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Research Article

A Multiple Fixed Point Result for (θ, ϕ, ψ)-Type Contractions in the Partially Ordered s-Distance Spaces with an Application

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In this manuscript, the aim is to prove a multiple fixed point (FP) result for partially ordered s-distance spaces under (θ, ϕ, ψ)-type weak contractive condition. The result will generalize some well-known results in literature such as coupled FP (Guo and Lakshmikantham, 1987), triple fixed point (Berinde and Borcut, 2011), and quadruple FP results (Karapinar, 2011). Moreover, to validate the result, an application for the existence of solution of a system of integral equations is also provided.

1. Introduction

In pure mathematics, the Banach fixed-point theorem [1] (contractive mapping theorem or also known as the contraction mapping theorem) is main result in the study of metric spaces; it assurances the uniqueness and existence of FP of certain self-maps of metric spaces and requires a constructive technique to discover those FP. It can be understood as an abstract formulation of Picard's method of successive approximations. The theorem is named after Stefan Banach (1892-1945) who first stated it in 1922. It has numerous applications in different fields such as computer science, physics, engineering, and various branches of mathematics itself. FP theory as a whole got an upward flight after this celebrated result.

Many authors started working in this field, and soon it became a hot field of research. A number of authors have extended this fundamental result to nonlinear analysis [2]. Following this streak, Guo and Lakshmikantham [3] established the idea of doubled FP. This is considered to be the first of its nature and was extended to *triple FP* by Berinde and Borcut [4]. Continuing in this direction, Karapinar [5] used four variable to strengthen the idea of *quadruple FP* in partially ordered metric spaces. In 2012, Berzig and Samet [6] discussed the existence of the fixed point of N-order for m-mixed monotone mappings in complete ordered metric spaces. In the same year, Roldán et al. [7] extended the notion of the FP of N-order to the ϕ-fixed point and obtained some ϕ-fixed point theorems for a mixed monotone mapping in partially ordered complete metric spaces. In [8], Karapinar et al. studied the existence and uniqueness of a FP of the multidimensional operators which satisfy Meir-Keeler type contraction condition. Soon after, a number of articles were published to discussed the concept of a " multidimensional FP" or "an m-tuples fixed point." For applications of such results, we refer the reader to [9] and the references cited therein.

In 2016, Choban and Berinde [10] generalized metric spaces to distance spaces. They established multidimensional FP results for ordered spaces with distance under certain contractive conditions [4, 11]. They pointed out that the concept allowed them to reduce the multidimensional case of FP and coincidence points to the one dimensional case. Recently, Rashid et al. [12] established some multiple FP findings for the C-distance spaces in the existence of various contractive mapping.

The abovementioned ideas serve as motivation of the work in the present paper. We have extended the results of [10] for partially ordered s-distance space, in terms of a significant multiple FP result under (θ, ϕ, ψ)-type contractive condition [13]. In support of this result, an example is also given.

Since, in Section 1, introduction and historical background of generalized metric spaces is given. In Section 2, preliminaries and some basic definitions are stated. In Section 3, main result is stated, and in Section 2, some consequences and examples of our main result will be described. In Section 3, an application is stated to support our main result. In the last section, article is concluded.

2. Preliminaries

Definition 1 [11]. Consider the $\mathbb{M}$, as a nonempty set and a function $\sigma : \mathbb{M} \times \mathbb{M} \longrightarrow \mathbb{R}$ is called s-distance on $\mathbb{M}$ if for all $\varsigma, \xi, \eta \in \mathbb{M}$, σ satisfies the following axioms:

(1) $\sigma(\varsigma, \xi) \geq 0$, $\forall \varsigma, \xi \in \mathbb{M}$

(2) $\sigma(\varsigma, \xi) + \sigma(\xi, \varsigma) = 0$ if and only if $\varsigma = \xi$, $\forall \varsigma, \xi \in \mathbb{M}$

(3) For a positive real number $s > 0$

$$\sigma(\varsigma, \xi) \leq s[\sigma(\varsigma, \eta) + \sigma(\eta, \xi)], \forall \varsigma, \xi, \eta \in \mathbb{M}. \quad (1)$$

An s-distance space $(\mathbb{M}, \sigma)$ is said to be a symmetric s-distance space if $\sigma(\varsigma, \xi) = \sigma(\xi, \varsigma)$, $\forall \varsigma, \xi \in \mathbb{M}$. Now, some remarks and examples as given below.

Remark 2.

(1) Every b-metric space is an s-distance space but not conversely

(2) In s-distance space, distance σ is not necessarily a continuous function, i.e., if $\varsigma_n \longrightarrow \varsigma$ and $\xi_n \longrightarrow \xi$ then $\sigma(\varsigma_n, \xi_n) \nrightarrow \sigma(\varsigma, \xi)$

(3) In an s-distance space, the limit of a convergent sequence may not be unique

Example 3. Let $\mathbb{M} = \{\alpha_1, \alpha_2, \alpha_3\}$, $\sigma : \mathbb{M} \times \mathbb{M} \longrightarrow [0,\infty)$ and

$$\begin{aligned}
&\sigma(\alpha_1, \alpha_2) = \frac{1}{4}, \sigma(\alpha_2, \alpha_1) = \frac{1}{2}, \\
&\sigma(\alpha_1, \alpha_3) = \sigma(\alpha_2, \alpha_3) = 1, \sigma(\alpha_i, \alpha_i) = 0, \\
&\sigma(\alpha_3, \alpha_1) = \sigma(\alpha_3, \alpha_2)\frac{1}{3}, \\
&\sigma(\alpha_i, \alpha_j) \geq 0, \\
&\sigma(\alpha_i, \alpha_j) + \sigma(\alpha_j, \alpha_i) = 0 \Leftrightarrow \alpha_i = \alpha_j,
\end{aligned} \quad (2)$$

for all $i, j \in \{1, 2, 3\}$.

$$\begin{aligned}
&\sigma(\alpha_1, \alpha_2) < [\sigma(\alpha_1, \alpha_3) + \sigma(\alpha_3, \alpha_2)], \\
&\sigma(\alpha_1, \alpha_3) < [\sigma(\alpha_1, \alpha_2) + \sigma(\alpha_2, \alpha_3)], \\
&\sigma(\alpha_2, \alpha_3) < [\sigma(\alpha_2, \alpha_1) + \sigma(\alpha_1, \alpha_3)], \\
&\sigma(\alpha_2, \alpha_1) < [\sigma(\alpha_2, \alpha_3) + \sigma(\alpha_3, \alpha_1)], \\
&\sigma(\alpha_3, \alpha_1) < [\sigma(\alpha_3, \alpha_2) + \sigma(\alpha_2, \alpha_1)], \\
&\sigma(\alpha_2, \alpha_3) < [\sigma(\alpha_2, \alpha_1) + \sigma(\alpha_1, \alpha_3)].
\end{aligned} \quad (3)$$

Hence, $(\mathbb{M}, \sigma)$ is an s-distance space with $s = 1$. However, it is not a b-metric space.

Fix $r \in \mathbb{N}$ and $\Gamma = (\Gamma_1, \cdots, \Gamma_r)$ is said to be a collection of mappings such that

$$\{\Gamma_{\acute{i}} : \{1, 2, 3, \cdots, r\} \longrightarrow \{1, 2, 3, \cdots, r\} ; 1 \leq i \leq r\}. \quad (4)$$

Let $(\mathbb{M}, \sigma)$ be a distance space and also $G : \mathbb{M}^r \longrightarrow \mathbb{M}$ be a mapping. The mapping $\Gamma G : \mathbb{M}^r \longrightarrow \mathbb{M}^r$ which is a composition of G and Γ is gvien as

$$\begin{aligned}
&\Gamma G(\varsigma_1, \cdots, \varsigma_r) = (\xi_1, \cdots, \xi_r), \\
&\xi_i = G\left(\varsigma_{\Gamma_i(1)}, \cdots, \varsigma_{\Gamma_i(r)}\right),
\end{aligned} \quad (5)$$

for any point $(\varsigma_1, \cdots, \varsigma_r) \in \mathbb{M}^r$ and $\acute{i} \in \{1, 2, \cdots, r\}$. A point $a = (a_1, \cdots, a_r) \in \mathbb{M}^r$ is considered as a Γ *-multiple FP* of G if it is a FP of ΓG, i.e., $a = \Gamma G(a)$ and

$$a_{\acute{i}} = G\left(a_{\Gamma_{\acute{i}}(1)}, \cdots, a_{\Gamma_{\acute{i}}(r)}\right) \text{ for any } \acute{i} \in \{1, 2, 3, \cdots, r\}. \quad (6)$$

Let $(\mathbb{M}, \sigma)$ be a distance space, $r \in \mathbb{N} = \{1, 2, \cdots\}$. On $\mathbb{M}^r$, consider the distance

$$\sigma^r((\varsigma_1, \cdots, \varsigma_r), (\xi_1, \cdots, \xi_r)) = \sup \left\{\sigma(\varsigma_{\acute{i}}, \xi_{\acute{i}}) : \acute{i} \leq r\right\}. \quad (7)$$

Obviously, $(\mathbb{M}^r, \sigma^r)$ is a distance space, too.

