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Research Article

Approximating Fixed Points of Enriched Nonexpansive Mappings in Geodesic Spaces

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In this paper, we consider the class of enriched nonexpansive mappings in the setting of geodesic spaces. We obtain a number of fixed point theorems for these mappings in geodesic spaces. Further, we employ the SP iterative method and present some new convergence theorems for the class of enriched nonexpansive mappings under different assumptions. We present some results concerning Δ and strong convergence.

1. Introduction

Nonexpansive mappings are those class of nonlinear mappings which have Lipschitz constant equal to one. A nonexpansive mapping needs not to admit a fixed point in a complete space. However, Browder [1], Göhde [2], and Kirk [3] independently ensured the existence of fixed points of nonexpansive mappings in Banach spaces under certain geometric assumptions. Many mathematicians have generalized and extended these results and considered a number of nonlinear mappings, see [4–9] (see also the references therein).

In 2019, Berinde [10] considered a new class of nonlinear mappings by enriching nonexpansive mappings, known as enriched nonexpansive mappings. He obtained some fixed point theorems for these classes of mappings in Hilbert spaces. It was observed in [10, 11] that class of enriched nonexpansive mappings has strong relations with averaged and nonexpansive mappings.

On the other hand, in 1970, Takahashi [12] considered the structure of convexity outside linear spaces. These spaces are fruitful in the context of fixed point theory. Goebel and Kirk [13] employed Krasnosel'skiĭ-Mann iterative method to find fixed points of nonexpansive mappings in hyperbolic type spaces. In the recent years, a number of papers have appeared in the literature dealing with the fixed point theorems in nonlinear spaces, see [14–24].

The class of enriched nonexpansive mappings has been studied only in linear spaces. Now it is natural to extend this class of mappings outside of linear spaces (or in nonlinear spaces) and ensure the existence of fixed points. The aim of this paper is to study the class of enriched nonexpansive mappings in geodesic spaces. We observe that for every b-enriched nonexpansive mapping, one can define a nonexpansive mapping, and the set of fixed points of both the mappings remains the same. Therefore, the existence of fixed points for b-enriched nonexpansive mappings is equivalent to existence of fixed points for nonexpansive mappings. However, the convergence of fixed points for b-enriched nonexpansive mappings is slightly different than the convergence of fixed points for nonexpansive mappings. We prove that Krasnosel'skiĭ method converges to fixed point of mapping. Further, we use SP iterative method to reckon fixed points of b-enriched nonexpansive mappings under certain assumptions. These results are new even in Hilbert spaces. Our results extend, complement, and generalize some results from [10, 11, 16, 19, 25–27].

2. Preliminaries

Let (Γ, ρ) be a metric space and $[0, c] \subset \mathbb{R}$. A mapping $g : [0, c] \longrightarrow \Gamma$ is called as geodesic path from ζ to ξ if

$$g(0) = \zeta, g(c) = \xi, \quad \rho\left(g(s), g\left(s'\right)\right) = \left|s - s'\right|, \tag{1}$$

for all $s, s' \in [0, c]$. The image $g([0, c])$ of g forms a geodesic joining ζ and ξ. It is noted that the geodesic segment joining ζ and ξ is not unique, in general. For more details of geodesic spaces, see [14, 21].

Definition 1 (see [28]). A triplet (Γ, ρ, Ω) is called as a hyperbolic metric space if (Γ, ρ) is a metric space, and function $\Omega : \Gamma \times \Gamma \times [0, 1] \longrightarrow \Gamma$ satisfies the following assumptions for all $\zeta, \xi, \upsilon, w \in \Gamma$ and $\mu, \theta \in [0, 1]$

(W1) $\rho(\upsilon, \Omega(\zeta, \xi, \mu)) \leq (1 - \mu)\rho(\upsilon, \zeta) + \mu\rho(\upsilon, \xi)$
(W2) $\rho(\Omega(\zeta, \xi, \mu), \Omega(\zeta, \xi, \theta)) = |\mu - \theta|\rho(\zeta, \xi)$
(W3) $\Omega(\zeta, \xi, \mu) = \Omega(\xi, \zeta, 1 - \mu)$
(W4) $\rho(\Omega(\zeta, \upsilon, \mu), \Omega(\xi, w, \mu)) \leq (1 - \mu)\rho(\zeta, \xi) + \mu\rho(\upsilon, w)$

Remark 2. If $\Omega(\zeta, \xi, \mu) = (1 - \mu)\zeta + \mu\xi$ for all $\zeta, \xi \in \Gamma, \mu \in [0, 1]$, then it can be seen that all normed linear spaces are hyperbolic metric space.

Remark 3. If conditions (W1)–(W3) are satisfied, then (Γ, ρ, Ω) is hyperbolic type space considered by Goebel and Kirk [13]. Reich and Shafrir [22] also obtained some important results in hyperbolic metric spaces.

We shall write

$$\Omega(\zeta, \xi, \mu) \coloneqq (1 - \mu)\zeta \oplus \mu\xi, \tag{2}$$

to denote a point $\Omega(\zeta, \xi, \mu)$ of (Γ, ρ, Ω) space. For $\zeta, \xi \in \Gamma$,

$$[\zeta, \xi] = \{(1 - \mu)\zeta \oplus \mu\xi : \mu \in [0, 1]\}, \tag{3}$$

indicates geodesic segments. A subset $\mathscr{Z}$ of hyperbolic metric space (or hyperbolic space) (Γ, ρ, Ω) is called convex if $[\zeta, \xi] \subset \mathscr{Z}$ whenever $\zeta, \xi \in \mathscr{Z}$.

Remark 4. Leustean [20] proved that the class of CAT(0) spaces is the class of complete uniformly convex hyperbolic spaces (in short, complete UCΩ-hyperbolic space), see the definition of UCΩ-hyperbolic space in [19].

