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Research Article

An Operational Matrix Technique Based on Chebyshev Polynomials for Solving Mixed Volterra-Fredholm Delay Integro-Differential Equations of Variable-Order

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In this work, an algorithm for finding numerical solutions of linear fractional delay-integro-differential equations (LFDIDEs) of variable-order (VO) is introduced. The operational matrices are used as discretization technique based on shifted Chebyshev polynomials (SCPs) of the first kind with the spectral collocation method. The proposed VO-LFDIDEs have multiterms of integer, fractional-order derivatives for delayed or nondelayed and mixed Volterra-Fredholm integral terms. The introduced model is a more general form of linear fractional VO pantograph, neutral, and mixed Fredholm–Volterra integro-differential equations with delay arguments. Caputo's VO fractional derivative operator is used to generate the matrices of the derivative. Operational matrices are presented for all terms. The reliability and efficiency of the proposed scheme are demonstrated by some numerical experiments. Also, some examples are included to improve the validity and applicability of the techniques. Finally, comparisons between the proposed method and other methods were used to solve this kind of equation.

1. Introduction

In recent few decades, fractional calculus has shown to be a useful implement to formulate many problems in science and engineering where the fractional derivatives and integrals can be used for the description of the properties of various real materials in different branches of science [1–7]. In 1993, Samko and Ross [8] have introduced the variable order derivative (VOD) operator just as a generalization of the fractional-order derivative, and they provided some of its main properties. In this introduced operator, the order of the derivative is a function of the independent variables such as time and space variables. With this generalization, many applications have been established in mechanics, physics, signal processing, and control [9–12]. One of the most important tools in studying the prime numbers is the Riemann zeta-function which embodies both additive and multiplicative structures in a single function, the fractional generalization of it is also found in [13–17].

Consequently, a generalized kind of differential equation appears named VO fractional differential equations (VOFDEs) [18–24]. In VOFDEs, the differential operator is VO, and the derivative changes in general concerning the independent variable or concerning an external functional behavior [8, 25]. Analytical solutions for the VOFDEs are difficult to obtain because the kernel of the VO's operator has a variable exponent, hence, there will be increasingly rapid developments in numerical approaches to VOFDEs [26–29]. The delay, neutral delay FDEs, and fractional integro-differential equations (FIDEs) are considered as a generalization and development of FDEs, and dealing with them analytically in most of the cases is difficult [30–34]. The VO fractional delay differential equations (VOFDDEs) are a kind of generalization for the fractional delay differential equations (FDDEs) [35–38]. VOFDDEs did not receive the attention that the FDDEs accomplished; however, the potential to describe complicated behavior by the VO of differentiation or integration is clear, all this was a motivation to start studying this type of equation. Soon after, a variety of definitions have been offered for variable order integral and derivative operators such as Riemann–Liouville (RL) [39], Grünwald–Letnikov [40], Caputo, and Fabrizio [41] derivatives.

In this work, we use operational matrices discretization technique based on shifted Chebyshev polynomials (SCPs) of the first kind with the spectral collocation method to solve the following type of linear fractional delay-integro-differential equations of variable-order (LFDIDEs):

$$\sum_{i=0}^{n_1} Q_i(x)\, D^{\nu_i(x)} y(x) + \sum_{j=0}^{n_2} P_j(x)\, D^{\alpha_j(x)} y(x-\tau_j) + \sum_{s=0}^{n_3} R_s(x)\, y^{(s)}(x-\varepsilon_s) = g(x) + \int_a^b \mathrm{K}(x,t)\, y(t)\, dt + \int_a^x \mathrm{V}(x,t)\, y(t)\, dt, x \in [a,b], \quad (1)$$

under the conditions

$$y^{(i)}(\rho_i) = \mu_i, i = 0, 1, 2 \cdots, m-1, \quad (2)$$

where $0 < \alpha_j(x) \le 1$, $0 < \nu_i(x) \le 1$, and τ_j, ε_s are positive real delay arguments, and the known functions $Q_i(x), P_j(x), R_s(x)$, $K(x,t)$, $V(x,t)$, and $g(x)$ are well defined, additionally, $y(x)$ is an unknown function to be determined. The symbole D denotes the variable-order derivative (VOD) operator in the Caputo sense.

Corollary 1. *The independent variable x of (1) belongs to $[a, b]$, which is the intersection of the intervals of the different delayed arguments and $[0,h]$, i.e., $x \in [a,b] = [\tau_j, h+\tau_j] \bigcap [\varepsilon_s, h+\varepsilon_s] \bigcap [0,h]$, and $\tau_j, \varepsilon_s < h$.*

The introduced model (1) is a more general form of linear fractional VO pantograph, neutral, and mixed Fredholm–Volterra integro-differential equations with delay arguments. Several methods have been presented for solving VO differential equations [42–46]. For example, Ganji et al. applied the first-kind CPs to obtain the solution of variable order differential equations [47]. Doha et al. solved variable-order fractional Volterra integro-differential equations by shifted Legendre–Gauss–Lobatto collocation method [39]. A numerical method based on the Jacobi polynomials for solving variable-order differential equations has been proposed in [48]. Furthermore, Ganji et al. have introduced a numerical scheme based on the Bernstein polynomials to solve variable order diffusion-wave equations [49]. As a notation, the proposed model (1) is a generalization of our previous reports [50–54] and other work [55–57] as well.

2. Basic Definitions

In this preliminary section, the definitions and properties of Caputo's VO fractional derivative and the required mathematical tools will be introduced. Also, briefly, the SCPs of the first kind, we will present what we need from their properties.

2.1. Caputo Variable-Order's Operator

Definition 2. The Caputo's VO fractional derivative operator of order $n-1 < \alpha(x) \le n$ operates a function $g(x) \in C^n[0,h]$ in [42, 43] as

$$D^{\alpha(x)} g(x) = \begin{cases} \dfrac{1}{\Gamma[n-\alpha(x)]} \displaystyle\int_0^x \dfrac{g(s)^{(n)}}{(x-s)^{\alpha(x)-n+1}} ds, & \text{for } n-1 < \alpha(x) < n, \\ g^{(n)}(x), & \text{for } \alpha(x) = n, \end{cases} \quad (3)$$

and $n \in \aleph$.

Remark 3. In this work, we write the Variable-Order Caputo's fractional symbol $D^{\alpha(x)}$ instead of ${}_0^C D_x^{\alpha(x)}$ for short.

Remark 4. The following useful property according to Definition 2:

$$D^{\alpha(x)} x^m = \begin{cases} \dfrac{\Gamma(m+1)}{\Gamma(m+1-\alpha(x))} x^{m-\alpha(x)}, & \text{for } m \ge n, \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

where

$$n-1 < \alpha(x) \le n, m \in \aleph_0 \text{ and } \aleph_0 = \aleph \cup \{0\}. \quad (5)$$

Definition 5. The Caputo's VO fractional derivative operator of order $\alpha(x), 0 < \alpha(x) \le 1$; operates $g(x)$, where $g(x) \in C^n[0,h]$ can be defined as [42, 58]

$$D^{\alpha(x)} g(x) = \frac{1}{\Gamma(1-\alpha(x))} \int_0^x \frac{g'(x)}{(x-t)^{\alpha(x)}} dt + \frac{g(0^+) - g(0^-)}{\Gamma(1-\alpha(x))} x^{-\alpha(x)}, \quad (6)$$

and $0 < \alpha(x) \le 1$. Definition 5 can be reformulated if we assume that; the starting time in a perfect situation, then:

$$D^{\alpha(x)} g(x) = \frac{1}{\Gamma(1-\alpha(x))} \int_0^x \frac{g'(x)}{(x-t)^{\alpha(x)}} dt, 0 < \alpha(x) < 1. \quad (7)$$

Remark 6. By (7), we can get the following formulas:

(i) The Caputo's VO operator is linear

$$D^{\alpha(x)} (\mu f(x) + \lambda g(x)) = \mu D^{\alpha(x)} f(x) + \lambda D^{\alpha(x)} g(x), \quad (8)$$

where μ and λ are any two constants.

(ii) As the ordinary differentiation operator the Caputo's VO operator as well, such that $D^{\alpha(x)} K = 0$, and K is constant.

2.2. Shifted First Kind of Chebyshev Polynomials. Next, let us introduce some properties of the SCPs [59]. It is well known that the classical CPs are defined on $[-1, 1]$ by the three-term recurrence relation:

$$T_{n+1}(t) = 2tT_n(t) - T_{n-1}(t), T_0(t) = 1, T_1(t) = t, n = 1, 2, \cdots. \tag{9}$$

Let the SCPs $T_n(2t/h - 1)$ be denoted by $T_n^h(x)$, which are defined on $[0, h]$. Then, $T_n^h(x)$ can be generated by using the following recurrence relation:

$$T_{n+1}^h(x) = 2\left(\frac{2x}{h} - 1\right)T_n^h(x) - T_{n-1}^h(x), n = 1, 2, \cdots, \tag{10}$$

where

$$T_0^h(x) = 1, T_1^h(x) = \frac{2x}{h} - 1. \tag{11}$$

The analytic form of the SCPs T_i^h of degree i is given by:

$$T_i^h(x) = \sum_{k=0}^{i} (-1)^{i-j} \frac{i(i+j-1)!2^{2j}}{(i-j)!(2j)!h^j} x^k. \tag{12}$$

The orthogonality condition of the SCPs is

$$\int_0^h T_i^h(x) T_j^h(x) w_h dx = \delta_j, \tag{13}$$

where

$$w_h = 1/\sqrt{hx - x^2}, \delta_j = \begin{cases} 0, & \text{for } i \neq j, \\ \phi_j/2\pi, & \text{for } i = j\, \phi_0 = 2, \phi_j = 1, j \geq 1. \end{cases} \tag{14}$$

A function $y(x)$, square integrable in $[0, h]$, may be expressed in terms of shifted Chebyshev polynomials as

$$y(x) = \sum_{i=0}^{\infty} a_i T_i^h(x), \tag{15}$$

where the coefficients a_i are given by

$$a_i = \frac{1}{h_i} \int_0^h y(x) T_i^h(x) w_h dx, i = 0, 1, 2, \cdots. \tag{16}$$

In practice, only the first $(1 + N)$-terms shifted Chebyshev polynomials are considered. The special values

$$T_{i,j}^h = (-1)^{i-j} \frac{i(i+j-1)!2^{2j}}{(i-j)!(2j)!h^j}, j \leq i, \tag{17}$$

will be of important use later.

