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Research Article

Finite Time Stability of 2D Fractional Hyperbolic System with Time Delay

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In this work, a class of two-dimensional fractional hyperbolic differential linear system (2D-FHDLS) with time delay is investigated. By using generalized Gronwall's inequality, sufficient conditions for the finite time stability (FTS) of two-dimensional fractional hyperbolic differential system with time delay are given. Numerical examples are also given to illustrate the stability result.

1. Introduction

Since twenty years, the area of fractional calculus has gained much attentions by the researchers, and numerous works have been published in this context. In fact, in [1], for a magnetic resonance imaging, a robust corner detection is developed. Authors have made a comparative experiment between the proposed methods and integer-order one. Furthermore, for the Hilfer stochastic delay fractional differential equations with the Poisson jumps, authors in [2] have analyzed the averaging principle. The author in [3] introduced a new approach for solving diffusive systems governed by the Caputo operator. Also, in [4], a hyperchaotic economic system was studied using fractional differential operator. Bayrak et al. in [5] established a novel approach for solving diffusive problems with conformable derivative. A new extension of the Hermite-Hadamard inequalities via generalized fractional integral has been given in [6]. Nagy and Ben Makhlouf in [7] studied the finite time stability of the linear Caputo-Katugampola fractional time delay systems.

Fractional differential equations have recently proved to be valuable tools in the modeling of many phenomena in different domain applications, whether in control theory, diffusion [5], viscoelasticity [8], or biology [9–11]. For example, in regard to the biology field, the pandemic transmission model of fractional-order COVID-19 type has been studied numerically by Higazy et al. in [9]. Regarding control theory field, a new adaptive surface control method based on fractional calculus is developed by Zouari et al. in [12]. It was found that all the variables, errors, and signals are practical finite time stability with an asymptotic convergence to zero of the tracking errors. Also, a regional observability for linear fractional systems has been studied in [13]. In [14], Xu et al. proved a global asymptotic stability for fractional neural networks with multiple time varying delay. A finite time stability for a class of fractional fuzzy neural networks with delay has been described and studied in [15]. In addition, authors in [16] have studied the FTS for fractional-order time delay systems.

For some basic results in the theory of fractional partial differential equations, the reader is referred to many various works. For example, for a perturbed partial fractional-order differential equations with finite delay, the Darboux problem is proposed by Abbas and Benchohra in [17]. A nonlinear fractional optimal control problem has been solved by generalized Bernoulli polynomials [18]. Wang and Zhang in

[19] studied a Lyapunov inequality for PDE with mixed Caputo derivative. Also, Benchohra and Hellal in [20] proved a global uniqueness results for fractional partial hyperbolic differential equations with delay.

Motivated by the above interpretations, the main objective of this paper is to study the FTS for the linear Darboux fractional partial differential equations with delay or simply, as mentioned above, the 2D-FHDLS with delay. In fact, we were able to establish a new result for the FTS of Caputo 2D-FHDLS with delay. Indeed, thanks to the generalized Gronwall's inequality, we have determined sufficient conditions for the FTS of the 2D-FHDLS with delay. Recall that in [21], we have proved a similar result, but using a fixed point approach. By comparing the two methods, we have shown by numerical tests that the generalized Gronwall's inequality method gives a wider stability interval than that given by fixed point method which proves that generalized Gronwall's inequality method gives very satisfactory stability results.

The paper is organized as follows. In Section 2, some preliminaries are given. In Section 3, FTS results are presented. In Section 4, some numerical examples which show the efficiency of the results were presented.

2. Basic Results

Definition 1 (see [22]). The Riemann-Liouville Fractional (RLF) integral of order $\gamma = (\gamma_1, \gamma_2)$ of w is defined by

$$I_c^{\gamma} w(\xi,\zeta) = [\Gamma(\gamma_1)\Gamma(\gamma_2)]^{-1}\int_{c_1}^{\xi}\int_{c_2}^{\zeta}(\xi-s)^{\gamma_1-1}(\zeta-t)^{\gamma_2-1}w(s,t)dtds, \tag{1}$$

where $c=(c_1,c_2)\in\mathbb{R}^2$, γ_1,γ_2 are strictly positive, and $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2 (see [22]). The RLF derivative of order $\gamma = (\gamma_1, \gamma_2)$ of w is defined by

$$\begin{aligned} D_c^{\gamma} w(\xi,\zeta) = D^2_{\xi,\zeta} I_c^{1-\gamma} w(\xi,\zeta), &= [\Gamma(1-\gamma_1)\Gamma(1-\gamma_2)]^{-1} \\ &\cdot D^2_{\xi,\zeta}\int_{c_1}^{\xi}\int_{c_2}^{\zeta}(\xi-s)^{-\gamma_1}(\zeta-t)^{-\gamma_2}w(s,t)\,dtds, \end{aligned} \tag{2}$$

where $c=(c_1,c_2)\in\mathbb{R}^2$, $(\gamma_1,\gamma_2)\in(0,1)^2$, and $D^2_{\xi,\zeta}=\partial^2/\partial\xi\partial\zeta$.