If (Γ, ρ, Ω) is a Busemann space, then there is a unique convexity mapping Ω in such a way that (Γ, ρ, Ω) is Ω-hyperbolic space with unique geodesics. In other words, for all $\zeta \neq \xi \in \Gamma$ and any $\mu \in [0, 1]$, there is an element $\upsilon \in \Gamma$ which is unique (say $\upsilon = \Omega(\zeta, \xi, \mu)$) in such a way

$$\rho(\zeta, \upsilon) = \mu\rho(\zeta, \xi) \text{ and } \rho(\xi, \upsilon) = (1 - \mu)\rho(\zeta, \xi). \tag{4}$$

Let ζ, ξ, υ be three points in metric space (Γ, ρ); the point ξ is said to lie between ζ and υ if

$$\rho(\zeta, \upsilon) = \rho(\zeta, \xi) + \rho(\xi, \upsilon), \tag{5}$$

and these points are distinct pairwise. Thus, if ξ lies between ζ and υ, then ξ lies between υ and ζ.

Lemma 5 (see [14]). *Let Γ be a uniquely geodesic space. Let $\zeta, \xi, \upsilon \in \Gamma$ be pairwise distinct points. A point ξ lies between ζ and υ if and only if $\xi \in [\zeta, \upsilon]$.*

Proposition 6 (see [14]). *Let Γ be a metric space and $\zeta, \xi, \upsilon, w \in \Gamma$ be pairwise distinct points. The following are equivalent:*

(a) ξ lies between ζ and υ, and υ lies between ζ and w

(b) ξ lies between ζ and w, and υ lies between ξ and w

Let $\{\zeta_n\}$ be a bounded sequence in a hyperbolic space (Γ, ρ, Ω) and $\mathscr{Z} \subseteq \Gamma$ with $\mathscr{Z} \neq \emptyset$. A functional $r(., \{\zeta_n\})$: $\Gamma \longrightarrow [0,\infty)$ can be defined as follows:

$$r(\xi, \{\zeta_n\}) = \limsup_{n \longrightarrow \infty} \rho(\xi, \zeta_n). \tag{6}$$

The asymptotic radius of $\{\zeta_n\}$ with respect to (in short, wrt) $\mathscr{Z}$ is defined as

$$r(\mathscr{Z}, \{\zeta_n\}) = \inf \{r(\xi, \{\zeta_n\}) | \xi \in \mathscr{Z}\}. \tag{7}$$

A point ζ in $\mathscr{Z}$ is called an asymptotic center of $\{\zeta_n\}$ wrt $\mathscr{Z}$ if

$$r(\zeta, \{\zeta_n\}) = r(\mathscr{Z}, \{\zeta_n\}). \tag{8}$$

$A(\mathscr{Z}, \{\zeta_n\})$ is denoted as set of all asymptotic centers of $\{\zeta_n\}$ wrt $\mathscr{Z}$. A bounded sequence $\{\zeta_n\}$ in a hyperbolic space (Γ, ρ, Ω) is said to Δ-converge to ζ if ζ is the unique asymptotic center for every subsequence $\{u_n\}$ of $\{\zeta_n\}$. A sequence $\{\zeta_n\} \subseteq \Gamma$ is called Fejér monotone wrt $\mathscr{Z}$ if for all $\zeta^\dagger \in \mathscr{Z}$

$$\rho\left(\zeta^\dagger, \zeta_{n+1}\right) \leq \rho\left(\zeta^\dagger, \zeta_n\right), \tag{9}$$

for all $n \geq 0$.

Definition 7 (see [29]). A mapping $F : \mathscr{Z} \longrightarrow \mathscr{Z}$ is called quasi-nonexpansive if

$$\rho\left(F(\zeta), \zeta^\dagger\right) \leq \rho\left(\zeta, \zeta^\dagger\right) \forall \zeta \in \mathscr{Z}, \quad \zeta^\dagger \in Fix(F) \neq \emptyset, \tag{10}$$

where $\text{Fix}(F) = \{\zeta^\dagger \in \mathscr{Z} | F(\zeta^\dagger) = \zeta^\dagger\}$.

Definition 8 (see [30]). The mapping $F : \mathscr{Z} \longrightarrow \mathscr{Z}$ with Fix $(F) \neq \emptyset$ is said to have Condition (I) if the following assumptions are satisfied:

(a) $\exists$ a function $f : [0,\infty) \longrightarrow [0,\infty)$ which is nondecreasing

(b) For $r \in (0,\infty)$, $f(r) > 0$ and $f(0) = 0$

(c) For all $\zeta \in \mathscr{Z}$, $\rho(\zeta, F(\zeta)) \geq f(\rho(\zeta, \mathrm{Fix}(F)))$

where $\rho(\zeta, \mathrm{Fix}(F)) = \inf \{\rho(\zeta, \xi): \xi \in \mathrm{Fix}(F)\}$.

Definition 9. Let (Γ, ρ) be a metric space and $\mathscr{Z} \subseteq \Gamma$ with $\mathscr{Z} \neq \emptyset$. A mapping $F : \mathscr{Z} \longrightarrow \mathscr{Z}$ is called as compact if $F(\mathscr{Z})$ has a compact closure.

