Research Article

Solution and Stability of Quartic Functional Equations in Modular Spaces by Using Fatou Property

N. Uthirasamy, K. Tamilvanan, Hemant Kumar Nashine, and Reny George

1Department of Mathematics, K.S. Rangasamy College of Technology, Tiruchengode, 637 215 Tamil Nadu, India
2Department of Mathematics, School of Advanced Sciences, Kalasalingam Academy of Research and Education, Srivilliputhur, 626 126 Tamil Nadu, India
3Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, 632014, Vellore, India
4Department of Mathematics and Applied Mathematics, University of Johannesburg, Kingsway Campus, Auckland Park 2006, South Africa
5Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam Bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia

Correspondence should be addressed to Hemant Kumar Nashine; drhknashine@gmail.com

Received 20 February 2022; Accepted 19 April 2022; Published 6 May 2022

Academic Editor: Richard I. Avery

Copyright © 2022 N. Uthirasamy et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We propose a novel generalized quartic functional equation and investigate its Hyers–Ulam stability in modular spaces using a fixed point technique and the Fatou property in this paper.

1. Introduction

The idea of functional equation stability occurs when a functional equation is replaced by an inequality that acts as a perturbation on the equation. Take into account that the topic of functional equation stability was prompted by a query posed by Ulam in 1940 [1], and Hyers response was published in [2]. By considering an unbounded Cauchy difference, Aoki [3] and Rassias [4] extended Hyers’ theorem for additive mappings and linear mappings, respectively.

In [5–8], the authors examined the Hyers–Ulam-Rassias stability findings for functional equations involving many variables. They discussed the approximate solution of the septic functional equation in [9]. They established that this equation is stable in quasi-\( \beta \)-Banach spaces and in \( (\beta, \rho) \)-Banach spaces. Additionally, they established the instability of the preceding radical functional equation in a pertinent example. Khamsi investigated the notion of quasi-contraction mappings in modular function spaces in the absence of the \( \Delta_2 \)-condition, establishing the presence of fixed points and thoroughly analysing their uniqueness in [10].

Kim and Tamilvanan [11] introduced a novel class of quartic functional equations and examined their Hyers–Ulam stability in fuzzy normed spaces using both the direct and fixed point methods. They demonstrated how sums and products of powers of norms may be used to manage the stability of this quartic functional equation. Wongkum et al. [12] studied the extended Ulam–Hyers–Rassias stability of quadratic functional equations extensively using fixed point theory. Their conclusions are achieved in the context of modular spaces whose modules are lower semicontinuous but do not meet any \( \Delta_2 \)-related requirements.

Nakano established the theory of modulars on linear spaces and the accompanying theory of modular linear spaces, which was further refined by Koshi and Shimogaki [13] and Yamamuro [14] and others. Luxemburg [15], Musielak [16], and Turpin [17] and their collaborators contributed the most comprehensive elaboration of these ideas. At the moment, modulars and modular spaces theory is widely employed, most notably in the study of different Orlicz spaces [18] and interpolation theory [19, 20], both of which have several applications [16].
structure of modular function spaces, which, in addition to being Banach spaces (or $F$-spaces in a more generic setting), contain modular equivalents of norm or metric concepts.

Notably, Nakano pioneered modular spaces with relation to order spaces in 1950 [21]. The spaces were produced using the Orlicz spaces theory, which substitutes a particular, integrated nonlinear functionality for the abstract functional that regulates the expansion of space members. They established several features of modular space in [22] and shown that every uniformly composition operator mapping this space onto itself must satisfy the so-called Matkowski’s requirements. In 2014, Sadeghi [23] explored the stability of functional equations in modular space using a fixed point technique.

In [24], the existence and Hyers-Ulam stability of the nearly periodic solution to the fractional differential equation with impulse and fractional Brownian motion were studied under nonlocal circumstances. The research was conducted using the semigroups of operators and Mönch fixed point technique, as well as the basic theory of Hyers-Ulam stability. In [25], the authors examined the existence and Hyers-Ulam stability of random impulsive stochastic functional equations with finite delays. They demonstrated that moderate solutions to the equations exist using Krasnoselskii’s fixed point approach. They then examined the Hyers-Ulam stability results under the Lipschitz condition on a bounded and closed interval. Finally, they demonstrated their findings with an example. The reader is recommended to see [26–28] and the references therein for further information on fixed point theorems in modular spaces.

**Definition 1 (see [29]).** Let $V$ be a vector space over $K$ ($C$ or $R$). A generalized functional $\rho : V \to [0,\infty]$ is called a modular if for arbitrary $u, v \in V$, $\rho$ satisfies

(a) $\rho(u) = 0$ if and only if $u = 0$

(b) $\rho(\beta u) = \rho(u)$ for every scalar $\beta$ with $|\beta| = 1$

(c) $\rho(\beta u + v) \leq \rho(u) + \rho(v)$, whenever $\beta, \gamma \geq 0$ and $\beta + \gamma = 1$

If we replace (c) by,

(c’) $\rho(\beta u + \gamma v) \leq \beta \rho(u) + \gamma \rho(v)$, whenever $\beta, \gamma \geq 0$ and $\beta + \gamma = 1$, then the modular $\rho$ is called convex. A modular $\rho$ defines a corresponding modular space, i.e., the vector space $V_\rho$ given by:

$$V_\rho = \{ u \in V \mid \rho(cu) \to 0 \text{ as } c \to 0 \}. \quad (1)$$

**Definition 2 (see [29]).** If $V_\rho$ is a modular space and the sequence $\{v_i\}$ in $V_\rho$, then

(i) $v_n \xrightarrow{\rho} v$ if $\rho(v_n - v) \to 0$ as $n \to \infty$

(ii) $\{v_n\}$ is known as $\rho$-Cauchy if $\rho(v_l - v_n) \to 0$ as $l, n \to \infty$