Definition 3 (see [22]). The Caputo fractional derivative (CFD) of order $\gamma = (\gamma_1, \gamma_2)$ of w is defined by

$$\begin{aligned} {}^{C}D_c^{\gamma} w(\xi,\zeta) &= D_c^{\gamma}[w(\xi,\zeta)-w(\xi,c_2)-w(c_1,\zeta)+w(c_1,c_2)], \\ &= [\Gamma(1-\gamma_1)\Gamma(1-\gamma_2)]^{-1} D^2_{\xi,\zeta}\int_{c_1}^{\xi}\int_{c_2}^{\zeta}(\xi-s)^{-\gamma_1}(\zeta-t)^{-\gamma_2} \\ &\quad\times[w(s,t)-w(s,c_2)-w(c_1,t)+w(c_1,c_2)]dtds, \end{aligned} \tag{3}$$

where $c=(c_1,c_2)\in\mathbb{R}^2$, $(\gamma_1,\gamma_2)\in(0,1)^2$, and $D^2_{\xi,\zeta}=\partial^2/\partial\xi\partial\zeta$.

Definition 4 (see [23]). Let $m\in\mathbb{N}$ and $m\neq 0$, $\alpha_j,\beta_j,z,\rho\in\mathbb{C}$, such that $\mathrm{Re}\,(\alpha_j), \mathrm{Re}\,(\beta_j)>0$ for $j=1,2,\cdots,m$. The generalized Mittag-Leffler function (MLF) is defined by

$$\mathbb{E}_{\rho}\left(\left(\alpha_j,\beta_j\right)_{j=1,m};(z)\right)=\sum_{k=0}^{+\infty}\frac{(\rho)_k}{\prod_{j=1}^{m}\Gamma\left(k\alpha_j+\beta_j\right)}\frac{z^k}{k!}, \tag{4}$$

where

$$(\rho)_k=\rho(\rho+1)\cdots(\rho+k-1)=\frac{\Gamma(\rho+k)}{\Gamma(\rho)}. \tag{5}$$

If $m=2$ and $\rho=1$, we get

$$\begin{aligned} \mathbb{E}_{\rho}\left(\left(\alpha_j,\beta_j\right)_{j=1,2};(z)\right) &= \mathbb{E}\left(\left(\alpha_j,\beta_j\right)_{j=1,2};(z)\right), \\ &= \sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(k\alpha_1+\beta_1)\Gamma(k\alpha_2+\beta_2)}. \end{aligned} \tag{6}$$

Lemma 5 (see [24]). *Let $\tilde{u},\tilde{v}$ be two integrable function and $\tilde{g}$ be a continuous function with domain J. Assume that*

(1) $\tilde{u}$ and $\tilde{v}$ are nonnegative

(2) $\tilde{v}$ is nondecreasing in each of its variables

(3) $\tilde{g}$ is nonnegative and nondecreasing in each of its variables

If

$$\tilde{u}(r,\mu)\le\tilde{v}(r,\mu)+\tilde{g}(r,\mu)\int_{a_1^+}^{r}\int_{a_2^+}^{\mu}(r-\zeta)^{\alpha_1-1}(\mu-\xi)^{\alpha_2-1}\tilde{u}(\zeta,\xi)d\xi d\zeta. \tag{7}$$

Then

$$\begin{aligned} \tilde{u}(r,\mu)\le\tilde{v}(r,\mu)\mathbb{E}((\alpha_1,1),(\alpha_2,1);\tilde{g}(r,\mu)\Gamma(\alpha_1)\Gamma(\alpha_2) \\ \cdot(r-a_1)^{\alpha_1}(\mu-a_2)^{\alpha_2}). \end{aligned} \tag{8}$$

3. Main Result

In this paper, we are interested on the study of the initial value fractional-order linear system defined on the bounded domain $\mathcal{Q}=[0,\mathcal{T}]\times[0,\mathbb{T}]$ as follows:

$${}^{C}D_0^{\nu}\vartheta(\xi,\zeta)=G\vartheta(\xi,\zeta)+K\vartheta(\xi-e_1(\xi),\zeta-e_2(\zeta))+L\delta(\xi,\zeta), \tag{9}$$

for all $(\xi,\zeta)\in\mathcal{Q}$. The initial condition

$$\vartheta(\xi,\zeta)=\psi(\xi,\zeta),\text{ for all }(\xi,\zeta)\in\mathcal{U}, \tag{10}$$

where ${}^{C}D_0^{\nu}(\cdot)$ is the CFD of order $\nu=(\nu_1,\nu_2)$, $0<\nu_1,\nu_2<1$. The functions e_1,e_2 are positive and continuous on $[0,\mathcal{T}]$ and $[0,\mathbb{T}]$, respectively. The matrices $G,K\in\mathbb{R}^{n\times n}$ and $L\in$

