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Research Article

Rough Fuzzy Ideals Induced by Set-Valued Homomorphism in Ternary Semigroups

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The main objective of this paper is to characterize rough approximations of fuzzy ideals in ternary semigroups. Rough fuzzy ideals are used to deal with vague and incomplete information in decision-making problems. In this research, approximations for fuzzy prime ideals in ternary semigroups are studied. It is proved that generalized lower approximations and generalized upper approximations of $(\in, \in\vee q)$-fuzzy prime (resp., semiprime) ideals of ternary semigroups are $(\in, \in\vee q)$-fuzzy prime (resp., semiprime) ideals. For this, the concept of SSVH (strong set-valued homomorphism) and SVH (set-valued homomorphism) is used. Also, it is shown by examples that the lower approximations of fuzzy subsemigroups and fuzzy ideals are not fuzzy subsemigroups and fuzzy ideals, respectively, for SVH.

1. Introduction and Motivation

In classical mathematics, the membership of components in a set is calculated by crisp set. According to this, a component either belongs or does not belong to the set, and a statement is true or false. In real life, there are many midway situations of a complicated problem. After a long effort, many theories are introduced by researchers to handle such problems such as fuzzy set theory, rough set theory, and soft set theory etc. Historically, Zadeh in 1965 [1] initiated the concept of fuzzy set theory, and he provided the idea to study ambiguity complications. After that, this concept has been used in different fields of our world. Fuzzy mathematics has stayed as a vital branch of mathematics until now. One of the foremost vital directions in fuzzy mathematics is the study of fuzzy algebras. Fuzzy set theory has many applications in robotics, artificial intelligence, control engineering, and many more branches of applied and pure mathematics.

In 1982, Pawlak [2] initiated the theory of rough set. Rough set is characterized using the lower and upper approximations. Based on the known attributes, a rough set contains all the information. Let C be any nonempty subset of universal set W, and then, lower and upper approximations of C are shown in Figure 1, where each rectangular region represents an equivalence class, since, in same equivalence class, elements are indiscernible from each other and equivalence classes are the small granularity in the whole data. Then, to define exactly C, a pair of approximation is used. The lower approximation consists of those equivalence classes that are completely enclosed in C, while the upper approximation (grey rectangles) consists of all equivalence classes that are partly inside the C and in the lower approximation.

1.1. Applications of the Proposed Model. In mathematics, algebraic structures play very important role. In different fields, these structures have solid applications, for example, in computer science, theoretical physics, information sciences, control engineering, topological spaces, and coding theory. Lehmer [3] was the first who furnished the idea of ternary algebraic systems. In the 19th century, cubic associations and ternary algebraic operations were measured by

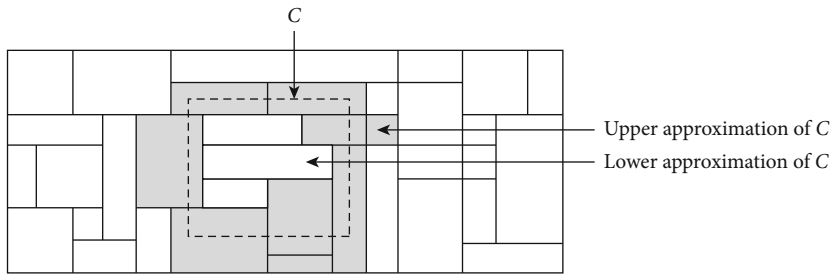

Figure 1: Upper and lower approximations.

numerous mathematicians. Similarly, rough set theory is used to determine the generalized rules that describe the connection between acoustical parameters of concert halls and sound processing approaches. This theory is a useful tool in the areas of artificial intelligence, such as pattern recognition, learning algorithms, inductive reasoning, and automatic classification. Also, it has many applications in measurement theory, classification theory, taxonomy, cluster analysis, etc. In medical field, a serious challenge is the abdominal pain in children. There are many reasons of this disease, and it is hard to detect the main cause. This theory helps the doctors to detect by discharge observations (for more applications, see [4–6]).

1.2. Innovative Contribution. There are many structures which are not closed under binary multiplication. For example, the subset $\mathbb{R}^+$ of real numbers $\mathbb{R}$ is closed with respect to the binary multiplication of semigroup, while in $\mathbb{R}^-$, the closure law does not hold, but it is closed with respect to ternary product. To handle such kind of problems in algebraic structures, we study ternary structures (for more applications of ternary operation, see [7, 8]). Ternary semigroups along with their algebraic structures are deliberated through few other authors as well, including Sioson, who examined ideals along with ternary semigroup [9]. In ternary semigroups, bi-ideals and quasi-ideals were examined by Dixit and Dewan [10]. Shabir and Bashir studied prime ideals in ternary semigroups [11]. Ternary semigroups in the form of minimal and maximal lateral ideals were investigated by Iampan in 2007 [12]. Kar and Maity in [13] deal with the study of congruences of ternary semigroups. Bashir et al. studied three-dimensional congruence relation in rough fuzzy ternary semigroups [14].

