_\rho\right\}, \tag{18}$$

and define the function $\bar{\rho}$ on Λ as follows:

$$\bar{\rho}(p) =: \inf\left\{\theta > 0 : \rho(p(v)) \leq \theta\psi(v, 0, \cdots, 0), \forall v \in V\right\}. \tag{19}$$

Now, we show that $\bar{\rho}$ is a convex modular on Λ. It is easy to verify that $\bar{\rho}$ satisfies the axioms (a) and (b) of a modular. Next, we will show that $\bar{\rho}$ is convex, and hence, (c') is satisfied. Let $\varepsilon > 0$ be given, then, there exist real constants $\theta_1 > 0$ and $\theta_2 > 0$ such that

$$\bar{\rho}(p) \leq \theta_1 \leq \bar{\rho}(p) + \varepsilon \text{ and } \bar{\rho}(q) \leq \theta_2 \leq \bar{\rho}(q) + \varepsilon. \tag{20}$$

Also

$$\rho(p(v)) \leq \theta_1\psi(v, 0, \cdots, 0), \rho(q(v)) \leq \theta_2\psi(v, 0, \cdots, 0), \tag{21}$$

for all $v \in V$. If $\beta + \gamma = 1$ and $\beta, \gamma \geq 0$, then, we get

$$\rho(\beta p(v) + \gamma q(v)) \leq \beta\rho(p(v)) + \gamma\rho(q(v)) \leq (\theta_1\beta + \theta_2\gamma)\psi(v, 0, \cdots, 0), \tag{22}$$

so we get

$$\bar{\rho}(\beta p + \gamma q) \leq \beta\bar{\rho}(p) + \gamma\bar{\rho}(q) + (\beta + \gamma)\varepsilon. \tag{23}$$

This concludes that $\bar{\rho}$ is convex modular on Λ. Now, we show that $\Lambda_{\bar{\rho}}$ is $\bar{\rho}$-complete.

Let $\{p_n\}$ is a $\bar{\rho}$-Cauchy sequence in $\Lambda_{\bar{\rho}}$ and let $\varepsilon > 0$. Then, there exists a positive integer $n_0 \in \mathbb{N}$ such that

$$\bar{\rho}(p_n - p_m) < \varepsilon, \tag{24}$$

for all $n, m \geq n_0$. Then

$$\rho(p_n(v) - p_m(v)) \leq \varepsilon\psi(v, 0, \cdots, 0), \tag{25}$$

for all $v \in V$ and all $n, m \geq n_0$. Therefore, $\{p_n(v)\}$ is a ρ-Cauchy sequence in W_ρ. Since W_ρ is ρ-complete, so $\{p_n(v)\}$ is convergent in W_ρ, for each $v \in V$. Hence, we can define a function $p : V \longrightarrow W_\rho$ by

$$p(v) \coloneqq \lim_{n\longrightarrow\infty} p_n(v), \tag{26}$$

for all $v \in V$. Since ρ satisfies the Fatou property, it follows from (25) that

$$\rho(p_n(v) - p(v)) \leq \liminf_{m\longrightarrow\infty} \rho(p_n(v) - p_m(v)) \leq \varepsilon\psi(v, 0, \cdots, 0), \tag{27}$$

so

$$\bar{\rho}(p_n - p) \leq \varepsilon, \tag{28}$$

for all $n \geq n_0$. Thus, $\{p_n\}$ is $\bar{\rho}$-converges. Hence, $\Lambda_{\bar{\rho}}$ is $\bar{\rho}$-complete.

Next, we show that $\bar{\rho}$ satisfies the Fatou property. Suppose that $\{p_n\}$ is a sequence in $\Lambda_{\bar{\rho}}$ which is $\bar{\rho}$-convergent to an element $p \in \Lambda_{\bar{\rho}}$.