The following are some basic concepts from [14]:

Consider a partially ordered distance space $(\mathbb{M}, \sigma, \preccurlyeq)$, r be a positive integer and $\{J, K\}$ be a partition of $J_r = \{1, 2, \cdots, r\}$, i.e., $J, K \neq \phi$, $J \cup K = J_r$, and $J \cap K = \phi$. Define $\mathbb{M}^r = \mathbb{M} \times \mathbb{M} \times \ldots \times \mathbb{M}(r \text{ times})$ the Cartesian product of the set $\mathbb{M}$. Define a partial order $\preccurlyeq_r$ over $\mathbb{M}^r$ as follows:

For any $\omega = (\varsigma_1, \varsigma_2, \cdots, \varsigma_r), \nu = (\xi_1, \xi_2, \cdots, \xi_r) \in \mathbb{M}^r$, $\omega \preccurlyeq_r \nu$ if and only if $\varsigma_{\acute{i}} \preccurlyeq_{\acute{i}} \xi_{\acute{i}}$ for all $\acute{i} \in J_r$, where

$$\varsigma \preccurlyeq_{\acute{i}}{}' \xi \text{ iff} \begin{cases} \varsigma \preccurlyeq \xi, & \text{if } \acute{i} \in J, \\ \varsigma \succcurlyeq \xi, & \text{if } \acute{i} \in K. \end{cases} \quad (8)$$

The function $\sigma^r : \mathbb{M}^r \times \mathbb{M}^r \longrightarrow [0,+\infty)$ given by

$$\sigma^r(\varsigma, \xi) = \sup_{1 \leq \acute{i} \leq r} \{\sigma(\varsigma_{\acute{i}}, \xi_{\acute{i}})\}, \quad (9)$$

defines a distance on $\mathbb{M}^r$, where $\varsigma = (\varsigma_1, \varsigma_2, \cdots, \varsigma_r)$ and $\xi = (\xi_1, \xi_2, \cdots, \xi_r)$. Obviously, $(\mathbb{M}^r, \sigma^r, \preccurlyeq_r)$ is a partially ordered distance space, so that it inherits the properties of $(\mathbb{M}, \sigma, \preccurlyeq)$ and $\sigma^r(\varsigma^\vartheta, \varsigma) \longrightarrow 0$ if and only if $\sigma(\varsigma_{\acute{i}}^\vartheta, \varsigma_{\acute{i}}) \longrightarrow 0$ as $\vartheta \longrightarrow \infty$ for all $\acute{i} \in J_r$.

Definition 4. [15]. Let g be a self mapping on $\mathbb{M}$. A mapping G has the mixed g -monotone property with respect to the partition $\{J, K\}$, if G is g-monotone nondecreasing in arguments of J and g -monotone nonincreasing in arguments of K, i.e., $\forall \varsigma_1, \varsigma_2, \cdots, \varsigma_r, \xi, \eta \in \mathbb{M}$, $\acute{i} \in J_r$, and

$$\begin{aligned} g(\xi) \preccurlyeq g(\eta) \Rightarrow G(\varsigma_1, \cdots, \varsigma_{\acute{i}-1}, \xi, \varsigma_{\acute{i}+1}, \cdots, \varsigma_r) \\ \preccurlyeq_{\acute{i}} G(\varsigma_1, \cdots, \varsigma_{\acute{i}-1}, \eta, \varsigma_{\acute{i}+1}, \cdots, \varsigma_r). \end{aligned} \tag{10}$$

If g is the identity mapping on $\mathbb{M}$, then the mapping G has the mixed monotone property with respect to the partition $\{J, K\}$. Define a set of mappings by

$$\begin{aligned} \Omega_{J,K} &= \{\Gamma_{\acute{i}} : J_r \longrightarrow J_r : \Gamma_{\acute{i}}(J) \subseteq J, \Gamma_{\acute{i}}(K) \subseteq K\}, \\ \Omega'_{J,K} &= \{\Gamma_{\acute{i}} : J_r \longrightarrow J_r : \Gamma_{\acute{i}}(J) \subseteq K, \Gamma_{\acute{i}}(K) \subseteq J\}, \end{aligned} \tag{11}$$

such that $\Gamma_{\acute{i}} \in \Omega_{J,K}$ if $\acute{i} \in J$ and $\Gamma_{\acute{i}} \in \Omega'_{J,K}$ if $\acute{i} \in K$.

Definition 5. If a function $\psi : [0,\infty) \longrightarrow 0,\infty)$ is continuous, nondecreasing, and $\psi(0) = 0$, then it is called an altering distance function.

Definition 6. The metric space $(\mathbb{M}, \sigma, \preccurlyeq)$ is called regular if it satisfies the following properties:

(1) If $\{\varsigma_r\}$ is a nondecreasing sequence and $\varsigma_r \longrightarrow \varsigma$ then $\varsigma_r \preccurlyeq \varsigma$ for all $r \geq 1$

(2) If $\{\varsigma_r\}$ is a nonincreasing sequence and $\varsigma_r \longrightarrow \varsigma$ then $\varsigma_r \pm \varsigma$ for all $r \geq 1$

3. Main Result

Berinde and Borcut [4] extended the concept of multidimensional FP to ordered distance spaces by utilizing the properties of contractive type mappings. Keeping ourselves in touch with all these concepts, we are extending these results to symmetric s-distance spaces by using a combination of altering distance functions. This result will generalize the main theorems of [14], in which the space considered is a metric space. It is also valid for b-metric spaces.

Theorem 7. *Consider a complete partially ordered symmetric s-distance space.*

$(\mathbb{M}, \sigma, \preccurlyeq)$ *and* $\Gamma = (\Gamma_1, \Gamma_2, \cdots, \Gamma_r)$ *be collection of mappings verifying* $\Gamma_{\acute{i}} \in \Omega_{J,K}$ *if* $\acute{i} \in J$ *and* $\Gamma_{\acute{i}} \in \Omega'_{J,K}$ *if* $\acute{i} \in K$. *If the mapping* $G : \mathbb{M}^r \longrightarrow \mathbb{M}$ *satisfies the following conditions:*

(a) For $\mu > 0$

$\psi(\mu) - \theta(\mu) - \varphi(\mu) > 0, (*)$

where ψ *is an altering distance function and* $\theta : [0,\infty) \longrightarrow 0,\infty)$ *and* $\varphi : [0,\infty) \longrightarrow 0,\infty)$ *are upper semicontinuous and increasing functions such that* $\theta(0) = \varphi(0) = 0$ *satisfying*

$$\psi(\sigma(G\varsigma, G\xi)) \leq \theta\left(\frac{\sigma^r(\varsigma, \xi)}{s+1}\right) + \varphi\left(\frac{\sigma^r(\varsigma, \xi)}{s+1}\right), \tag{12}$$

for all $\varsigma, \xi \in \mathbb{M}^r$ *with* $\varsigma \preccurlyeq_r \xi$.

(b) There is $\varsigma^0 = (\varsigma_1^0, \varsigma_2^0, \cdots, \varsigma_r^0)$ *such that* $\varsigma_{\acute{i}}^0 \preccurlyeq_{\acute{i}} G(\varsigma^0_{\Gamma_{\acute{i}}(1)}, \cdots, \varsigma^0_{\Gamma_{\acute{i}}(r)})$ *for all* $\acute{i} \in J_r$

(c) G has mixed monotone property with respect to $\{J, K\}$

(d) G is continuous, or $(\mathbb{M}, \sigma, \preccurlyeq)$ *is regular; then, G has* Γ *-multiple FP*

(e) Moreover, if for ς *and* ξ *in* $\mathbb{M}^r$, *there is* $\eta \in \mathbb{M}^r$ *such that* $\varsigma \preccurlyeq_r \eta$ *and* $\xi \preccurlyeq_r \eta$; *then, G has a unique* Γ*-multiple FP*

Proof. Step 1. Let $\varsigma^\vartheta = (\Gamma G)^\vartheta(\varsigma^0)$ be the nth Picard iterate of ς^0 under ΓG, i.e., $\varsigma^\vartheta = (\Gamma G)^\vartheta(\varsigma^0) = (\varsigma_1^\vartheta, \varsigma_2^\vartheta, \cdots, \varsigma_r^\vartheta)$, where

$$\begin{aligned} \varsigma_1^\vartheta &= G\left(\varsigma^{\vartheta-1}_{\Gamma_1(1)}, \varsigma^{\vartheta-1}_{\Gamma_1(2)}, \cdots \varsigma^{\vartheta-1}_{\Gamma_1(r)}\right), \\ \varsigma_2^\vartheta &= G\left(\varsigma^{\vartheta-1}_{\Gamma_2(1)}, \varsigma^{\vartheta-1}_{\Gamma_2(2)}, \cdots \varsigma^{\vartheta-1}_{\Gamma_2(r)}\right), \\ &\vdots \\ \varsigma_r^\vartheta &= G\left(\varsigma^{\vartheta-1}_{\Gamma_r(1)}, \varsigma^{\vartheta-1}_{\Gamma_r(2)}, \cdots \varsigma^{\vartheta-1}_{\Gamma_r(r)}\right). \end{aligned} \tag{13}$$

By condition (b) and definition of ΓG, it follows that $\varsigma^0 \preccurlyeq_r \varsigma^1$. Since, G has mixed monotone property so ΓG is monotone nondecreasing [15]; therefore

$$\varsigma^{\vartheta-1} \preccurlyeq_r \varsigma^\vartheta \forall \vartheta \geq 1. \tag{14}$$