In the approximation theory, the series in (15) can be approximated by taking the first $(N + 1)$ terms as follows:

$$y(x) \simeq y_N(x) = \sum_{i=0}^{N} a_i T_i^h(x) = A^T T^h(x), \tag{18}$$

where $A = [a_0, a_1, \cdots, a_N]^T$ is a vector, $T^h(x) = [T_0^h(x)\ T_1^h(x)\ \cdots\ T_N^h(x)]^T$ and if we assume that $X(x) = [\,x^0 \quad x^1 \quad \cdots \quad x^N\,]$.

From (12) and (17), $T^h(x)$ can be written as the following form:

$$T^h(x) = X(x) M^T, \tag{19}$$

where M is square lower triangle matrix with size $(N + 1) \times (N + 1)$ given by:

$$M = \begin{bmatrix} m_{00} & 0 & 0 & \cdots & 0 \\ m_{10} & m_{11} & 0 & \cdots & 0 \\ m_{20} & m_{21} & m_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{N+1,0} & m_{N+1,1} & m_{N+1,2} & \cdots & m_{N+1,N+1} \end{bmatrix}. \tag{20}$$

and its entries elements are given by:

$$m_{ij} = \begin{cases} 1, & \text{if } i = j = 0, \\ (-1)^{i-j} \dfrac{i(i+j-1)!2^{2j}}{(i-j)!(2j)!h^j}, & \text{if } j \leq i, \\ 0, & \text{if } j > i. \end{cases} \tag{21}$$

For example, if $N = 4$, then the square matrix M is given by:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 1 & -8 & 8 & 0 & 0 \\ -1 & 18 & -48 & 32 & 0 \\ 1 & -32 & 160 & -256 & 128 \end{pmatrix}. \tag{22}$$

Now, from (19), we can obtain the k^{th} derivative of the

Table 1: The comparisons between the absolute errors of different N for Example 1.

x	Absolute errors		
	Our method $N=3$	Our method $N=4$	Our method $N=5$
0.2	1.66×10^{-16}	2.77×10^{-16}	1.94×10^{-16}
0.4	1.66×10^{-16}	3.33×10^{-16}	2.77×10^{-16}
0.6	1.94×10^{-16}	2.77×10^{-16}	2.22×10^{-16}
0.8	1.11×10^{-16}	3.88×10^{-16}	2.77×10^{-16}
1	1.38×10^{-16}	3.40×10^{-16}	1.75×10^{-16}

matrix $T(x)$ as

$$T^{(k)}(x)=X^{(k)}(x)M^T, k=0,1,2,\cdots. \tag{23}$$

3. Operational Matrices

In this section, we introduce the generalized operational matrices for $T^{h(k)}(x)$, $T^{h(s)}(x-\tau_s)$, $D^{\nu_i(x)}T^h(x)$, and $D^{\alpha_i(x)}T^h(x-\tau_i)$ according to fractional calculus.

Lemma 7. *The $(k)^{th}$ order derivative of the row vector $T^h(x)$ can be in the following relation form [50, 60]:*

$$T^{h(k)}(x)=X(x)B^kM^T, \tag{24}$$

where B is the $(N+1)\times(N+1)$ operational matrix of derivative for $X(x)$ and can be obtained from

$$B=(\varepsilon_{ij})=\begin{cases}0, & \text{for otherwise},\\ j+1, & \text{for } i=j+1,\end{cases}, j=0,1,\cdots,N. \tag{25}$$

Proof (see [50, 60]). The row vector $T(x-\tau)$ represents in terms of the vector $X(x)$ in the following form [50, 60]:

$$T(x-\tau)=X(x)B_{-\tau}M^T, \tag{26}$$

where $B_{-\tau}$ is square upper triangle matrix with size $(N+1)\times(N+1)$ given by:

$$B_{-\tau}=(\beta_{ij})=\begin{cases}\begin{pmatrix}j\\ i\end{pmatrix}(-\tau)^{j-i}, & \text{for } j\geq i,\\ 0, & \text{for otherwise},\end{cases}, i,j=0,1,\cdots,N. \tag{27}$$

□

Corollary 8. *The $(s)^{th}$ order derivative of the delay row vector $T^h(x-\tau_s)$ can be represented as [50, 60]*

$$T^{h(s)}(x-\tau)=X^{(s)}(x-\tau_s)M^T=X(x)B^sB_{-\tau_s}M^T. \tag{28}$$

According to the previous lemmas with the fractional calculus by using the Caputo's variable-order fractional derivative, we introduce the following theorem.

Theorem 9. *The $\nu_i(x)$th variable-order fractional derivative of the vector $T(x)$ can be written as*

$$D^{\nu_i(x)}T^h(x)=X_{\nu_i(x)}(x)B_{\nu_i(x)}(x)M^T, \tag{29}$$

where

$$X_{\nu_i(x)}(x)=\left[0, x^{1-\nu_i(x)}, x^{2-\nu_i(x)}, \cdots, x^{N-\nu_i(x)}\right], 0<\nu_i(x)<1, \tag{30}$$

where $B_{\nu_i(x)}(x)$ diagonal matrix with size $(N+1)\times(N+1)$, its elements λ_{ij} given by:

$$\lambda_{ij}=\begin{cases}\dfrac{\Gamma(i+1)}{\Gamma(i+1-\nu_i(x))}, & \text{for } i=j\neq 0.,\\ 0, & \text{for otherwise}.\end{cases}, i=0,1,\cdots,N. \tag{31}$$

Proof.

$$\begin{aligned}D^{\nu_i(x)}T^h(x)&=D^{\nu_i(x)}X(x)M^T=D^{\nu_i(x)}\left[1,x,x^2,\cdots,x^N\right]M^T, 0<\nu_i(x)<1,\\ &=\left[0, \frac{\Gamma(2)}{\Gamma(2-\nu_i(x))}x^{1-\nu_i(x)}, \frac{\Gamma(3)}{\Gamma(3-\nu_i(x))}x^{2-\nu_i(x)}, \cdots, \frac{\Gamma(N+1)}{\Gamma(N+1-\nu_i(x))}x^{N-\nu_i(x)}\right]M^T\\ &=X_{\nu_i(x)}(x)B_{\nu_i}(x)M^T.\end{aligned} \tag{32}$$

□

Corollary 10. *Relation (29) satisfies for $n-1<\nu_i(x)<n$ according to Definition 2 by induction with*

$$X_{\nu_i(x)}(x)=\left[0,0,0,\cdots,x^{n-\nu_i(x)},\cdots,x^{N-\nu_i(x)}\right], n-1<\nu_i(x)<n, \tag{33}$$

and

$$B_{\nu_i(x)}(x)=(\lambda_{ij})=\begin{cases}\dfrac{\Gamma(i+1)}{\Gamma(i+1-\nu_i(x))}, & \text{for } i=j\geq n,\\ 0, & \text{for otherwise}.\end{cases}, i=0,1,\cdots,N, n<N. \tag{34}$$

Corollary 11. *The $\alpha_i(x)$th variable-order fractional derivative of the delay vector $T(x-\tau_i)$ can be written in the following form:*

$$D^{\alpha_i(x)}T^h(x-\tau_i)=X_{\alpha_i(x)}(x)B_{\alpha_i(x)}(x)B_{-\tau_i}M^T. \tag{35}$$

Form (18), we get

$$y_N(x)=T^h(x)A, \tag{36}$$

$$y_N^{(k)}(x)=T^{h(k)}(x)A, \tag{37}$$

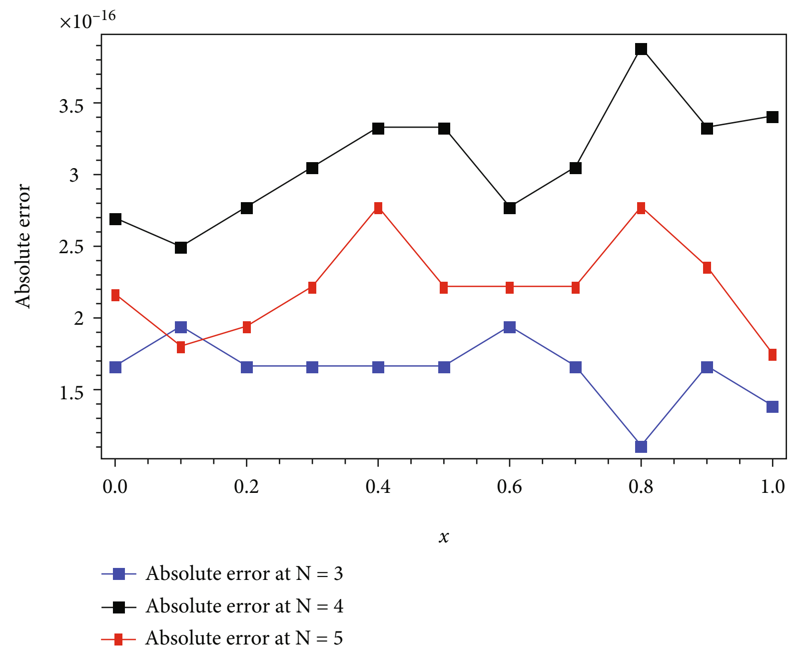

Figure 1: The behavior of the absolute errors for the proposed method with $N = 3, 4, 5$ for Example 1.