Proposition 10 (see [20]). *Let (Γ, ρ, Ω) be a complete UCΩ-hyperbolic space, $\mathscr{Z} \subseteq \Gamma$ with $\mathscr{Z} \neq \emptyset$. Suppose that $\mathscr{Z}$ is convex and closed, and $\{\zeta_n\}$ is bounded sequence in Γ. Then, $\{\zeta_n\}$ has a unique asymptotic center with respect to $\mathscr{Z}$.*

Lemma 11 (see [17]). *Let (Γ, ρ, Ω) be same as in Proposition 10. Let $w \in \Gamma$ and $\{\omega_n\}$ be a sequence with $\{\omega_n\} \subseteq [a, b] \subseteq (0, 1)$. For some $r \geq 0$, if $\{\zeta_n\}$ and $\{\xi_n\}$ are sequences in Γ with $\limsup_{n\longrightarrow\infty} \rho(\zeta_n, w) \leq r$, $\limsup_{n\longrightarrow\infty} \rho(\xi_n, w) \leq r$, and $\lim_{n\longrightarrow\infty} \rho(\omega_n \xi_n \oplus (1-\omega_n)\zeta_n, w) = r$. Then, $\lim_{n\longrightarrow\infty} \rho(\xi_n, \zeta_n) = 0$.*

Lemma 12. *Let (Γ, ρ, Ω) and $\mathscr{Z}$ be same as in Proposition 10. Let $F : \mathscr{Z} \longrightarrow \mathscr{Z}$ be a mapping. For $\lambda \in (0, 1)$, consider $\Psi : \mathscr{Z} \longrightarrow \mathscr{Z}$ as follows:*

$$\Psi(\zeta) = (1-\lambda)\zeta \oplus \lambda F(\zeta), \tag{11}$$

for all $\zeta \in \mathscr{Z}$. Then, $\mathrm{Fix}(\Psi) = \mathrm{Fix}(F)$.

Lemma 13 (see [20]). *Let $\{\zeta_n\}$ be a bounded sequence in Γ and $A(\mathscr{Z}, \{\zeta_n\}) = \{v\}$. Let $\{\kappa_n\}$ and $\{\nu_n\}$ be two sequences in $\mathbb{R}$ with for all $n \in \mathbb{N}, \kappa_n \in [0,\infty)$, $\limsup \kappa_n \leq 1$ and $\limsup \nu_n \leq 0$. Suppose that $\xi \in \mathscr{Z}$ and there exists m, $q \in \mathbb{N}$ such that*

$$\rho(\xi, \zeta_{n+m}) \leq \kappa_n \rho(v, \zeta_n) + \nu_n \forall n \geq q. \tag{12}$$

Then, $\xi = v$.

Lemma 14 (see [14]). *Let (Γ, ρ, Ω) be a metric space, $\mathscr{Z} \subseteq \Gamma$ such that $\mathscr{Z} \neq \emptyset$. If $\{\zeta_n\}$ is Fejér monotone wrt $\mathscr{Z}$, $A(\mathscr{Z}, \{\zeta_n\}) = \{\zeta\}$ and $A(\Gamma, \{u_n\}) \subseteq \mathscr{Z}$ for every subsequence $\{u_n\}$ of $\{\zeta_n\}$. Then, the sequence $\{\zeta_n\}$ Δ-converges to $\zeta \in \mathscr{Z}$.*

Lemma 15 (see [16]). *Let (Γ, ρ, Ω) be a complete UCΩ-hyperbolic space and $\mathscr{Z} \subseteq \Gamma$ such that $\mathscr{Z} \neq \emptyset$ and $\mathscr{Z}$ is closed convex. Let $\{\zeta_n\}$ be a bounded sequence in Γ and $\tau : \mathscr{Z} \longrightarrow [0,\infty)$ a function defined as follows:*

$$\tau(\zeta) = \limsup_{n\longrightarrow\infty} \rho(\zeta_n, \zeta), \tag{13}$$

for any $\zeta \in \mathscr{Z}$. τ is called as type function, and it is unique.

Then, there is a minimum point (unique) $w \in \mathscr{Z}$ and $\tau(w) = \inf \{\tau(\zeta): \zeta \in \mathscr{Z}\}$.

Proposition 16 (see [13]). *Let (Γ, ρ, Ω) and $\mathscr{Z}$ be same as in Lemma 15 with $\mathscr{Z}$ is bounded. Let $F : \mathscr{Z} \longrightarrow \mathscr{Z}$ be a nonexpansive mapping. Let $\zeta_0 \in \mathscr{Z}$ and $\vartheta \in (0, 1)$. Define a sequence $\{\zeta_n\}$ in $\mathscr{Z}$ by Krasnosel'skiĭ iterative method* [31].

$$\zeta_{n+1} = (1-\vartheta)\zeta_n \oplus \vartheta F(\zeta_n), n \in \mathbb{N} \cup \{0\}. \tag{14}$$

Then, $\lim_{n\longrightarrow\infty} \rho(\zeta_n, F(\zeta_n)) = 0$.

The proof of the following theorem is motivated from [16].

Theorem 17. *Let (Γ, ρ, Ω), $\mathscr{Z}$, and F be same as in Proposition 16. Then, $\mathrm{Fix}(F) \neq \emptyset$.*

Proof. For a given $\zeta_0 \in \mathscr{Z}$ and for any $\omega \in (0, 1)$, a sequence can be defined:

$$\zeta_{n+1} = (1-\omega)\zeta_n \oplus \omega \Psi(\zeta_n). \tag{15}$$

From Proposition 16, it implies that

$$\lim_{n\longrightarrow\infty} \rho(\zeta_n, \Psi(\zeta_n)) = 0. \tag{16}$$

From Lemma 15, there is a minimum point (unique) $v^\dagger \in \mathscr{Z}$ in such a way that

$$\tau\left(v^\dagger\right) = \inf \{\tau(\omega): \omega \in \mathscr{Z}\}. \tag{17}$$

From the definition of mapping Ψ,

$$\begin{aligned} \tau\left(\Psi\left(v^\dagger\right)\right) &= \limsup_{n\longrightarrow\infty} \rho\left(\zeta_n, \Psi\left(v^\dagger\right)\right) \\ &\leq \limsup_{n\longrightarrow\infty} \rho(\zeta_n, \Psi(\zeta_n)) + \limsup_{n\longrightarrow\infty} \rho\left(\Psi(\zeta_n), \Psi\left(v^\dagger\right)\right) \\ &\leq \limsup_{n\longrightarrow\infty} \rho\left(\zeta_n, v^\dagger\right). \end{aligned} \tag{18}$$

Then, $\Psi(v^\dagger) = v^\dagger$. □

3. Main Results

In 2019, Berinde [10] considered a new class of mappings which is defined below.