(iii) A subset $A \subseteq V_\rho$ is known as $\rho$-complete iff every $\rho$-Cauchy sequence is $\rho$-convergent in $A$

**Definition 3 (see [29]).** Let $V_\rho$ be a modular space and a nonempty subset $A \subseteq V_\rho$. The mapping $J : A \to A$ is referred to as a quasicontraction, if there is $k < 1$ satisfies

$$\rho(Jl - Jm) \leq k \max \{\rho(l - m), \rho(l - Jl), \rho(m - Jm), \rho(m - ml), \rho(m - ml)\}, \quad (2)$$

for any $l, m \in A$.

**Definition 4 (see [29]).** Let $V_\rho$ be a modular space, a nonempty subset $A \subseteq V_\rho$, and a function $J : A \to A$, the $J$ orbit around a point $v$ is

$$O(J) = \{ u, J(t), J^2(u), \ldots \}, \quad (3)$$

the quantity

$$Y_\rho(J) = \sup \{ \rho(p - q) \mid p, q \in O(J) \}, \quad (4)$$

is then related to $J$ and is referred to as the orbital diameter of $J$ at $v$. If $Y_\rho(J) < \infty$, in particular, one says that $J$ has an orbit of $v$ that is limited to $v$.

Fatou property: the $\rho$-modular will have the Fatou property iff $\rho(v) \leq \lim_{m \to \infty} \inf \rho(v_m)$ whenever $\{v_m\} \xrightarrow{\rho} v$. A modular function is stated to fulfill the conditions $\Delta_3$ if there is $k > 0$ which satisfies $\rho(3v) \leq k \rho(v)$, for every $v \in V_\rho$.

In this work, we introduce new generalized quartic functional equation

$$\sum_{1 \leq j < k \leq n} \phi(v_j + v_k + v_l) = (n - 8) \sum_{1 \leq j < k \leq n} \phi(v_j + v_k + v_l)$$
$$- (n^2 - 12n + 28) \sum_{1 \leq j < k \leq n} \phi(v_j + v_k + v_l)$$
$$+ \left( \frac{n^3 - 15n^2 + 60n - 68}{2} \right) \sum_{1 \leq j < k \leq n} \phi(v_j + v_k)$$
$$+ 2 \sum_{1 \leq j < k \leq n} \phi(v_j - v_k) + \sum_{1 \leq j \leq n} (3v_i)$$
$$- \left( \frac{n^4 - 17n^3 + 86n^2 - 148n + 558}{6} \right) \sum_{j = 1}^{n} \phi(v_j). \quad (5)$$

where $n \geq 5$, and investigate Hyers-Ulam stability of this quartic functional equation in modular space by using the fixed point method with the help of Fatou property.

2. Solution

We denote $V$ and $W$ as two real vector spaces.

**Theorem 5.** If an even mapping $\phi : V \to W$ satisfies the functional equation (5) for all $v_1, v_2, \ldots, v_n \in V$, then, the function $\phi$ is quartic.
Proof. In the view of evenness, we obtain \( \phi(-v) = \phi(v) \) for all \( v \) in \( V \). Now, setting \( v_1 = v_2 = \cdots = v_n = 0 \) in equation (5), we have \( \phi(0) = 0 \). Replacing \( (v_1, v_2, \ldots, v_n) \) by \( (0, 0, \ldots, 0) \) in equation (5), we get

\[
\phi(3v) = 3^4\phi(v),
\]

for all \( v \in V \). Replacing \( v \) by \( 3v \) in equation (6), we have

\[
\phi(3^2v) = 3^{4(2)}\phi(v),
\]

for all \( v \in V \). Again, replacing \( v \) by \( 3v \) in equation (7), we obtain

\[
\phi(3^3v) = 3^{4(3)}\phi(v),
\]

for all \( v \in V \). For any nonnegative integer \( n \geq 0 \), we can generalize the result that

\[
\phi(3^n v) = 3^{4n}\phi(v),
\]

for all \( v \in V \). Similarly, we have

\[
\phi(3^{-n} v) = 3^{-4n}\phi(v),
\]

for all \( v \in V \). Next, replacing \( (v_1, v_2, \ldots, v_n) \) by \( (v_1, v_1, v_2, 0, \ldots, 0) \), we obtain

\[
\phi(2v_1 + v_2) + \phi(2v_1 - v_2) = 4\phi(v_1 - v_2) + 4\phi(v_1 + v_2) + 24\phi(v_1) - 6\phi(v_2),
\]

for all \( v_1, v_2 \in V \). Hence, the function \( \phi \) is quartic. \( \square \)

3. Hyers-Ulam Stability

We assume that \( \rho \) is a convex modular on \( \rho \)-complete modular spaces \( W_\rho \), with the Fatou property that meets the \( \Delta_3 \)-condition with \( 0 < k \leq 3 \) in this section. Let \( V \) be a linear space as well. For convenience, we call a function \( \phi : V \rightarrow W_\rho \) by the shorthand \( \phi : V \rightarrow W_\rho \).