1.3. Related Works. Rosenfeld [15] studied fuzzification of algebraic structures and then approached the fuzzy groups. P. Ming and L. Ming in 1980 studied the fuzzy point [16]. This fundamental notion of fuzzy sets applied to semigroup was initially considered by Kuroki in 1991 [17]. Bhakat and Das studied the fuzzy subgroups [18]. Among these $(\in, \in \vee q)$-fuzzy ideals in semigroups are most significant (see [19]). The idea of rough semigroups was popularized by Biswas and Nanda [20]. Petchkhaew and Chinram studied rough fuzzy ideals in ternary semigroups in 2009 [21]. Rameez et al. worked on the generalization of roughness in $(\in, \in \vee q)$-fuzzy ideals of hemirings [22]. In 2021, Anwar et al. worked on roughness and multigranulation roughness of intuitionistic fuzzy sets in terms of soft relations [23, 24]. In 2022, Anwar et al. studied pessimistic multigranulation roughness of intuitionistic fuzzy sets based on soft relations [25]. Recently, Shabir et al. approximate fuzzy ideals of semirings using bipolar techniques [26]. In this paper, we generalize the structure of [22] to ternary semigroups.

1.4. Organization of the Paper. This paper is arranged as follows: in Section 2, some basic material related to ternary semigroups, fuzzy sets, rough sets, and $(\in, \in \vee q)$-fuzzy ideals in ternary semigroup is given. Section 3> is about to our main work in which the lower approximation and upper approximation of fuzzy ternary subsemigroups, fuzzy ideals, fuzzy semiprime ideals, and fuzzy prime ideals are studied. Approximations of $(\in, \in \vee q)$-fuzzy ternary subsemigroups and $(\in, \in \vee q)$-fuzzy ideals are studied in Section 4. Lastly, comparative study and conclusions are given. The list of acronyms used in the research article is given in Abbreviations.

2. Preliminaries

Here, we introduce some basic notions about ternary semigroups, fuzzy ternary semigroups, and rough sets.

A nonempty set is called a ternary semigroup, if there exists a ternary operation $A \times A \times A \longrightarrow A$ written as $(d, e, f) \longrightarrow (def)$ satisfying the following identity $(def)gh = d(efg)h = de(fgh)$ for any $d,e,f,g,h \in A$. By a subset, we always mean a nonempty subset. A subset M of a ternary semigroup A is known as a left ideal of A if $AAM \subseteq M$, a lateral ideal of A if $AMA \subseteq M$, and a right ideal of A if $MAA \subseteq M$. A proper ideal P of a ternary semigroup A is said to be a prime ideal of A if $LMN \subseteq P$ implies $L \subseteq P$ or $M \subseteq P$ or $N \subseteq P$, for all ideals L, M, N of A, and it is semiprime if $I^3 \subseteq P$ implies $I \subseteq P$ for all ideal I of A [11].

A fuzzy set defined on ternary semigroup such that $\mu : A \longrightarrow [0, 1]$ is known as fuzzy ternary subsemigroup; here, $[0, 1]$ represents a unit section of a real number line. In this paper, A stands for ternary semigroup and μ is fuzzy subset (FS) of A, unless stated otherwise.

Definition 1 (see [16]). A fuzzy point is defined as

$$\mu(d) = \begin{cases} t & \text{if } d = e \\ 0 & \text{if } d \neq e \end{cases}, \tag{1}$$

where d is support of an FS μ of A and $t \in (0, 1]$. It is denoted as e_t.

Definition 2 (see [27]). If d_i is a fuzzy point, then $d_i \in \mu$ ($d_i q \mu$), respectively, as follows.

(1) If $\mu(d) \geq i$, then d_i belongs to μ and is denoted as $d_i \in \mu$

(2) If $\mu(d) + i > 1$, then d_i is said to be quasi-coincident with μ and is written as $d_i q \mu$

(3) If $\mu(d) \geq i$ or $\mu(d) + i > 1$, then $d_i \in \vee q \mu$

When any of $d_i \in \mu$, $d_i q \mu$, or $d_i \in \vee q \mu$ does not hold, then we write $d_i \bar{\in} \mu$, $d_i \bar{q} \mu$ or $d_i \bar{\in} \bar{\vee} q \mu$, respectively.

Definition 3 (see [27]). The fuzzy ternary subsemigroup μ of A is defined as for every $d,e,f \in A$, $\mu(def) \geq \min\{\mu(d), \mu(e), \mu(f)\}$.

Definition 4 (see [27]). The fuzzy left (resp., right, lateral) ideal μ of A is defined as $\mu(def) \geq \mu(f)$ (resp. $\mu(def) \geq \mu(d)$ and $\mu(def) \geq \mu(e)$) for all $d,e,f \in A$.

Definition 5 (see [21]). The fuzzy prime ideal μ of A is defined as for all $d,e,f \in A, \mu(def) = \mu(d)$ or $\mu(def) = \mu(e)$ or $\mu(def) = \mu(f)$ and fuzzy semiprime if $\mu(ddd) = \mu(d)$.