Definition 18. Let $(\Gamma, \|.\|)$ be a Banach space and $F : \Gamma \longrightarrow \Gamma$ a mapping. The mapping F is called b-enriched nonexpansive if $\exists b \in [0,\infty)$ in such a way that

$$\|b(\zeta - \xi) + F(\zeta) - F(\xi)\| \leq (b+1)\|\zeta - \xi\|, \tag{19}$$

for all $\zeta, \xi \in \Gamma$.

It can be noted that 0-enriched mapping is nonexpansive mapping. Even both the class of mappings, that is, quasi-nonexpansive and b-enriched nonexpansive, are independent in nature, cf. [27].

Remark 19. Take $b \neq 0$, and it is straight forward from (19) that

$$\begin{aligned}
&\left\|\frac{b}{b+1}(\zeta - \xi) + \frac{1}{b+1}(F(\zeta) - F(\xi))\right\| \\
&\quad \leq \|\zeta - \xi\| \Leftrightarrow \left\|\left(1 - \frac{1}{b+1}\right)(\zeta - \xi) + \frac{1}{b+1}(F(\zeta) - F(\xi))\right\| \\
&\quad \leq \|\zeta - \xi\| \Leftrightarrow \left\|\left(1 - \frac{1}{b+1}\right)\zeta + \frac{1}{b+1}F(\zeta)\right. \\
&\qquad \left. - \left\{\left(1 - \frac{1}{b+1}\right)\xi + \frac{1}{b+1}F(\xi)\right\}\right\| \\
&\quad \leq \|\zeta - \xi\|.
\end{aligned} \tag{20}$$

Take, $\lambda_b = 1/(b+1) \in (0,1)$ then

$$\|(1-\lambda_b)\zeta + \lambda_b F(\zeta) - \{(1-\lambda_b)\xi + \lambda_b F(\xi)\}\| \leq \|\zeta - \xi\|. \tag{21}$$

From the above inequality, we can take convex combination of F and the identity mappings.

In view of Remark 19, we consider Definition 18 in Ω-hyperbolic spaces.

Definition 20. Let (Γ, ρ, Ω) be a Ω-hyperbolic space, $\mathscr{Z}$ a subset of Γ such that $\mathscr{Z} \neq \varnothing$, and $F : \mathscr{Z} \longrightarrow \mathscr{Z}$ a mapping. The mapping F is called b-enriched nonexpansive if $\exists b \in [0,\infty)$ in such a way that

$$\rho((1-\lambda_b)\zeta \oplus \lambda_b F(\zeta), (1-\lambda_b)\xi \oplus \lambda_b F(\xi)) \leq \rho(\zeta, \xi), \tag{22}$$

for all $\zeta, \xi \in \mathscr{Z}$, where $\lambda_b = 1/(b+1)$.

We prove the following important lemma which will be utilized throughout this paper.

Lemma 21. *Let (Γ, ρ, Ω) be a uniquely geodesic space. For some $\lambda, \omega \in (0,1)$, let $\zeta, \xi, \upsilon \in \Gamma$ be pairwise distinct points with $\xi = (1-\lambda)\zeta \oplus \lambda\upsilon$ and $\upsilon = (1-\omega)\zeta \oplus \omega w$. Then*

$$\xi = (1-\vartheta)\zeta \oplus \vartheta w, \tag{23}$$

where $\vartheta = \omega\lambda$.

Proof. From Lemma 5, ξ lies between ζ and υ. And υ lies between ζ and w. From Proposition 6, ξ lies between ζ and w. Thus, $\xi \in [\zeta, w]$ and

$$\xi = (1-\vartheta)\zeta \oplus \vartheta w, \tag{24}$$

for some $\vartheta \in (0,1)$. Since Γ is uniquely geodesic space, we have

$$\rho(\zeta, \xi) = \vartheta\rho(\zeta, w), \tag{25}$$

$$\rho(\xi, w) = (1-\vartheta)\rho(\zeta, w). \tag{26}$$

Since $\xi = (1-\lambda)\zeta \oplus \lambda\upsilon$, we have

$$\rho(\zeta, \xi) = \lambda\rho(\zeta, \upsilon). \tag{27}$$

Again, since $\upsilon = (1-\omega)\zeta \oplus \omega w$, we have

$$\rho(\zeta, \upsilon) = \omega\rho(\zeta, w). \tag{28}$$

From (25), (27), and (28), one can conclude

$$\rho(\zeta, \xi) = \lambda\rho(\zeta, \upsilon) = \omega\lambda\rho(\zeta, w) = \frac{\omega\lambda}{\vartheta}\rho(\zeta, \xi). \tag{29}$$

Therefore, $\omega\lambda/\vartheta = 1$, and $\vartheta = \omega\lambda$. □

Theorem 22. *Let (Γ, ρ, Ω) be a complete UCΩ-hyperbolic space and $\mathscr{Z} \subseteq \Gamma$ such that $\mathscr{Z} \neq \varnothing$. Assume that $\mathscr{Z}$ is closed, bounded, and convex. Let $F : \mathscr{Z} \longrightarrow \mathscr{Z}$ be a b-enriched nonexpansive mapping. Then, $Fix(F) \neq \varnothing$. Moreover, for given $\zeta_0 \in \mathscr{Z}$, any $\omega \in (0,1)$, there exists $\omega_b = \omega/(b+1)$ such that the sequence $\{\zeta_n\}$ generated by (Krasnosel'skiĭ method)*

$$\zeta_{n+1} = (1-\omega_b)\zeta_n + \omega_b F(\zeta_n) \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{30}$$

Δ-converges to an element of $Fix(F)$.