Step 2. We need to show that $\lim_{\vartheta \longrightarrow \infty} \sigma^r(\varsigma^{\vartheta-1}, \varsigma^\vartheta) = 0$. Set $D_{\acute{i}}^\vartheta = \sigma(\varsigma_{\acute{i}}^{\vartheta-1}, \varsigma_{\acute{i}}^\vartheta)$ and $D^\vartheta = \sup_{\acute{i} \in J_r}(D_{\acute{i}}^\vartheta) = \sigma^r(\varsigma^{\vartheta-1}, \varsigma^\vartheta)$. If $D^\vartheta = 0$ for some $\vartheta \geq 1$ then $\varsigma^{\vartheta-1} = \Gamma G(\varsigma^{\vartheta-1})$ which means G has Γ-multiple FP which completes the proof. Assume $D^\vartheta > 0$ for all $\vartheta \geq 1$. Since $\varsigma^{\vartheta-1} \preccurlyeq_r \varsigma^\vartheta$ and $\Gamma_{\acute{i}}(J_r) \subseteq J_r$, it follows that

$$\left(\varsigma^{\vartheta-1}_{\Gamma_{\acute{i}}(1)}, \varsigma^{\vartheta-1}_{\Gamma_{\acute{i}}(2)}, \cdots, \varsigma^{\vartheta-1}_{\Gamma_{\acute{i}}(r)}\right) \preccurlyeq_r \left(\varsigma^{\vartheta}_{\Gamma_{\acute{i}}(1)}, \varsigma^{\vartheta}_{\Gamma_{\acute{i}}(2)}, \cdots \varsigma^{\vartheta}_{\Gamma_{\acute{i}}(r)}\right), \tag{15}$$

for any $\acute{i} \in J_r$ and $\vartheta \geq 1$. Now, using condition (12)

$$\begin{aligned} \psi\left(D_{i'}^\vartheta\right) = \psi\Big(\sigma\Big(G\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}}(1)}, \varsigma^{\vartheta-2}_{\Gamma_{\acute{i}}(2)}, \cdots, \varsigma^{\vartheta-2}_{\Gamma_{\acute{i}}(r)}\right), \\ \cdot G\left(\varsigma^{\vartheta-1}_{\Gamma_{\acute{i}}(1)}, \varsigma^{\vartheta-1}_{\Gamma_{\acute{i}}(2)}, \cdots, \varsigma^{\vartheta-1}_{\Gamma_{\acute{i}}(r)}\right)\Big)\Big), \end{aligned}$$

$$\le \theta\left(\frac{\sup_{j\in J_r}\sigma\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}}(j)}, \varsigma^{\vartheta-1}_{\Gamma_{\acute{i}}(j)}\right)}{s+1}\right) + \varphi\left(\frac{\sup_{j\in J_r}\sigma\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}}(j)}, \varsigma^{\vartheta-1}_{\Gamma_{\acute{i}}(j)}\right)}{s+1}\right). \tag{16}$$

Since, θ and φ both are increasing and $s > 0$ so that

$$\psi\left(D^{\vartheta}_{\acute{i}}\right) \le \theta\left(\sup_{j\in J_r}\left\{\sigma\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}}(j)}, \varsigma^{\vartheta-1}_{\Gamma_{\acute{i}}(j)}\right)\right\}\right) + \varphi\left(\sup_{j\in J_r}\left\{\sigma\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}}(j)}, \varsigma^{\vartheta-1}_{\Gamma_{\acute{i}}(j)}\right)\right\}\right), \tag{17}$$

for any $\acute{i} \in J_r$, since J_r is a finite set, there is an index $\acute{i}(\vartheta) \in J_r$ such that $\sup_{\acute{i}\in J_r}\{D^{\vartheta}_{\acute{i}}\} = D^{\vartheta}_{\acute{i}(\vartheta)}$. From above inequality, it follow that

$$\begin{aligned}
&\psi\left(D^{\vartheta}_{\acute{i}(\vartheta)}\right) \\
&= \psi\left(D^{\vartheta}_{\acute{i}(\vartheta)}\right), = \psi\left(\sigma\left(\begin{array}{c} G\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}(\vartheta)}(1)}, \varsigma^{\vartheta-2}_{\Gamma_{\acute{i}(\vartheta)}(2)}, \cdots, \varsigma^{\vartheta-2}_{\Gamma_{\acute{i}(\vartheta)}(r)}\right), \\ G\left(\varsigma^{\vartheta-1}_{\Gamma_{\acute{i}(\vartheta)}(1)}, \varsigma^{\vartheta-1}_{\Gamma_{\acute{i}(\vartheta)}(2)}, \cdots, \varsigma^{\vartheta-1}_{\Gamma_{\acute{i}(\vartheta)}(r)}\right)\end{array}\right)\right), \\
&\le \theta\left(\frac{\sup_{j\in J_r}\sigma\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}(\vartheta)}(j)}, \varsigma^{\vartheta-1}_{\Gamma_{\acute{i}(\vartheta)}(j)}\right)}{s+1}\right) + \varphi\left(\frac{\sup_{j\in J_r}\sigma\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}(\vartheta)}(j)}, \varsigma^{\vartheta-1}_{\Gamma_{\acute{i}(\vartheta)}(j)}\right)}{s+1}\right), \\
&\le \theta\left(\sup_{j\in J_r}\sigma\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}(\vartheta)}(j)}, \varsigma^{\vartheta-1}_{\Gamma_{\acute{i}(\vartheta)}(j)}\right)\right) + \varphi\left(\sup_{j\in J_r}\sigma\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}(\vartheta)}(j)}, \varsigma^{\vartheta-1}_{\Gamma_{\acute{i}(\vartheta)}(j)}\right)\right).
\end{aligned} \tag{18}$$

Since

$$0 < \sup_{j\in J_r}\left\{\sigma\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}(\vartheta)}(j)}, \varsigma^{\vartheta-1}_{\Gamma_{\acute{i}(\vartheta)}(j)}\right)\right\} \le D^{\vartheta-1}, \tag{19}$$

for all $\vartheta \ge 1$. Therefore, from the inequality $(*)$, we get

$$\begin{aligned}
&\theta\left(\sup_{j\in J_r}\left\{\sigma\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}(\vartheta)}(j)}, \varsigma^{\vartheta-1}_{\Gamma_{\acute{i}(\vartheta)}(j)}\right)\right\}\right) + \varphi\left(\sup_{j\in J_r}\left\{\sigma\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}(\vartheta)}(j)}, \varsigma^{\vartheta-1}_{\Gamma_{\acute{i}(\vartheta)}(j)}\right)\right\}\right) \\
&< \psi\left(\sup_{j\in J_r}\left\{\sigma\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}(\vartheta)}(j)}, \varsigma^{\vartheta-1}_{\Gamma_{\acute{i}(\vartheta)}(j)}\right)\right\}\right) \\
&\le \psi\left(\sup_{j\in J_r}\left\{\sigma\left(\varsigma^{\vartheta-2}_{(j)}, \varsigma^{\vartheta-1}_{(j)}\right)\right\}\right).
\end{aligned} \tag{20}$$

Combining (19) and (20), we have

$$\psi\left(D^{\vartheta}\right) < \psi\left(\sup_{j\in J_r}\left\{\sigma\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}(\vartheta)}(j)}, \varsigma^{\vartheta-1}_{\Gamma_{\acute{i}(\vartheta)}(j)}\right)\right\}\right) \le \psi\left(D^{\vartheta-1}\right), \tag{21}$$

for all $\vartheta \ge 1$. Since ψ is an altering distance function, it follows that

$$D^{\vartheta} < \sup_{j\in J_r}\left\{\sigma\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}(\vartheta)}(j)}, \varsigma^{\vartheta-1}_{\Gamma_{\acute{i}(\vartheta)}(j)}\right)\right\} \le D^{\vartheta-1}. \tag{22}$$

Hence, the sequence D^{ϑ} and $\sup_{j\in J_r}\{\sigma(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}(\vartheta)}(j)}, \varsigma^{\vartheta-1}_{\Gamma_{\acute{i}(\vartheta)}(j)})\}$ are monotone decreasing and bounded below. So, we have $\tau \ge 0$ such that

$$\lim_{\vartheta\longrightarrow\infty} D^{\vartheta} = \lim_{\vartheta\longrightarrow\infty}\sup_{j\in J_r}\left\{\sigma\left(\varsigma^{\vartheta-2}_{\Gamma_{\acute{i}(\vartheta)}(j)}, \varsigma^{\vartheta-1}_{\Gamma_{\acute{i}(\vartheta)}(j)}\right)\right\} = \tau. \tag{23}$$

We need to show that $\tau = 0$. Suppose $\tau > 0$, then by applying limit as $\vartheta \longrightarrow \infty$ in (19) and utilizing the properties of ψ, θ and φ, we have $\psi(\tau) - \theta(\tau) - \varphi(\tau) \le 0$, which contradicts $(*)$. Hence

$$\lim_{\vartheta\longrightarrow\infty} D^{\vartheta} = \lim_{\vartheta\longrightarrow\infty}\sigma^{r}\left(\varsigma^{\vartheta-1}, \varsigma^{\vartheta}\right) = 0. \tag{24}$$

Step 3. Now to prove that the sequence $\{\varsigma^{\vartheta}\}_{\vartheta\in\mathbb{N}}$ is Cauchy. Suppose on contrary that it is not Cauchy, then there exists $\varepsilon > 0$ for which there are subsequences $\{\varsigma^{\vartheta_\zeta}\}$ and $\{\varsigma^{r_\zeta}\}$ of $\{\varsigma^{\vartheta}\}$ with $\vartheta_\zeta > r_\zeta > \zeta$ such that

$$\sigma^{r}\left(\varsigma^{r_\varsigma}, \varsigma^{\vartheta_\varsigma}\right) \ge \varepsilon. \tag{25}$$

Let $\vartheta_\zeta$be the smallest integer satisfying above, then

$$\sigma^{r}\left(\varsigma^{r_\varsigma}, \varsigma^{\vartheta_\varsigma-1}\right) < \varepsilon. \tag{26}$$

Assume that $s < 1$, then consider

$$\begin{aligned}
\sigma^{r}\left(\varsigma^{r_\zeta}, \varsigma^{\vartheta_\zeta}\right) &\le s\left[\sigma^{r}\left(\varsigma^{r_\zeta}, \varsigma^{\vartheta_\zeta-1}\right) + \sigma^{r}\left(\varsigma^{\vartheta_\zeta-1}, \varsigma^{\vartheta_\zeta}\right)\right] \\
&< \left[\varepsilon + \sigma^{r}\left(\varsigma^{\vartheta_\zeta-1}, \varsigma^{\vartheta_\zeta}\right)\right].
\end{aligned} \tag{27}$$