Definition 6 (see [27]). The $(\in, \in \vee q)$ -fuzzy ternary subsemigroup μ of A is defined as for every $d,e,f \in A$ and $t,r,s \in (0, 1]$, $d_t, e_r, f_s \in \mu \longrightarrow (def)_{\min(t,r,s)} \in \vee q \mu$.

Definition 7 (see [27]). The $(\in, \in \vee q)$ -fuzzy left (resp., right, lateral) ideal μ of A is defined as for $d_i \in \mu$ and $e, f \in A \longrightarrow (efd)_i \in \vee q \mu \{resp.\ (def)_i \in \vee q \mu\ and\ (edf)_i \in \vee q \mu\}$ for all $d,e,f \in A$ and $i \in (0, 1]$.

Definition 8. The $(\in, \in \vee q)$ -fuzzy ideal μ of A is said to be a semiprime if $(d^3)_i \in \mu$ implies $d_i \in \vee q \mu$, for all $d \in A$ and $i \in (0, 1]$.

Definition 9. If μ is $(\in, \in \vee q)$ -fuzzy ideal of A, then it is said to be a prime if for all $d, e, f \in A$, $i \in (0, 1]$, $(def)_i \in \mu$ implies $d_i \in \vee q \mu$ *or* $e_i \in \vee q \mu$ or $f_i \in \vee q \mu$ and semiprime if $(d^3)_i \in \mu$ implies $d_i \in \vee q \mu$.

Theorem 10 (see [27]). *If μ is a $(\in, \in \vee q)$ -fuzzy ternary subsemigroup of A, then $\mu(def) \geq \wedge\{\mu(d), \mu(e), \mu(f), 0.5\}$ for all $d,e,f \in A$.*

Theorem 11 (see [27]). *If μ is a $(\in, \in \vee q)$ -fuzzy left ideal of A. Then, $\mu(def) \geq \wedge\{\mu(f), 0.5\}$ for every $d,e,f \in A$.*

Theorem 12. *If μ is a $(\in, \in \vee q)$ -fuzzy semiprime ideal of A, then for any $d \in A, \mu(d) \geq \{\mu(d^3), 0.5\}$.*

Proof. Proof is same as Theorem 11. □

Theorem 13. *If μ is a $(\in, \in \vee q)$ -fuzzy prime ideal of A, then for any $d, e, f \in A, \vee\{\mu(d), \mu(e), \mu(f)\} \geq \wedge\{\mu(def), 0.5\}$.*

Proof. Proof is same as Theorem 11. □

Now, we give some concepts on rough set. Let L be an equivalence relation on universe W, and then, for any $s \in W$, an equivalence class of s is the collection of elements of W that are related to s and is written as $[s]_L$. A pair (W, L) is said to be an approximation space.

Definition 14. Suppose μ is FS of A. Then, there are the following sets: $\underline{L}(\mu) = \wedge\{\mu(t) | t \in [s]_L\}$ and $\bar{L}(\mu) = \vee\{\mu(t) | t \in [s]_L\}$ as a lower approximation and upper approximation, respectively.

Definition 15. Let μ be any FS of A. Then, for every $d \in A$, we define fuzzy subsets $\underline{J(\mu)}(d) = \bigwedge_{m \in J(d)} \mu(m)$ and $\overline{J(\mu)}(d) = \bigvee_{m \in J(d)} \mu(m)$. $\underline{J(\mu)}$ is called the lower approximation and $J(\bar{\mu})$ is called the upper approximation of the FS μ with respect to J. The combination of $(\underline{J(\mu)}, \overline{J(\mu)})$ is called the rough FS of A if $\underline{J(\mu)} \neq J(\bar{\mu})$.

Definition 16. Let A and A' be two ternary semigroups, and then, set valued map $J : A \longrightarrow P^*(A')$ is called an SVH if for all $d,e,f \in A, J(d)J(e)J(f) \subseteq J(def)$.

Definition 17. Let A and A' be two ternary semigroups, and then, set valued map $J : A \longrightarrow P^*(A')$ is called an SSVH if for all $d,e,f \in A, J(d)J(e)J(f) = J(def)$.

Here, $P^*(A')$ means the collection of all nonempty subsets of A'.

In this paper, for all $d \in A$, the image $J(d)$ is always a nonempty subset of A. This mapping is natural. In case of groups, canonical maps are set-valued maps in which image of an element is a coset. Also, J is a function $J : A \longrightarrow P^*(A)$, unless stated otherwise.

3. Lower and Upper Approximations of Fuzzy Ideals

Initially in this section, approximations of fuzzy ternary subsemigroups are studied. Then, it is shown that the approximations of fuzzy semiprime ideals and fuzzy prime ideals are fuzzy semiprime ideals and fuzzy prime ideals, respectively.

Theorem 18. *If J is SSVH and μ is fuzzy ternary subsemigroup of A, $\underline{J}(\mu)$ is fuzzy ternary subsemigroup of A.*

Proof. Since μ is a fuzzy ternary subsemigroup of A, then $\mu(def) \geq \wedge\{\mu(d), \mu(e), \mu(f)\}$ for all $d,e,f \in A$.