\[
D\phi(v_1, v_2, \ldots, v_n) = \sum_{1 \leq i < j \leq n} \phi \left( -v_i + \sum_{j=1 \atop j \neq i}^n v_j \right) - (n - 8) \sum_{1 \leq i < j < k \leq n} \phi \left( v_i + v_j + v_k + v_1 \right) + \left( n^2 - 12n + 28 \right) \sum_{1 \leq i < j < k \leq n} \phi \left( v_i + v_j + v_k \right) - \left( n^3 - 15n^2 + 6n - 68 \right),
\]

for all \( v_1, v_2, \ldots, v_n \in V \).

\[
\sum_{1 \leq i < j \leq n} \phi(v_i + v_j) - 2 \sum_{1 \leq i < j \leq n} \phi(v_i - v_j) - \sum_{i=1}^n \phi(3v_i) + \left( n^4 - 17n^3 + 86n^2 - 148n + 558 \right) \sum_{i=1}^n \phi(v_i) = 0,
\]

for all \( v_1, v_2, \ldots, v_n \in V \).

Theorem 6. Let \( \psi : V^n \rightarrow [0, +\infty) \) be a function such that

\[
\lim_{m \to +\infty} \frac{1}{3^m} \psi (3^m v_1, 3^m v_2, \ldots, 3^m v_n) = 0,
\]

\[
\psi (3v_1, 3v_2, \ldots, 3v_n) \leq \frac{3^4}{(1 - L)^3} \psi (v_1, v_2, \ldots, v_n),
\]

for all \( v_1, v_2, \ldots, v_n \in V \), with \( L < 1 \). If an even mapping \( \phi : V \rightarrow W_\rho \) with \( \phi(0) = 0 \) and such that

\[
\rho(D\phi(v_1, v_2, \ldots, v_n)) \leq C_{0,1} \psi (v_1, v_2, \ldots, v_n),
\]

for all \( v_1, v_2, \ldots, v_n \in V \), then, there exists a unique quartic mapping \( Q_4 : V \rightarrow W_\rho \) satisfying

\[
\rho (Q_4 (v) - \phi (v)) \leq \frac{1}{3^4 (1 - L)^3} \psi (v, 0, \ldots, 0),
\]

for all \( v \in V \).

Proof. We consider the set

\[
\Lambda = \{ \rho : V \rightarrow W_\rho \},
\]

and define the function \( \bar{\rho} \) on \( \Lambda \) as follows:

\[
\bar{\rho}(\rho) = \inf \{ \theta > 0 : \rho(\rho) \leq \theta \psi (v, 0, \ldots, 0), \forall v \in V \}.
\]

Now, we show that \( \bar{\rho} \) is a convex modular on \( \Lambda \). It is easy to verify that \( \bar{\rho} \) satisfies the axioms (a) and (b) of a modular. Next, we will show that \( \bar{\rho} \) is convex, and hence, \( (c') \) is satisfied. Let \( \varepsilon > 0 \) be given, then, there exist real constants \( \theta_1 > 0 \) and \( \theta_2 > 0 \) such that

\[
\bar{\rho}(\rho) \leq \theta_1 \rho(p) + \varepsilon \text{ and } \bar{\rho}(q) \leq \theta_2 \rho(q) + \varepsilon.
\]

Also

\[
\rho (\rho (v)) \leq \theta_2 \psi (v, 0, \ldots, 0), \rho (q (v)) \leq \theta_3 \psi (v, 0, \ldots, 0),
\]

for all \( v \in V \). If \( \beta + \gamma = 1 \) and \( \beta, \gamma \geq 0 \), then, we get

\[
\rho (\beta \rho (v) + \gamma q (v)) \leq \beta \rho (\rho (v)) + \gamma q (q (v)) \leq (\theta, \beta + \theta, \gamma) \psi (v, 0, \ldots, 0).
\]

so we get

\[
\bar{\rho}(\beta \rho + \gamma q) \leq \beta \bar{\rho}(\rho) + \gamma \bar{\rho}(q) + (\beta + \gamma) \varepsilon.
\]
This concludes that \( \tilde{\rho} \) is convex modular on \( \Lambda \). Now, we show that \( \Lambda_{\tilde{\rho}} \) is \( \tilde{\rho} \)-complete.

Let \( \{p_n\} \) be a \( \tilde{\rho} \)-Cauchy sequence in \( \Lambda_{\tilde{\rho}} \) and let \( \varepsilon > 0 \). Then, there exists a positive integer \( n_0 \in \mathbb{N} \) such that

\[
\tilde{\rho}(p_n - p_m) < \varepsilon,
\]
for all \( n, m \geq n_0 \). Then

\[
\rho(p_n(v) - p_m(v)) \leq \varepsilon \psi(v, 0, \ldots, 0),
\]
for all \( v \in V \) and all \( n, m \geq n_0 \). Therefore, \( \{p_n(v)\} \) is a \( \rho \)-Cauchy sequence in \( W_{\rho} \). Since \( W_{\rho} \) is \( \rho \)-complete, so \( \{p_n(v)\} \) is convergent in \( W_{\rho} \) for each \( v \in V \). Hence, we can define a function \( \rho : V \to W_{\rho} \) by

\[
p(v) = \lim_{n \to \infty} p_n(v),
\]
for all \( v \in V \). Since \( \rho \) satisfies the Fatou property, it follows from (25) that

\[
\rho(p_n(v) - p(v)) \leq \lim_{m \to \infty} \inf \rho(p_n(v) - p_m(v)) \leq \varepsilon \psi(v, 0, \ldots, 0),
\]
so

\[
\tilde{\rho}(p_n - p) \leq \varepsilon,
\]
for all \( n \geq n_0 \). Thus, \( \{p_n\} \) is \( \tilde{\rho} \)-converges. Hence, \( \Lambda_{\tilde{\rho}} \) is \( \tilde{\rho} \)-complete.