On letting $\zeta \longrightarrow \infty$ and using condition (24), we get

$$\varepsilon \le \lim\sup_{\zeta\longrightarrow\infty}\sigma^{r}\left(\varsigma^{r_\varsigma}, \varsigma^{\vartheta_\zeta}\right) < \varepsilon, \tag{28}$$

which leads to a contradiction. Now consider $s \ge 1$, then

$$\begin{aligned}
\sigma^{r}\left(\varsigma^{r_\zeta}, \varsigma^{\vartheta_\zeta}\right) &\le s\left[\sigma^{r}\left(\varsigma^{r_\zeta}, \varsigma^{\vartheta_\zeta-1}\right) + \sigma^{r}\left(\varsigma^{\vartheta_\zeta-1}, \varsigma^{\vartheta_\zeta}\right)\right] \\
&< s\left[\varepsilon + \sigma^{r}\left(\varsigma^{\vartheta_\zeta-1}, \varsigma^{\vartheta_\zeta}\right)\right].
\end{aligned} \tag{29}$$

Again letting $\zeta \longrightarrow \infty$ and using condition (24), we get

$$\varepsilon \le \lim_{\zeta \longrightarrow \infty} \sup \sigma^r\left(\varsigma^{r_\zeta}, \varsigma^{\vartheta_\zeta}\right) < s\varepsilon. \tag{30}$$

Now

$$\begin{gathered}\sigma^r\left(\varsigma^{r_\zeta - 1}, \varsigma^{\vartheta_\zeta - 1}\right) \le s\left[\sigma^r\left(\varsigma^{r_\zeta - 1}, \varsigma^{r_\zeta}\right) + \sigma^r\left(\varsigma^{r_\zeta}, \varsigma^{\vartheta_\zeta - 1}\right)\right], \\ \lim_{\zeta \longrightarrow \infty} \sup \sigma^r\left(\varsigma^{r_\zeta - 1}, \varsigma^{\vartheta_\zeta - 1}\right) < s\varepsilon.\end{gathered} \tag{31}$$

From $\varsigma^{\vartheta-1} \preccurlyeq_r \varsigma^{\vartheta}$, it follows that

$$\varsigma^{r_\zeta} \preccurlyeq_r \varsigma^{r_\zeta + 1} \preccurlyeq_r \cdots \preccurlyeq_r \varsigma^{\vartheta_\zeta - 1} \preccurlyeq_r \varsigma^{\vartheta_\zeta} \preccurlyeq_r \varsigma^{\vartheta_\zeta + 1}. \tag{32}$$

Since $\varsigma^{r_\zeta - 1} \preccurlyeq_r \varsigma^{\vartheta_\zeta - 1}$ and $\Gamma_{\acute{i}}(J_r) \subseteq J_r$, we have

$$\left(\varsigma^{r_\zeta - 1}_{r_{\acute{i}}(1)}, \varsigma^{r_\zeta - 1}_{r_{\acute{i}}(2)}, \cdots, \varsigma^{r_\zeta - 1}_{r_{\acute{i}}(r)}\right) \preccurlyeq_r \left(\varsigma^{\vartheta_\zeta - 1}_{r_{\acute{i}}(1)}, \varsigma^{\vartheta_\zeta - 1}_{r_{\acute{i}}(2)}, \cdots, \varsigma^{\vartheta_\zeta - 1}_{\Gamma_{\acute{i}}(r)}\right), \tag{33}$$

for any $\acute{i} \in J_r$ and $\zeta \ge 1$. By condition (a), it follows that

$$\begin{aligned}\psi\left(\sigma\left(\varsigma^{r_\zeta}_{\acute{i}}, \varsigma^{\vartheta_\zeta}_{\acute{i}}\right)\right) &\le \psi\left(\sigma\left(G\left(\varsigma^{r_\zeta - 1}_{\Gamma_{\acute{i}}(1)}, \varsigma^{r_\zeta - 1}_{\Gamma_{\acute{i}}(2)}, \cdots \varsigma^{r_\zeta - 1}_{\Gamma_{\acute{i}}(r)}\right), \right.\right. \\ &\qquad \left.\left. \cdot\, G\left(\varsigma^{r_\zeta - 1}_{\Gamma_{\acute{i}}(1)}, \varsigma^{r_\zeta - 1}_{\Gamma_{\acute{i}}(2)}, \cdots \varsigma^{r_\zeta - 1}_{\Gamma_{\acute{i}}(r)}\right)\right)\right), \\ &\le \theta\left(\frac{\sup_{j \in J_r}\left\{\sigma\left(\varsigma^{r_\zeta - 1}_{\Gamma_{\acute{i}}(j)}, \varsigma^{\vartheta_\zeta - 1}_{\Gamma_{\acute{i}}(j)}\right)\right\}}{s+1}\right) \\ &\quad + \varphi\left(\frac{\sup_{j \in J_r}\left\{\sigma\left(\varsigma^{r_\zeta - 1}_{\Gamma_{\acute{i}}(j)}, \varsigma^{\vartheta_\zeta - 1}_{\Gamma_{\acute{i}}(j)}\right)\right\}}{s+1}\right),\end{aligned} \tag{34}$$

for any $\acute{i} \in J_r$. Since J_r is finite, there will be an index $\acute{i}(\zeta)$ in J_r, so

$$\sup_{\acute{i} \in J_r}\left\{\sigma\left(\varsigma^{r_\zeta}_{\acute{i}}, \varsigma^{\vartheta_\zeta}_{\acute{i}}\right)\right\} = \sigma\left(\varsigma^{r_\zeta}_{\acute{i}(\zeta)}, \varsigma^{\vartheta_\zeta}_{\acute{i}(\zeta)}\right). \tag{35}$$

From inequality (29)

$$\begin{aligned}\psi(\varepsilon) &\le \psi\left(\sigma^r\left(\varsigma^{r_\zeta}, \varsigma^{\vartheta_\zeta}\right)\right), \\ &\le \psi\left(\sigma\left(\begin{array}{c} G\left(\varsigma^{r_\zeta - 1}_{\Gamma_{i(\zeta)}(1)}, \varsigma^{r_\zeta - 1}_{\Gamma_{i(\zeta)}(2)}, \cdots \varsigma^{r_\zeta - 1}_{\Gamma_{i(\zeta)}(r)}\right), \\ G\left(\varsigma^{r_\zeta - 1}_{\Gamma_{i(\zeta)}(1)}, \varsigma^{r_\zeta - 1}_{\Gamma_{i(\zeta)}(2)}, \cdots \varsigma^{r_\zeta - 1}_{\Gamma_{i(\zeta)}(r)}\right)\end{array}\right)\right), \\ &\le \theta\left(\frac{\sup_{j \in J_r} \sigma\left(\varsigma^{r_\zeta - 1}_{\Gamma_{i(\zeta)}(j)}, \varsigma^{\vartheta_\zeta - 1}_{\Gamma_{i(\zeta)}(j)}\right)}{s+1}\right) + \varphi\left(\frac{\sup_{j \in J_r} \sigma\left(\varsigma^{r_\zeta - 1}_{\Gamma_{i(\zeta)}(j)}, \varsigma^{\vartheta_\zeta - 1}_{\Gamma_{i(\zeta)}(j)}\right)}{s+1}\right).\end{aligned} \tag{36}$$

Since θ and φ are increasing functions and $\Gamma_i{}'(J_r) \subseteq J_r$, so

$$\begin{aligned}\psi(\varepsilon) &\le \theta\left(\frac{\sup_{j \in J_r}\left\{\sigma\left(\varsigma^{r_\zeta - 1}_{\Gamma_{i(\zeta)}(j)}, \varsigma^{\vartheta_\zeta - 1}_{\Gamma_{i(\zeta)}(j)}\right)\right\}}{s}\right) \\ &\quad + \varphi\left(\frac{\sup_{j \in J_r}\left\{\sigma\left(\varsigma^{r_\zeta - 1}_{\Gamma_{i(\zeta)}(j)}, \varsigma^{\vartheta_\zeta - 1}_{\Gamma_{i(\zeta)}(j)}\right)\right\}}{s}\right), \\ &\le \theta\left(\frac{\sup_{j \in J_r}\left\{\sigma\left(\varsigma^{r_\zeta - 1}_{(j)}, \varsigma^{\vartheta_\zeta - 1}_{(j)}\right)\right\}}{s}\right) \\ &\quad + \varphi\left(\frac{\sup_{j \in J_r}\left\{\sigma\left(\varsigma^{r_\zeta - 1}_{(j)}, \varsigma^{\vartheta_\zeta - 1}_{(j)}\right)\right\}}{s}\right), \\ &\le \theta\left(\frac{\sigma^r\left(\varsigma^{r_\zeta - 1}, \varsigma^{\vartheta_\zeta - 1}\right)}{s}\right) + \varphi\left(\frac{\sigma^r\left(\varsigma^{r_\zeta - 1}, \varsigma^{\vartheta_\zeta - 1}\right)}{s}\right).\end{aligned} \tag{37}$$

Applying $\limsup_{\zeta \longrightarrow \infty}$ over above inequality and then using (29), we get

$$\psi(\varepsilon) < \theta\left(\frac{s\varepsilon}{\varepsilon}\right) + \varphi\left(\frac{s\varepsilon}{\varepsilon}\right), < \theta(\varepsilon) + \varphi(\varepsilon), \tag{38}$$

which is a contradiction to the condition $(*)$. Hence, $(\varsigma^\vartheta)_{\vartheta \in \mathbb{N}}$ is a Cauchy sequence.