Let $\varepsilon > 0$ be given. For each $n \in \mathbb{N}$, let θ_n be a real constant such that

$$\bar{\rho}(p_n) \leq \theta_n \leq \bar{\rho}(p_n) + \varepsilon. \tag{29}$$

So

$$\rho(p_n(v)) \leq \theta_n\psi(v, 0, \cdots, 0), \tag{30}$$

for all $v \in V$. Since ρ satisfies the Fatou property, we get

$$\begin{aligned}\rho(p(v)) &\leq \liminf_{n\longrightarrow\infty} \rho(p_n(v)) \leq \liminf_{n\longrightarrow\infty} \theta_n\psi(v, 0, \cdots, 0)\\ &\leq \left[\liminf_{n\longrightarrow\infty} \bar{\rho}(p_n) + \varepsilon\right]\psi(v, 0, \cdots, 0).\end{aligned} \tag{31}$$

Thus, we obtain

$$\bar{\rho}(p) \leq \liminf_{n\longrightarrow\infty} \bar{\rho}(p_n) + \varepsilon. \tag{32}$$

Hence, $\bar{\rho}$ satisfies the Fatou property. Consider the function $\Psi : \Lambda_{\bar{\rho}} \longrightarrow \Lambda_{\bar{\rho}}$ by

$$\Psi p(v) = \frac{1}{3^4} p(3v), \tag{33}$$

for all $v \in V$ and all $p \in \Lambda_{\bar{\rho}}$. Let $p, q \in \Lambda_{\bar{\rho}}$ and let $\theta \in [0, 1]$ be an arbitrary constant with $\bar{\rho}(p - q) < \theta$. From the definition of $\bar{\rho}$, we obtain

$$\rho(p(v) - q(v)) \leq \theta\psi(v, 0, \cdots, 0), \tag{34}$$

for all $v \in V$. By inequality (15) and the above inequality, we get

$$\rho\left(\frac{p(3v)}{3^4} - \frac{q(3v)}{3^4}\right) \leq \frac{1}{3^4}\rho(p(3v) - q(3v)) \leq \frac{1}{3^4}\theta\psi(3v, 0, \cdots, 0) \leq \theta L\psi(v, 0, \cdots, 0), \tag{35}$$

for all $v \in V$. Hence,

$$\bar{\rho}(\Psi p - \Psi q) \leq L\bar{\rho}(p - q), \forall p, q \in \Lambda_{\bar{\rho}}, \tag{36}$$

i.e., Ψ is a $\bar{\rho}$-contraction. Next, we show that Ψ has a bounded orbit at ϕ. Replacing $(v_1, v_2, \cdots, v_n)$ by $(v, 0, \cdots, 0)$ in (16), we get

$$\rho\left(\phi(3v) - 3^4\phi(v)\right) \leq \psi(v, 0, \cdots, 0) \Rightarrow \rho\left(\frac{\phi(3v)}{3^4} - \phi(v)\right) \leq \frac{1}{3^4}\psi(v, 0, \cdots, 0), \tag{37}$$

for all $v \in V$. Replacing v with $3v$ in (37), we get

$$\rho\left(\frac{\phi(3^2v)}{3^{4(2)}} - \frac{\phi(3v)}{3^4}\right) \leq \frac{1}{3^{4(2)}}\psi(3v, 0, \cdots, 0), \tag{38}$$

for all $v \in V$. By using (37) and (38), we get

$$\begin{aligned}\rho\left(\frac{\phi(3^2v)}{3^{4(2)}} - \phi(v)\right) &\leq \rho\left(\frac{\phi(3^2v)}{3^{4(2)}} - \frac{\phi(3v)}{3^4} + \frac{\phi(3v)}{3^4} - \phi(v)\right)\\ &\leq \rho\left(\frac{\phi(3^2v)}{3^{4(2)}} - \frac{\phi(3v)}{3^4}\right) + \rho\left(\frac{\phi(3v)}{3^4} - \phi(v)\right)\\ &\leq \frac{1}{3^{4(2)}}\psi(3v, 0, \cdots, 0) + \frac{1}{3^4}\psi(v, 0, \cdots, 0),\end{aligned} \tag{39}$$

for all $v \in V$. By induction, we can easily see that

$$\begin{aligned}\rho\left(\frac{\phi(3^nv)}{3^{4n}} - \phi(v)\right) &\leq \sum_{i=1}^{n} \frac{1}{3^{4i}}\psi\left(3^{i-1}v, 0, \cdots, 0\right)\\ &\leq \frac{1}{L3^4}\psi(v, 0, \cdots, 0)\sum_{i=1}^{n} L^i\\ &\leq \frac{1}{3^4(1 - L)}\psi(v, 0, \cdots, 0),\end{aligned} \tag{40}$$