Next, we show that \( \tilde{\rho} \) satisfies the Fatou property. Suppose that \( \{p_n\} \) is a sequence in \( \Lambda_{\tilde{\rho}} \) which is \( \tilde{\rho} \)-convergent to an element \( p \in \Lambda_{\tilde{\rho}} \).

Let \( \varepsilon > 0 \) be given. For each \( n \in \mathbb{N} \), let \( \theta_n \) be a real constant such that

\[
\tilde{\rho}(p_n) \leq \theta_n \leq \tilde{\rho}(p_n) + \varepsilon.
\]
So

\[
\rho(p_n(v)) \leq \theta_n \psi(v, 0, \ldots, 0),
\]
for all \( v \in V \). Since \( \rho \) satisfies the Fatou property, we get

\[
\rho(p(v)) \leq \lim_{n \to \infty} \inf \rho(p_n(v)) \leq \lim_{n \to \infty} \inf \theta_n \psi(v, 0, \ldots, 0)
\]

\[
\leq \lim_{n \to \infty} \inf \tilde{\rho}(p_n) + \varepsilon \psi(v, 0, \ldots, 0).
\]
Thus, we obtain

\[
\tilde{\rho}(p) \leq \lim_{n \to \infty} \inf \tilde{\rho}(p_n) + \varepsilon.
\]
Hence, \( \tilde{\rho} \) satisfies the Fatou property. Consider the function \( \Psi : \Lambda_{\tilde{\rho}} \to \Lambda_{\tilde{\rho}} \) by

\[
\Psi p(v) = \frac{1}{3^\tau} \rho(3^\tau v),
\]
for all \( v \in V \) and all \( p \in \Lambda_{\tilde{\rho}} \). Let \( \rho, q \in \Lambda_{\tilde{\rho}} \) and let \( \theta \in [0, 1] \) be an arbitrary constant with \( \tilde{\rho}(p - q) < \theta \). From the definition of \( \rho \), we obtain

\[
\rho(p(v) - q(v)) \leq \theta \psi(v, 0, \ldots, 0),
\]
for all \( v \in V \). By inequality (15) and the above inequality, we get

\[
\rho \left( \frac{\rho(3^\tau v) - q(3^\tau v)}{3^\tau} \right) \leq \frac{1}{3^\tau} \rho(p(3^\tau v) - q(3^\tau v)) \leq \frac{1}{3^\tau} \theta \psi(3^\tau v, 0, \ldots, 0) \leq \theta L \psi(v, 0, \ldots, 0),
\]
for all \( v \in V \). Hence,

\[
\tilde{\rho}(\Psi p - \Psi q) \leq \theta \rho(p - q), \forall q \in \Lambda_{\tilde{\rho}},
\]
i.e., \( \Psi \) is a \( \tilde{\rho} \)-contraction. Next, we show that \( \Psi \) has a bounded orbit at \( \Phi \). Replacing \( (v_1, v_2, \ldots, v_n) \) by \( (v, 0, \ldots, 0) \) in (16), we get

\[
\rho(\phi(3^\tau v) - 3^\tau \phi(v)) \leq \psi(3^\tau v, 0, \ldots, 0) \Rightarrow \rho \left( \frac{\phi(3^\tau v)}{3^\tau} - \phi(v) \right) \leq \frac{1}{3^\tau} \psi(3^\tau v, 0, \ldots, 0),
\]
for all \( v \in V \). Replacing \( v \) with \( 3^\tau v \) in (37), we get

\[
\rho \left( \frac{\phi(3^\tau v)}{3^\tau} - \phi(v) \right) \leq \frac{1}{3^\tau} \psi(3^\tau v, 0, \ldots, 0),
\]
for all \( v \in V \). By using (37) and (38), we get

\[
\rho \left( \frac{\phi(3^\tau v)}{3^\tau} - \phi(v) \right) \leq \rho \left( \frac{\phi(3^\tau v)}{3^\tau} - \phi(3^\tau v) + \phi(3^\tau v) - \phi(v) \right)
\]

\[
\leq \rho \left( \frac{\phi(3^\tau v)}{3^\tau} - \phi(3^\tau v) \right) + \rho \left( \phi(3^\tau v) - \phi(v) \right)
\]

\[
\leq \frac{1}{3^\tau} \psi(3^\tau v, 0, \ldots, 0) + \frac{1}{3^\tau} \psi(v, 0, \ldots, 0),
\]
for all \( v \in V \). By induction, we can easily see that

\[
\rho \left( \frac{\phi(3^\tau v)}{3^\tau} - \phi(v) \right) \leq \frac{1}{3^\tau} \psi(3^\tau v, 0, \ldots, 0)
\]