Consider

$$\underline{J}(\mu)(def) = \bigwedge_{m\in J(def)} \mu(m),$$

$$= \bigwedge_{m\in J(d)J(e)J(f)} \mu(m), \text{ as } J \text{ is SSVH},$$

$$= \bigwedge_{rst\in J(d)J(e)J(f)} \mu(rst), \text{ take } m$$

$$= rst \text{ where } r \in J(d), s \in J(e), \text{ and } t \in J(f),$$

$$\geq \bigwedge_{r\in J(d), s\in J(e), t\in J(f)} \{\mu(r)\wedge\mu(s)\wedge\mu(t)\},$$

$$= \left\{\bigwedge_{r\in J(d)} \mu(r)\right\}\wedge\left\{\bigwedge_{s\in J(e)} \mu(s)\right\}\wedge\left\{\bigwedge_{t\in J(f)} \mu(t)\right\},$$

$$= \underline{J}(\mu)(d)\wedge\underline{J}(\mu)(e)\wedge\underline{J}(\mu)(f). \quad (2)$$

This implies $\underline{J}(\mu)(def) \geq \underline{J}(\mu)(d)\wedge\underline{J}(\mu)(e)\wedge\underline{J}(\mu)(f)$.

Hence, $\underline{J}(\mu)$ is a fuzzy ternary subsemigroup of A.

In general, the lower approximation of a fuzzy ternary subsemigroup is not a fuzzy ternary subsemigroup for SVH. □

Example 1. Let $H = \{d, e, f, \zeta\}$ be a ternary semigroup under ternary multiplication defined as $(def) = (de)f$ in Table 1.

Let $J : A \longrightarrow P^*(A)$ be defined as $J(d) = \{e\}$,$J(e) = \{d\}$, and $J(f) = J(\zeta) = \{f, \zeta\}$, and then, J is SVH. Now, let μ be an FS of A given by $\mu(d) = 0.8 = \mu(e)$,$\mu(f) = 0.9$, and $\mu(\zeta) = 0.5$, and then, μ is a fuzzy ternary subsemigroup of A. Then, $\underline{J}(\mu)(d) = 0.8 = \underline{J}(\mu)(e)$; also, $\underline{J}(\mu)(f) = 0.5 = \underline{J}(\mu)(\zeta)$. Then, $\underline{J}(\mu)$ is not a fuzzy ternary subsemigroup of A because $\underline{J}(\mu)(def) \geq \underline{J}(\mu)(d)\wedge\underline{J}(\mu)(e)\wedge\underline{J}(\mu)(f)$ does not hold, as $\underline{J}(\mu)(dee) = 0.5$ and $\underline{J}(\mu)(d)\wedge\underline{J}(\mu)(e)\wedge\underline{J}(\mu)(e) = 0.8$.

Theorem 19. *If J is SVH and μ is fuzzy ternary subsemigroup of A, $\bar{J}(\mu)$ is a fuzzy ternary subsemigroup of A.*

Proof. As μ is a fuzzy ternary subsemigroup of A, then $\mu(def) \geq \wedge\{\mu(d), \mu(e), \mu(f)\}$ for all $d,e,f \in A$. Consider

$$\bar{J}(\mu)(def) = \bigvee_{m\in J(def)} \mu(m),$$

$$\geq \bigvee_{m\in J(d)J(e)J(f)} \mu(m), \text{ since } J \text{ is SVH},$$

$$= \bigvee_{rst\in J(d)J(e)J(f)} \mu(rst), \text{ take } m$$

$$= rst \text{ where } r \in J(d), s \in J(e), \text{ and } t \in J(f),$$

$$\geq \bigvee_{r\in J(d), s\in J(e), t\in J(f)} \{\mu(r)\wedge\mu(s)\wedge\mu(t)\},$$

$$= \left\{\bigvee_{r\in J(d)} \mu(r)\right\}\wedge\left\{\bigvee_{s\in J(e)} \mu(s)\right\}\wedge\left\{\bigvee_{t\in J(f)} \mu(t)\right\},$$

$$= \bar{J}(\mu)(d)\wedge\bar{J}(\mu)(e)\wedge\bar{J}(\mu)(f). \quad (3)$$

This implies $\bar{J}(\mu)(def) \geq \bar{J}(\mu)(d)\wedge\bar{J}(\mu)(e)\wedge\bar{J}(\mu)(f)$.

Hence, proved $\bar{J}(\mu)$ is a fuzzy ternary subsemigroup of A. □

Table 1: Ternary multiplication.

$\cdot$	d	e	f	ζ
d	f	f	f	ζ
e	f	f	f	ζ
f	f	f	f	ζ
ζ	ζ	ζ	ζ	ζ

Theorem 20. *If J is SSVH and μ is fuzzy left (resp., right, lateral) ideal of A, $\underline{J}(\mu)$ is a fuzzy left (resp., right, lateral) ideal of A.*

Proof. Let μ be a fuzzy left ideal of A. We have to show that $\underline{J}(\mu)(def) \geq \underline{J}(\mu)(f)$. Consider

$$\underline{J}(\mu)(def) = \bigwedge_{m\in J(def)} \mu(m),$$

$$= \bigwedge_{m\in J(d)J(e)J(f)} \mu(m), \text{ as } J \text{ is SSVH},$$

$$= \bigwedge_{rst\in J(d)J(e)J(f)} \mu(rst), \text{ by taking } m$$

$$= rst \text{ such that } r \in J(d), s \in J(e), \text{ and } t \in J(f),$$

$$\geq \bigwedge_{t\in J(f)} \mu(t),$$

$$= \underline{J}(\mu)(f). \quad (4)$$

So, $\underline{J}(\mu)(def) \geq \underline{J}(\mu)(f)$. □

Proofs for right and lateral ideals are in similar way.