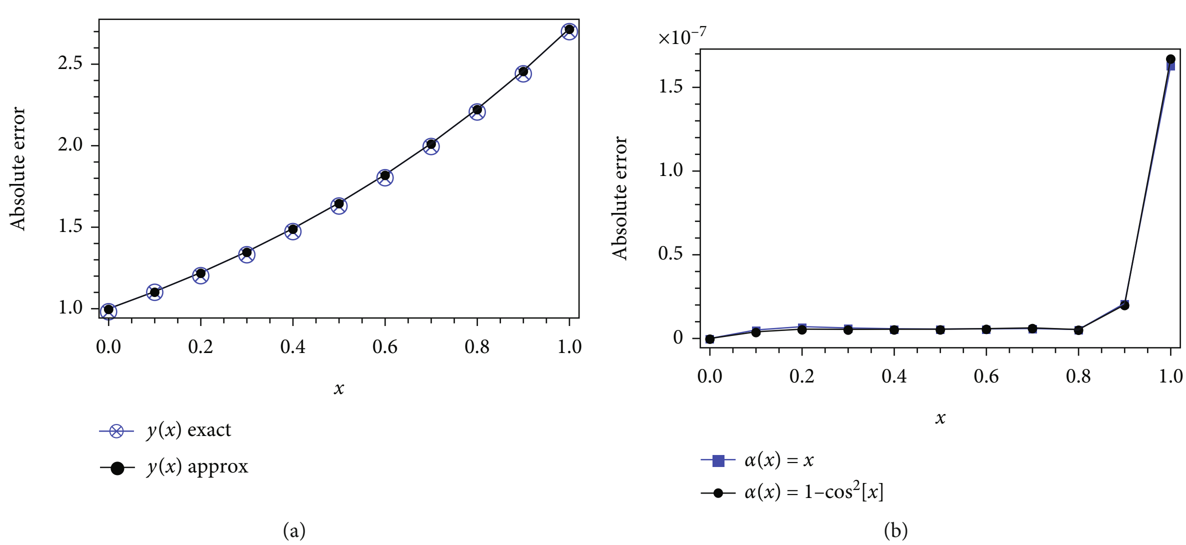

Figure 2: Exact solution and numerical solutions (a), absolute error at two different cases of $\alpha(x)$ (b) for Example 2 of $N = 7$, $\tau = 0$, and $h = 1$
.

$$D^{\nu_i(x)} y_N(x) = T^{h(\nu_i(x))}(x)A, \tag{38}$$

and

$$D^{\alpha_i(x)} y_N(x-\tau_i) = T^{(\alpha_i(x))}(x-\tau_i)A, i = 0, 1, \cdots, N. \tag{39}$$

By putting $x \longrightarrow x - \varepsilon_s$ in the relation (37), we obtain the matrix form

$$y_N^{(s)}(x-\varepsilon_s) = T^{h(s)}(x-\varepsilon_s)A. \tag{40}$$

Consequently, by substituting the matrix form (24) into (37), we have the matrix relation

$$y_N^{(k)}(x) = X^{(k)}(x)B^k M^T A. \tag{41}$$

By substituting the matrix forms (29) into (38), we have the matrix relation

$$D^{\nu_i(x)} y_N(x) = X_{\nu_i(x)}(x)B_{\nu_i(x)}(x)M^T A. \tag{42}$$

Table 2: The comparisons between the absolute errors of the method given in [44] the presented method with different values of $\alpha(x)$, $\tau_0 = 0$ and $N = 7$ for Example 2.

x	Absolute errors Our method $\alpha(x) = x$	Method [44] $\alpha(x) = x$	Our method $\alpha(x) = 1 - \cos^2(x)$	Method [44] $\alpha(x) = 1 - \cos^2(x)$
0.1	3.97×10^{-9}	5.04×10^{-6}	5.08×10^{-9}	7.34×10^{-6}
0.3	5.43×10^{-9}	1.37×10^{-6}	6.27×10^{-9}	2.86×10^{-6}
0.5	5.53×10^{-9}	1.63×10^{-6}	5.64×10^{-9}	3.55×10^{-6}
0.7	6.32×10^{-9}	9.62×10^{-7}	5.94×10^{-9}	3.17×10^{-7}
0.9	2.01×10^{-8}	8.21×10^{-7}	2.09×10^{-8}	6.10×10^{-7}

Table 3: The comparisons between the absolute errors of the presented method for Example 2 with different values of τ and $N = 7$ in case $\alpha(x) = 1 - \cos^2(x)$.

x	Absolute errors Our method $\tau = 0.02$	Our method $\tau = 0.002$	Our method $\tau = 0.0002$
0.1	5.73×10^{-9}	5.14×10^{-9}	5.09×10^{-9}
0.3	6.86×10^{-9}	6.33×10^{-9}	$6.28 \times 10^{-}9$
0.5	6.47×10^{-9}	6.54×10^{-9}	6.70×10^{-9}
0.7	6.76×10^{-9}	5.98×10^{-9}	5.94×10^{-9}
0.9	2.22×10^{-8}	2.13×10^{-8}	2.09×10^{-8}

By using (35) and (39), we get

$$D^{\alpha_j(x)} y_N(x - \tau_j) = X_{\alpha_j(x)}(x) B_{\alpha_j(x)}(x) B_{-\tau_j} M^T A. \tag{43}$$

Similar form (28) and (40), we obtain

$$y_N^{(s)}(x - \varepsilon_s) = X(x) B^s B_{-\varepsilon_s} M^T A. \tag{44}$$

3.1. Matrix Representation for Fredholm Integral Terms. Now, we try to find the matrix form corresponding to the integral term. Assume that $K(x, t)$ can be expanded to univariate Chebyshev series concerning t, as follows

$$K(x, t) \cong \sum_{r=0}^{N} u_r(x) T_r(t). \tag{45}$$

Then the matrix representation of the kernel function $K(x, t)$ is given by

$$K(x, t) \cong U(x) T^T(t), \tag{46}$$

where

$$U(x) = [u_0(x) u_1(x) \cdots u_N(x)]. \tag{47}$$

Substituting the relations (29) and (46) in the present integral part, we obtained

$$\begin{aligned} \int_a^b K(x, t) y(t)\, dt &= \int_a^b U(x) T^T(t) \mathrm{T}(t) A dt = \int_a^b U(x) M X^T(t) \mathrm{X}(t) M^T A dt \\ &= \int_a^b U(x) M [t^0 t^1 \cdots t^N]^T [t^0 t^1 \cdots t^N] M^T A\, dt \\ &= U(x) M \left(\int_a^b t^p t^q\, dt \right) M^T A = U(x) M \left(\int_a^b t^{p+q}\, dt \right) M^T A \\ &= U(x) M Z M^T \mathrm{A}, p, q = 0, 1, \cdots, N, \end{aligned} \tag{48}$$

where

$$Z = \int_a^b t^{p+q}\, dt, p, q = 0, 1, \cdots, N, \tag{49}$$

or

$$Z = [z_{pq}] = \frac{(b)^{p+q+1} - (a)^{p+q+1}}{p + q + 1}, p, q = 0, 1, \cdots, N. \tag{50}$$

So, the present integral term can be written as

$$\int_a^b K(x, t) y(t)\, dt = U(x) M Z M^T A = U(x) M Z M^T A. \tag{51}$$

3.2. Matrix Representation for Volterra Integral Terms. Now, we try to find the matrix form corresponding to the integral term. By the same way, $V_c(x, t)$ can be expanded as (45)