\[
\leq \frac{1}{3^\tau} \psi(v, 0, \ldots, 0),
\]
for all \( v \in V \). It follows from inequality (40) that
\[
\rho\left(\frac{\phi(3^nv)}{3^m} - \frac{\phi(3^nv)}{3^m}\right) \leq \frac{1}{2} \rho\left(\frac{2 \phi(3^nv)}{3^m} - 2\phi(v)\right) + \frac{1}{2} \rho\left(\frac{2 \phi(3^nv)}{3^m} - 2\phi(v)\right)
\]
\[
\leq \frac{k}{2} \rho\left(\frac{\phi(3^nv)}{3^m} - \phi(v)\right) + \frac{k}{2} \rho\left(\frac{\phi(3^nv)}{3^m} - \phi(v)\right)
\]
\[
\leq \frac{k}{2} \frac{1}{3^m} \psi(0, 0, \ldots, 0) + \frac{k}{2} \frac{1}{3^m} \psi(0, 0, \ldots, 0)
\]
\[
\leq \frac{k}{3^m} \psi(0, 0, \ldots, 0),
\]
(41)
for all \( v \in V \) and all \( n, m \in \mathbb{N} \). By the definition of \( \bar{\rho} \), we conclude that
\[
\bar{\rho}(\Psi^n \phi - \Psi^m \phi) \leq \frac{k}{3^m(1 - L)}
\]
(42)
which implies the boundedness of an orbit of \( \Psi \) at \( \phi \). It follows from Theorem 1.5 [29] that the sequence \( \{\Psi^n \phi\} \bar{\rho} \)-converges to \( \bar{Q}_4 \in A_{\bar{\rho}} \). Now, by the \( \bar{\rho} \)-contractivity of \( \Psi \), we have
\[
\bar{\rho}(\Psi^n \phi - \Psi^m \phi) \leq L \bar{\rho}(\Psi^{n-1} \phi - Q_4).
\]
(43)
Passing to the limit \( n \to \infty \) and applying the Fatou property of \( \rho \), we obtain that
\[
\bar{\rho}(\Psi Q_4 - Q_4) \leq \liminf \nrightarrow_{n \to \infty} \bar{\rho}(\Psi^n Q_4 - \Psi^n \phi) \leq \liminf \nrightarrow_{n \to \infty} \bar{\rho}(Q_4 - \Psi^n \phi) = 0.
\]
(44)
Therefore, \( Q_4 \) is a fixed point of \( \Psi \). Replacing \((v_1, v_2, \ldots, v_n)\) by \((3^j v_1, 3^j v_2, \ldots, 3^j v_n)\) in (16), we get
\[
\rho\left(D\phi\left(3^j v_1, 3^j v_2, \ldots, 3^j v_n\right)\right) \leq \psi\left(3^j v_1, 3^j v_2, \ldots, 3^j v_n\right),
\]
(45)
for all \( v_1, v_2, \ldots, v_n \in V \). Therefore
\[
\rho\left(\frac{1}{3^m} D\phi\left(3^j v_1, 3^j v_2, \ldots, 3^j v_n\right)\right) \leq \frac{1}{3^m} \psi\left(3^j v_1, 3^j v_2, \ldots, 3^j v_n\right).
\]
(46)
Employing the limit \( l \to \infty \), we get
\[
DQ_4(v_1, v_2, \ldots, v_n) = 0,
\]
(47)
for all \( v_1, v_2, \ldots, v_n \in V \). It follows from Theorem 2 that \( Q_4 \) is quartic. By using (40), we get (17). To prove the uniqueness of \( Q_4 \), let \( Q_4' : V \to W_{\bar{\rho}} \) be another quartic mapping satisfying (17). Then, \( Q_4' \) is a fixed point of \( \Psi \). Therefore, the function \( Q_4 \) is unique. This completes the proof.

**Corollary 7.** Let a mapping \( \psi : V^n \to [0, +\infty) \) such that
\[
\lim_{n \to \infty} \frac{1}{n^{4/3}} \psi\left(3^n v_1, 3^n v_2, \ldots, 3^n v_n\right) = 0,
\]
(49)
\[
\psi(3^n v_1, 3^n v_2, \ldots, 3^n v_n) \leq L^3 \psi(v_1, v_2, \ldots, v_n),
\]
(50)
for all \( v_1, v_2, \ldots, v_n \in V \) with \( L < 1 \). Suppose that a mapping \( \phi : V \to W \) with \( \phi(0) = 0 \) and such that
\[
\|D\phi(v_1, v_2, \ldots, v_n)\| \leq \psi(v_1, v_2, \ldots, v_n),
\]
(51)
for all \( v_1, v_2, \ldots, v_n \in V \), then, there exists a unique quartic mapping \( Q_4 : V \to W \) satisfying
\[
\|Q_4(v) - \phi(v)\| \leq \frac{1}{3^{4/3}} \psi(v, 0, \ldots, 0),
\]
(52)
for all \( v \in V \).

**Proof.** It is known that every normed space is modular space with the modular \( \rho(v) = \|v\| \) and satisfies the \( \Delta_2 \)-condition with \( k = 3 \).

**Remark 8.** If we replace \( \psi(v_1, v_2, \ldots, v_n) \) by \( \alpha(\sum_{i=1}^n \|v_i\|^p) \) and letting \( L = 3^{p-1} \) in Corollary 7, we obtain the stability results for the sum of norms that
\[
\|Q_4(v) - \phi(v)\| \leq \frac{\alpha\|v\|^p}{(3^4 - 3^p)},
\]
(53)
for all \( v \in V \), where \( \alpha \) and \( p \) are constants with \( p < 4 \).

**Remark 9.** If we replace \( \psi(v_1, v_2, \ldots, v_n) \) by \( \alpha(\sum_{i=1}^n \|v_i\|^p + \prod_{i=1}^n \|v_i\|^p) \) and letting \( L = 3^{np-4} \) in Corollary 7, we obtain the stability results for the sum of product of norms that
\[
\|Q_4(v) - \phi(v)\| \leq \frac{\alpha\|v\|^p}{(3^{4} - 3^{np})},
\]
(54)
for all \( v \in V \), where \( \alpha \) and \( p \) are constants with \( np < 4 \).