for all $v \in V$. It follows from inequality (40) that

$$\begin{aligned}\rho\left(\frac{\phi(3^n v)}{3^{4n}}-\frac{\phi(3^m v)}{3^{4m}}\right) &\le \frac{1}{2}\rho\left(2\frac{\phi(3^n v)}{3^{4n}}-2\phi(v)\right)+\frac{1}{2}\rho\left(2\frac{\phi(3^m v)}{3^{4m}}-2\phi(v)\right)\\ &\le \frac{k}{2}\rho\left(\frac{\phi(3^n v)}{3^{4n}}-\phi(v)\right)+\frac{k}{2}\rho\left(\frac{\phi(3^m v)}{3^{4m}}-\phi(v)\right)\\ &\le \frac{k}{2}\frac{1}{3^4(1-L)}\psi(v,0,\cdots,0)+\frac{k}{2}\frac{1}{3^4(1-L)}\psi(v,0,\cdots,0)\\ &\le \frac{k}{3^4(1-L)}\psi(v,0,\cdots,0),\end{aligned} \tag{41}$$

for all $v \in V$ and all $n, m \in \mathbb{N}$. By the definition of $\bar{\rho}$, we conclude that

$$\bar{\rho}(\Psi^n\phi-\Psi^m\phi) \le \frac{k}{3^4(1-L)}, \tag{42}$$

which implies the boundedness of an orbit of Ψ at ϕ. It follows from Theorem 1.5 [29] that the sequence $\{\Psi^n\phi\}\bar{\rho}$ -converges to $Q_4 \in \Lambda_{\bar{\rho}}$. Now, by the $\bar{\rho}$-contractivity of Ψ, we have

$$\bar{\rho}(\Psi^n\phi-\Psi Q_4) \le L\bar{\rho}\left(\Psi^{n-1}\phi-Q_4\right). \tag{43}$$

Passing to the limit $n \longrightarrow \infty$ and applying the Fatou property of $\bar{\rho}$, we obtain that

$$\bar{\rho}(\Psi Q_4-Q_4) \le \lim_{n\longrightarrow\infty}\inf \bar{\rho}(\Psi Q_4-\Psi^n\phi) \le L\lim_{n\longrightarrow\infty}\inf \bar{\rho}\left(Q_4-\Psi^{n-1}\phi\right)=0. \tag{44}$$

Therefore, Q_4 is a fixed point of Ψ. Replacing $(v_1, v_2,\cdots,v_n)$ by $(3^l v_1, 3^l v_2,\cdots,3^l v_n)$ in (16), we get

$$\rho\left(D\phi\left(3^l v_1, 3^l v_2,\cdots,3^l v_n\right)\right) \le \psi\left(3^l v_1, 3^l v_2,\cdots,3^l v_n\right), \tag{45}$$

for all $v_1, v_2, \cdots, v_n \in V$. Therefore

$$\rho\left(\frac{1}{3^{4l}}D\phi\left(3^l v_1, 3^l v_2,\cdots,3^l v_n\right)\right) \le \frac{1}{3^{4l}}\psi\left(3^l v_1, 3^l v_2,\cdots,3^l v_n\right). \tag{46}$$

Employing the limit $l \longrightarrow \infty$, we get

$$DQ_4(v_1, v_2,\cdots,v_n)=0, \tag{47}$$

for all $v_1, v_2, \cdots, v_n \in V$. It follows from Theorem 2 that Q_4 is quartic. By using (40), we get (17).