$\mathbb{R}^{n\times p}$ and the function $\psi \in C(\mathscr{U}, \mathbb{R}^n)$. Here, the domain $\mathscr{U}$ is given by

$$\mathscr{U} = ([-m_1, \mathscr{T}] \times [-m_2, \mathbb{T}]) \setminus ((0, \mathscr{T}] \times (0, \mathbb{T}]), \tag{11}$$

where the constants m_1, m_2 are given by

$$\begin{aligned} m_1 &= \max_{t\in[0,\mathscr{T}]} (e_1(t)), \\ m_2 &= \max_{t\in[0,\mathbb{T}]} (e_2(t)). \end{aligned} \tag{12}$$

The function δ is a perturbation. We assume that the function $\delta \in C(\mathbb{R}_+^2, \mathbb{R}^p)$ and satisfies

$$\exists \rho > 0 : \delta^T(\xi, \zeta)\delta(\xi, \zeta) \le \rho^2. \tag{13}$$

Let us introduce the following constants a, b, c which are defined by

$$\begin{aligned} a &= \|G\|, \\ b &= \|K\|, \\ c &= \|L\|. \end{aligned} \tag{14}$$

Definition 6. Let $\varepsilon > 0$ and $\gamma > 0$ such that $\varepsilon < \gamma$. System (9) is robustly FTS with respect to $\{\varepsilon, \gamma, \rho, \mathscr{T}, \mathbb{T}\}$, if the following relation is satisfied:

$$\|\psi\| \le \varepsilon \Rightarrow \|\vartheta(\xi, \zeta)\| \le \gamma, \forall(\xi, \zeta) \in \mathcal{Q}, \tag{15}$$

for all perturbation δ satisfying equation (9) and condition (13).

Recall that the solution of system (9) is defined by

$$\vartheta(\xi,\zeta) = \begin{cases} \psi(\xi,\zeta), \forall(\xi,\zeta) \in \mathscr{U}, \\ \rho(\xi,\zeta) + \dfrac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \displaystyle\int_0^\xi\int_0^\zeta (\xi-u)^{\nu_1-1}(\zeta-v)^{\nu_2-1} \times \Psi(u,v)\, dv du, \forall(\xi,\zeta) \in \mathcal{Q}, \end{cases} \tag{16}$$

where the functions Ψ, ρ are defined by

$$\rho(\xi, \zeta) = \psi(\xi, 0) + \psi(0, \zeta) - \psi(0, 0), \tag{17}$$

$$\Psi(u, v) = G\vartheta(u, v) + K\vartheta(u - e_1(u), v - e_2(v)) + L\delta(u, v). \tag{18}$$

The main result in this work is as follows.

Theorem 7. *System (9) is FTS with respect to $\{\varepsilon, \gamma, \rho, \mathscr{T}, \mathbb{T}\}$, $\varepsilon < \gamma$, if the following inequality holds:*

$$\left(3\varepsilon + c\rho \frac{\mathscr{T}^{\nu_1}\mathbb{T}^{\nu_2}}{\Gamma(\nu_1+1)\Gamma(\nu_2+1)}\right) E_{(\nu_1,1);(\nu_2,1)}((a+b)\mathscr{T}^{\nu_1}\mathbb{T}^{\nu_2}) \le \gamma, \tag{19}$$

where $E_{(\cdot,\cdot);(\cdot,\cdot)}$ is the generalized MLF given in equation (9) and the constants ρ, a, b, c are given in (13) and (14).

Proof. The solution of system (9) is given by relation (16). Then, we can deduce the following estimation: for all $(\xi, \zeta) \in \mathcal{Q}$

$$\begin{aligned} \|\vartheta(\xi,\zeta)\| \le{}& \|\rho(\xi,\zeta)\| + \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \\ &\cdot \int_0^\xi\int_0^\zeta (\xi-u)^{\nu_1-1}(\zeta-v)^{\nu_2-1}\|\Psi(u,v)\|\, dv du \le 3\|\psi\| \\ &+ \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_0^\xi\int_0^\zeta (\xi-u)^{\nu_1-1}(\zeta-v)^{\nu_2-1} \\ &\times [a\|\vartheta(u,v)\| + b\|\vartheta(u-e_1(u), v-e_2(v))\| \\ &+ c\|\delta(u,v)\|]\, dv du \le \left[3\|\psi\| + c\rho \frac{\mathscr{T}^{\nu_1}\mathbb{T}^{\nu_2}}{\Gamma(\nu_1+1)\Gamma(\nu_2+1)}\right] \\ &+ \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_0^\xi\int_0^\zeta (\xi-u)^{\nu_1-1}(\zeta-v)^{\nu_2-1} \\ &\times [a\|\vartheta(u,v)\| + b\|\vartheta(u-e_1(u), v-e_2(v))\|]\, dv du. \end{aligned} \tag{20}$$

Let us consider the function $y(\xi, \zeta)$ defined on the extended bounded domain $J_{\xi\zeta} = [-m_1, \xi] \times [-m_2, \zeta]$ as follows:

$$y(\xi, \zeta) = \sup_{(r,\mu)\in J_{\xi\zeta}} \|\vartheta(r, \mu)\|. \tag{21}$$