$$V(x, t) \cong \sum_{r=0}^{N} f_r(x) T_r(t). \tag{52}$$

Then, the matrix representation of the kernel function $V_c(x, t)$ is given by

$$V(x, t) \cong F(x) T^T(t), \tag{53}$$

where

$$F(x) = [f_0(x) f_1(x) \cdots f_N(x)]. \tag{54}$$

Substituting the relations (29) and (53) in the present

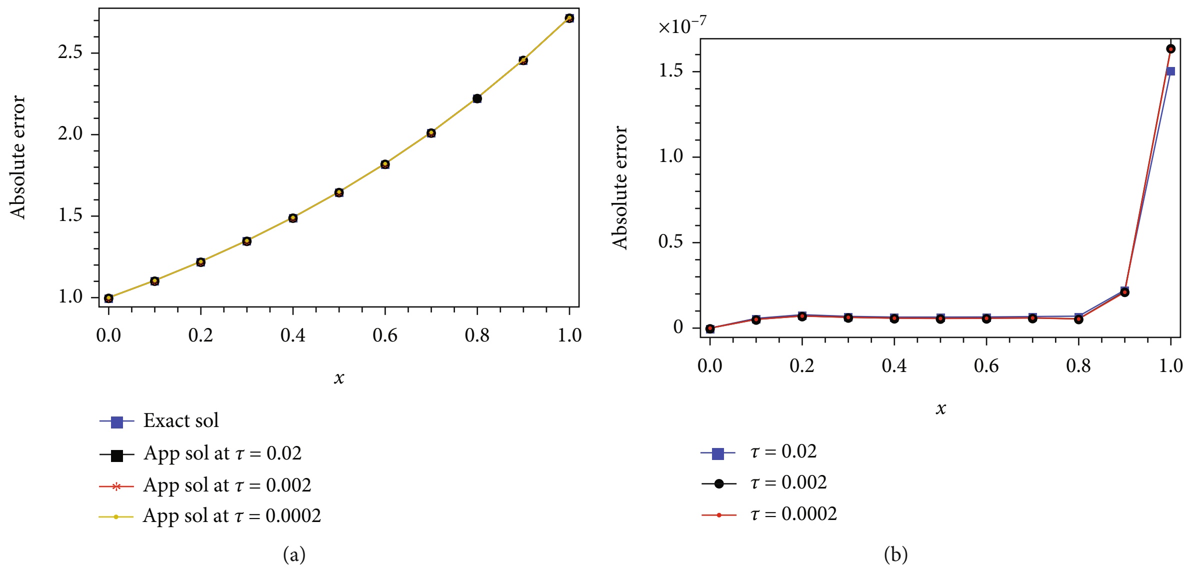

FIGURE 3: Exact solution and numerical solutions (a), absolute error (b) for Example 2 with three different values of τ and $N = 7$ in case $\alpha(x) = 1 - \cos^2(x)$ and $h = 1$.

TABLE 4: Comparison of the maximum absolute error of the method given in [45] and the presented method for Example 3 with different values of N and $h = 1$ in $\alpha(x) = x/2$ for Example 3.

	Absolute errors			
x	Our method at $N = 7$	[45] at $N = 7$.	Our method at $N = 5$	[45] at $N = 5$
0.1	9.05×10^{-16}	2.10×10^{-6}	1.56×10^{-4}	2.88×10^{-3}
0.3	1.80×10^{-15}	5.45×10^{-6}	5.58×10^{-4}	4.72×10^{-4}
0.5	3.49×10^{-15}	3.50×10^{-6}	6.27×10^{-4}	1.12×10^{-4}
0.7	3.49×10^{-15}	5.25×10^{-4}	6.94×10^{-4}	2.01×10^{-3}
0.9	9.54×10^{-15}	4.03×10^{-2}	4.18×10^{-3}	4.39×10^{-2}

integral part, we obtained

$$\begin{aligned}\int_a^x V(x,t)y(t)\,dt &= \int_a^x F(x)T^T(t)\,T(t)\,Adt = \int_a^x F(x)M\,X^T(t)X(t)M^T\,Adt \\ &= \int_a^x F(x)\,M\left[t^0 t^1 \cdots t^N\right]^T \left[t^0 t^1 \cdots t^N\right] M^T Adt \\ &= F(x)M\left(\int_a^x t^p\,t^q\,dt\right)M^T A = F(x)\,M\left(\int_a^x t^{p+q}\,dt\right)M^T A \\ &= F(x)\,ME(x)M^T A, p, q = 0, 1, \cdots, N,\end{aligned} \tag{55}$$

where

$$E(x) = \int_a^x t^{p+q}\,dt, p, q = 0, 1, \cdots, N, \tag{56}$$

or

$$E(x) = \left[e_{pq}(x)\right] = \frac{(x)^{p+q+1} - (a)^{p+q+1}}{p+q+1}, p, q = 0, 1, \cdots, N. \tag{57}$$

So, the present integral term can be written as:

$$\int_a^x V(x,t)y(t)\,dt = F(x)\,M\,E(x)\,M^T\,A. \tag{58}$$

3.3. Conditions' Matrix Relation. Finally, we can obtain the matrix form for condition (2) by using (36) on the form:

$$X(b_i)B^i M^T A = \mu_i, i = 0, 1, 2 \cdots, m-1, \tag{59}$$

Via, Equations (41), (42), (43), (44), (51), and (58), then, Equation (1) converted to

$$\begin{aligned}&\left[\sum_{i=0}^{n_1} Q_i(x)\,X_{v_i(x)}(x)B_{v_i(x)}(x)M^T + \sum_{j=0}^{n_2} P_j(x)\,X_{\alpha_j(x)}(x)B_{\alpha_j(x)}(x)B_{-\tau_j}M^T\right. \\ &\left.+ \sum_{s=0}^{n_3} R_s(x)\,X(x)B^s B_{-\varepsilon_s}M^T - U(x)\,M\,Z\,M^T - F(x)\,M\,E(x)\,M^T\right]A = g(x).\end{aligned} \tag{60}$$

We can write (60) in the form:

$$OA = G, \text{or}[O\,;G], \tag{61}$$

where

$$\begin{aligned}O = &\sum_{i=0}^{n_1} Q_i(x)\,X_{v_i(x)}(x)B_{v_i(x)}(x)M^T + \sum_{j=0}^{n_2} P_j(x)\,X_{\alpha_j(x)}(x)B_{\alpha_j(x)}(x)B_{-\tau_j}M^T \\ &+ \sum_{s=0}^{n_3} R_s(x)\,X(x)B^s B_{-\varepsilon_s}M^T - U(x)\,M\,Z\,M^T - F(x)\,M\,E(x)\,M^T,\end{aligned} \tag{62}$$

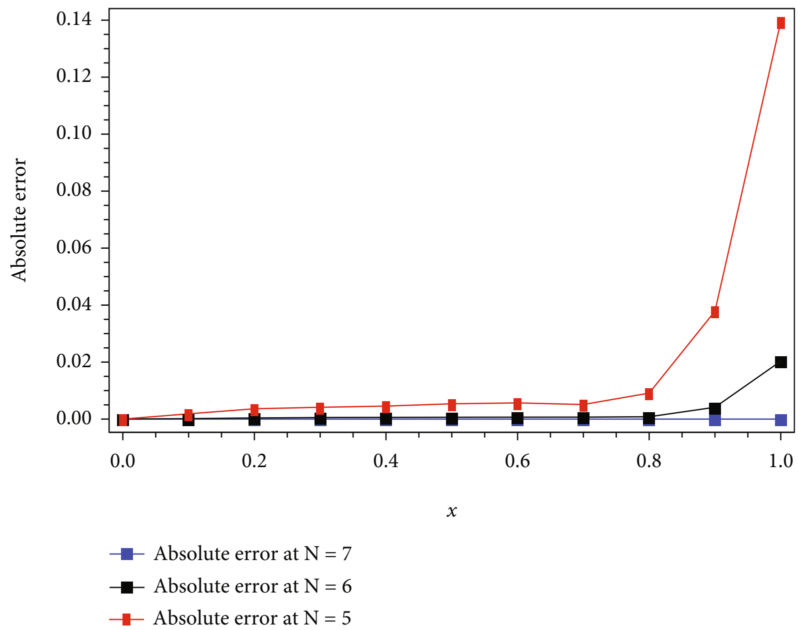

FIGURE 4: The absolute error when $h = 1$ for three different values of N for Example 3.

The collocation points are defined in this form:

$$x_l = \frac{h(2l+1)}{2N+2}, l = 0, 1, 2, \cdots, N. \quad (63)$$

By substituting (63) in (60), then (60) can be turned into the following system:

$$\left[\sum_{i=0}^{n_1} Q_i(x_l)\bar{X}_{v_i(x_l)}(x_l)\bar{B}_{v_i(x_l)}(x_l)M^T + \sum_{j=0}^{n_2} P_j(x_l)\bar{X}_{\alpha_j(x_l)}(x_l)\bar{B}_{\alpha_j(x_l)}(x_l)B_{-\tau_j}M^T + \sum_{s=0}^{n_3} R_s(x_l)X(x_l)B^s B_{-\varepsilon_s}M^T - U(x_l)M\,Z\,M^T - \bar{F}\,M\,\bar{E}\,M^T\right]A = G, \quad (64)$$

where $\bar{X}_{v_i(x_l)}(x_l)$, $\bar{B}_{v_i(x_l)}(x_l)$, $\bar{X}_{\alpha_j(x_l)}(x_l)$, and $\bar{B}_{\alpha_j(x_l)}(x_l)$ are block matrices, also $\bar{F}$ and $\bar{E}$ are block matrices defined in this form:

$$\bar{F} = \begin{pmatrix} f(x_0) & 0 & 0\cdots & 0 \\ 0 & f(x_1) & 0\cdots & 0 \\ 0 & 0 & f(x_2)\cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0\cdots & f(x_N) \end{pmatrix}, \bar{E} = \begin{pmatrix} e(x_0) & 0 & 0\cdots & 0 \\ 0 & e(x_1) & 0\cdots & 0 \\ 0 & 0 & e(x_2)\cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0\cdots & e(x_N) \end{pmatrix},$$

$$G = \begin{pmatrix} g(x_1) \\ g(x_2) \\ \vdots \\ g(x_N) \end{pmatrix}. \quad (65)$$

Here, the system in (64) in addition to the supplementary conditions can be solved numerically (the inverse matrix method is preferred when O be invertible) to determine the unknown vector A. Hence, the approximate analytical solution defined in series (18) can be calculated.