To prove the uniqueness of Q_4, let $Q_4' : V \longrightarrow W_\rho$ be another quartic mapping satisfying (17). Then, $Q4'$ is a fixed point of Ψ.

$$\bar{\rho}\left(Q_4-Q_4'\right)=\bar{\rho}\left(\Psi Q_4-\Psi Q_4'\right) \le L\bar{\rho}\left(Q_4-Q_4'\right), \tag{48}$$

which implies that $\bar{\rho}(Q_4-Q_4')=0$. This proves that $Q_4=Q_4'$. Therefore, the function Q_4 is unique. This completes the proof. □

Corollary 7. *Let a mapping $\psi : V^n \longrightarrow [0,+\infty)$ such that*

$$\lim_{l\longrightarrow\infty}\frac{1}{3^{4l}}\psi\left(3^l v_1, 3^l v_2,\cdots,3^l v_n\right)=0, \tag{49}$$

$$\psi(3v_1, 3v_2,\cdots,3v_n) \le L3^4\psi(v_1, v_2,\cdots,v_n), \tag{50}$$

for all $v_1, v_2, \cdots, v_n \in V$ with $L<1$. Suppose that a mapping $\phi : V \longrightarrow W$ with $\phi(0)=0$ and such that

$$\|D\phi(v_1, v_2,\cdots,v_n)\| \le \psi(v_1, v_2,\cdots,v_n), \tag{51}$$

for all $v_1, v_2, \cdots, v_n \in V$, then, there exists a unique quartic mapping $Q_4 : V \longrightarrow W$ satisfying

$$\|Q_4(v)-\phi(v)\| \le \frac{1}{3^4(1-L)}\psi(v,0,\cdots,0), \tag{52}$$

for all $v \in V$.

Proof. It is known that every normed space is modular space with the modular $\rho(v)=\|v\|$ and satisfies the Δ_3-condition with $k=3$. □

Remark 8. If we replace $\psi(v_1, v_2,\cdots,v_n)$ by $\alpha(\sum_{i=1}^n\|v_i\|^p)$ and letting $L=3^{p-4}$ in Corollary 7, we obtain the stability results for the sum of norms that

$$\|Q_4(v)-\phi(v)\| \le \frac{\alpha\|v\|^p}{\left(3^4-3^p\right)}, \tag{53}$$

for all $v \in V$, where α and p are constants with $p<4$.

Remark 9. If we replace $\psi(v_1, v_2,\cdots,v_n)$ by $\alpha(\sum_{i=1}^n\|v_i\|^{np}+\prod_{i=1}^n\|v_i\|^p)$ and letting $L=3^{np-4}$ in Corollary 7, we obtain the stability results for the sum of product of norms that

$$\|Q_4(v)-\phi(v)\| \le \frac{\alpha\|v\|^{np}}{\left(3^4-3^{np}\right)}, \tag{54}$$

for all $v \in V$, where α and p are constants with $np<4$.

Theorem 10. *Let $\psi : V^n \longrightarrow [0,+\infty)$ be a function such that*

$$\lim_{m\longrightarrow\infty}3^{4m}\psi\left(\frac{v_1}{3^m}, \frac{v_2}{3^m},\cdots,\frac{v_n}{3^m}\right)=0, \tag{55}$$

$$\psi\left(\frac{v_1}{3}, \frac{v_2}{3},\cdots,\frac{v_n}{3}\right) \le \frac{L}{3^4}\psi(v_1, v_2,\cdots,v_n), \tag{56}$$

for all $v_1, v_2, \cdots, v_n \in V$ with $L<1$. Suppose that $\phi : V \longrightarrow W_\rho$ with $\phi(0)=0$ and satisfies (16), then, there exists a *unique quartic mapping* $Q_4 : V \longrightarrow W_\rho$ *satisfying*

$$\rho(Q_4(v) - \phi(v)) \le \frac{L}{3^4(1-L)}\psi(v,0,\cdots,0), \tag{57}$$

for all $v \in V$.

Proof. We consider the set

$$\Lambda = \{p : V \longrightarrow W_\rho\}, \tag{58}$$

and define the function $\bar{\rho}$ on Λ as follows:

$$\bar{\rho}(p) =: \inf\{\theta > 0 : \rho(p(v)) \le \theta\psi(v,0,\cdots,0), \forall v \in V\}. \tag{59}$$