In general, the lower approximation of a fuzzy ideal is not a fuzzy ideal of A for SVH.

Example 2. Let $A = \{0, m, n\}$ be a ternary semigroup under ternary multiplication defined in Table 2.

Let $J : A \longrightarrow P^*(A)$ be defined as $J(0) = \{0, n\}$, $J(m) = \{n\}$, and $J(n) = \{m, n\}$, and then, J is SVH. Let μ be an FS of A given as $\mu(0) = 1$,$\mu(m) = 0.4$, and $\mu(n) = 0.7$, and then, μ is fuzzy left ideal of A.

So, by the definition of lower approximation, $\underline{J}(\mu)(0) = 0.7$,$\underline{J}(\mu)(m) = 0.7$, and $\underline{J}(\mu)(n) = 0.4$. This implies $\underline{J}(\mu)$ is not a fuzzy ideal of A.$\underline{J}(\mu)(nnm) \geq \underline{J}(\mu)(m)$ is not satisfied because $\underline{J}(\mu)(nnm) = \underline{J}(\mu)(n) = 0.4$ and $\underline{J}(\mu)(m) = 0.7$.

Theorem 21. *If J is an SVH and μ is a fuzzy left (resp., right, lateral) ideal of A,$\bar{J}(\mu)$ is a fuzzy left (resp., right, lateral) ideal of A.*

Table 2: Ternary multiplication.

$\cdot$	0	m	n
0	0	0	0
m	0	m	n
n	0	n	n

Proof. Let μ be a fuzzy left ideal of A; we have to prove that $\bar{J}(\mu)(def) \geq \bar{J}(\mu)(f)$. Consider

$$\begin{aligned}
\bar{J}(\mu)(def) &= \bigvee_{m\in J(def)} \mu(m),\\
&\geq \bigvee_{m\in J(d)J(e)J(f)} \mu(m), \text{ as } J \text{ is SVH},\\
&= \bigvee_{rst\in J(d)J(e)J(f)} \mu(rst), \text{ by taking } m\\
&\quad = rst \text{ such that } r\in J(d), s\in J(e), \text{ and } t\in J(f),\\
&\geq \bigvee_{t\in J(f)} \mu(t),\\
&= \bar{J}(\mu)(f).
\end{aligned} \tag{5}$$

So, $\bar{J}(\mu)(def) \geq \bar{J}(\mu)(f)$. □

Proofs for right and lateral ideals are in same way.

Theorem 22. *If J is an SSVH and μ is a fuzzy semiprime ideal of A, $\underline{J}(\mu)$ is a fuzzy semiprime ideal.*

Proof. Since μ is a fuzzy semiprime ideal of A, then by Theorem 20, $\underline{J}(\mu)$ is fuzzy ideal. Also, $\mu(d^3) = \mu(d)$ for all $d \in A$. To prove that $\underline{J}(\mu)$ is fuzzy semiprime, we have to show that for all $d \in A, \underline{J}(\mu)(d^3) = \underline{J}(\mu)(d)$. Consider

$$\begin{aligned}
\underline{J}(\mu)(d^3) &= \bigwedge_{m\in J(d^3)} \mu(m),\\
&= \bigwedge_{m\in J(d)J(d)J(d)} \mu(m) \text{ as } J \text{ is SSVH},\\
&= \bigwedge_{rrr\in J(d)J(d)J(d)} \mu(r^3), \text{ for } m = r^3 \text{ such that } r\in J(d),\\
&= \bigwedge_{r\in J(d)} \mu(r),\\
&= \underline{J}(\mu)(d).
\end{aligned} \tag{6}$$

□

Theorem 23. *If J is an SSVH and μ is a fuzzy semiprime ideal of A, then $\bar{J}(\mu)$ is fuzzy semiprime ideal.*

Proof. . Same as Theorem 22. □

Theorem 24. *Let J be an SSVH and μ be a fuzzy prime ideal of A, $\underline{J}(\mu)$ is a fuzzy prime ideal.*