3.4. Convergence Analysis

Theorem 12. *Assume that a function $y(x) \in [0, h]$ be nth times continuously differentiable and $y_n(x)$ be the best square approximation of $y(x)$ defined in (15). Then, we have*

$$\|y(x) - y_N(x)\| \le \frac{MH^{(N+1)}\sqrt{l}}{(N+1)!}, \quad (66)$$

where

$$M = \max_{x\in[0,h]} y^{(n+1)}(x), H = \max\{h - x_0, x_0\} \text{ and } l = \int_0^h \frac{1}{\sqrt{x(h-x)}}dt. \quad (67)$$

Proof. Using Taylor expansion for $y(x)$ as follows:

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \cdots + \frac{(x - x_0)^N}{N!}y^{(n)}(x_0) + \frac{(x - x_0)^{N+1}}{(N+1)!}y^{(N+1)}(\beta), \quad (68)$$

where $x_0 \in [0, h]$ and $\beta \in]x_0, x[$. Assume

$$\bar{y}_N(x) = y(x_0) + (x - x_0)y'(x_0) + \cdots + \frac{(x - x_0)^N}{N!}y^{(N)}(x_0) + \frac{(x - x_0)^{N+1}}{(N+1)!}y^{(N+1)}(\beta), \quad (69)$$

Table 5: Comparison of the absolute error of the method given in [45] and the presented method for Example 4 with different values of N, $h = 1$, and $\tau_0 = 0$.

x	Absolute errors Our method at $N = 7$	[45] at $N = 7$	Our method at $N = 5$	[45] at $N = 5$
0.1	2.95×10^{-6}	8.24×10^{-4}	8.95×10^{-6}	1.31×10^{-3}
0.3	2.31×10^{-6}	9.11×10^{-5}	1.89×10^{-5}	1.28×10^{-4}
0.5	3.76×10^{-6}	1.25×10^{-3}	2.39×10^{-5}	3.72×10^{-3}
0.7	5.88×10^{-6}	5.25×10^{-4}	9.57×10^{-3}	2.64×10^{-2}
0.9	2.64×10^{-5}	9.57×10^{-3}	8.65×10^{-2}	1.81×10^{-1}

then

$$\|y(x) - y_N(x)\| = \left| \frac{(x - x_0)^{N+1}}{(N+1)!} y^{(N+1)}(\beta) \right|. \tag{70}$$

According to, $y_N(x)$ that given in (18), we obtain

$$\begin{aligned} \|y(x) - y_N(x)\|^2 &\le \|y(x) - \bar{y}_N(x)\|^2 = \int_0^h \omega(x) \, [y(x) - \bar{y}_N(x)]^2 dx \\ &= \int_0^h \omega(x) \left[\frac{(x - x_0)^{N+1}}{(N+1)!} y^{(N+1)}(\beta) \right]^2 dx \\ &\le \frac{M^2}{[(N+1)!]^2} \int_0^h \omega(x) \left[(x - x_0)^{N+1} \right]^2 dx. \end{aligned} \tag{71}$$

Now, let $H = \max \{h - x_0, x_0\}$, thus (71) rewritten as

$$\|y(x) - y_N(x)\|^2 \le \frac{M^2 \left[H^{N+1} \right]^2}{[(N+1)!]^2} \int_0^h \omega(x) dx. \tag{72}$$

Since, $\omega = 1/\sqrt{x(h-x)}$,
Then

$$\|y(x) - y_N(x)\|^2 \le \frac{M^2 \left[H^{N+1} \right]^2}{[(N+1)!]^2} \int_0^h \frac{1}{\sqrt{x(h-x)}} dx. \tag{73}$$

Hence, the proof is completed. □

4. Numerical Examples and Results Analysis

In this section, we introduce some numerical examples for VO fractional delay-differential equation to illustrate the above results. All results are coded and obtained by using Mathematica package programming. Moreover, the absolute maximum error $E_{\max}$ will be used in our computational results for the comparison between the exact and approximate solutions, where $E_{\max}$ is defined as the below formula:

$$E_{\max} = \max_{l=0,1,2,\cdots,N} |y(x_l) - y_N(x_l)|, x_l = \frac{h(2l+1)}{2N+2}, l = 0, 1, 2, \cdots, N, \tag{74}$$

where x, $y(x)$, and $y_N(x)$ are the space vectors, exact, and numerical solutions, respectively.

Example 1. Consider the following linear variable-order fractional delay integro-differential equation:

$$\begin{aligned} &D^{\nu(x)} y(x) + D^{\alpha(x)} y(x - 0.001) + y''(x - 0.002) - y(x) \\ &\quad = g(x) + \int_0^h (x + t) \, y(t) \, dt + \int_0^x (3t - 2x) \, y(t) \, dt, x \in [0, h]. \end{aligned} \tag{75}$$

The initial conditions are $y(0) = 0$, $y'(0) = -1$, and the exact solution is $y(x) = x^2 - x$ where $h = 1$, $\nu(x) = 1 - \text{Sin}[x]/9$, $\alpha(x) = 2 - \text{Cos}[x]/7$, $Q_0(x) = 1$, $R_0(x) = -1$,$R_2(x) = 1$, $P_0(x) = 1$, and $g(x) = 25/12 + 7x/6 - x^2 - x^4/12 + x^{1-\alpha[x]}(1.002/-1.+\alpha[x] + 2.x/2 - 3\alpha[x] + \alpha[x]^2)/\text{Gamma}[1 - \alpha[x]] + x^{1-\upsilon[x]}(-2 + 2x + \upsilon[x])/\text{Gamma}[1 - \upsilon[x]](-2 + \upsilon[x])(-1 + \upsilon[x])$. The fundamental matrix equation of the problem (75) at $N = 3$ is defined by:

$$\begin{aligned} &\left[Q_0 \bar{X}_{\nu(x_l)} \bar{B}_{\nu(x_l)} M^T + P_0 \bar{X}_{\alpha(x_l)} \bar{B}_{\alpha(x_l)} B_{-\tau} M^T + R_2 X B^2 B_\varepsilon M^T \right. \\ &\quad \left. - R_0 X M^T - U \, M \, Z \, M^T - \bar{F} \, M \, \bar{E} \, M^T \right] A = G, \end{aligned} \tag{76}$$

where

$$Q_0 = R_0 = P_0 = \text{idintity}, B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & -8 & 8 & 0 \\ -1 & 18 & -48 & 32 \end{pmatrix},$$

$$X = \begin{pmatrix} 1 & 0.125 & 0.015625 & 0.00195313 \\ 1 & 0.375 & 0.140625 & 0.0527344 \\ 1 & 0.625 & 0.390625 & 0.244141 \\ 1 & 0.875 & 0.765625 & 0.669922 \end{pmatrix}, G = \begin{pmatrix} 1.9211 \\ 1.85111 \\ 1.94491 \\ 2.15722 \end{pmatrix},$$

$$\bar{B}_{\nu(x)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1.03874 & 0 & 0 \\ 0 & 0 & 1.09183 & 0 \\ 0 & 0 & 0 & 1.12841 \\ 0 & 0 & 0 & 0 \\ 0 & 1.02855 & 0 & 0 \\ 0 & 0 & 1.06608 & 0 \\ 0 & 0 & 0 & 1.0917 \\ 0 & 0 & 0 & 0 \\ 0 & 1.01898 & 0 & 0 \\ 0 & 0 & 1.04302 & 0 \\ 0 & 0 & 0 & 1.0593 \\ 0 & 0 & 0 & 0 \\ 0 & 1.01076 & 0 & 0 \\ 0 & 0 & 1.02399 & 0 \\ 0 & 0 & 0 & 1.03288 \end{pmatrix}, \bar{B}_{\alpha(x)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1.05546 & 0 & 0 \\ 0 & 0 & 1.13733 & 0 \\ 0 & 0 & 0 & 1.19466 \\ 0 & 0 & 0 & 0 \\ 0 & 1.05846 & 0 & 0 \\ 0 & 0 & 1.14601 & 0 \\ 0 & 0 & 0 & 1.2075 \\ 0 & 0 & 0 & 0 \\ 0 & 1.06414 & 0 & 0 \\ 0 & 0 & 1.1629 & 0 \\ 0 & 0 & 0 & 1.2327 \\ 0 & 1.07187 & 0 & 0 \\ 0 & 0 & 1.1871 & 0 \\ 0 & 0 & 0 & 1.26924 \end{pmatrix},$$

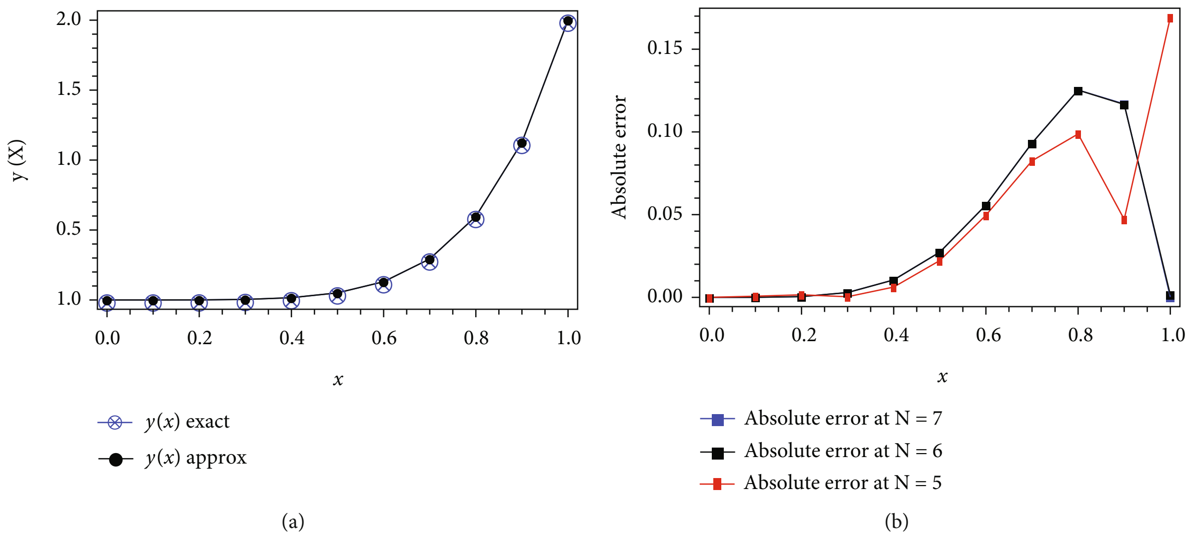

Figure 5: Exact solution and numerical solutions (a), absolute error (b) for Example 4 with different values of N and $h = 1$ in case of $h = 1$.