Similar to the proof of Theorem 6, we have

(1) $\bar{\rho}$ is a convex modular on Λ

(2) $\Lambda_{\bar{\rho}}$ is $\bar{\rho}$-complete

(3) $\bar{\rho}$ satisfies the Fatou property

Now, we consider the function $\Psi : \Lambda_{\bar{\rho}} \longrightarrow \Lambda_{\bar{\rho}}$ defined by

$$\Psi p(v) = 3^4 p\left(\frac{v}{3}\right), \tag{60}$$

for all $v \in V$ and all $p \in \Lambda_{\bar{\rho}}$. Let $p, q \in \Lambda_{\bar{\rho}}$ and let $\theta \in [0,1]$ be an arbitrary constant with $\bar{\rho}(p-q) < \theta$. From the definition of $\bar{\rho}$, we have

$$\rho(p(v) - q(v)) \le \theta\psi(v,0,\cdots,0), \tag{61}$$

for all $v \in V$. By the assumption and the last inequality, we get

$$\begin{aligned}\rho\left(3^4 p\left(\frac{v}{3}\right) - 3^4 q\left(\frac{v}{3}\right)\right) &\le k^4\rho\left(p\left(\frac{v}{3}\right) - q\left(\frac{v}{3}\right)\right) \le k^4\theta\psi\left(\frac{v}{3},0,\cdots,0\right)\\ &\le \theta L\psi(v,0,\cdots,0), v \in V.\end{aligned} \tag{62}$$

Hence,

$$\bar{\rho}(\Psi p - \Psi q) \le L\bar{\rho}(p-q), p, q \in \Lambda_{\bar{\rho}}, \tag{63}$$

i.e., Ψ is a $\bar{\rho}$-contraction.

Next, we prove then that Ψ has a bounded orbit at ϕ. Replacing $(v_1, v_2, \cdots, v_n)$ by $(v, 0, \cdots, 0)$ in (16), we get

$$\rho\left(3^4\phi(v) - \phi(3v)\right) \le \psi(v,0,\cdots,0), \tag{64}$$

for all $v \in V$. Replacing v with $v/3$ in (64), we get

$$\rho\left(3^4\phi\left(\frac{v}{3}\right) - \phi(v)\right) \le \psi\left(\frac{v}{3},0,\cdots,0\right), \tag{65}$$

for all $v \in V$. Replacing v with $v/3$ in (65), we get

$$\rho\left(3^4\phi\left(\frac{v}{3^2}\right) - \phi\left(\frac{v}{3}\right)\right) \le \psi\left(\frac{v}{3^2},0,\cdots,0\right), \tag{66}$$

for all $v \in V$. By using (64), (65), and (66), we get

$$\begin{aligned}\rho\left(3^{4(2)}\phi\left(\frac{v}{3^2}\right) - \phi(v)\right) &\le \rho\left(3^{4(2)}\phi\left(\frac{v}{3^2}\right) - 3^4\phi\left(\frac{v}{3}\right)\right) + \rho\left(3^4\phi\left(\frac{v}{3}\right) - \phi(v)\right)\\ &\le k^4\rho\left(3^4\phi\left(\frac{v}{3^2}\right) - \phi\left(\frac{v}{3}\right)\right) + \rho\left(3^4\phi\left(\frac{v}{3}\right) - \phi(v)\right)\\ &\le 3^4\psi\left(\frac{v}{3^2},0,\cdots,0\right) + \psi\left(\frac{v}{3},0,\cdots,0\right),\end{aligned} \tag{67}$$

for all $v \in V$. By induction, we can easily see that

$$\begin{aligned}\rho\left(3^{4n}\phi\left(\frac{v}{3^n}\right) - \phi(v)\right) &\le \frac{1}{3^4}\sum_{i=1}^{n} 3^{4i}\psi\left(\frac{v}{3^i},0,\cdots,0\right)\\ &\le \frac{1}{3^4}\psi(v,0,\cdots,0)\sum_{i=1}^{n} L^i\\ &\le \frac{L}{3^4(1-L)}\psi(v,0,\cdots,0),\end{aligned} \tag{68}$$