We have, for all $(r, \mu) \in [0, \xi] \times [0, \zeta]$, the following estimations:

$$\begin{aligned} \|\vartheta(r, \mu)\| &\le y(r, \mu), \\ \|\vartheta(r - e_1(r), \mu - e_2(\mu))\| &\le y(r, \mu). \end{aligned} \tag{22}$$

Then, for all $(\xi, \zeta) \in \mathcal{Q}$, we obtain

$$\|\vartheta(\xi,\zeta)\| \le \mathscr{M} + \frac{a+b}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_0^\xi\int_0^\zeta (\xi-u)^{\nu_1-1}(\zeta-v)^{\nu_2-1} y(u,v)\, dv du, \tag{23}$$

where the constant $\mathscr{M}$ is given by

$$\mathscr{M} = 3\|\psi\| + c\rho \frac{\mathscr{T}^{\nu_1}\mathbb{T}^{\nu_2}}{\Gamma(\nu_1+1)\Gamma(\nu_2+1)}. \tag{24}$$

Let us notice that the function

$$(\xi,\zeta) \mapsto \int_0^\xi\int_0^\zeta (\xi-u)^{\nu_1-1}(\zeta-v)^{\nu_2-1} y(u,v)\, dv du \tag{25}$$

is nondecreasing with respect to each of its variables, because $y(u, v)$ is nondecreasing with respect to each of its variables. Then, for all $(r,\mu) \in [0,\xi] \times [0,\zeta]$:

$$\|\vartheta(r,\mu)\| \le \mathscr{M} + \frac{a+b}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_0^\xi \int_0^\zeta (\xi-u)^{\nu_1-1}(\zeta-v)^{\nu_2-1} y(u,v)\,dvdu. \tag{26}$$

Then, we get

$$\begin{aligned} y(\xi,\zeta) &= \sup_{(r,\mu)\in J_{\xi\zeta}} \|\vartheta(r,\mu)\|, \\ &\le \max\left\{ \sup_{(r,\mu)\in J_{\xi\zeta}\setminus(0,\xi]\times(0,\zeta]} \|\vartheta(r,\mu)\|, \sup_{(r,\mu)\in[0,\xi]\times[0,\zeta]} \|\vartheta(r,\mu)\| \right\} \\ &\le \max\left\{ \|\psi\|, \mathscr{M} + \frac{a+b}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_0^\xi \int_0^\zeta (\xi-u)^{\nu_1-1}(\zeta-v)^{\nu_2-1} y(u,v)\,dvdu \right\} \\ &\le \mathscr{M} + \frac{a+b}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_0^\xi \int_0^\zeta (\xi-u)^{\nu_1-1}(\zeta-v)^{\nu_2-1} y(u,v)\,dvdu \Big\}. \end{aligned} \tag{27}$$

Now, using the generalized Gronwall inequality, we get

$$\begin{aligned} y(\xi,\zeta) &\le \mathscr{M} E_{(\nu_1,1);(\nu_2,1)}\left((a+b)\xi^{\nu_1}\zeta^{\nu_2}\right) \\ &\le \left(3\|\psi\| + c\rho \frac{\mathscr{T}^{\nu_1}\mathbb{T}^{\nu_2}}{\Gamma(\nu_1+1)\Gamma(\nu_2+1)}\right) \\ &\quad \cdot E_{(\nu_1,1);(\nu_2,1)}\left((a+b)\mathscr{T}^{\nu_1}\mathbb{T}^{\nu_2}\right) \le \gamma. \end{aligned} \tag{28}$$

for all $(\xi,\zeta) \in \mathscr{Q}$. The proof is completed. □

Remark 8. Note that a similar result, to that given in Theorem 7, has been proved in ([21], Theorem 2) by a fixed point method.

4. Numerical Scheme

From relation (16), we have

$$\vartheta(\xi,\zeta) = \rho(\xi,\zeta) + \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_0^\xi \int_0^\zeta (\xi-u)^{\nu_1-1}(\zeta-v)^{\nu_2-1} \Psi(u,v)\,dvdu, \tag{29}$$

for all $(\xi,\zeta) \in [0,\mathscr{T}] \times [0,\mathbb{T}]$, where the state ϑ is the solution of system (9), and the functions Ψ, ρ are given by relations (17) and (18). In this section, we study system (9) where $\vartheta, \rho, \Psi \in \mathbb{R}^2$. Then, let us assume that the solution ϑ is of the following form:

$$\vartheta(\xi,\zeta) = (\vartheta_1(\xi,\zeta), \vartheta_2(\xi,\zeta))^T \in \mathbb{R}^2. \tag{30}$$