Table 6: The comparisons between the absolute errors of the presented method for Example 5 with value of $N = 4$.

x	Absolute errors		
	Our method	Method [46]	Method [39]
0.1	1.11×10^{-16}	3.52×10^{-13}	4.99×10^{-16}
0.3	1.11×10^{-16}	2.33×10^{-13}	1.22×10^{-15}
0.5	0	1.35×10^{-13}	1.05×10^{-15}
0.7	0	4.86×10^{-14}	6.66×10^{-16}
0.9	0	2.99×10^{-14}	5.55×10^{-16}

$$\bar{X}_{\alpha(x)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0.288108 & 0 & 0 & 0 \\ 0.0360135 & 0 & 0 & 0 \\ 0.00450169 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.562781 & 0 & 0 \\ 0 & 0.211043 & 0 & 0 \\ 0 & 0.079141 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.767795 & 0 \\ 0 & 0 & 0.479872 & 0 \\ 0 & 0 & 0.29992 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.931899 \\ 0 & 0 & 0 & 0.815411 \\ 0 & 0 & 0 & 0.713485 \end{pmatrix}^T,$$

$$\bar{X}_{\nu(x)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0.153018 & 0 & 0 & 0 \\ 0.0191273 & 0 & 0 & 0 \\ 0.00239091 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.401814 & 0 & 0 \\ 0 & 0.15068 & 0 & 0 \\ 0 & 0.0565052 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.878023 & 0 \\ 0 & 0 & 0.399181 & 0 \\ 0 & 0 & 0.29992 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.931899 \\ 0 & 0 & 0 & 0.76827 \\ 0 & 0 & 0 & 0.672236 \end{pmatrix}^T,$$

$$Z = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix}, B_{0.001} = \begin{pmatrix} 1 & -0.001 & 1.1 \times 10^{-6} & -1.0 \times 10^{-9} \\ 0 & 1 & -0.002 & 3.0 \times 10^{-6} \\ 0 & 0 & 1 & -0.003 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (77)$$

In Table 1, we list the absolute errors between the exact solutions and the approximate solutions using the proposed method with $\nu(x) = 1 - \mathrm{Sin}[x]/9$, $\alpha(x) = 2 - \mathrm{Cos}[x]/7$ and three choices of N. In Figure 1 is the behavior of the absolute errors for the proposed method of different N.

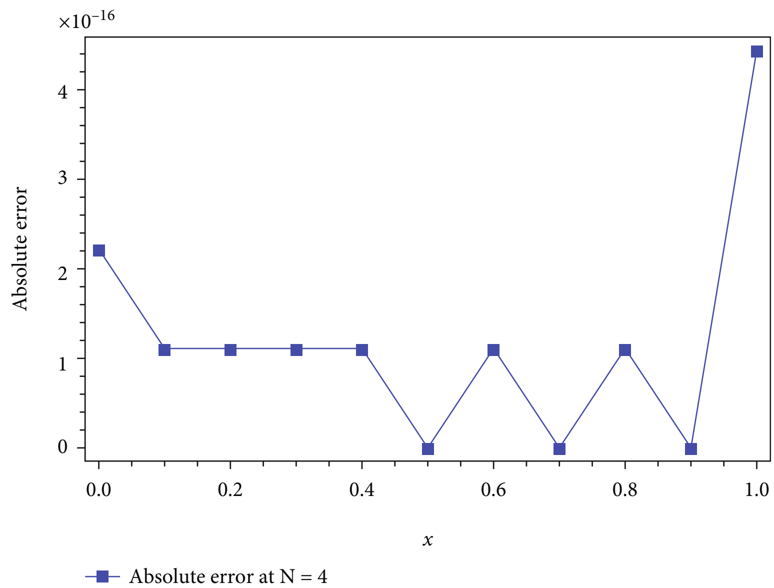

Figure 6: Exact solution and numerical solutions of Example 4-case 1 for different values of N, h.

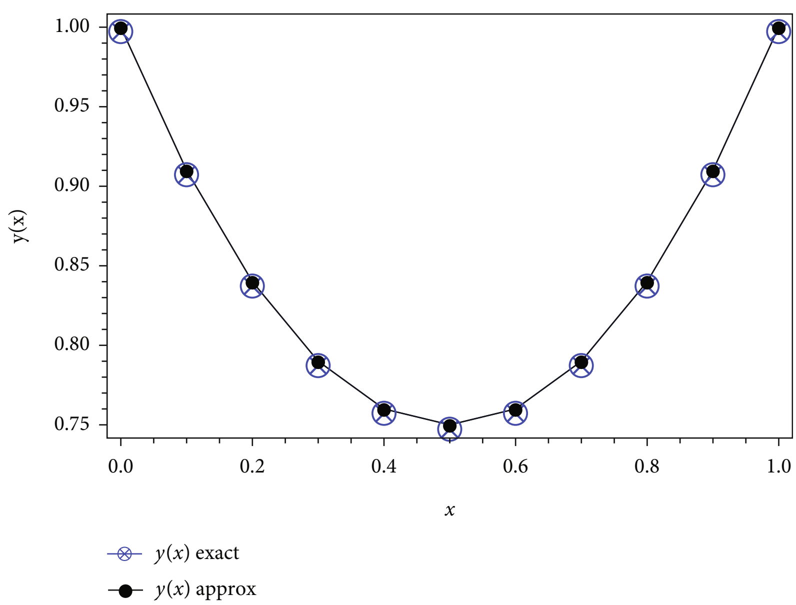

Figure 7: Comparison between the approximate solutions and the exact solutions of Example 5 for $N = 4$.

Example 2. Consider the following linear variable-order fractional delay integro-differential equation [44]:

$$D^{\alpha(x)}y(x-\tau_0) = g(x) + \int_0^h (x-t)\,y(t)\,dt + \int_0^x (t+x)\,y(t)\,dt, x \in [0,h]. \tag{78}$$

with the initial condition $y(0) = 1$, $y'(0) = 1$. Problem (78) found in [44] with the exact solution given as $y(x) = e^x$ at $\tau_0 = 0$. Now, by applying the suggested technique introduced in the previous section with finite terms $(N+1)$ to (78), then, we have

$$\left[P_0 X_{\alpha(x)} B_{\alpha(x)} B_{-\tau_0} M^T - U(x)\,M\,Z\,M^T - F(x)\,M\,E(x)\,M^T\right] A = g(x), \tag{79}$$

Hence, using the collocation method with the collection points x_l that is given in the previous section. Then, the previous equation can be rewritten as follows:

$$\left[P_0 \bar{X}_{\alpha(x_l)} \bar{B}_{\alpha(x_l)} B_{-\tau_0} M^T - U\,M\,Z\,M^T - \bar{F}\,M\,\bar{E}\,M^T\right] A = G, \tag{80}$$

The numerical solutions obtained by different values of $\alpha(x)$ together with the exact solution are displayed in Figure 2. From this figure, we see that the numerical solution converges to the exact one by increasing the number of basis functions. In Table 2, the comparisons between the absolute errors of the method are given in [44] and the presented method for different values of α and $N = 7$ and choices of $\tau_0 = 0$ and $h = 1$.

In Table 3 shows the absolute errors between the exact and approximate solutions for $\nu = 1$ with $N = 6$ and various choices of τ_0. In Figure 3 is the exact solution and numerical solutions (a), absolute error (b) at $N = 6$, $\tau_0 = 0.001$, $\tau_0 = 0.0001$, and $h = 1$.

Example 3. Consider the following linear variable-order fractional delay integro-differential equation [45]:

$$D^{\alpha(x)}y(x-\tau) = g(x) + \frac{1}{3}\int_0^h (x+t)y(t)\,dt + \frac{1}{10}\int_0^x (t\,x)y(t)\,dt, x \in [0, h]. \tag{81}$$

where $\alpha(x) = x/2$, $g(x) = -17/216 - 5x/56 - 1/720x^9(9+8x) + 720x^{6-\alpha[x]}(7+7x-\alpha[x])/\text{Gamma}[8-\alpha[x]]$ and the initial condition is equal to $y(0) = 0$. The exact solution as given in [45] is given by $y(x) = x^7 + x^6$. Now, by applying the suggested technique that introduced in the previous section with finite terms $(N+1)$ to (53) then we have

$$\left[P_0 X_{\alpha(x)} B_{\alpha(x)} B_{-\tau} M^T - U(x)\, M\, Z\, M^T - F(x)\, M\, E(x)\, M^T\right] A = g(x), \tag{82}$$

hence, using the collocation method with the collection points x_l that given in the previous section. Then, the previous equation can be rewritten as follows:

$$\left[P_0 \bar{X}_{\alpha(x_l)} \bar{B}_{\alpha(x_l)} B_{-\tau} M^T - U\, M\, Z\, M^T - \bar{F}\, M\, \bar{E}\, M^T\right] A = G. \tag{83}$$

In Table 4, the comparison is the maximum absolute error of the method given in [45] and the presented method for different values of N and $h = 1$ in $\alpha(x) = x/2$. The absolute error when $h = 1$ for different values of N is shown in Figure 4.