Proof. By the definition of mapping F, we get

$$\rho((1-\lambda_b)\zeta \oplus \lambda_b F(\zeta), (1-\lambda_b)\xi \oplus \lambda_b F(\xi)) \leq \rho(\zeta, \xi), \tag{31}$$

for all $\zeta, \xi \in \mathscr{Z}$ and $\lambda_b = 1/(b+1)$. Set the mapping Ψ as follows:

$$\Psi(\zeta) = \left(1 - \frac{1}{b+1}\right)\zeta \oplus \frac{1}{b+1}F(\zeta) \text{ for all } \zeta \in \mathscr{Z}. \tag{32}$$

Thus, from (31), we get, for all $\zeta, \xi \in \mathscr{Z}$,

$$\rho(\Psi(\zeta), \Psi(\xi)) \leq \rho(\zeta, \xi), \tag{33}$$

and Ψ is a nonexpansive mapping. For any $\omega \in (0,1)$ and a given $\zeta_0 \in \mathscr{Z}$, we can define a sequence

$$\zeta_{n+1} = (1-\omega)\zeta_n \oplus \omega\Psi(\zeta_n). \tag{34}$$

From Proposition 16, it follows that

$$\lim_{n\longrightarrow\infty}\rho(\zeta_n, \Psi(\zeta_n)) = 0. \tag{35}$$

From Theorem 17, $\mathrm{Fix}(\Psi) \neq \emptyset$; thus, from Lemma 12, $\mathrm{Fix}(\Psi) = \mathrm{Fix}(F) \neq \emptyset$. Further, for any $v^\dagger \in \mathrm{Fix}(\Psi)$,

$$\rho\left(\Psi(\zeta_n), v^\dagger\right) \leq \rho\left(\zeta_n, v^\dagger\right) \text{ for all } n \geq 0. \tag{36}$$

Thus, from (W1)

$$\begin{aligned}\rho\left(\zeta_{n+1}, v^\dagger\right) &= \rho\left((1-\omega)\zeta_n \oplus \omega\Psi(\zeta_n), v^\dagger\right)\\ &\leq (1-\omega)\rho\left(\zeta_n, v^\dagger\right) + \omega\rho\left(\Psi(\zeta_n), v^\dagger\right)\\ &\leq \rho\left(\zeta_n, v^\dagger\right).\end{aligned} \tag{37}$$

Hence, the sequence $\{\rho(\zeta_n, v^\dagger)\}$ is monotone nonincreasing. It implies that $\{\zeta_n\}$ is Fejér monotone sequence wrt $\mathrm{Fix}(F)$. In view of Proposition 10, the sequence $\{\zeta_n\}$ has unique asymptotic center $w^\dagger$ wrt $\mathrm{Fix}(F)$. Suppose $\{u_n\}$ is a subsequence of $\{\zeta_n\}$ and $u^\dagger$ is unique asymptotic center of $\{u_n\}$ wrt $\mathrm{Fix}(F)$. Now,

$$\begin{aligned}\rho\left(u_n, \Psi\left(u^\dagger\right)\right) &\leq \rho\left(\Psi(u_n), \Psi\left(u^\dagger\right)\right) + \rho(\Psi(u_n), u_n)\\ &\leq \rho\left(u_n, u^\dagger\right) + \rho(\Psi(u_n), u_n).\end{aligned} \tag{38}$$

From (35) and Lemma 13, it follows that $\Psi(u^\dagger) = u^\dagger$. From Lemma 14, the sequence $\{\zeta_n\}\Delta$-converges to an element of $\mathrm{Fix}(F)$. From Lemma 21 with $v = \Psi(\zeta)$ and $w = F(\zeta)$, we have

$$(1-\omega)\zeta \oplus \omega\Psi(\zeta) = (1-\omega\lambda_b)\zeta \oplus \omega\lambda_b F(\zeta), \tag{39}$$

for all $\zeta \in \mathscr{Z}$ since $\omega \in (0,1)$ and $\lambda_b = 1/(b+1)$. It follows that $\omega\lambda_b \in (0, 1/(b+1))$. Thus, for any $\omega_b = \omega\lambda_b \in (0, 1/(b+1))$, the sequence $\{\zeta_n\}$ defined by (30) Δ-converges to a point in $\mathrm{Fix}(F)$. □

Remark 23. It can be seen that Theorem 22 generalizes the results in [10] (Theorem 3.3) from Hilbert spaces to hyperbolic spaces.

Theorem 24. *Let (Γ, ρ, Ω) and F be same as in Theorem 22. Suppose $\mathscr{Z} \subseteq \Gamma$ such that $\mathscr{Z} \neq \emptyset$, and $\mathscr{Z}$ is closed and convex. Assume F satisfies Condition (I) with $\mathrm{Fix}(F) \neq \emptyset$. For fixed $\zeta_0 \in \mathscr{Z}$ and any $\omega \in (0,1)$, there exists $\omega_b = \omega/(b+1)$ such that the sequence $\{\zeta_n\}$ generated by (Krasnosel'skiĭ method)*

$$\zeta_{n+1} = (1-\omega_b)\zeta_n + \omega_b F(\zeta_n) \forall n \in \mathbb{N} \cup \{0\}, \tag{40}$$

strongly converges to an element of $\mathrm{Fix}(F)$.