In this section, we use the same techniques of discretization and approximations that we have already used for the numerical resolution of the nonlinear problem in [21]. Thus, we build an uniform grid on the domain $[-m_1,\mathscr{T}] \times [-m_2,\mathbb{T}]$. Let $r, h \in \mathbb{R}$ and $\mathscr{K}, \mathscr{J}, n, m \in \mathbb{N}$ such that

$$\begin{aligned} r &= \frac{\mathscr{T}}{\mathscr{K}} = \frac{m_1}{n}, \\ h &= \frac{\mathbb{T}}{\mathscr{J}} = \frac{m_2}{m}. \end{aligned} \tag{31}$$

Then, we introduce two sequences $(\xi_i)_i$ and $(\zeta_j)_j$ defined by

$$\begin{aligned} \xi_i &= ir, \forall i = -n, -n+1, -n+2.\cdots, -1, 0, \cdots, \mathscr{K}, \\ \zeta_j &= jh, \forall j = -m, -m+1, -m+2.\cdots, -1, 0, \cdots, \mathscr{J}. \end{aligned} \tag{32}$$

So, the state ϑ can be expressed at the point (ξ_i,ζ_j) as follows:

$$\begin{aligned} \vartheta(\xi_i,\zeta_j) &= \rho(\xi_i,\zeta_j) + \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \\ &\quad \cdot \int_0^{\xi_i} \int_0^{\zeta_j} (\xi_i-u)^{\nu_1-1}(\zeta_j-v)^{\nu_2-1} \Psi(u,v)\,dvdu, \end{aligned} \tag{33}$$

where $\rho(\xi_i,\zeta_j) = \psi(0,\zeta_j) + \psi(\xi_i,0) - \psi(0,0)$. By considering the following approximations,

$$\begin{aligned} &\vartheta(\xi_i,\zeta_j) \approx \vartheta_{ij}, \rho(\xi_i,\zeta_j) \approx \rho_{ij}, \\ &\psi(0,\zeta_j) \approx \psi_{0j}, \psi(\xi_i,0) \approx \psi_{i0}, \psi(0,0) \approx \psi_{00}, \end{aligned} \tag{34}$$

we can rewrite equation (33) as follows:

$$\vartheta_{ij} = \rho_{ij} + \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_0^{\xi_i} \int_0^{\zeta_j} (\xi_i-u)^{\nu_1-1}(\zeta_j-v)^{\nu_2-1} \Psi(u,v)\,dvdu. \tag{35}$$

Then, we deduce that

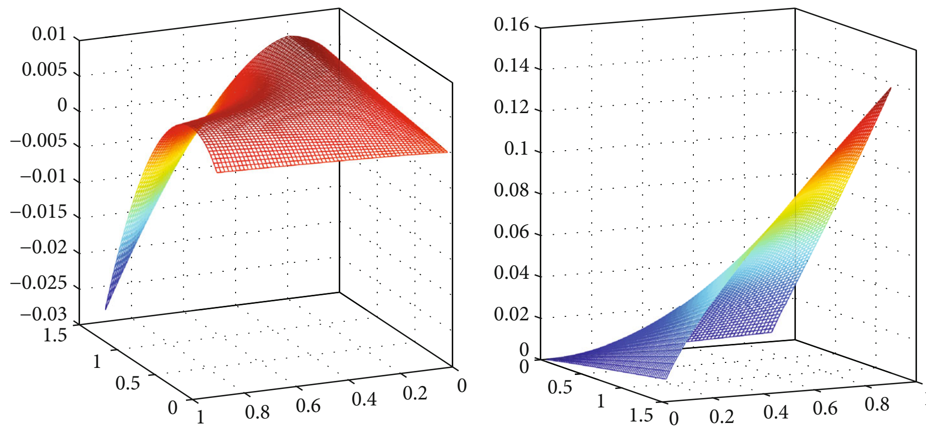
$$\begin{aligned} \vartheta_{0j} &= \rho_{0j} = \psi_{0j}, \\ \vartheta_{i0} &= \rho_{i0} = \psi_{i0}. \end{aligned} \tag{36}$$

By using the properties of integration, we can rewrite equation (35) as follows:

$$\vartheta_{ij} = \rho_{ij} + \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \sum_{k=0}^{i-1} \sum_{l=0}^{j-1} \int_{\xi_k}^{\xi_{k+1}} \int_{\zeta_l}^{\zeta_{l+1}} (\xi_i-u)^{\nu_1-1}(\zeta_j-v)^{\nu_2-1} \Psi(u,v)\,dvdu. \tag{37}$$