Proof. As μ is fuzzy prime ideal A, so $\mu(def) = \mu(d)$ or $\mu(def) = \mu(e)$ or $\mu(def) = \mu(f)$ for all $d,e,f \in A$. Also, by Theorem 20, $\underline{J}(\mu)$ is a fuzzy ideal. Consider

$$\begin{aligned}
\underline{J}(\mu)(def) &= \bigwedge_{m\in J(def)} \mu(m),\\
&= \bigwedge_{m\in J(d)J(e)J(f)} \mu(m) \text{ as } J \text{ is SSVH},\\
&= \bigwedge_{rst\in J(d)J(e)J(f)} \mu(rst), \text{ for } m\\
&\quad = rst \text{ such that } r\in J(d), s\in J(e), \text{ and } t\in J(f),\\
&= \bigwedge_{r\in J(d), s\in J(e), t\in J(f)} \mu(rst),\\
&= \bigwedge_{r\in J(d)} \mu(r) \text{or} \bigwedge_{s\in J(e)} \mu(s) \text{or} \bigwedge_{t\in J(f)} \mu(t),\\
&= \underline{J}(\mu)(d) \text{or} \underline{J}(\mu)(e) \text{or} \underline{J}(\mu)(f).
\end{aligned} \tag{7}$$

□

Theorem 25. *If J is an SSVH and μ is a fuzzy prime ideal of A, $\bar{J}(\mu)$ is a fuzzy prime ideal.*

Proof. Same as Theorem 24. □

4. Approximation of $(\in, \in\vee q)$-Fuzzy Ideals

This section is generalization of Section 3 because $(\in, \in\vee q)$-fuzzy ideals are actually generalization of fuzzy ideals. Here, we have discussed the lower and upper approximation of $(\in, \in\vee q)$-fuzzy ternary subsemigroup and $(\in, \in\vee q)$-fuzzy ideals (prime and semiprime) of ternary semigroups.

Theorem 26. *If J is an SSVH and μ is a $(\in, \in\vee q)$-fuzzy ternary subsemigroup of A, $\underline{J}(\mu)$ is a $(\in, \in\vee q)$-fuzzy ternary subsemigroup of A.*

Proof. Let $d_r, e_s, f_t \in \underline{J}(\mu)$, where $d,e,f \in A$ and $r,s,t \in (0,1]$, and then, $\underline{J}(\mu)(d) \geq r$, $\underline{J}(\mu)(e) \geq s$, and $\underline{J}(\mu)(f) \geq t$. Consider

$$\begin{aligned}
\underline{J}(\mu)(def) &= \bigwedge_{u\in J(def)} \mu(u),\\
&= \bigwedge_{u\in J(d)J(e)J(f)} \mu(u),\\
&= \bigwedge_{jkl\in J(d)J(e)J(f)} \mu(jkl), \text{ here } u\\
&\quad = jkl \text{ such that } j\in J(d), k\in J(e), \text{ and } l\in J(f),\\
&\geq \bigwedge_{j\in J(d), k\in J(e), l\in J(f)} \min\{\mu(j), \mu(k), \mu(l), 0.5\}, \text{ using Theorem 10},\\
&= \min\left\{\left(\bigwedge_{j\in J(d)} \mu(j)\right), \left(\bigwedge_{k\in J(d)} \mu(k)\right), \left(\bigwedge_{l\in J(d)} \mu(l)\right), 0.5\right\},\\
&= \min\{\underline{J}(\mu)(d), \underline{J}(\mu)(e), \underline{J}(\mu)(f), 0.5\},\\
&\geq \min(r, s, t, 0.5).
\end{aligned} \tag{8}$$

Then, $\underline{J}(\mu)(def) \geq \min(r,s,t)$ when $\min(r,s,t) \leq 0.5$. So, $(def)_{\min(r,s,t)} \in \underline{J}(\mu) \longrightarrow (1)$.

When $\min(r,s,t) > 0.5$, we have, $\underline{J}(\mu)(def) \geq 0.5$.

Then, $\underline{J}(\mu)(def) + \min(r,s,t) > 1$, so $(def)_{\min(r,s,t)} q \underline{J}(\mu) \longrightarrow (2)$. From (1) and (2), we get $\underline{J}(def)_{\min(r,s,t)} \in \vee q \underline{J}(\mu)$. □

Theorem 27. *If J is an SVH and μ is a $(\in, \in\vee q)$-fuzzy ternary subsemigroup of A, $\bar{J}(\mu)$ is a $(\in, \in\vee q)$-fuzzy ternary subsemigroup of A.*

Proof. Take $d_r, e_s, f_t \in \bar{J}(\mu)$, where $d,e,f \in A$ and $r,s,t \in (0,1]$, and then, $\bar{J}(\mu)(d) \geq r$, $\bar{J}(\mu)(e) \geq s$, and $\bar{J}(\mu)(f) \geq t$. Consider

$$\begin{aligned}
\bar{J}(\mu)(def) &= \bigwedge_{u \in J(def)} \mu(u), \\
&\geq \bigwedge_{u \in J(d)J(e)J(f)} \mu(u), \\
&= \bigwedge_{jkl \in J(d)J(e)J(f)} \mu(jkl), \text{ here } u \\
&= jkl \text{ such that } j \in J(d),\ k \in J(e), \text{ and } l \in J(f), \\
&\geq \bigwedge_{j \in J(d), k \in J(e), l \in J(f)} \min\{\mu(j), \mu(k), \mu(l), 0.5\}, \\
&= \min\left\{\left(\bigwedge_{j \in J(d)} \mu(j)\right), \left(\bigwedge_{k \in J(d)} \mu(k)\right), \left(\bigwedge_{l \in J(d)} \mu(l)\right), 0.5\right\}, \\
&= \min\{\bar{J}(\mu)(d), \bar{J}(\mu)(e), \bar{J}(\mu)(f), 0.5,\} \\
&\geq \min(r,s,t,0.5). \qquad (9)
\end{aligned}$$

Now, $\bar{J}(\mu)(def) \geq \min(r,s,t)$ when $\min(r,s,t) \leq 0.5$. So, $(def)_{\min(r,s,t)} \in \bar{J}(\mu) \longrightarrow (1)$.