Proof. By the similar technique in proof of Theorem 22, one can set a mapping Ψ as in (32), and Ψ is nonexpansive. Let $\omega \in (0,1)$ and define

$$\zeta_{n+1} = (1-\omega)\zeta_n \oplus \omega\Psi(\zeta_n). \tag{41}$$

For all $v^\dagger \in \mathrm{Fix}(\Psi)$

$$\rho\left(\Psi(\zeta_n), v^\dagger\right) \leq \rho\left(\zeta_n, v^\dagger\right) \text{ for all } n \geq 1. \tag{42}$$

From (41), we have

$$\rho\left(\zeta_{n+1}, v^\dagger\right) \leq \rho\left(\zeta_n, v^\dagger\right). \tag{43}$$

Thus, $\{\rho(\zeta_n, v^\dagger)\}$ and $\{\rho(\zeta_n, \mathrm{Fix}(\Psi))\}$ are monotone nonincreasing sequences and $\lim_{n\longrightarrow\infty}\rho(\zeta_n, v^\dagger)$ and $\lim_{n\longrightarrow\infty}\rho(\zeta_n, \mathrm{Fix}(\Psi))$ exist. Let

$$\lim_{n\longrightarrow\infty}\rho\left(\zeta_n, v^\dagger\right) = r. \tag{44}$$

From (42)

$$\lim_{n\longrightarrow\infty}\rho\left(\Psi(\zeta_n), v^\dagger\right) \leq r. \tag{45}$$

By (44), we have

$$r = \lim_{n\longrightarrow\infty}\rho\left(\zeta_{n+1}, v^\dagger\right) = \lim_{n\longrightarrow\infty}\rho\left((1-\omega)\zeta_n \oplus \omega\Psi(\zeta_n), v^\dagger\right). \tag{46}$$

In view of (44), (45), (46), and Lemma 11, it implies:

$$\lim_{n\longrightarrow\infty}\rho(\zeta_n, \Psi(\zeta_n)) = 0. \tag{47}$$

Since

$$\Psi(\zeta) = \left(1 - \frac{1}{b+1}\right)\zeta \oplus \frac{1}{b+1}F(\zeta), \tag{48}$$

we have

$$\rho(\zeta, \Psi(\zeta)) = \frac{1}{b+1}\rho(\zeta, F(\zeta)) \text{ for } all\, \zeta \in \mathscr{Z}. \tag{49}$$

Since F satisfies Condition (I) and (49), we obtain

$$\begin{aligned}(b+1)\rho(\zeta_n, \Psi(\zeta_n)) = \rho(\zeta_n, F(\zeta_n)) &\geq f(\rho(\zeta_n, \mathrm{Fix}(F)))\\ &= f(\rho(\zeta_n, \mathrm{Fix}(\Psi))).\end{aligned} \tag{50}$$

By (47), $\lim_{n\longrightarrow\infty} f(\rho(\zeta_n, \mathrm{Fix}(\Psi))) = 0$ and

$$\lim_{n\longrightarrow\infty}\rho(\zeta_n, \mathrm{Fix}(\Psi)) = 0. \tag{51}$$

One can easily show that $\{\zeta_n\}$ is a Cauchy sequence. For the sake of completeness, we prove this claim. From (51), for given $\varepsilon > 0$, $\exists$ a $n_0 \in \mathbb{N}$ in such a way that

$$\rho(\zeta_n, \mathrm{Fix}(\Psi)) < \frac{\varepsilon}{4}, \tag{52}$$

for all $n \geq n_0$. Hence,

$$\inf \left\{\rho\left(\zeta_{n_0}, \upsilon^\dagger\right): \upsilon^\dagger \in \mathrm{Fix}(\Psi)\right\} < \frac{\varepsilon}{4}, \tag{53}$$

so there is $\upsilon^\dagger \in \mathrm{Fix}(\Psi)$ in such a way

$$\rho\left(\zeta_{n_0}, \upsilon^\dagger\right) < \frac{\varepsilon}{2}. \tag{54}$$

Therefore, for all $m, n \geq n_0$,

$$\rho(\zeta_{n+m}, \zeta_n) \leq \rho\left(\zeta_{n+m}, \upsilon^\dagger\right) + \rho\left(\upsilon^\dagger, \zeta_n\right) \leq 2\rho\left(\zeta_{n_0}, \upsilon^\dagger\right) < 2\frac{\varepsilon}{2} = \varepsilon, \tag{55}$$

and $\{\zeta_n\}$ is a Cauchy sequence. By the closedness of $\mathscr{X}$ in Γ, $\{\zeta_n\}$ converges to a point $\zeta^\dagger \in \mathscr{X}$. Now

$$\begin{aligned}\rho\left(\zeta^\dagger, \Psi\left(\zeta^\dagger\right)\right) &\leq \rho\left(\zeta^\dagger, \zeta_n\right) + \rho(\zeta_n, \Psi(\zeta_n)) + \rho\left(\Psi(\zeta_n), \Psi\left(\zeta^\dagger\right)\right) \\ &\leq 2\rho\left(\zeta^\dagger, \zeta_n\right) + \rho(\zeta_n, \Psi(\zeta_n)).\end{aligned} \tag{56}$$

From (47), $\zeta^\dagger = \Psi(\zeta^\dagger)$. Therefore, the sequence $\{\zeta_n\}$ strongly converges to a point in $\mathrm{Fix}(F)$. Further,

$$(1-\omega)\zeta \oplus \omega\Psi(\zeta) = (1-\omega_b)\zeta \oplus \omega_b F(\zeta), \tag{57}$$

for all $\zeta \in \mathscr{X}$ with $\omega_b = \omega/(b+1)$. □

Remark 25. Theorem 24 generalizes the results in [11] (Theorem 3.2) from Hilbert spaces to hyperbolic spaces.

Theorem 26. *Let (Γ, ρ, Ω) and F be same as in Theorem 22. Let $\mathscr{X} \subseteq \Gamma$ such that $\mathscr{X} \neq \emptyset$ and $\mathscr{X}$ be a closed and convex. Suppose that F is compact mapping with $\mathrm{Fix}(F) \neq \emptyset$. For fixed $\lambda \in (0, 1/(b+1))$, $\{\zeta_n\}$ is a sequence generated as follows:*

$$\zeta_{n+1} = (1-\lambda)\zeta_n \oplus \lambda F(\zeta_n), \tag{58}$$

strongly converges to an element of $\mathrm{Fix}(F)$.