Step 4. Since the space $(\mathbb{M}, \sigma)$ is complete so $(\mathbb{M}^r, \sigma^r)$ is complete. Therefore, we have $\nu \in \mathbb{M}^r$ such that $(\varsigma^\vartheta)_{\vartheta \in \mathbb{N}} \longrightarrow \nu$, i.e.,

$$\begin{aligned}\lim_{\vartheta \longrightarrow \infty} \varsigma^\vartheta_1 &= G\left(\varsigma^{\vartheta-1}_{\Gamma_1(1)}, \varsigma^{\vartheta-1}_{\Gamma_1(2)}, \cdots, \varsigma^{\vartheta-1}_{\Gamma_1(r)}\right) = \nu_1, \\ \lim_{\vartheta \longrightarrow \infty} \varsigma^\vartheta_2 &= G\left(\varsigma^{\vartheta-1}_{\Gamma_2(1)}, \varsigma^{\vartheta-1}_{\Gamma_2(2)}, \cdots, \varsigma^{\vartheta-1}_{\Gamma_2(r)}\right) = \nu_2, \\ &\vdots \\ \lim_{\vartheta \longrightarrow \infty} \varsigma^\vartheta_r &= G\left(\varsigma^{\vartheta-1}_{\Gamma_r(1)}, \varsigma^{\vartheta-1}_{\Gamma_r(2)}, \cdots, \varsigma^{\vartheta-1}_{\Gamma_r(r)}\right) = \nu_r.\end{aligned} \tag{39}$$

Next, we show that $\nu = (\nu_1, \nu_2, \cdots, \nu_r) \in \mathbb{M}^r$ is a Γ-multiple FP of operator G. If condition (c) holds and G is continuous, then

$$\begin{aligned}\lim_{\vartheta \longrightarrow \infty} G\left(\varsigma^{\vartheta-1}_{\Gamma_1(1)}, \varsigma^{\vartheta-1}_{\Gamma_1(2)}, \cdots, \varsigma^{\vartheta-1}_{\Gamma_1(r)}\right) &= G\left(\nu_{\Gamma_1(1)}, \nu_{\Gamma_1(2)}, \cdots, \nu_{\Gamma_1(r)}\right), \\ \lim_{\vartheta \longrightarrow \infty} G\left(\varsigma^{\vartheta-1}_{\Gamma_2(1)}, \varsigma^{\vartheta-1}_{\Gamma_2(2)}, \cdots, \varsigma^{\vartheta-1}_{\Gamma_2(r)}\right) &= G\left(\nu_{\Gamma_2(1)}, \nu_{\Gamma_2(2)}, \cdots, \nu_{\Gamma_2(r)}\right), \\ &\vdots \\ \lim_{\vartheta \longrightarrow \infty} G\left(\varsigma^{\vartheta-1}_{\Gamma_r(1)}, \varsigma^{\vartheta-1}_{\Gamma_r(2)}, \cdots, \varsigma^{\vartheta-1}_{\Gamma_r(r)}\right) &= G\left(\nu_{\Gamma_i(1)}, \nu_{\Gamma_i(2)}, \cdots, \nu_{\Gamma_i(r)}\right).\end{aligned} \tag{40}$$

Above shows that

$$\nu_{\acute{i}} = G\left(\nu_{\Gamma_{\acute{i}}(1)}, \nu_{\Gamma_{\acute{i}}(2)}, \cdots, \nu_{\Gamma_{\acute{i}}(r)}\right), \text{for all } \acute{i} \in J_r, \tag{41}$$

which means the point $\nu = (\nu_1, \nu_2, \cdots, \nu_r)$ is a Γ-FP of G. Next, suppose $(\mathbb{M}, \sigma \preccurlyeq)$ is regular. The above relation implies $\varsigma^\vartheta = (\varsigma_1^\vartheta, \varsigma_2^\vartheta, \cdots, \varsigma_r^\vartheta) \preccurlyeq_r \nu = (\nu_1, \nu_2, \cdots, \nu_r)$. On the other hand

$$\left(\varsigma^\vartheta_{\Gamma_{\acute{i}}(1)}, \varsigma^\vartheta_{\Gamma_{\acute{i}}(2)}, \cdots, \varsigma^\vartheta_{\Gamma\acute{i}(r)}\right) \preccurlyeq_r \left(\nu_{\Gamma_{\acute{i}}(1)}, \nu_{\Gamma_{\acute{i}}(2)}, \cdots, \nu_{\Gamma_{\acute{i}}(r)}\right). \tag{42}$$

Since, $\Gamma_{\acute{i}}(J_r) \subseteq J_r$ for any $\acute{i} \in J_r$. By using (a)

$$\begin{aligned}
&\psi\left(\sigma\left(G\left(\nu_{\Gamma_{\acute{i}}(1)}, \nu_{\Gamma_{\acute{i}}(2)}, \cdots, \nu_{\Gamma_{\acute{i}}(r)}\right), G\left(\varsigma^\vartheta_{\Gamma_{\acute{i}}(1)}, \varsigma^\vartheta_{\Gamma_{\acute{i}}(2)}, \cdots, \varsigma^\vartheta_{\Gamma\acute{i}(r)}\right)\right)\right),\\
&\le \theta\left(\frac{\sup_{j\in J_r}\left\{\sigma\left(\nu_{\Gamma_{\acute{i}}(j)}, \varsigma^\vartheta_{\Gamma_{\acute{i}}(j)}\right)\right\}}{s+1}\right) + \varphi\left(\frac{\max_{j\in J_r}\left\{\sigma\left(\nu_{\Gamma_{\acute{i}}(j)}, \varsigma^\vartheta_{\Gamma_{\acute{i}}(j)}\right)\right\}}{s+1}\right),\\
&\le \theta\left(\sup_{j\in J_r}\left\{\sigma\left(\nu_{\Gamma_{\acute{i}}(j)}, \varsigma^\vartheta_{\Gamma_{\acute{i}}(j)}\right)\right\}\right) + \varphi\left(\sup_{j\in J_r}\left\{\sigma\left(\nu_{\Gamma_{\acute{i}}(j)}, \varsigma^\vartheta_{\Gamma_{\acute{i}}(j)}\right)\right\}\right).
\end{aligned} \tag{43}$$

Taking into account above and letting $\vartheta \longrightarrow \infty$ in the last inequality, it implies

$$\psi\left(\sigma\left(G\left(\nu_{\Gamma_{\acute{i}}(1)}, \nu_{\Gamma_{\acute{i}}(2)}, \cdots, \nu_{\Gamma_{\acute{i}}(r)}\right), \nu_{\acute{i}}\right)\right) = 0. \tag{44}$$

Hence

$$G\left(\nu_{\Gamma_{\acute{i}}(1)}, \nu_{\Gamma_{\acute{i}}(2)}, \cdots, \nu_{\Gamma_{\acute{i}}(r)}\right) = \nu_{\acute{i}} \text{ for all } \acute{i} \in J_r. \tag{45}$$

This completes the existence of Γ-multiple FP.

Step 5. In this step, we will show that the FP of G is unique.

Suppose $w \in \mathbb{M}^r$ be another FP of G. By condition (e), there exists $\eta = (\eta_1, \eta_2, \cdots, \eta_r) \in \mathbb{M}^r$ such that $\nu \preccurlyeq_r \eta$ and $w \preccurlyeq_r \eta$.

Put $\eta^0 = \eta$ and define

$$\begin{aligned}
\eta_1^\vartheta &= G\left(\eta^{\vartheta-1}_{\Gamma_1(1)}, \eta^{\vartheta-1}_{\Gamma_1(2)}, \cdots, \eta^{\vartheta-1}_{\Gamma_1(r)}\right),\\
\eta_2^\vartheta &= G\left(\eta^{\vartheta-1}_{\Gamma_2(1)}, \eta^{\vartheta-1}_{\Gamma_2(2)}, \cdots, \eta^{\vartheta-1}_{\Gamma_2(r)}\right),\\
&\vdots\\
\eta_r^\vartheta &= G\left(\eta^{\vartheta-1}_{\Gamma_r(1)}, \eta^{\vartheta-1}_{\Gamma_r(2)}, \cdots, \eta^{\vartheta-1}_{\Gamma_r(r)}\right).
\end{aligned} \tag{46}$$

By the induction method, we have

$$\nu \preccurlyeq_r \eta^\vartheta, w \preccurlyeq_r \eta^\vartheta, \tag{47}$$

for all $\vartheta \ge 0$. By condition (e), $\nu \preccurlyeq_r \eta^0$. Assume that above condition holds for $\vartheta - 1$.

Using the procedure of Step 1, it can be shown that

$$\begin{aligned}
\nu &= G\left(\nu_{\Gamma_{\acute{i}}(1)}, \nu_{\Gamma_{\acute{i}}(2)}, \cdots, \nu_{\Gamma_{\acute{i}}(r)}\right)\\
&\preccurlyeq_{\acute{i}} G\left(\eta^{\vartheta-1}_{\Gamma\acute{i}(1)}, \eta^{\vartheta-1}_{\Gamma_{\acute{i}}(2)}, \cdots, \eta^{\vartheta-1}_{\Gamma\acute{i}(r)}\right) = \eta^\vartheta_{\acute{i}},
\end{aligned} \tag{48}$$

for all $\acute{i} \in J_r$; that is, $\nu \preccurlyeq_r \eta^\vartheta$. Similarly, we can prove the second inequality. Further, we prove that

$$\lim_{\vartheta\longrightarrow\infty} \sigma^r\left(\nu, \eta^\vartheta\right) = 0. \tag{49}$$