for all $v \in V$. It follows from inequality (68) that

$$\begin{aligned}\rho\left(3^{4n}\phi\left(\frac{v}{3^n}\right) - 3^{4m}\phi\left(\frac{v}{3^m}\right)\right) &\le \frac{1}{2}\rho\left(2\left(3^{4n}\right)\phi\left(\frac{v}{3^n}\right) - 2\phi(v)\right)\\ &\quad + \frac{1}{2}\rho\left(2\left(3^{4m}\right)\phi\left(\frac{v}{3^m}\right) - 2\phi(v)\right)\\ &\le \frac{kL}{3^4(1-L)}\psi(v,0,\cdots,0),\end{aligned} \tag{69}$$

for all $v \in V$ and all $n, m \in \mathbb{N}$. By the definition of $\bar{\rho}$, we conclude that

$$\bar{\rho}(\Psi^n\phi - \Psi^m\phi) \le \frac{kL}{3^4(1-L)}, \tag{70}$$

which implies the boundedness of an orbit of Ψ at ϕ. It follows from Theorem 1.5 [29] that the sequence $\{\Psi^n\phi\}\bar{\rho}$ -converges to $Q_4 \in \Lambda_{\bar{\rho}}$.

Now, by the $\bar{\rho}$-contractivity of Ψ, we have

$$\bar{\rho}(\Psi^n\phi - \Psi Q_4) \le L\bar{\rho}\left(\Psi^{n-1}\phi - Q_4\right). \tag{71}$$

Employing the limit $n \longrightarrow \infty$ and applying the Fatou property of $\bar{\rho}$, we obtain that

$$\begin{aligned}\bar{\rho}(\Psi Q_4 - Q_4) &\le \lim\inf_{n\longrightarrow\infty}\bar{\rho}(\Psi Q_4 - \Psi^n\phi)\\ &\le L\lim\inf_{n\longrightarrow\infty}\bar{\rho}\left(Q_4 - \Psi^{n-1}\phi\right) = 0.\end{aligned} \tag{72}$$

Therefore, Q_4 is a fixed point of Ψ. Replacing $(v_1, v_2, \cdots, v_n)$

by $(v_1/3^l, v_2/3^l,\cdots,v_n/3^l)$ in (16), we get

$$\rho\left(D\phi\left(3^{-l}v_1, 3^{-l}v_2,\cdots,3^{-l}v_n\right)\right) \le \psi\left(3^{-l}v_1, 3^{-l}v_2,\cdots,3^{-l}v_n\right), \tag{73}$$

for all $v_1, v_2, \cdots, v_n \in V$. Therefore

$$\rho\left(3^{4l}D\phi\left(\frac{v_1}{3^l}, \frac{v_2}{3^l},\cdots,\frac{v_n}{3^l}\right)\right) \le k^{4l}\psi\left(\frac{v_1}{3^l}, \frac{v_2}{3^l},\cdots,\frac{v_n}{3^l}\right). \tag{74}$$

Passing to the limit $l \longrightarrow \infty$, we get

$$DQ_4(v_1, v_2,\cdots,v_n) = 0, \tag{75}$$

for all $v_1, v_2, \cdots, v_n \in V$. It follows from Theorem 2 that Q_4 is quartic. By using (68), we get (57).

In order to prove the uniqueness of Q_4, consider another quartic solution $Q_4' : V \longrightarrow W_\rho$ that satisfy the inequality (17). Then, Q_4' is a fixed point of Ψ.

$$\bar{\rho}\left(Q_4 - Q_4'\right) = \bar{\rho}\left(\Psi Q_4 - \Psi Q_4'\right) \le L\bar{\rho}\left(Q_4 - Q_4'\right), \tag{76}$$

which implies that $\bar{\rho}(Q_4 - Q_4') = 0$ or $Q_4 = Q_4'$. Hence, the proof is now completed. □

Corollary 11. *Let a mapping $\psi : V^n \longrightarrow [0,+\infty)$ such that*

$$\lim_{l\longrightarrow\infty} 3^{4l}\psi\left(\frac{v_1}{3^l}, \frac{v_2}{3^l},\cdots,\frac{v_n}{3^l}\right) = 0, \tag{77}$$

$$\psi\left(\frac{v_1}{3}, \frac{v_2}{3},\cdots,\frac{v_n}{3}\right) \le \frac{L}{3^4}\psi(v_1, v_2,\cdots,v_n), \tag{78}$$

for all $v_1, v_2, \cdots, v_n \in V$, with $L < 1$. Suppose that $\phi : V \longrightarrow W$ with $\phi(0) = 0$ and satisfies (51), then there exists a unique quartic mapping $Q_4 : V \longrightarrow W$ satisfying

$$\|Q_4(v) - \phi(v)\| \le \frac{L}{3^4(1-L)}\psi(v, 0,\cdots,0), \tag{79}$$

for all $v \in V$.