Example 4. Consider the following linear variable-order fractional delay integro-differential equation [45]:

$$D^{\alpha(x)}y(x-\tau_0) = g(x) + \int_0^1 \sin(x) t y(t)\,dt + \int_0^x (t-x)y(t)\,dt, x \in [0, h], \tag{84}$$

where $P_0 = 1$, $g(x) = -16x^{27/4}/621 - 25x^{41/5}/1476 + x^{19/4-\alpha[x]}(\text{Gamma}[23/4]/\text{Gamma}[23/4-\alpha[x]] + x^{29/20}\text{Gamma}[36/5]/\text{Gamma}[36/5-\alpha[x]]) - 299\text{Sin}[x]/1107$, $\alpha(x) = x$ and the initial condition is equal to $y(0) = 0$. The exact solution of this equation is given by $y(x) = x^{19/4} + x^{31/5}$ at $\tau_0 = 0$ (see [45]). Now, by applying the suggested technique that introduced in the previous section with finite terms $(N+1)$ to (8), then, we have

$$\left[P_0 X_{\alpha(x)} B_{\alpha(x)} B_{-\tau_0} M^T - U(x)\, M\, Z\, M^T - F(x)\, M\, E(x)\, M^T\right] A = g(x), \tag{85}$$

Hence, using the collocation method with the collection points x_l that given in the previous section. Then, the previous equation can be rewritten as follows:

$$\left[P_0 \bar{X}_{\alpha(x_l)} \bar{B}_{\alpha(x_l)} B_{-\tau_0} M^T - U\, M\, Z\, M^T - \bar{F}\, M\, \bar{E}\, M^T\right] A = G. \tag{86}$$

By solving the algebraic system (86), we can obtain the vector $A = [a_0, a_1, \cdots, a_n]^T$. Subsequently, numerical solution according to (18) is obtained. Table 5 gives the comparison of the absolute error at some points obtained by the present method and method of [45] with different values of N, and it is seen that our method gives more accurate results than the method of [45] at $\tau_0 = 0$. In Figure 5, the exact solution and numerical solutions given (a), where the absolute error shown (b), are all with different values of N and $h = 1$.

Example 5. Consider the following linear variable-order fractional integro-differential equation [39, 46]:

$$D^{\nu(x)}y(x) + \sin(x)y'(x) + (x-1)y(x) = g(x) - \int_0^x e^x y(t)\,dt, x \in [0, 1], \tag{87}$$

where $Q_0 = 1$, $R_0(x) = (x-1)$, $R_1(x) = (\sin(x))$, and $g(x) = (-1+x)(1+(-1+x)x) + 1/6e^x x(6+x(-3+2x)) + (-1+2x)\text{Sin}[x] + x^{1-\upsilon[x]}(-2+2x+\upsilon[x])/\text{Gamma}[3-\upsilon[x]]$ and the initial conditions is equal to $y(0) = 0, y'(0) = -1$. The exact solution of (87) according to [39, 46] is given by $y(x) = x^2 - x + 1$ at $\nu(x) = x/3$. Now, by applying the suggested technique that introduced in the previous section with finite terms $(N+1)$ to (8), then, we have

$$\left[Q_0 X_{\nu_x} B_{\nu(x)} M^T + R_0\, X(x) M^T + R_1\, X(x) B M^T + F(x)\, M\, E(x)\, M^T\right] A = g(x), \tag{88}$$

Hence, using the present collocation method with the collection points x_l that given in the previous section. Then, the previous equation can be rewritten as follows:

$$\left[Q_0 \bar{X}_{\nu_{x_l}} \bar{B}_{\nu_{x_l}} M^T + R_0\, X(x_l) M^T + R_1\, X(x_l) B M^T + \bar{F}\, M\, \bar{E}\, M^T\right] A = G. \tag{89}$$

By solving the algebraic system (89), we can obtain the vector $A = [a_0, a_1, \cdots, a_n]^T$. Subsequently, numerical solution (18) is obtained. In Table 6, a comparison between the AEs is obtained in [39, 46], and the results are obtained in this work for $N = 4$ of for example (12). In Figure 6, we see the matching of the AEs values in this figure and that in Table 6. While Figure 7 displays the curves of exact and approximate solutions for example (12) with values of parameters listed in their caption.

5. Conclusion

In this work, the general form of mixed Fredholm–Volterra integro-differential equations with delay arguments of variable-order based on the operational matrix method with the Shifted Chebyshev polynomials (CPs) of the first kind is presented. The spectral collocation technique with the aid of CPs is used as an operational matrix method for solving the proposed model, which is reduced by the operational matrices to the matrix form. Caputo's VO fractional derivative operator is used to generate the matrices of the derivative. The accuracy of the proposed technique is obtained by many numerical examples. Finally, we used codes written with the Mathematica package to calculate our numerical results and graphs.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors carried out the proofs and conceived the study. All authors read and approved the final manuscript.

References

[1] D. Baleanu and A. Mendes Lopes, *Handbook of Fractional Calculus with Applications, in: Applications in Engineering, Life and Social Sciences, Part A, Southampton: Comput Mech Publicat*, vol. 7, De Gruyter, Berlin, Boston, 2019.

[2] X. J. Yang, *General Fractional Derivatives: Theory*, CRC Press, New York, Methods and Applications, 2019.

[3] Z. Hammouch and T. Mekkaoui, "Circuit design and simulation for the fractional-order chaotic behavior in a new dynamical system," *Complex & Intelligent Systems*, vol. 4, no. 4, pp. 251–260, 2018.

[4] X. J. Yang, "New general calculi with respect to another functions applied to describe the newton-like dashpot models in anomalous viscoelasticity," *Thermal Science*, vol. 23, no. 6, Part B, pp. 3751–3757, 2019.

[5] A. Atangana and S. Jain, "The role of power decay, exponential decay and Mittag-Leffler function's waiting time distribution: application of cancer spread," *Physica A*, vol. 512, pp. 330–351, 2018.

[6] X. J. Yang, F. Gao, and H. W. Jing, "New mathematical models in anomalous viscoelasticity from the derivative with respect to another function view point," *Thermal Science*, vol. 23, no. 3, Part A, pp. 1555–1561, 2019.

[7] A. Debbouche and J. J. Nieto, "Relaxation in controlled systems described by fractional integro-differential equations with nonlocal control conditions," *Electronic Journal of Differential Equations*, vol. 2015, no. 89, pp. 1–18, 2015.

[8] S. G. Samko and B. Ross, "Integration and differentiation to a variable fractional order," *Integral Transforms and Special Functions*, vol. 1, no. 4, pp. 277–300, 1993.

[9] H. Hassani and E. Naraghirad, "A new computational method based on optimization scheme for solving variable- order time fractional Burgers' equation," *Mathematics and Computers in Simulation*, vol. 162, pp. 1–17, 2019.

[10] T. Odzijewicz, A. B. Malinowska, and D. F. M. Torres, *Fractional Variational Calculus of Variable Order, in: Advances in Harmonic Analysis and Operator Theory, Operator Theory: Advances and Applications*, vol. 229, Birkhauser/Springer Basel AG, Basel, 2013.

[11] R. Yan, M. Han, Q. Ma, and X. Ding, "A spectral collocation method for nonlinear fractional initial value problems with a variable-order fractional derivative," *Computational and Applied Mathematics*, vol. 38, no. 2, pp. 1–25, 2019.

[12] A. Atangana, *Fractional Operators with Constant and Variable Order with Application to Geo-Hydrology*, Academic Press, 2018.

[13] P. Závada, "Operator of fractional derivative in the complex plane," *Communications in Mathematical Physics*, vol. 192, no. 2, pp. 261–285, 1998.

[14] E. Guariglia, "Riemann zeta fractional derivative–"functional equation and link with primes"," *Advances in Difference Equations*, vol. 2019, no. 1, 2019.

[15] A. Torres-Hernandez and F. Brambila-Paz, "An approximation to zeros of the Riemann zeta function using fractional calculus," 2020, http://arxiv.org/abs/2006.14963.

[16] E. Guariglia, "Fractional calculus, zeta functions and Shannon entropy," *Open Mathematics*, vol. 19, no. 1, pp. 87–100, 2021.

[17] E. Guariglia, "Harmonic symmetry of the Riemann zeta fractional derivative," *AIP Conference Proceedings*, no. 1, article 020035, 2018AIP Publishing LLC, 2018.

[18] G. Diaz and C. F. M. Coimbra, "Nonlinear dynamics and control of a variable order oscillator with application to the van der pol equation," *Nonlinear Dynamics*, vol. 56, no. 1–2, pp. 145–157, 2009.

[19] L. E. S. Ramirez and C. F. M. Coimbra, "On the variable order dynamics of the nonlinear wake caused by a sedimenting particle," *Physica D: Nonlinear Phenomena*, vol. 240, no. 13, pp. 1111–1118, 2011.

[20] D. Ingman and J. Suzdalnitsky, "Control of damping oscillations by fractional differential operator with time- dependent order," *Computer Methods in Applied Mechanics and Engineering*, vol. 193, no. 52, pp. 5585–5595, 2004.