For this, we first show that if $\sigma^r(\nu, \eta^\vartheta) = 0$ for some ϑ_0 then $\sigma^r(\nu, \eta^\vartheta) = 0$, for all $\vartheta \ge \vartheta_0$. Indeed, from above condition, it follows that

$$\left(\nu_{\Gamma_{\acute{i}}(1)}, \nu_{\Gamma_{\acute{i}}(2)}, \cdots, \nu_{\Gamma_{\acute{i}}(r)}\right) \preccurlyeq_r \left(\eta^\vartheta_{\Gamma\acute{i}(1)}, \eta^\vartheta_{\Gamma_{\acute{i}}(2)}, \cdots, \eta^\vartheta_{\Gamma\acute{i}(r)}\right), \tag{50}$$

for all $\acute{i} \in J_r$ and $\vartheta \ge 1$. From (a), it follows that

$$\begin{aligned}
\psi\left(\sigma\left(\nu_{\acute{i}}, \eta^\vartheta_{\acute{i}}\right)\right) &= \psi\left(\sigma\left(G\left(\nu_{\Gamma_{\acute{i}}(1)}, \nu_{\Gamma_{\acute{i}}(2)}, \cdots, \nu_{\Gamma_{\acute{i}}(r)}\right),\right.\right.\\
&\quad \left.\left.\cdot\, G\left(\eta^{\vartheta-1}_{\Gamma\acute{i}(1)}, \eta^{\vartheta-1}_{\Gamma_{\acute{i}}(2)}, \cdots, \eta^{\vartheta-1}_{\Gamma\acute{i}(r)}\right)\right)\right),\\
&\le \theta\left(\frac{\sup_{j\in J_r}\left(\sigma\left(\nu_{\Gamma_{\acute{i}}(j)}, \eta^{\vartheta-1}_{\Gamma\acute{i}(j)}\right)\right)}{s+1}\right)\\
&\quad + \varphi\left(\frac{\sup_{j\in J_r}\left(\sigma\left(\nu_{\Gamma_{\acute{i}}(j)}, \eta^{\vartheta-1}_{\Gamma\acute{i}(j)}\right)\right)}{s+1}\right),\\
&\le \theta\left(\sup_{j\in J_r}\left(\sigma\left(\nu_{\Gamma_{\acute{i}}(j)}, \eta^{\vartheta-1}_{\Gamma i'(j)}\right)\right)\right)\\
&\quad + \varphi\left(\sup_{j\in J_r}\left(\sigma\left(\nu_{\Gamma_{\acute{i}}(j)}, \eta^{\vartheta-1}_{\Gamma\acute{i}(j)}\right)\right)\right),
\end{aligned} \tag{51}$$

for all $\acute{i} \in J_r$ and $\vartheta \ge 1$. Recall that $\Gamma_{\acute{i}}(J_r) \subseteq J_r$. Hence

$$\sup_{j\in J_r}\left(\sigma\left(\nu_{\Gamma_{\acute{i}}(j)}, \eta^{\vartheta-1}_{\Gamma\acute{i}(j)}\right)\right) \le \sigma^r\left(\nu, \eta^{\vartheta-1}\right), \tag{52}$$

for all $\vartheta \ge 1$. Taking into account $(*)$, it follows

$$\begin{aligned}
&\theta\left(\sup_{j\in J_r}\left(\sigma\left(\nu_{\Gamma_{\acute{i}}(j)}, \eta^{\vartheta-1}_{\Gamma\acute{i}(j)}\right)\right)\right) + \varphi\left(\sup_{j\in J_r}\left(\sigma\left(\nu_{\Gamma_{\acute{i}}(j)}, \eta^{\vartheta-1}_{\Gamma\acute{i}(j)}\right)\right)\right),\\
&\le \psi\left(\sup_{j\in J_r}\left(\sigma\left(\nu_{\Gamma_{\acute{i}}(j)}, \eta^{\vartheta-1}_{\Gamma\acute{i}(j)}\right)\right)\right) \le \psi\left(\sigma^r\left(\nu, \eta^\vartheta\right)\right),
\end{aligned} \tag{53}$$

for all $\acute{i} \in J_r$ and $\vartheta \geq 1$. Combining (50) and (52), we get

$$\psi\left(\sup_{j\in J_r}\left\{\sigma\left(\nu_{\acute{i}}, \eta_{\acute{i}}^{\vartheta}\right)\right\}\right) \leq \psi\left(\sigma^r\left(\nu, \eta^{\vartheta-1}\right)\right), \quad (54)$$

for $\vartheta \geq 1$. Since, ψ is an altering distance function

$$\psi\left(\sigma^r\left(\nu, \eta^{\vartheta}\right)\right) \leq \psi\left(\sigma^r\left(\nu, \eta^{\vartheta-1}\right)\right). \quad (55)$$

Now, it is obvious that if $\sigma^r(\nu, \eta^{\vartheta_0}) = 0$, then $\sigma^r(\nu, \eta^{\vartheta}) = 0$ for all $\vartheta \geq \vartheta_0$.

Next, assume that $\sigma^r(\nu, \eta^{\vartheta}) > 0$ for all $\vartheta \geq 1$. Using the same manner adopted in Step 1, it can be proved that $\sigma^r(\nu, \eta^{\vartheta}) \leq \sigma^r(\nu, \eta^{\vartheta-1})$. Hence, there exists $r \geq 0$ such that $\lim_{\vartheta\longrightarrow\infty}\sigma^r(\nu, \eta^{\vartheta}) = r$. In similar way, it can be shown

$$\lim_{\vartheta\longrightarrow\infty} \sigma^r\left(\nu, \eta^{\vartheta}\right) = 0 \text{ and } \lim_{\vartheta\longrightarrow\infty} \sigma^r\left(w, \eta^{\vartheta}\right) = 0. \quad (56)$$

On the other hand,

$$\sigma^r(\nu, w) \leq s\left[\sigma^r\left(\nu, \eta^{\vartheta}\right) + \sigma^r\left(\eta^{\vartheta}, w\right)\right]. \quad (57)$$

Applying $\lim\vartheta \longrightarrow \infty$ over the above inequality, it follows

$$\sigma^r(\nu, w) = 0. \quad (58)$$

This implies $\nu = w$. Hence, G has a unique multidimensional FP.□ □

4. Some Consequences and Example of Theorem 7

In the following section, some important concept regarding the consequences of Theorem 7 are discussed which are in terms of the main results represented in the articles [4, 16]. An illustrative example is also added in this section which will be helpful for the readers to understand the structure of multidimensional FP under the weak contractions for partially ordered s-distance spaces.

Corollary 8. *Let $(\mathbb{M}, \sigma, \preccurlyeq)$ is known as complete partially ordered s-distance space and $\Gamma = (\Gamma_1, \Gamma_2, \cdots, \Gamma_r)$ be a collection of mappings verifying $\Gamma_{\acute{i}} \in \Omega_{J,K}$ if $\acute{i} \in J$ and $\Gamma_{\acute{i}} \in \Omega'_{J,K}$ if $\acute{i} \in K$. Assume that the mapping $G : \mathbb{M}^r \longrightarrow \mathbb{M}$ satisfies the following conditions:*

(1) If there exists $\gamma \in (0, 1)$ such that

$$\sigma(G(\varsigma_1, \varsigma_2, \cdots, \varsigma_r), G(\xi_1, \xi_2, \cdots, \xi_r)) \leq \gamma\sigma^r(\varsigma, \xi), \quad (59)$$

for all $\varsigma = (\varsigma_1, \varsigma_2, \cdots, \varsigma_r) \in \mathbb{M}^r$ and $\xi = (\xi_1, \xi_2, \cdots, \xi_r) \in \mathbb{M}^r$ with $\varsigma \preccurlyeq_r \xi$ satisfies conditions.

(2) To (e) of the above theorem then G has a unique multidimensional FP

Proof. We can prove this corollary easily by taking $\psi(\varsigma) = \varsigma$, $\theta(\varsigma) = (s+1)\gamma\varsigma$, and $\varphi(\varsigma) = 0$ in the above theorem.□ □

Remark 9. Theorems 2.1 and 2.2 of [16], in s-distance spaces, follow from Corollary 8. In [16], contractive condition is

$$\sigma(F(\varsigma_1, \varsigma_2), F(\xi_1, \xi_2)) \leq \frac{\delta}{2}[\sigma(\varsigma_1, \xi_1) + \sigma(\varsigma_2, \xi_2)], \quad (60)$$

and for $\varsigma, \xi \in \mathbb{M}^2$ such that $\varsigma \preccurlyeq_2 \xi$, it implies

$$\sigma(F(\varsigma_1, \varsigma_2), F(\xi_1, \xi_2)) \leq \frac{\delta}{2}[\sigma(\varsigma_1, \xi_1) + \sigma(\varsigma_2, \xi_2)] \leq \delta\sigma^2(\varsigma, \xi). \quad (61)$$

Applying Corollary 8, we get the desired result in s-distance space.

Remark 10. Corollary 8 also generalizes the main triple FP result of [4], in which the space under consideration is a metric space. We generalize it for b -metric space and for an s-distance space. In [4], $\Gamma = \{\Gamma_1, \Gamma_2, \Gamma_3\}$, A_3 is chosen as $A = \{1, 3\}$, $B = \{2\}$, collection of mappings is defined as

$$\begin{pmatrix} \Gamma_1(1) & \Gamma_1(2) & \Gamma_1(3) \\ \Gamma_2(1) & \Gamma_2(2) & \Gamma_2(3) \\ \Gamma_3(1) & \Gamma_3(2) & \Gamma_3(3) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad (62)$$

the contractive condition in [4] is

$$\begin{aligned} &\sigma(F(\varsigma_1, \varsigma_2, \varsigma_3), F(\xi_1, \xi_2, \xi_3)) \\ &\quad \leq \delta_1\sigma(\varsigma_1, \xi_1) + \delta_2\sigma(\varsigma_2, \xi_2) + \delta_3\sigma(\varsigma_3, \xi_3), \end{aligned} \quad (63)$$

for any $\varsigma = (\varsigma_1, \varsigma_2, \varsigma_3), \xi = (\xi_1, \xi_2, \xi_3)$ with $\varsigma \preccurlyeq_i \xi$ and $\delta_1, \delta_2, \delta_3 \geq 0$ and $\delta_1 + \delta_2 + \delta_3 < 1$. Obviously

$$\sigma(F(\varsigma_1, \varsigma_2, \varsigma_3), F(\xi_1, \xi_2, \xi_3)) \leq (\delta_1 + \delta_2 + \delta_3)\sigma^3(\varsigma, \xi). \quad (64)$$

Applying Corollary 8, we get the desired result in s-distance space.