**Theorem 10.** Let \( \psi : V^n \to [0, +\infty) \) be a function such that
\[
\lim_{m \to \infty} 3^m \psi\left(\frac{v_1}{3^m}, \frac{v_2}{3^m}, \ldots, \frac{v_n}{3^m}\right) = 0,
\]
(55)
\[
\psi\left(\frac{v_1}{3}, \frac{v_2}{3}, \ldots, \frac{v_n}{3}\right) \leq \frac{L}{3} \psi(v_1, v_2, \ldots, v_n),
\]
(56)
for all \( v_1, v_2, \ldots, v_n \in V \) with \( L < 1 \). Suppose that \( \phi : V \to W_{\rho} \) with \( \phi(0) = 0 \) and satisfies (16), then, there exists a
unique quartic mapping \( Q_\nu : V \rightarrow W_\rho \) satisfying
\[
\rho(Q_\nu(v) - \phi(v)) \leq \frac{L}{3^4(1-L)} \psi(v, 0, \ldots, 0),
\]
for all \( v \in V \).

**Proof.** We consider the set
\[
\Lambda = \{ \rho : V \rightarrow W_\rho \},
\]
and define the function \( \bar{\rho} \) on \( \Lambda \) as follows:
\[
\bar{\rho}(\rho) = \inf \{ \theta > 0 : \rho(p(v)) \leq \theta \psi(v, 0, \ldots, 0), \forall v \in V \}. \quad (59)
\]
Similar to the proof of Theorem 6, we have
(1) \( \bar{\rho} \) is a convex modular on \( \Lambda \)
(2) \( \Lambda_\rho \) is \( \bar{\rho} \)-complete
(3) \( \bar{\rho} \) satisfies the Fatou property

Now, we consider the function \( \Psi : \Lambda_\rho \rightarrow \Lambda_\rho \) defined by
\[
\Psi(\rho) = 3^4 \rho \left( \frac{V}{3} \right),
\]
for all \( v \in V \) and all \( \rho, q \in \Lambda_\rho \). Let \( \rho, q \in \Lambda_\rho \) and let \( \theta \in [0, 1] \) be an arbitrary constant with \( \bar{\rho}(\rho - q) < \theta \). From the definition of \( \bar{\rho} \), we have
\[
\rho(p(v) - q(v)) \leq \theta \psi(v, 0, \ldots, 0),
\]
for all \( v \in V \). By the assumption and the last inequality, we get
\[
\rho(3^4 \rho \left( \frac{V}{3} \right) - q(3^4 \frac{V}{3})) \leq k^4 \rho \left( \frac{V}{3} \right) - q(3^4 \frac{V}{3}) \leq k^4 \theta \psi(v, 0, \ldots, 0), v \in V.
\]
Hence,
\[
\bar{\rho}(\Psi(\rho) - \Psi(q)) \leq L \bar{\rho}(\rho - q), \rho, q \in \Lambda_\rho,
\]
i.e., \( \Psi \) is a \( \bar{\rho} \)-contraction.

Next, we prove then that \( \Psi \) has a bounded orbit at \( \phi \). Replacing \( (v_1, v_2, \ldots, v_n) \) by \( (v, 0, \ldots, 0) \) in (16), we get
\[
\rho(3^4 \phi(v) - \phi(3v)) \leq \psi(v, 0, \ldots, 0),
\]
for all \( v \in V \). Replacing \( v \) with \( v/3 \) in (64), we get
\[
\rho \left( 3^4 \phi \left( \frac{V}{3} \right) - \phi(v) \right) \leq \psi \left( \frac{V}{3}, 0, \ldots, 0 \right),
\]
for all \( v \in V \). Replacing \( v \) with \( v/3 \) in (65), we get
\[
\rho \left( 3^4 \phi \left( \frac{V}{3} \right) - \phi(v) \right) \leq \psi \left( \frac{V}{3}, 0, \ldots, 0 \right),
\]
for all \( v \in V \). Replacing \( v \) with \( v/3 \) in (65), we get
\[
\rho \left( 3^4 \phi \left( \frac{V}{3} \right) - \phi(v) \right) \leq \psi \left( \frac{V}{3}, 0, \ldots, 0 \right),
\]
for all \( v \in V \). By using (64), (65), and (66), we get
\[
\rho \left( 3^4 \phi \left( \frac{V}{3} \right) - \phi(v) \right) \leq \psi \left( \frac{V}{3}, 0, \ldots, 0 \right),
\]
for all \( v \in V \). By induction, we can easily see that
\[
\rho \left( 3^4 \phi \left( \frac{V}{3} \right) - \phi(v) \right) \leq \frac{1}{3} \sum_{i=1}^{n} 3^4 \psi \left( \frac{V}{3^i}, 0, \ldots, 0 \right)
\]
\[
\leq \frac{L}{3^4(1-L)} \psi(v, 0, \ldots, 0),
\]
for all \( v \in V \). It follows from inequality (68) that
\[
\rho \left( 3^{4n} \phi \left( \frac{V}{3^n} \right) - 3^{4n} \phi \left( \frac{V}{3^m} \right) \right) \leq \frac{1}{2} \rho \left( 2 \left( 3^4 \phi \left( \frac{V}{3^n} \right) - 2\phi(v) \right) \right)
\]
\[
+ \frac{1}{2} \rho \left( 2 \left( 3^4 \phi \left( \frac{V}{3^m} \right) - 2\phi(v) \right) \right)
\]
\[
\leq \frac{KL}{3^4(1-L)} \psi(v, 0, \ldots, 0),
\]
for all \( v \in V \) and all \( n, m \in \mathbb{N} \). By the definition of \( \bar{\rho} \), we conclude that
\[
\bar{\rho}(\Psi^n \phi - \Psi^n \phi) \leq \frac{KL}{3^4(1-L)},
\]
which implies the boundedness of an orbit of \( \Psi \) at \( \phi \). It follows from Theorem 1.5 [29] that the sequence \( \{ \Psi^n \phi \} \bar{\rho} \)-converges to \( Q_\nu \in \Lambda_\rho \).