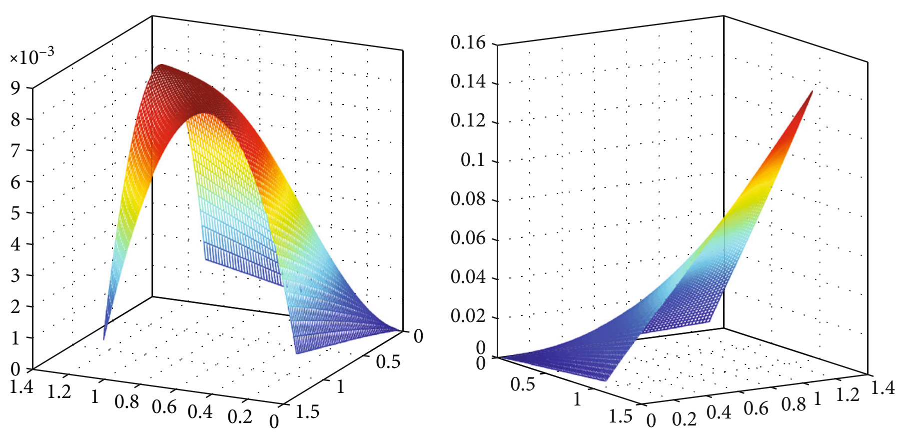
Now, using approximation proposed in [25], we obtain

$$\begin{aligned} \vartheta_{ij} &= \rho_{ij} + \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \sum_{k=0}^{i-1} \sum_{l=0}^{j-1} \int_{\xi_k}^{\xi_{k+1}} \int_{\zeta_l}^{\zeta_{l+1}} (\xi_i-u)^{\nu_1-1}(\zeta_j-v)^{\nu_2-1} \Psi(\xi_k,\zeta_l)\,dvdu \\ &= \rho_{ij} + \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \sum_{k=0}^{i-1} \sum_{l=0}^{j-1} \Psi^{kl} \int_{\xi_k}^{\xi_{k+1}} \int_{\zeta_l}^{\zeta_{l+1}} (\xi_i-u)^{\nu_1-1}(\zeta_j-v)^{\nu_2-1}\,dvdu, \end{aligned} \tag{38}$$



(a) (b)

Figure 1: The solution $\vartheta = (\vartheta_1, \vartheta_2)^T$, $\mathscr{T} = 1.5$, $\mathbb{T} = 0.9$, and $\nu = (0.9, 0.7)$. The curves of ϑ_1 and ϑ_2 are represented in (a) and (b), respectively. The norm $\|\vartheta\| = 3.473$.



(a) (b)

Figure 2: The solution $\vartheta = (\vartheta_1, \vartheta_2)^T$, $\mathscr{T} = 1.092$, $\mathbb{T} = 1.3$, and $\nu = (0.9, 0.9)$. The curves of ϑ_1 and ϑ_2 are represented in (a) and (b), respectively. The norm $\|\vartheta\| = 3.1271$.

where we have the approximation $\Psi(\xi_k, \zeta_l) \approx \Psi^{kl}$ and

$$\Psi(\xi_k, \zeta_l) = G\vartheta(\xi_k, \zeta_l) + K\vartheta(\xi_k - m_1, \zeta_l - m_2) + L\delta(\xi_k, \zeta_l), \tag{39}$$

and the term

$$\begin{aligned}\vartheta(\xi_k - m_1, \zeta_l - m_2) &= \vartheta(kr - nr, lh - mh),\\ &= \vartheta((k-n)r, (l-m)h),\\ &= \vartheta(\xi_{k-n}, \zeta_{l-m}) \approx \vartheta_{k-n,l-m}.\end{aligned} \tag{40}$$

So, we deduce that

$$\Psi^{kl} = G\vartheta_{kl} + K\vartheta_{k-n,l-m} + L\delta_{kl}. \tag{41}$$

Calculating and simplifying the integral in equation (38) gives the following expression:

$$\vartheta_{ij} = \rho_{ij} + \frac{r^{\nu_1} h^{\nu_2}}{\Gamma(\nu_1 + 1)\Gamma(\nu_2 + 1)} \sum_{k=0}^{i-1} \sum_{l=0}^{j-1} \sigma_{ik} \tau_{lj} \Psi^{kl}, \tag{42}$$

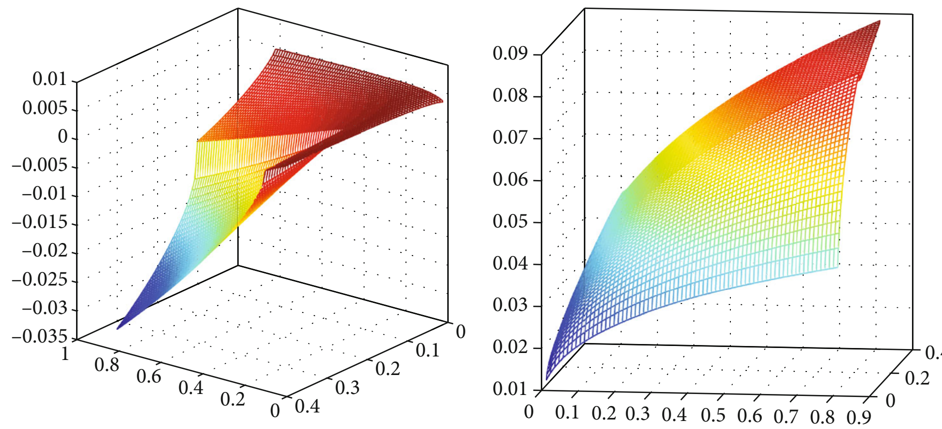
where σ_{ik}, τ_{lj} are given by

$$\begin{aligned}\sigma_{ik} &= (i-k-1)^{\nu_1} - (i-k)^{\nu_1},\\ \tau_{lj} &= (j-l-1)^{\nu_2} - (j-l)^{\nu_2}.\end{aligned} \tag{43}$$

Remark 9. More details about the convergence, the consistency, and behavior of the error $\|\vartheta(\xi_i, \zeta_j) - \vartheta_{ij}\|$ with respect to r, h, ν_1, ν_2 can be deduced from [25].