Corollary 11. *Consider $(\mathbb{M}, \sigma, \preccurlyeq)$ as a complete partially ordered b-metric space and $\Gamma = (\Gamma_1, \Gamma_2, \cdots, \Gamma_r)$ be collection of mappings verifying $\Gamma_{\acute{i}} \in \Omega_{J,K}$ if $\acute{i} \in J$ and $\Gamma_{\acute{i}} \in \Omega'_{J,K}$ if $\acute{i} \in K$. Assume that the mapping $G : \mathbb{M}^r \longrightarrow \mathbb{M}$ satisfies conditions (a) to (e), then G has a unique multidimensional FP.*

Example 12. Let $\mathbb{M} = \mathbb{R}$, $\mathbb{M}^r = \mathbb{R}^r$. Define $\sigma^r(\omega, \nu) = \sup_{\acute{i}\in J_r}\{\sigma(\varsigma_{\acute{i}}, \xi_{\acute{i}})\}$ and $\sigma(\varsigma, \xi) = |\varsigma - \xi|$, for all $\omega = (\varsigma_1, \varsigma_2, \cdots, \varsigma_r), \nu = (\xi_1, \xi_2, \cdots, \xi_r) \in \mathbb{M}^r$. Now, define $G : \mathbb{M}^r \longrightarrow \mathbb{M}$, as $G(\varsigma_1, \varsigma_2, \cdots, \varsigma_r) = \varsigma_1/2$.

Since "≤" is a partial order on $\mathbb{M}$, therefore

$$\begin{aligned} &\omega \preccurlyeq_r \nu \text{ if and only if } \varsigma_1 \preccurlyeq_i \xi_i, \\ &\varsigma \preccurlyeq_i \xi \text{ iff } \left\{ \begin{array}{ll} \varsigma \leq \xi & i \in J = \{1, 2, \cdots, k\} \\ \varsigma \leq \xi & i \in K = \{k+1, k+2, \cdots, r\} \end{array} \right\}, \end{aligned} \tag{65}$$

where J and K partitions $J_r = \{1, 2, \cdots, r\}$ such that $J \cup K = J_r$ and $J \cap K = \phi$.

A function $\psi : [0,\infty) \longrightarrow 0,\infty)$ defined as $\psi(t) = (1/3)t$ is increasing and continuous, $\theta : [0,\infty) \longrightarrow [0,\infty)$ defined as $\theta(t) = (1/9)t$, θ is increasing and upper semicontinuous function, and $\varphi : [0,\infty) \longrightarrow [0,\infty)$ defined as $\varphi(t) = (2/9)t$, which is increasing and lower semicontinuous function, for all $t \in [0,\infty)$.

$$\begin{aligned} \sigma(G\omega, G\nu) &= \left| \frac{\varsigma_1 - \xi_1}{2} \right|, \psi(\sigma(G\omega, G\nu)) \\ &= \left| \frac{\varsigma_1 - \xi_1}{6} \right|, \theta\left(\frac{\sigma^r(\omega, \nu)}{s+1} \right) \\ &= \frac{\sigma^r(\omega, \nu)}{9(s+1)}, \varphi\left(\frac{\sigma^r(\omega, \nu)}{s+1} \right) = \frac{2\sigma^r(\omega, \nu)}{9(s+1)}. \end{aligned} \tag{66}$$

Consider

$$\begin{aligned} &\theta\left(\frac{\sigma^r(\omega, \nu)}{s+1} \right) + \varphi\left(\frac{\sigma^r(\omega, \nu)}{s+1} \right) \\ &\quad = \frac{\sigma^r(\omega, \nu)}{9(s+1)} + \frac{2\sigma^r(\omega, \nu)}{9(s+1)} = \frac{\sigma^r(\omega, \nu)}{2(s+1)}, \\ &\quad \leq \frac{\sigma^r(\omega, \nu)}{2} = \frac{\sup_{\acute{i} \in J_r} \{|\varsigma_{\acute{i}} - \xi_{\acute{i}}|\}}{2}. \end{aligned} \tag{67}$$

Thus, we have

$$\left| \frac{\varsigma_1 - \xi_1}{6} \right| \leq \frac{\sup_{\acute{i} \in J_r} \{|\varsigma_{\acute{i}} - \xi_{\acute{i}}|\}}{2}, \tag{68}$$

which follows

$$\psi(\sigma(G\omega, G\nu)) \leq \theta\left(\frac{\sigma^r(\omega, \nu)}{s+1} \right) + \varphi\left(\frac{\sigma^r(\omega, \nu)}{s+1} \right). \tag{69}$$

Now to show G has mixed monotone property, if $\xi \preccurlyeq \eta \Rightarrow \xi \leq \eta$, consider

$$\begin{aligned} G(\varsigma_1, \varsigma_2, \cdots, \varsigma_{i-1}, \xi, \cdots, \varsigma_r) &= \frac{\varsigma_1}{2}, \\ G(\varsigma_1, \varsigma_2, \cdots, \varsigma_{i-1}, \eta, \cdots, \varsigma_r) &= \frac{\varsigma_1}{2}. \end{aligned} \tag{70}$$

So

$$G(\varsigma_1, \varsigma_2, \cdots, \varsigma_{\acute{i}-1}, \xi, \cdots, \varsigma_r) \preccurlyeq_{\acute{i}} G(\varsigma_1, \varsigma_2, \cdots, \varsigma_{\acute{i}-1}, \eta, \cdots, \varsigma_r), \tag{71}$$

which implies G has mixed monotone property with respect to $\{J, K\}$. Set of mappings is defined as

$$\begin{pmatrix} \Gamma_1(1) & \Gamma_1(2) & \cdots & \Gamma_1(r) \\ \Gamma_2(1) & \Gamma_2(2) & \cdots & \Gamma_2(r) \\ \vdots & \vdots & & \vdots \\ \Gamma_r(1) & \Gamma_r(2) & \cdots & \Gamma_r(r) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & \cdots & r \\ 2 & 3 & 4 & \cdots & 1 \\ 3 & 4 & 5 & \cdots & 2 \\ \vdots & \vdots & \vdots & & \vdots \\ r & 1 & 2 & \cdots & r-1 \end{pmatrix}. \tag{72}$$

Let $(\varsigma_1^0, \varsigma_2^0, \cdots, \varsigma_r^0) = (\varsigma_1, \varsigma_2, \cdots, \varsigma_r) \in \mathbb{M}^r$ and

$$\begin{aligned} \varsigma_1^\vartheta &= G\left(\varsigma_1^{\vartheta-1}, \varsigma_2^{\vartheta-1}, \cdots, \varsigma_r^{\vartheta-1} \right) = \frac{\varsigma_1^{\vartheta-1}}{2}, \\ \varsigma_2^\vartheta &= G\left(\varsigma_2^{\vartheta-1}, \varsigma_3^{\vartheta-1}, \cdots, \varsigma_r^{\vartheta-1}, \varsigma_1^{\vartheta-1} \right) = \frac{\varsigma_2^{\vartheta-1}}{2}, \\ &\vdots \\ \varsigma_r^\vartheta &= G\left(\varsigma_r^{\vartheta-1}, \varsigma_1^{\vartheta-1}, \cdots, \varsigma_{r-2}^{\vartheta-1}, \varsigma_{r-1}^{\vartheta-1} \right) = \frac{\varsigma_r^{\vartheta-1}}{2}, \end{aligned} \tag{73}$$

for $\vartheta = 1$

$$\begin{aligned} \varsigma_1^1 &= \frac{\varsigma_1^0}{2} = \frac{\varsigma_1}{2}, \\ \varsigma_2^1 &= \frac{\varsigma_2^0}{2} = \frac{\varsigma_2}{2}, \\ &\vdots \\ \varsigma_r^1 &= \frac{\varsigma_r^0}{2} = \frac{\varsigma_r}{2}, \end{aligned} \tag{74}$$

for $\vartheta = 2$

$$\begin{aligned} \varsigma_1^2 &= \frac{\varsigma_1^1}{2} = \frac{\varsigma_1}{2^2}, \\ \varsigma_2^2 &= \frac{\varsigma_2^1}{2} = \frac{\varsigma_2}{2^2}, \\ &\vdots \\ \varsigma_r^2 &= \frac{\varsigma_r^1}{2} = \frac{\varsigma_r}{2^2}. \end{aligned} \tag{75}$$

Continuing this process, we have

$$\begin{aligned} \lim_{\vartheta \longrightarrow \infty} \left(\varsigma_1^\vartheta, \varsigma_2^\vartheta, \cdots, \varsigma_r^\vartheta \right) &= \lim_{\vartheta \longrightarrow \infty} \left(\frac{\varsigma_1}{2^\vartheta}, \frac{\varsigma_2}{2^\vartheta}, \cdots, \frac{\varsigma_r}{2^\vartheta} \right) \\ &= (0, 0, \cdots, 0) = (O_1, \cdots, O_r). \end{aligned} \tag{76}$$

Thus

$$O_i{}' = G\left(O_{\Gamma_i(1)}, O_{\Gamma_i(2)}, \cdots, O_{\Gamma_i(r)}\right), \tag{77}$$

Hence, $O_i{}'$ is a unique multidimensional FP of G.

5. An Application

To give the broad impact of FP results to the different areas of study like physics, biological sciences, engineering, game theory, and economics has always been an interest of various mathematicians. The applications can be found in various directions (like [17, 18]). In the following section, we have also included an application of the main result to find the solution of a system of integral equations.

Consider the point $a, b \in \mathbb{R}$ with $a < b$ and let $I = [a, b]$. Let the space $\mathbb{M} = C(I)$ of all continuous real valued functions defined on I, which define a partial order $\preccurlyeq$ on $\mathbb{M}$ by

$$\lambda \preccurlyeq \mu \text{ if and only if } \lambda(t) \le \mu(t), \text{for all } t \in [a, b], \tag{78}$$

and the distance

$$\sigma(\lambda, \mu) = \max_{t\in I}|\lambda(t) - \mu(t)|^p \text{ for all } \lambda, \mu \in \mathbb{M}, p \ge 1, \tag{79}$$

then $(\mathbb{M}, \sigma)$ is a complete partially ordered s-distance space with $s = 2^{p-1}$.