Now, by the \( \bar{\rho} \)-contractivity of \( \Psi \), we have
\[
\bar{\rho}(\Psi^n \phi - Q_\nu) \leq L \bar{\rho}(\Psi^{n-1} \phi - Q_\nu).
\]
Employing the limit \( n \rightarrow \infty \) and applying the Fatou property of \( \bar{\rho} \), we obtain that
\[
\bar{\rho}(\Psi Q_\nu - Q_\nu) \leq \lim_{n \rightarrow \infty} \inf \bar{\rho}(\Psi Q_\nu - \Psi^n \phi) \leq L \lim_{n \rightarrow \infty} \inf \bar{\rho}(Q_\nu - \Psi^{n-1} \phi) = 0.
\]
Therefore, \( Q_\nu \) is a fixed point of \( \Psi \). Replacing \( (v_1, v_2, \ldots, v_n) \)
by \((v_1/3^l, v_2/3^l, \ldots, v_n/3^l)\) in (16), we get
\[
\rho \left( D\Phi \left( 3^{-l}v_1, 3^{-l}v_2, \ldots, 3^{-l}v_n \right) \right) \leq \Phi \left( 3^{-l}v_1, 3^{-l}v_2, \ldots, 3^{-l}v_n \right),
\]
for all \(v_1, v_2, \ldots, v_n \in V\). Therefore
\[
\rho \left( 3^l D\Phi \left( v_1/3^l, v_2/3^l, \ldots, v_n/3^l \right) \right) \leq k^l \Phi \left( v_1/3^l, v_2/3^l, \ldots, v_n/3^l \right).
\]
Passing to the limit \(l \to \infty\), we get
\[
DQ_4(v_1, v_2, \ldots, v_n) = 0,
\]
for all \(v_1, v_2, \ldots, v_n \in V\). It follows from Theorem 2 that \(Q_4\) is quartic. By using (68), we get (57).

In order to prove the uniqueness of \(Q_4\), consider another quartic solution \(Q_4' : V \to W\), that satisfy the inequality (17). Then, \(Q_4'\) is a fixed point of \(\Psi\).
\[
\bar{\rho} \left( Q_4 - Q_4' \right) = \bar{\rho} \left( \Psi Q_4 - \Psi Q_4' \right) \leq L \bar{\rho} \left( Q_4 - Q_4' \right),
\]
which implies that \(\bar{\rho}(Q_4 - Q_4') = 0\) or \(Q_4 = Q_4'\). Hence, the proof is now completed.

**Corollary 11.** Let a mapping \(\Psi : V^n \to [0, +\infty)\) such that
\[
\lim_{l \to \infty} 3^l \Psi \left( \frac{v_1}{3^l}, \frac{v_2}{3^l}, \ldots, \frac{v_n}{3^l} \right) = 0,
\]
\[
\Psi \left( \frac{v_1}{3^l}, \frac{v_2}{3^l}, \ldots, \frac{v_n}{3^l} \right) \leq \frac{L}{3^l} \Psi \left( v_1, v_2, \ldots, v_n \right),
\]
for all \(v_1, v_2, \ldots, v_n \in V\), with \(L < 1\). Suppose that \(\Phi : V \to W\) with \(\Phi(0) = 0\) and satisfies (51), then there exists a unique quartic mapping \(Q_4 : V \to W\) satisfying
\[
\|Q_4(v) - \Phi(v)\| \leq \frac{L}{3^l(l - L)} \Psi(v_1, 0, \ldots, 0),
\]
for all \(v \in V\).

**Proof.** It is known that every normed space is modular space with the modular \(\rho(v) = \|v\|\) and satisfies the \(\Delta_2\)-condition with \(k = 3\). □

**Remark 12.** If we replace \(\Phi(v_1, v_2, \ldots, v_n)\) by \(\alpha(\sum_{j=1}^n |v_j|^p)\) and letting \(L = 3^{1-p}\) in Corollary 11, we obtain the stability results for the sum of norms that
\[
\|Q_4(v) - \Phi(v)\| \leq \frac{\alpha\|v\|^p}{(3^p - 3^p)},
\]
for all \(v \in V\), where \(\alpha\) and \(p\) are constants with \(p > 4\).

### 4. Counterexample

We present a counterexample to show instability of a particular condition of the equality (5) using modified example of Gajda [7].