When $\min(r,s,t) > 0.5$ we have, $\bar{J}(def) \geq 0.5$.

Also, $\bar{J}(def) + \min(r,s,t) > 1$, so $(def)_{\min(r,s,t)} q \bar{J}(\mu) \longrightarrow (2)$. From (1) and (2), we get $\bar{J}(def)_{\min(r,s,t)} \in \vee q \bar{J}(\mu)$. □

Theorem 28. *If J is SSVH and μ is a $(\in, \in\vee q)$-fuzzy left (resp., right, lateral) ideal of A, $\bar{J}(\mu)$ is a $(\in, \in\vee q)$ fuzzy left (resp., right, lateral) ideal of A.*

Proof. Let μ be a $(\in, \in\vee q)$-fuzzy left ideal of A. Take $f_i \in \bar{J}(\mu)$ and $d, e \in A$, and then, $\bar{J}(\mu)(f) \geq i$. Consider

$$\begin{aligned}
\bar{J}(\mu)(def) &= \bigvee_{u \in J(def)} \mu(u), \\
&= \bigvee_{u \in J(d)J(e)J(f)} \mu(u), \\
&= \bigvee_{jkl \in J(d)J(e)J(f)} \mu(jkl), \text{ for } u \\
&= jkl \text{ where } j \in J(d), k \in J(e), \text{ and } l \in J(f), \\
&\geq \bigvee_{l \in J(f)} \min\{\mu(l), 0.5\}, \text{ by Theorem 11}, \\
&= \min\{\bar{J}(\mu)(f), 0.5\}, \\
&\geq \min(i, 0.5). \qquad (10)
\end{aligned}$$

When $i \leq 0.5$, then we have $\bar{J}(\mu)(def) \geq i$. □

This implies $(def)_i \in \bar{J}(\mu) \longrightarrow (1)$.

When $i > 0.5$, we have $\bar{J}(\mu)(def) + i > 1$. This implies $(def)_i \in \vee q \bar{J}(\mu)$.

Similarly, we can prove this for right and lateral ideals in same way.

Theorem 29. *If J is SSVH and μ is a $(\in, \in\vee q)$-fuzzy left resp., (right, lateral) ideal of A, $\underline{J}(\mu)$ is a fuzzy left (resp., right, lateral) ideal of A.*

Proof. Same as Theorem 28. □

Theorem 30. *If J is an SSVH and μ is a $(\in, \in\vee q)$-fuzzy semiprime ideal of A, $\bar{J}(\mu)$ is a $(\in, \in\vee q)$-fuzzy semiprime ideal of A.*

Proof. Suppose $(d^3)_r \in \bar{J}(\mu)$ for $d \in A$ and $r \in (0,1]$, and then, $\bar{J}(\mu)(d^3) \geq r$. Consider

$$\begin{aligned}
\bar{J}(\mu)(d) &= \bigvee_{u \in J(d)} \mu(u), \\
&\geq \bigvee_{uuu \in J(d)J(d)J(d)} \min\{\mu(u^3), 0.5\}, \text{ by Theorem 12}, \\
&= \bigvee_{u^3 \in J(d^3)} \min\{\mu(u^3), 0.5\}, \qquad (11) \\
&= \min\{\bar{J}(\mu)(d^3), 0.5\}, \\
&\geq \min(r, 0.5).
\end{aligned}$$

When $r \leq 0.5$, then $\bar{J}(\mu)(d) \geq r$. So, $d_r \in \bar{J}(\mu)$. When $r > 0.5$, we have $\bar{J}(\mu)(d) \geq 0.5$, so $\bar{J}(\mu)(d) + r > r$ and then $d_r q \bar{J}(\mu)$. Therefore, we get $d_r \in \vee q \bar{J}(\mu)$. □

Theorem 31. *If J is SSVH and μ is a $(\in, \in\vee q)$-fuzzy semiprime ideal of A, $\underline{J}(\mu)$ is a $(\in, \in\vee q)$-fuzzy semiprime ideal of A.*

Proof. Similar to Theorem 30. □

Theorem 32. *If J is SSVH and μ is a $(\in, \in\vee q)$-fuzzy prime ideal of A, $\bar{J}(u)$ is $(\in, \in\vee q)$-fuzzy prime ideal of A.*