[21] L. E. S. Ramirez and C. F. M. Coimbra, "A variable order constitutive relation for viscoelasticity," *Annalen der Physik*, vol. 16, no. 7-8, pp. 543–552, 2007.

[22] Y. C. H. Sheng, H. Sun, and T. Qiu, "Synthesis of multifractional Gaussian noises based on variable-order fractional operators," *Signal Processing*, vol. 91, no. 7, pp. 1645–1650, 2011.

[23] P. Ostalczyk and T. Rybicki, "Variable-fractional-order deadbeat control of an electromagnetic servo," *Journal of Vibration and Control*, vol. 14, no. 9-10, pp. 1457–1471, 2008.

[24] J. Orosco and C. F. M. Coimbra, "On the control and stability of variable-order mechanical systems," *Nonlinear Dynamics*, vol. 86, pp. 695–710, 2016.

[25] S. Samko, “Fractional integration and differentiation of variable order: an overview,” *Nonlinear Dynamics*, vol. 71, no. 4, pp. 653–662, 2013.

[26] H. Sheng, H. Sun, C. Coopmans, Y. Chen, and G. W. Bohannan, “A physical experimental study of variable-order fractional integrator and differentiator,” *The European Physical Journal Special Topics*, vol. 193, no. 1, pp. 93–104, 2011.

[27] D. Tavares, R. Almeida, and D. F. M. Torres, “Caputo derivatives of fractional variable order: numerical approximations,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 35, pp. 69–87, 2016.

[28] A. H. Bhrawy and M. A. Zaky, “Numerical algorithm for the variable-order caputo fractional functional differential equation,” *Nonlinear Dynamics*, vol. 85, no. 3, pp. 1815–1823, 2016.

[29] A. H. Bhrawy and M. A. Zaky, “Numerical simulation for two-dimensional variable-order fractional nonlinear cable equation,” *Nonlinear Dynamics*, vol. 80, no. 1-2, pp. 101–116, 2015.

[30] N. I. Mahmudov and M. Aydın, “Representation of solutions of nonhomogeneous conformable fractional delay differential equations,” *Chaos, Solitons & Fractals*, vol. 150, article 111190, 2021.

[31] O. Moaaz, E. E. Mahmoud, and W. R. Alharbi, “Third-order neutral delay differential equations: new iterative criteria for oscillation,” vol. 2020, pp. 1–8, 2020.

[32] A. Boutiara, M. M. Matar, M. K. Kaabar, F. Martínez, S. Etemad, and S. Rezapour, “Some qualitative analyses of neutral functional delay differential equation with generalized Caputo operator,” *Journal of Function Spaces*, vol. 2021, 13 pages, 2021.

[33] R. Amin, K. Shah, M. Asif, I. Khan, and F. Ullah, “An efficient algorithm for numerical solution of fractional integro- differential equations via Haar wavelet,” *Journal of Computational and Applied Mathematics*, vol. 381, article 113028, 2021.

[34] I. Khan, M. Asif, R. Amin, Q. Al-Mdallal, and F. Jarad, “On a new method for finding numerical solutions to integro-differential equations based on Legendre multi-wavelets collocation,” *Alexandria Engineering Journal*, vol. 61, no. 4, pp. 3037–3049, 2022.

[35] A. Babakhani, D. Baleanu, and R. Khanbabaie, “Hopf bifurcation for a class of fractional dierential equations with delay,” *Nonlinear Dynamics*, vol. 69, no. 3, pp. 101–116, 2012.

[36] Z. Gao, “A graphic stability criterion for non-commensurate fractional-order time-delay systems,” *Nonlinear Dynamics*, vol. 78, no. 3, pp. 2101–2111, 2014.

[37] B. P. Moghaddam and Z. S. Mostaghim, “A novel matrix approach to fractional finite difference for solving models based on nonlinear fractional delay differential equations,” *Ain Shams Engineering Journal*, vol. 5, no. 2, pp. 585–594, 2014.

[38] B. P. Moghaddam and Z. S. Mostaghim, “Modified finite difference method for solving fractional delay differential equations,” *Boletim da Sociedade Paranaense de Matemtica*, vol. 35, no. 2, pp. 49–58, 2017.

[39] E. H. Doha, M. A. Abdelkawy, A. Z. M. Amin, and D. Baleanu, “Spectral technique for solving variable–order fractional Volterra integro–differential equations,” *Numerical Methods for Partial Differential Equations*, vol. 34, no. 5, pp. 1659–1677, 2018.

[40] D. Sierociuk, W. Malesza, and M. Macias, “Derivation, interpretation, and analog modelling of fractional variable order derivative definition,” *Applied Mathematical Modelling*, vol. 39, no. 13, pp. 3876–3888, 2015.

[41] X. J. Yang, “Fractional derivatives of constant and variable orders applied to anomalous relaxation models in heat transfer problems,” *Thermal Science*, vol. 21, no. 3, pp. 1161–1171, 2017.

[42] Y. Chen, L. Liu, B. Li, and Y. Sun, “Numerical solution for the variable order linear cable equation with Bernstein polynomials,” *Applied Mathematics and Computation*, vol. 238, pp. 329–341, 2014.

[43] S. Shen, F. Liu, J. Chen, I. Turner, and V. Anh, “Numerical techniques for the variable order time fractional diffusion equation,” *Applied Mathematics and Computation*, vol. 218, no. 22, pp. 10861–10870, 2012.

[44] R. M. Ganji, H. Jafari, and S. Nemati, “A new approach for solving integro-differential equations of variable order,” *Journal of Computational and Applied Mathematics*, vol. 379, article 112946, 2020.

[45] M. Yi, J. Huang, and L. Wang, “Operational matrix method for solving variable order fractional integro-differential equations,” *CMES-Computer Modeling in Engineering and Sciences*, vol. 96, pp. 361–377, 2013.

[46] J. Liu, T. Li, and L. Wu, “An operational matrix technique for solving variable order fractional differential-integral equation based on the second kind of Chebyshev polynomials,” *Advances in Mathematical Physics*, vol. 2016, 9 pages, 2016.

[47] R. M. Ganji, H. Jafari, and D. Baleanu, “A new approach for solving multi variable orders differential equations with Mittag-Leffler kernel,” *Chaos, Solitons and Fractals*, vol. 130, article 109405, 2020.

[48] R. M. Ganji and H. Jafari, “A numerical approach for multi-variable orders differential equations using Jacobi polynomials,” *International Journal of Applied and Computational Mathematics*, vol. 5, no. 2, pp. 1–9, 2019.

[49] R. M. Ganji, H. Jafari, and A. R. Adem, “A numerical scheme to solve variable order diffusion–wave equations,” *Thermal Science*, vol. 23, pp. 2063–2071, 2019.

[50] R. Kamal, “Spectral Tau method for solving general fractional order differential equations with linear functional argument,” in *Journal of the Egyptian Mathematical Society*, Mohamed, 2019.

[51] K. K. Ali, M. A. Abd El Salam, and E. M. Mohamed, “Chebyshev operational matrix for solving fractional order delay-differential equations using spectral collocation method,” *Arab Journal of Basic and Applied Sciences*, vol. 26, no. 1, pp. 342–353, 2019.

[52] E. M. Mohamed, K. R. Raslan, K. K. Ali, and M. A. Abd El Salam, “On general form of fractional delay integro-differential equations,” *Arab Journal of Basic and Applied Sciences*, vol. 27, no. 1, pp. 313–323, 2020.

[53] K. K. Ali, M. A. Abd El Salam, E. M. Mohamed, B. Samet, S. Kumar, and M. S. Osman, “Numerical solution for generalized nonlinear fractional integro-differential equations with linear functional arguments using Chebyshev series,” *Advances in Difference Equations*, vol. 2020, no. 1, 23 pages, 2020.

[54] K. K. Ali, M. A. Abd El Salam, and E. M. H. Mohamed, “A numerical technique for a general form of nonlinear fractional-order differential equations with the linear functional argument,” *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 22, no. 1, pp. 83–91, 2021.

[55] M. A. Abdelkawy, A. Z. Amin, A. H. Bhrawy, J. A. T. Machado, and A. M. Lopes, “Jacobi collocation approximation for

solving multi-dimensional Volterra integral equations," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 18, no. 5, pp. 411–425, 2017.

[56] K. Sun and M. Zhu, "Numerical algorithm to solve a class of variable order fractional integral-differential equation based on Chebyshev polynomials," *Mathematical Problems in Engineering*, vol. 2015, Article ID 902161, 10 pages, 2015.

[57] E. H. Doha, M. A. Abdelkawy, A. Z. M. Amin, and A. M. Lopes, "On spectral methods for solving variable-order fractional integro-differential equations," *Computational and Applied Mathematics*, vol. 37, no. 3, pp. 3937–3950, 2018.

[58] J. Liu, X. Li, and L. Wu, "An operational matrix of fractional differentiation of the second kind of Chebyshev polynomial for solving multiterm variable order fractional differential equation," *Mathematical Problems in Engineering*, vol. 2016, 10 pages, 2016.

[59] A. H. Bhrawy, M. A. Zaky, and J. A. T. Machado, "Numerical solution of the two-sided space–time fractional telegraph equation via Chebyshev tau approximation," *Journal of Optimization Theory and Applications*, vol. 174, no. 1, pp. 321–341, 2017.

[60] M. Gülsu, Y. Öztürk, and M. Sezer, "A new collocation method for solution of mixed linear integro-differential- difference equations," *Applied Mathematics and Computation*, vol. 216, no. 7, pp. 2183–2198, 2010.