**Remark 14.** If a function \(\phi : \mathbb{R} \to V\) satisfies the functional equation (5), then, the following assertions hold:

1. \(\phi(q^{1/4}v) = q^\phi(v), q \in Q, k \in Z\) and \(v \in \mathbb{R}\)
2. \(\phi(v) = v^4\phi(1), v \in \mathbb{R}\) if the function \(\phi\) is continuous

**Example 15.** Let a mapping \(\phi : \mathbb{R} \to \mathbb{R}\) be defined as follows:
\[
\phi(v) = \sum_{n=0}^{\infty} \chi \left( \frac{3^n v}{3^4} \right),
\]
where
\[
\chi(v) = \begin{cases} 
\theta v^4, & -1 < v < 1, \\
\theta, & \text{else},
\end{cases}
\]
then, the mapping \(\phi : \mathbb{R} \to \mathbb{R}\) satisfies
\[
|D\Phi(v_1, v_2, \ldots, v_n)| \leq \left( \frac{n^4 - 20n^3 + 47n^2 - 40n + 540}{6} \right) \left( \frac{\theta}{80} \right) \left( \sum_{j=1}^n |v_j|^4 \right),
\]
for all \(v_1, v_2, \ldots, v_n \in \mathbb{R}\), but a quartic mapping \(Q_4 : \mathbb{R} \to \mathbb{R}\) does not exist satisfies
\[
|\psi(v) - Q_4(v)| \leq \epsilon |v|^4,
\]
for all \(v \in \mathbb{R}\), where \(\theta\) and \(\epsilon\) are a constant.

**Proof.** It is easy to show that \(\phi\) is bounded by \(81/80\theta\) on \(\mathbb{R}\). If \(\sum_{j=1}^n |v_j|^4 \geq 1/3^4\) or 0, then
\[
|D\Phi(v_1, v_2, \ldots, v_n)| < \left( \frac{n^4 - 20n^3 + 47n^2 - 40n + 540}{6} \right) \frac{81}{80} \theta.
\]
Thus, (84) is valid. Next, suppose that

\[ 0 < \sum_{j=1}^{n} |v_j|^4 < \frac{1}{3^4}, \quad (87) \]

then, there exists an integer \( m > 0 \) satisfies

\[ \frac{1}{3^{4(m+2)}} \leq \sum_{j=1}^{n} |v_j|^4 < \frac{1}{3^{4(m+1)}}, \quad (88) \]

So that \( 3^{4m}|v_1| < 1/3^4, 3^{4m}|v_2| < 1/3^4, \cdots, 3^{4m}|v_n| < 1/3^4 \) and

\[ 3^4v_1, 3^4v_2, \cdots, 3^4v_n \]

\[ \sum_{i=1}^{n} \left( -3^a v_i + \sum_{j=1}^{n} 3^a v_j \right) \]

\[ \sum_{1 \leq i < j < k \leq n} \left( 3^a (v_i + v_j + v_k) \right) \]

\[ \sum_{1 \leq i < j \leq n} \left( 3^a (v_i + v_j) \right) \]

\[ \sum_{1 \leq i \leq n} \left( 3^a (v_i) \right) \]

Also, for \( a = 0, 1, \cdots, m - 1 \),

\[ D\chi(v_1, v_2, \cdots, v_n) = 0. \quad (90) \]

Next, by inequality (88), we obtain that

\[ |\Phi(v_1, v_2, \cdots, v_n)| \leq \sum_{j=1}^{n} |v_j|^4 \leq \frac{1}{3^{4m}} \sum_{j=1}^{n} |v_j|^4 \leq \frac{1}{3^{4m+1}} \sum_{j=1}^{n} |v_j|^4 \]

\[ + \left( \frac{n^4 - 12n + 28}{6} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \chi(3^a (v_i + v_j + v_k + v_l)) \right) \]

\[ + \left( \frac{n^4 - 17n^2 + 86n - 558}{6} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \chi(3^a (v_i + v_j + v_k + v_l)) \right) \]

\[ \leq \frac{1}{3^{4m+1}} \sum_{j=1}^{n} |v_j|^4 \]

\[ \leq \left( \frac{n^4 - 20n^3 + 47n^2 - 40n + 540}{6} \right)^{\frac{312}{80}} \left( \frac{\sum_{j=1}^{n} |v_j|^4}{80} \right)^{\frac{312}{80}} \quad (92) \]

It follows from (88) that

\[ |\Phi(v_1, v_2, \cdots, v_n)| \leq \left( \frac{n^4 - 20n^3 + 47n^2 - 40n + 540}{6} \right)^{\frac{312}{80}} \left( \frac{\sum_{j=1}^{n} |v_j|^4}{80} \right)^{\frac{312}{80}} \quad (92) \]

Thus, the function \( \Phi \) satisfies the inequality (84). Assume on a contrary that there exist a quartic solution \( Q_4 : \mathbb{R} \to \mathbb{R} \) satisfying (85). For every \( v \) in \( \mathbb{R} \), since \( \Phi \) is continuous and bounded, \( Q_4 \) is limited to an open interval of origin and continuous origin.

In the view of Remark 14, \( Q_4 \) must be \( Q_4(v) = cv^4, v \in \mathbb{R} \). So we obtain

\[ |\Phi(v)| \leq (\varepsilon + |c|)|v|^4, v \in \mathbb{R}. \quad (93) \]

Suppose, we can choose \( m > 0 \) with \( m\theta > \varepsilon + |c| \). If \( v \in (0, 1/3^{m-1}) \), then, \( 3^a v \in (0, 1) \) for all \( a = 0, 1, \cdots, m - 1 \), we obtain

\[ \varepsilon + |c| > (\varepsilon + |c|)|v|^4. \quad (94) \]

which contradicts. \( \square \)

**Data Availability**

No data were used to support this study.
Additional Points

Rights and Permissions. Open Access. This article is distributed under the terms of the Creative Commons Attribution.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors’ Contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

Acknowledgments

The authors like to express their gratitude to referees for their suggestions. The authors are thankful to the Deanship of Scientific Research at Prince Sattam bin Abdulaziz University, Al-Kharj, Kingdom of Saudi Arabia, for supporting this research.

References