Proof. It is known that every normed space is modular space with the modular $\rho(v) = \|v\|$ and satisfies the Δ_3-condition with $k = 3$. □

Remark 12. If we replace $\psi(v_1, v_2,\cdots,v_n)$ by $\alpha(\sum_{i=1}^n \|v_i\|^p)$ and letting $L = 3^{4-p}$ in Corollary 11, we obtain the stability results for the sum of norms that

$$\|Q_4(v) - \phi(v)\| \le \frac{\alpha\|v\|^p}{\left(3^p - 3^4\right)}, \tag{80}$$

for all $v \in V$, where α and p are constants with $p > 4$.

Remark 13. If we replace $\psi(v_1, v_2,\cdots,v_n)$ by $\alpha(\sum_{i=1}^n \|v_i\|^{np} + \prod_{i=1}^n \|v_i\|^p)$ and letting $L = 3^{4-np}$ in Corollary 11, we obtain the stability results for the sum of product of norms that

$$\|Q_4(v) - \phi(v)\| \le \frac{\alpha\|v\|^{np}}{\left(3^{np} - 3^4\right)}, \tag{81}$$

for all $v \in V$, where α and p are constants with $np > 4$.

4. Counterexample

We present a counterexample to show instability of a particular condition of the equality (5) using modified example of Gajda [7].

Remark 14. If a function $\phi : \mathbb{R} \longrightarrow V$ satisfies the functional equation (5), then, the following assertions hold:

(1) $\phi(q^{k/4}v) = q^k\phi(v), q \in \mathbb{Q}, k \in \mathbb{Z}$ and $v \in \mathbb{R}$

(2) $\phi(v) = v^4\phi(1), v \in \mathbb{R}$ if the function ϕ is continuous

Example 15. Let a mapping $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ be defined as follows:

$$\phi(v) = \sum_{n=0}^{\infty} \frac{\chi(3^n v)}{3^{4n}}, \tag{82}$$

where

$$\chi(v) = \begin{cases} \theta v^4, -1 < v < 1, \\ \theta, \text{else}, \end{cases} \tag{83}$$

then, the mapping $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies

$$|D\phi(v_1, v_2,\cdots,v_n)| \le \left(\frac{n^4 - 20n^3 + 47n^2 - 40n + 540}{6}\right)\left(\frac{3^{12}}{80}\right)\theta\left(\sum_{j=1}^n |v_j|^4\right), \tag{84}$$

for all $v_1, v_2, \cdots, v_n \in \mathbb{R}$, but a quartic mapping $Q_4 : \mathbb{R} \longrightarrow \mathbb{R}$ does not exist satisfies

$$|\phi(v) - Q_4(v)| \le \varepsilon|v|^4, \tag{85}$$

for all $v \in \mathbb{R}$, where θ and ε are a constant.

Proof. It is easy to show that ϕ is bounded by $81/80\theta$ on $\mathbb{R}$. If $\sum_{j=1}^n |v_j|^4 \ge 1/3^4$ or 0, then

$$|D\phi(v_1, v_2,\cdots,v_n)| < \left(\frac{n^4 - 20n^3 + 47n^2 - 40n + 540}{6}\right)\frac{81}{80}\theta. \tag{86}$$

Thus, (84) is valid. Next, suppose that

$$0 < \sum_{j=1}^{n} \left|v_j\right|^4 < \frac{1}{3^4}, \tag{87}$$

then, there exists an integer $m > 0$ satisfies

$$\frac{1}{3^{4(m+2)}} \le \sum_{j=1}^{n} \left|v_j\right|^4 < \frac{1}{3^{4(m+1)}}. \tag{88}$$