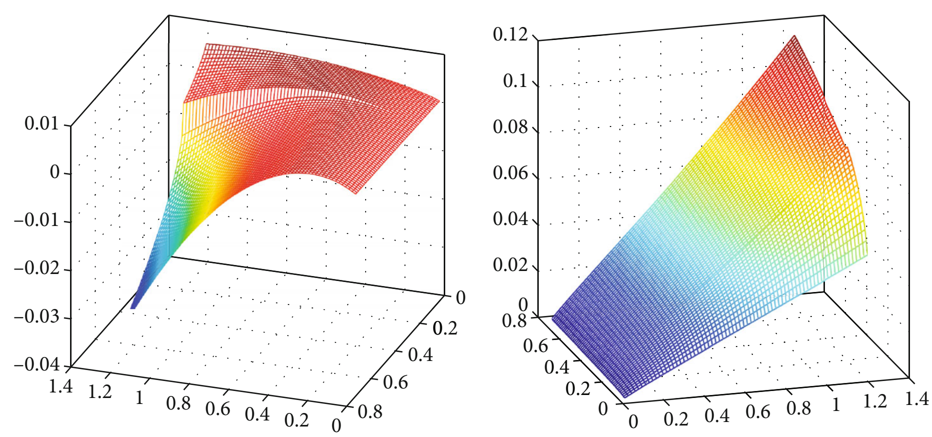
5. Numerical Simulation and Interpretation

In this section, we propose some numerical examples. With the help of these examples, we show that the solution of system (9) is consistent with Definition 6. Indeed, for any $\varepsilon > 0$,



(a) (b)

Figure 3: The solution $\vartheta = (\vartheta_1, \vartheta_2)^T$, $\mathcal{T} = 0.81$, $\mathbb{T} = 0.4$, and $\nu = (0.3{,}0.2)$. The curves of ϑ_1 and ϑ_2 are represented in (a) and (b), respectively. The norm $\|\vartheta\| = 3.8669$.



(a) (b)

Figure 4: The solution $\vartheta = (\vartheta_1, \vartheta_2)^T$, $\mathcal{T} = 1.2$, $\mathbb{T} = 0.7$, and $\nu = (0.9{,}0.2)$. The curves of ϑ_1 and ϑ_2 are represented in (a) and (b), respectively. The norm $\|\vartheta\| = 3.7642$.

$\gamma > 0$ such that $\varepsilon < \gamma$, we can verify that

$$\text{if } \|\psi\| \leq \varepsilon \text{ then } \|\vartheta(\xi,\zeta)\| \leq \gamma, \forall (\xi,\zeta) \in [0,\mathcal{T}] \times [0,\mathbb{T}]. \tag{44}$$

System (9) is defined by

$${}^{C}D_{0}^{\nu}\vartheta(\xi,\zeta) = G\vartheta(\xi,\zeta) + K\vartheta(\xi - m_1, \zeta - m_2) + L\delta(\xi,\zeta), \tag{45}$$

for all $(\xi,\zeta) \in [0,\mathcal{T}] \times [0,\mathbb{T}]$, $\nu = (\nu_1, \nu_2)$. The initial condition is given by

$$\vartheta(\xi,\zeta) = \psi(\xi,\zeta), \forall (\xi,\zeta) \in [-m_1, 0] \times [-m_2, 0]. \tag{46}$$

Recall that we have denoted the solution ϑ of system (45) as follows:

$$\vartheta(\xi,\zeta) = (\vartheta_1(\xi,\zeta), \vartheta_2(\xi,\zeta))^T, \forall (\xi,\zeta) \in [0,\mathcal{T}] \times [0,\mathbb{T}]. \tag{47}$$

Remark 10. The following examples are established under the condition where the parameters $\varepsilon, \gamma, \rho, \mathcal{T}, \mathbb{T}$ satisfy estimation (19).

We have chosen the following data:

$$\begin{aligned} G &= \begin{pmatrix} -0.3 & -0.5 \\ 0.2 & 0.4 \end{pmatrix}, \\ K &= \begin{pmatrix} 0.1 & -0.5 \\ -0.3 & -0.1 \end{pmatrix}, \\ L &= \begin{pmatrix} 0.3 & 1 \\ 2 & -0.5 \end{pmatrix}, \\ \delta(t,s) &= (0.04{,}0.01)^T. \end{aligned} \tag{48}$$

The initial condition

$$\psi(\xi,\zeta) = (0.002 \sin(11\pi\xi\zeta), 0)^T, \forall (\xi,\zeta) \in [-0.1{,}0] \times [-0.2{,}0], \tag{49}$$

Table 1: Variation of $\mathcal{T}$, $\mathbb{T}$, and $\|\vartheta\|$ versus variation of $\nu = (\nu_1, \nu_2)$.