Proof. By Theorem 28, $\bar{J}(\mu)$ is a $(\in, \in\vee q)$-fuzzy ideal. Further by Theorem 13, $\max\{\mu(d), \mu(e), \mu(f)\} \geq \min\{\mu(def), 0.5\}$. Let $(def)_r \in \bar{J}(\mu)$, and then, $\bar{J}(\mu)(def) \geq r$ for $d, e, f \in A$ and $r \in (0, 1]$. Consider

$$\begin{aligned}
&\bar{J}(\mu)(d)\vee\bar{J}(\mu)(e)\vee\bar{J}(\mu)(f)\\
&\quad=\left(\bigvee_{j\in J(d)}\mu(j)\right)\vee\left(\bigvee_{k\in J(e)}\mu(k)\right)\vee\left(\bigvee_{l\in J(f)}\mu(l)\right),\\
&\qquad\geq\bigvee_{j\in J(d),k\in J(e),l\in J(f)}\{\mu(jkl)\wedge 0.5\},\\
&\qquad=\bigvee_{jkl\in J(d)J(e)J(f)}\{\mu(jkl)\wedge 0.5\},\\
&\qquad=\bigvee_{jkl\in J(def)}\{\mu(jkl)\wedge 0.5\},\\
&\qquad=\bar{J}(\mu)(def)\wedge 0.5,\\
&\qquad\geq\min(r, 0.5).
\end{aligned}\tag{12}$$

□

When $r \leq 0.5$, we have $\bar{J}(\mu)(d)\vee\bar{J}(\mu)(e)\vee\bar{J}(\mu)(f) \geq r$, and then, $\bar{J}(\mu)(d) \geq r$ or $\bar{J}(\mu)(e) \geq r$ or $\bar{J}(\mu)(f) \geq r$. Therefore, $d_r \in \bar{J}(\mu)$ or $e_r \in \bar{J}(\mu)$ or $f_r \in \bar{J}(\mu)$.

When $r > 0.5$, we have $\bar{J}(\mu)(d)\vee\bar{J}(\mu)(e)\vee\bar{J}(\mu)(f) \leq 0.5$, so $\bar{J}(\mu)(d) \geq 0.5$ or $\bar{J}(\mu)(e) \geq 0.5$ or $\bar{J}(\mu)(f) \geq 0.5$. This implies that $\bar{J}(\mu)(d) + r > 1$ or $\bar{J}(\mu)(e) + r > 1$ or $\bar{J}(\mu)(f) + r > 1$. Hence, $d_r q\bar{J}(\mu)$ or $e_r q\bar{J}(\mu)$ or $f_r q\bar{J}(\mu)$. Therefore, $d_r \in \vee q\bar{J}(\mu)$ or $e_r \in \vee q\bar{J}(\mu)$ or $f_r \in \vee q\bar{J}(\mu)$.

Theorem 33. *If J is an SSVH and μ is a fuzzy prime ideal of A, $\underline{J}(\mu)$ is a $(\in, \in\vee q)$-fuzzy prime ideal of A.*

Proof. Similar to Theorem 32. □

5. Comparative Study

In this paper, we have extended the work of [22] to ternary structure, which is more generalized form than [22]. This work is better than [22] because there are many structures in which binary operation does not hold. For example, $W = \{0, i, -i\}$ is not closed under binary operation, while it is closed under ternary product. Also, the set of negative integers, set of negative rational numbers, and set of negative real numbers are closed under ternary multiplication but not closed under usual multiplication. Therefore, it is a need to study the ternary operation which is generalization of binary operation.

6. Conclusions

Here, we observe that the lower approximation of fuzzy ternary subsemigroups (resp., fuzzy ideals) using SSVH is fuzzy ternary subsemigroups (resp., fuzzy ideals). The upper approximation of fuzzy ternary subsemigroups (resp., fuzzy ideals) using SVH is fuzzy ternary subsemigroups (resp., fuzzy ideals). By examples, we show that the lower approximation of a fuzzy ternary subsemigroup (resp., fuzzy ideal) is not fuzzy ternary subsemigroup (resp., fuzzy ideal) for SVH. Also, it is proved that the approximations of fuzzy semiprime (resp., prime) ideals using SSVH is fuzzy semiprime (resp., prime) ideals.

The lower approximation of an $(\in, \in\vee q)$-fuzzy ternary subsemigroup using SSVH and the upper approximation of an $(\in, \in\vee q)$-fuzzy ternary subsemigroup using SVH are $(\in, \in\vee q)$-fuzzy ternary subsemigroups. It is also shown that the approximations of $(\in, \in\vee q)$-fuzzy ideals using SSVH are $(\in, \in\vee q)$-fuzzy ideals.

In fuzzy set, membership degree is limited to [0,1], but there was a difficulty to deliberate the irrelevancy of data. It is noticed that people may have bipolar responses at a time for the same qualities of an item or a plan. Bipolarity is also a significant theory, used in our life. In the future, we will study its bipolar and m-polar part also; we will extend this work to other algebraic structures such as ternary hemiring and LA-semigroup.

Abbreviations

FS: Fuzzy subset
SVH: Set-valued homomorphism
SSVH: Strong set-valued homomorphism.

Data Availability

We did not use any data for this research work.

Ethical Approval

Not applicable.

Consent

Not applicable.

Conflicts of Interest

The authors declare that they have no conflict of interest.

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