So that $3^{4m}|v_1| < 1/3^4, 3^{4m}|v_2| < 1/3^4, \cdots, 3^{4m}|v_n| < 1/3^4$ and

$$\left.\begin{array}{l} 3^a v_1, 3^a v_2, \cdots, 3^a v_n \\ \sum_{i=1}^{n}\left(-3^a v_i + \sum_{j=1}^{n} 3^a v_j\right) \\ \sum_{1\le i<j<k<l\le n} \left(3^a\left(v_i+v_j+v_k+v_l\right)\right) \\ \sum_{1\le i<j<k\le n} \left(3^a\left(v_i+v_j+v_k\right)\right) \\ \sum_{1\le i<j\le n} \left(3^a\left(v_i+v_j\right)\right) \\ \sum_{1\le i<j\le n} \left(3^a\left(v_i-v_j\right)\right) \end{array}\right\} \in \left]-1, 1\right[, a = 0, 1, \cdots, m-1. \tag{89}$$

Also, for $a = 0, 1, \cdots, m-1$,

$$D\chi(v_1, v_2, \cdots, v_n) = 0. \tag{90}$$

Next, by inequality (88), we obtain that

$$\begin{aligned} |D\phi(v_1, v_2, \cdots, v_n)| &\le \sum_{a=0}^{\infty} \frac{1}{3^{4a}} |D\chi(3^a v_1, 3^a v_2, \cdots, 3^a v_n)| \le \sum_{a=0}^{\infty} \frac{1}{3^{4a}} \left| \sum_{i=1}^{n} \chi\left(-3^a v_i + \sum_{j=1}^{n} 3^a v_j\right) - (n-8) \sum_{1\le i<j<k<l\le n} \chi\left(3^a\left(v_i+v_j+v_k+v_l\right)\right) \right. \\ &\quad + \left(n^2 - 12n + 28\right) \sum_{1\le i<j<k\le n} \chi\left(3^a\left(v_i+v_j+v_k\right)\right) - \left(\frac{n^3 - 15n^2 + 60n - 68}{2}\right) \sum_{1\le i<j\le n} \chi\left(3^a\left(v_i+v_j\right)\right) - 2 \sum_{1\le i<j\le n} \chi\left(3^a\left(v_i-v_j\right)\right) - \sum_{i=1}^{n} \chi(3^a(3v_i)) \\ &\quad \left. + \left(\frac{n^4 - 17n^3 + 86n^2 - 148n + 558}{6}\right) \sum_{i=1}^{n} \chi(3^a(v_i)) \right| \le \sum_{a=m}^{\infty} \frac{1}{3^{4a}} \left(\frac{n^4 - 20n^3 + 47n^2 - 40n + 540}{6}\right)\theta. \end{aligned} \tag{91}$$

It follows from (88) that

$$|D\phi(v_1, v_2, \cdots, v_n)| \le \left(\frac{n^4 - 20n^3 + 47n^2 - 40n + 540}{6}\right) \frac{3^{12}}{80} \theta \left(\sum_{j=1}^{n} \left|v_j\right|^4\right). \tag{92}$$

Thus, the function ϕ satisfies the inequality (84). Assume on a contrary that there exist a quartic solution $Q_4 : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying (85). For every v in $\mathbb{R}$, since ϕ is continuous and bounded, Q_4 is limited to an open interval of origin and continuous origin.

In the view of Remark 14, Q_4 must be $Q_4(v) = cv^4$, $v \in \mathbb{R}$. So we obtain

$$|\phi(v)| \le (\varepsilon + |c|)|v|^4, v \in \mathbb{R}. \tag{93}$$

Suppose, we can choose $m > 0$ with $m\theta > \varepsilon + |c|$. If $v \in (0, 1/3^{m-1})$, then, $3^a v \in (0, 1)$ for all $a = 0, 1, \cdots, m-1$, we obtain

$$\phi(v) = \sum_{a=0}^{\infty} \frac{\chi(3^a v)}{3^{4a}} \ge \sum_{a=0}^{m-1} \frac{\theta(3^a v)^4}{3^{4a}} = m\theta v^4 > (\varepsilon + |c|)\left|v^4\right|, \tag{94}$$

which contradicts. □

Data Availability

No data were used to support this study.

Additional Points

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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