Proof. We set the nonexpansive mapping Ψ as in the proof of Theorem 22. For given $\zeta_0 \in \mathscr{X}$ and for any $\omega \in (0, 1)$, define a sequence

$$\zeta_{n+1} = (1-\omega)\zeta_n \oplus \omega\Psi(\zeta_n). \tag{59}$$

Following largely as in Theorem 24 and from Lemma 11

$$\lim_{n\longrightarrow\infty} \rho(\zeta_n, \Psi(\zeta_n)) = 0. \tag{60}$$

Since

$$\Psi(\zeta) = \left(1 - \frac{1}{b+1}\right)\zeta \oplus \frac{1}{b+1}F(\zeta), \tag{61}$$

we get

$$\rho(\zeta, \Psi(\zeta)) = \frac{1}{b+1}\rho(\zeta, F(\zeta)) \text{ for all } \zeta \in \mathscr{X}. \tag{62}$$

From the above equation and (60)

$$\lim_{n\longrightarrow\infty} \rho(\zeta_n, F(\zeta_n)) = 0. \tag{63}$$

Since the range of $\mathscr{X}$ under F is subset of a compact set, there is a subsequence $\{F(\zeta_{n_j})\}$ of $\{F(\zeta_n)\}$ strongly converges to $\zeta^\dagger \in \mathscr{X}$. By (63), the subsequence $\{\zeta_{n_j}\}$ strongly converges to $\zeta^\dagger$. Since Ψ is nonexpansive mapping and by the triangle inequality, we obtain

$$\begin{aligned}\rho\left(\zeta_{n_j}, \Psi\left(\zeta^\dagger\right)\right) &\leq \rho\left(\zeta_{n_j}, \Psi\left(\zeta_{n_j}\right)\right) + \rho\left(\Psi\left(\zeta_{n_j}\right), \Psi\left(\zeta^\dagger\right)\right) \\ &\leq \rho\left(\zeta_{n_j}, \Psi\left(\zeta_{n_j}\right)\right) + \rho\left(\zeta_{n_j}, \zeta^\dagger\right).\end{aligned} \tag{64}$$

Thus, subsequence $\{\zeta_{n_j}\}$ strongly converges to $\Psi(\zeta^\dagger)$ and $\Psi(\zeta^\dagger) = \zeta^\dagger$. Since $\lim_{n\longrightarrow\infty} \rho(\zeta_n, \zeta^\dagger)$ exists, it follows that the sequence $\{\zeta_n\}$ strongly converges to an element of $\mathrm{Fix}(F)$. □

4. SP Iterative Method

In this section, we present some convergence results for SP iterative process. For a fix $\zeta_0 \in \mathscr{X}$ and the mapping $F : \mathscr{X} \longrightarrow \mathscr{X}$, the SP iterative method in the setting of hyperbolic metric spaces can be defined as follows [26]:

$$\begin{cases} w_n = (1-\sigma_n)\zeta_n \oplus \sigma_n F(\zeta_n), \\ \xi_n = (1-\vartheta_n)w_n \oplus \vartheta_n F(w_n), \\ \zeta_{n+1} = (1-\omega_n)\xi_n \oplus \omega_n F(\xi_n), \end{cases} \tag{65}$$

where $\{\sigma_n\}$, $\{\vartheta_n\}$, and $\{\omega_n\}$ are sequences in $[0, 1]$.

Similar to [25] (Lemma 4), we model the following lemma.

Lemma 27. *Let (Γ, ρ, Ω) and $\mathscr{X}$ be same as in Theorem 24. Let $\Psi : \mathscr{X} \longrightarrow \mathscr{X}$ be a nonexpansive mapping with $\mathrm{Fix}(F) \neq \emptyset$. For fixed $\zeta_0 \in \mathscr{X}$ and for all $n \in \mathbb{N} \cup \{0\}$, $\omega_n, \vartheta_n, \sigma_n \in [\omega, \vartheta]$ with $\omega, \vartheta \in (0, 1)$, the sequence $\{\zeta_n\}$ is defined by*

$$\begin{cases} w_n = (1-\sigma_n)\zeta_n \oplus \sigma_n \Psi(\zeta_n), \\ \xi_n = (1-\vartheta_n)w_n \oplus \vartheta_n \Psi(w_n), \\ \zeta_{n+1} = (1-\omega_n)\xi_n \oplus \omega_n \Psi(\xi_n). \end{cases} \tag{66}$$

Then, the following holds:

(1) $\lim_{n\longrightarrow\infty} \rho(\zeta_n, \upsilon)$ exists $\forall \upsilon \in \mathrm{Fix}(\Psi)$

(2) $\lim_{n\longrightarrow\infty} \rho(\zeta_n, \Psi(\zeta_n)) = 0$

Proof. From (W1), we get

$$\begin{aligned}\rho(w_n, \upsilon) &\le (1-\sigma_n)\rho(\zeta_n, \upsilon) + \sigma_n \rho(\Psi(\zeta_n), \upsilon) \\ &\le (1-\sigma_n)\rho(\zeta_n, \upsilon) + \sigma_n \rho(\zeta_n, \upsilon) = \rho(\zeta_n, \upsilon),\end{aligned} \quad (67)$$

$$\begin{aligned}\rho(\xi_n, \upsilon) &\le (1-\vartheta_n)\rho(w_n, \upsilon) + \vartheta_n \rho(\Psi(w_n), \upsilon) \\ &\le (1-\vartheta_n)\rho(w_n, \upsilon) + \vartheta_n \rho(w_n, \upsilon) = \rho(w_n, \upsilon).\end{aligned} \quad (68)$$