Consider the following system of equations

$$\varsigma_1(t) = \kappa + \int_a^t K(w)L(\varsigma_1(w), \varsigma_2(w), \cdots, \varsigma_r(w))dw, \tag{80}$$

$$\begin{aligned}\varsigma_i(t) = \kappa + \int_a^t K(w)L(&\varsigma_i(w), \varsigma_{i+1}(w), \cdots, \varsigma_r(w),\\ &\cdot \varsigma_1(w), \cdots, \varsigma_{i-1}(w))dw,\end{aligned} \tag{81}$$

for $i = 1, 2, \cdots, r$, where $L : \mathbb{R}^r \longrightarrow \mathbb{R}$ be a mapping verifying

(i) L is continuous

(ii) For all $(e_1, e_2, \cdots, e_r), (h_1, h_2, \cdots, h_r) \in \mathbb{R}^r$

$$|L(e_1, e_2, \cdots, e_r) - L(h_1, h_2, \cdots, h_r)| \le \frac{\left(\sup_{1\le i\le r}|e_i - h_i|^p\right)^{1/p}}{(b-a)(s+1)^{2/p}}, \tag{82}$$

and $K : I \longrightarrow \mathbb{R}$ be a continuous mapping such that $K(t) \ge 0$ and $\int_a^b K(t)dt \le (b - a)$.

Define a mapping $G : \mathbb{M}^r \longrightarrow \mathbb{M}$ for all $\varsigma = (\varsigma_1, \varsigma_2, \cdots, \varsigma_r)$ in $\mathbb{M}^r$ and $\kappa \in \mathbb{R}$ such that

$$G(\varsigma_1, \varsigma_2, \cdots, \varsigma_r)(t) = \kappa + \int_a^t K(w)L(\varsigma_1(w), \varsigma_2(w), \cdots, \varsigma_r(w))dw. \tag{83}$$

Clearly, $G \in C(I)$. To find solution of the system (79), it is required to show that G has a multiple FP. For this, consider

$$\begin{aligned}
&\sigma(G(\varsigma_1, \varsigma_2, \cdots, \varsigma_r), G(\nu_1, \nu_2, \cdots, \nu_r))\\
&\quad = \max_{t\in I}|G(\varsigma_1, \varsigma_2, \cdots, \varsigma_r)(t) - G(\nu_1, \nu_2, \cdots, \nu_r)(t)|^p,\\
&\quad = \max_{t\in I}\left|\begin{matrix}\left(\kappa + \int_a^t K(w)L(\varsigma_1(w), \varsigma_2(w), \cdots, \varsigma_r(w))dw\right)\\ -\left(\kappa + \int_a^t K(w)L(\nu_1(w), \nu_2(w), \cdots, \nu_r(w))dw\right)\end{matrix}\right|^p,\\
&\quad = \max_{t\in I}\left|\int_a^t K(w)\begin{pmatrix}L(\varsigma_1(w), \varsigma_2(w), \cdots, \varsigma_r(w))\\ -L(\nu_1(w), \nu_2(w), \cdots, \nu_r(w))\end{pmatrix}dw\right|^p,\\
&\quad \le \max_{t\in I}\left(\int_a^t |K(w)|\left|\begin{matrix}L(\varsigma_1(w), \varsigma_2(w), \cdots, \varsigma_r(w)),\\ -L(\nu_1(w), \nu_2(w), \cdots, \nu_r(w))\end{matrix}\right|dw\right)^p,\\
&\quad \le \max_{t\in I}\left(\int_a^t K(w)\frac{\left(\sup_{1\le i\le r}|\varsigma_i(w) - \nu_i(w)|^p\right)^{1/p}}{(b-a)(s+1)^{2/p}}dw\right)^p,\\
&\quad \le \max_{t\in I}\left(\begin{matrix}\left(\frac{1}{(b-a)(s+1)^{2/p}}\right)\times\\ \sup_{1\le i\le r}\int_a^t K(w)\left(\max_{t\in I}|\varsigma_i(w) - \nu_i(w)|^p\right)^{1/p}dw\end{matrix}\right)^p,\\
&\quad = \max_{t\in I}\left(\left(\frac{1}{(b-a)(s+1)^{2/p}}\right)\sup_{1\le i\le r}\int_a^t K(w)(\sigma(\varsigma_i, \nu_i))^{1/p}dw\right)^p,\\
&\quad \le \frac{\sup_{1\le i\le r}\{\sigma(\varsigma_i, \nu_i)\}}{(s+1)^2}\left(\frac{1}{(b-a)}\left(\max_{t\in I}\int_a^t K(w)dw\right)\right)^p,\\
&\quad \cdot \sigma(G(\varsigma_1, \varsigma_2, \cdots, \varsigma_r), G(\nu_1, \nu_2, \cdots, \nu_r))\\
&\quad \le \frac{\sigma^r((\varsigma_1, \varsigma_2, \cdots, \varsigma_r), (\nu_1, \nu_2, \cdots, \nu_r))}{(s+1)^2}.
\end{aligned} \tag{84}$$

Define functions $\psi, \theta, \varphi : [0,\infty) \longrightarrow [0,\infty)$ by

$$\psi(t) = (s+1)t, \theta(t) = t, \varphi(t) = 0, \tag{85}$$

such that for $t > 0$

$$\psi(t) - \theta(t) - \varphi(t) > 0. \tag{86}$$

Then, from above inequality, we get

$$\psi(\sigma(G\varsigma, G\nu)) \le \theta\left(\frac{\sigma^r(\varsigma, \nu)}{s+1}\right) + \varphi\left(\frac{\sigma^r(\varsigma, \nu)}{s+1}\right), \tag{87}$$

which by Theorem 7 implies that G has a unique multiple FP, which gives the required solution of system (79).

6. Conclusion

The main idea of this paper was to prove a multiple fixed point (FP) result for partially ordered s-distance spaces under (θ, ϕ, ψ)-type weak contractive condition which is the generalization of some well-known results in literature. Many other related and relevant results can be obtained in the same manner for partially ordered generalized distance spaces such as C-distance space, balanced distance space, and (s, q) distance space.

Data Availability

No data used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, no. 1, pp. 133–181, 1992.

[2] H. Alsamir, M. S. M. Noorani, and W. Shatanawi, "On new fixed point theorems for three types of (α,β)-(ψ,θ,ϕ)-multivalued contractive mappings in metric spaces," *Cogent Mathematics*, vol. 3, no. 1, article 1257473, 2016.

[3] D. Guo and V. Lakshmikantham, "Coupled fixed points of nonlinear operators with applications," *Coupled fixed points of nonlinear operators with applications*, vol. 11, no. 5, pp. 623–632, 1987.

[4] V. Berinde and M. Borcut, "Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 15, pp. 4889–4897, 2011.

[5] E. Karapinar, *Quadruple fixed point theorems for weak φ-contractions*, ISRN Mathematical Analysis, 2011.

[6] M. Berzig and B. Samet, "An extension of coupled fixed point's concept in higher dimension and applications," *Computers & Mathematcs with Applications*, vol. 63, no. 8, pp. 1319–1334, 2012.

[7] A. Roldán, J. Martnez-Moreno, and C. Roldán, "Multidimensional fixed point theorems in partially ordered complete metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 396, no. 2, pp. 536–545, 2012.

[8] E. Karapinar, A. Roldán, J. Martínez-Moreno, and C. Roldán, "Meir-Keeler type multidimensional fixed point theorems in partially ordered metric spaces," *Abstract and Applied Analysis*, vol. 2013, Article ID 406026, 9 pages, 2013.

[9] M. M. Choban and V. Berinde, "A general concept of multiple fixed point for mappings defined on spaces with a distance," *Carpathian Journal of Mathematics*, vol. 33, no. 3, pp. 275–286, 2017.

[10] M. M. Choban and V. Berinde, "Multiple fixed point theorems for contractive and Meir-Keeler type mappings defined on partially ordered spaces with a distance," 2016, https://arxiv.org/abs/1701.00518.

[11] M. M. Choban, "Fixed points of mappings defined on spaces with distance," *Carpathian Journal of Mathematics*, vol. 32, no. 2, pp. 173–188, 2016.

[12] M. Rashid, R. Bibi, A. Kalsoom, D. Baleanu, A. Ghaffar, and K. S. Nisar, "Multidimensional fixed points in generalized distance spaces," *Advances in Difference Equations*, vol. 2020, no. 1, 20 pages, 2020.

[13] H. Akhadkulov, R. Zuhra, A. B. Saaban, and F. Shaddad, "The existence of Υ-fixed point for the multidimensional nonlinear mappings satisfying (ψ, φ, θ)-weak contractive conditions," *Sains Malaysiana*, vol. 46, no. 8, pp. 1341–1346, 2017.

[14] H. Akhadkulov, S. M. Noorani, A. B. Saaban, F. M. Alipiah, and H. Alsamir, "Notes on multidimensional fixed-point theorems," *Demonstratio Mathematica*, vol. 50, no. 1, pp. 360–374, 2017.

[15] A. Roldán, J. Martinez-Moreno, C. Roldán, and E. Karapınar, "Some remarks on multidimensional fixed point theorems," *Fixed Point Theory*, vol. 15, no. 2, pp. 545–558, 2014.

[16] T. G. Bhaskar and V. Lakshmikantham, "Fixed point theorems in partially ordered metric spaces and applications," *Nonlinear analysis: theory, methods & applications*, vol. 65, no. 7, pp. 1379–1393, 2006.

[17] D. Baleanu, A. Jajarmi, H. Mohammadi, and S. Rezapour, "A new study on the mathematical modelling of human liver with Caputo-Fabrizio fractional derivative," *Chaos, Solitons & Fractals*, vol. 134, article 109705, 2020.

[18] S. Rezapour, H. Mohammadi, and A. Jajarmi, "A new mathematical model for Zika virus transmission," *Advances in Difference Equations*, vol. 2020, no. 1, 15 pages, 2020.