$\nu = (\nu_1, \nu_2)$	$\mathcal{T}$	$\mathbb{T}$	$\|\vartheta\|$
0.9, 0.5	1.3	0.815	3.3394
″, 0.7	″	0.965	3.1741
″, 0.9	″	1.0915	3.0458
0.5, 0.5	1.3	0.6335	3.5099
″, 0.7	″	0.837	3.4191
″, 0.9	″	1.0025	3.4392
0.9, 0.5	1.0026	1.3	3.4333
″, 0.7	1.0309	″	3.2607
″, 0.9	1.092	″	3.1271

where $(m_1, m_2) = (0.1,0.2)$ and we have $\|\psi\| \approx 0.002 < \varepsilon$. We consider the data: $\mathscr{K} = 70$, $\mathscr{J} = 60$, $\varepsilon = 0.1$, $\gamma = 10$, and $\rho = 0.55$. In the following, we have plotted the solution $\vartheta(\xi, \zeta) = (\vartheta_1(\xi, \zeta), \vartheta_2(\xi, \zeta))^T$, $\forall(\xi, \zeta) \in [0, \mathcal{T}] \times [0, \mathbb{T}]$ for different values of $\nu = (\nu_1, \nu_2)$ and $\mathcal{T}$, $\mathbb{T}$, (see Figures 1, 2, 3, and 4). Also, in Table 1, we study the variation of $\mathcal{T}$, $\mathbb{T}$, and $\|\vartheta\|$ versus variation of $\nu = (\nu_1, \nu_2)$. We conclude that the stability relation given in (44) is well satisfied $\|\vartheta(\xi, \zeta)\| < 10$.

By using a fixed point method, we have proved in our work in [21] that system (9) is FTS w.r.t. $\{\varepsilon, \gamma, \rho, \mathcal{T}, \mathbb{T}\}$ if there exists $\eta_1, \eta_2 > 0$ such that $a + b < \eta_1\eta_2$ and the following inequality holds:

$$\begin{aligned} &3\left[1 + (a+b)M_0(\eta_1, \eta_2)E_{\nu_1}(\eta_1\mathcal{T}^{\nu_1})E_{\nu_2}(\eta_2\mathbb{T}^{\nu_2})\right] \\ &\quad \cdot \varepsilon + \left[cM_0(\eta_1, \eta_2)E_{\nu_1}(\eta_1\mathcal{T}^{\nu_1})E_{\nu_2}(\eta_2\mathbb{T}^{\nu_2})\right]\rho \le \gamma, \end{aligned} \tag{50}$$

where $E_\cdot(\cdot)$ is the Mittag-Leffler function and $M_0(\eta_1, \eta_2)$ is given by

$$M_0(\eta_1, \eta_2) = \frac{\mathcal{T}^{\nu_1}\mathbb{T}^{\nu_2}}{[1 - (a + b/\eta_1\eta_2)]\Gamma(\nu_1 + 1)\Gamma(\nu_2 + 1)}. \tag{51}$$

Now, we will compare the time interval $[0, \mathcal{T}_{FP}] \times [0, \mathbb{T}_{FP}]$ obtained by the fixed point method relation (50) and the time interval $[0, \mathcal{T}_{GG}] \times [0, \mathbb{T}_{GG}]$ obtained by the generalized Gronwall's inequality relation (19). We consider the same data taken at the beginning of this section to which we add $(\nu_1, \nu_2) = (0.9,0.5)$, $\eta_1 = 1.5$, and $\eta_2 = 1$. Then, we fix $\mathcal{T}_{GG} = \mathcal{T}_{FP} = 1.3$. From Table 1, we deduce that $\mathbb{T}_{GG} = 0.815$. Now using relation (50), we can deduce by calculation that $\mathbb{T}_{FP} = 0.00615$.

$$\begin{aligned} [0, \mathcal{T}_{GG}] \times [0, \mathbb{T}_{GG}] &= [0,1.3] \times [0,0.815], \\ [0, \mathcal{T}_{FP}] \times [0, \mathbb{T}_{FP}] &= [0,1.3] \times [0,0.00615]. \end{aligned} \tag{52}$$

We conclude from this experiment that the generalized Gronwall's inequality gives a wider stability interval than that given by fixed point method which proves that generalized Gronwall's inequality method gives very satisfactory stability results.

6. Conclusion

In this work, we have proved the FTS for a class of two-dimensional fractional hyperbolic differential systems with time delay by using generalized Gronwall's inequality. Sufficient conditions for the FTS of such systems are given. Moreover, numerical examples are given to illustrate the stability result.

In the coming works, we aspire to generalize the FTS for the 2D-FHDLS to several other types of well-known fractional derivatives.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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