Further, from (67) and (68), we get

$$\begin{aligned}\rho(\zeta_{n+1}, \upsilon) &\le (1-\omega_n)\rho(\xi_n, \upsilon) + \omega_n \rho(\Psi(\xi_n), \upsilon) \\ &\le (1-\omega_n)\rho(\xi_n, \upsilon) + \omega_n \rho(\xi_n, \upsilon) \\ &= \rho(\xi_n, \upsilon) \le \rho(w_n, \upsilon) \le \rho(\zeta_n, \upsilon).\end{aligned} \quad (69)$$

Thus, $\{\rho(\zeta_n, \upsilon)\}$ is a monotone nonincreasing sequence. Hence, $\lim_{n\longrightarrow\infty} \rho(\zeta_n, \upsilon)$ exists. Let

$$\lim_{n\longrightarrow\infty} \rho(\zeta_n, \upsilon) = r > 0. \quad (70)$$

From (69) and (70), we have

$$\begin{aligned} r &\le \liminf_{n\longrightarrow\infty} \rho(w_n, \upsilon), \\ \limsup_{n\longrightarrow\infty} \rho(w_n, \upsilon) &\le r.\end{aligned} \quad (71)$$

Thus

$$\lim_{n\longrightarrow\infty} \rho(w_n, \upsilon) = r. \quad (72)$$

Since the mapping Ψ is nonexpansive

$$\limsup_{n\longrightarrow\infty} \rho(\Psi(\zeta_n), \upsilon) \le \lim_{n\longrightarrow\infty} \rho(\zeta_n, \upsilon) = r, \quad (73)$$

and from (72)

$$\lim_{n\longrightarrow\infty} \rho((1-\sigma_n)\zeta_n \oplus \sigma_n \Psi(\zeta_n), \upsilon) = \lim_{n\longrightarrow\infty} \rho(w_n, \upsilon) = r. \quad (74)$$

From (70), (73), (74), and Lemma 11, it follows:

$$\lim_{n\longrightarrow\infty} \rho(\zeta_n, \Psi(\zeta_n)) = 0. \quad (75)$$

□

Theorem 28. *Let (Γ, ρ, Ω) and $\mathcal{Z}$ be same as in Theorem 24. Let $F : \mathcal{Z} \longrightarrow \mathcal{Z}$ be a b-enriched nonexpansive mapping with $Fix(F) \neq \emptyset$. For fixed $\zeta_0 \in \mathcal{Z}$, for all $n \in \mathbb{N} \cup \{0\}$, ω_n, $\vartheta_n, \sigma_n \in [\omega/(b+1), \vartheta/(b+1)]$ with $\omega, \vartheta \in (0,1)$, the sequence $\{\zeta_n\}$ generated by (65) Δ-converges to an element of $Fix(F)$.*

Proof. For given $\zeta_0 \in \mathcal{Z}$ and for all $n \in \mathbb{N} \cup \{0\}$, $\omega_n^b, \vartheta_n^b, \sigma_n^b \in [\omega, \vartheta]$ with $\omega, \vartheta \in (0,1)$, we can consider a sequence $\{\zeta_n\}$:

$$\begin{cases} w_n = \left(1-\sigma_n^b\right)\zeta_n \oplus \sigma_n^b \Psi(\zeta_n), \\ \xi_n = \left(1-\vartheta_n^b\right) w_n \oplus \vartheta_n^b \Psi(w_n), \\ \zeta_{n+1} = \left(1-\omega_n^b\right)\xi_n \oplus \omega_n^b \Psi(\xi_n), \end{cases} \quad (76)$$

where Ψ is a mapping defined as in (32). Using Lemma 21, we have

$$\begin{cases} w_n = (1-\sigma_n)\zeta_n \oplus \sigma_n F(\zeta_n), \\ \xi_n = (1-\vartheta_n) w_n \oplus \vartheta_n F(w_n), \\ \zeta_{n+1} = (1-\omega_n)\xi_n \oplus \omega_n F(\xi_n), \end{cases} \quad (77)$$

where $\sigma_n = \sigma_n^b/(b+1)$, $\vartheta_n = \vartheta_n^b/(b+1)$, and $\omega_n = \omega_n^b/(b+1)$. By Lemma 27 and repeating the technique of proof of Theorem 22, one can complete the proof. □

Theorem 29. *Let (Γ, ρ, Ω), $\mathcal{Z}$, and F be same as in Theorem 24. For fixed $\zeta_0 \in \mathcal{Z}$, for all $n \in \mathbb{N} \cup \{0\}$, $\omega_n, \vartheta_n, \sigma_n \in [\omega/(b+1), \vartheta/(b+1)]$ with $\omega, \vartheta \in (0,1)$, the sequence $\{\zeta_n\}$ generated by (65) strongly converges to an element of $Fix(F)$.*

Proof. Using proof of Theorem 28, Lemma 27, and Theorem 24, one can complete the proof. □

Theorem 30. *Let (Γ, ρ, Ω) and $\mathcal{Z}$ be same as in Theorem 24. Let $F : \mathcal{Z} \longrightarrow \mathcal{Z}$ be a compact b-enriched nonexpansive mapping with $Fix(F) \neq \emptyset$. For fixed $\zeta_0 \in \mathcal{Z}$, for all $n \in \mathbb{N} \cup \{0\}$, $\omega_n, \vartheta_n, \sigma_n \in [\omega/(b+1), \vartheta/(b+1)]$ with $\omega, \vartheta \in (0,1)$, the sequence $\{\zeta_n\}$ generated by (65) strongly converges to a point in $Fix(F)$.*

Proof. Using proof of Theorem 28, Lemma 27, and Theorem 26, one can complete the proof. □

Remark 31. Theorems 28–30 are new even in Hilbert spaces.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors contributed equally to this work.

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