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Research Article

Some New Generalized Fractional Newton's Type Inequalities for Convex Functions

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In this paper, we establish some new Newton's type inequalities for differentiable convex functions using the generalized Riemann-Liouville fractional integrals. The main edge of the newly established inequalities is that these can be turned into several new and existing inequalities for different fractional integrals like Riemann-Liouville fractional integrals, *k*-fractional integrals, Katugampola fractional operators, conformable fractional operators, Hadamard fractional operators, and fractional operators with the exponential kernel without proving one by one. It is also shown that the newly established inequalities are the refinements of the previously established inequalities inside the literature.

1. Introduction

The fascinating idea of inequalities has long been a topic of discussion in various mathematical disciplines. Fractional calculus, quantum calculus, operator theory, numerical analysis, operator equations, network theory, and quantum information theory are just a few fascinating applications. This is a very active study topic right now, and the interplay between different areas has enriched it. Numerical integration and definite integral estimation are important aspects of applied sciences. Among the numerical techniques, Simpson's rules are crucial that can be stated as follows:

(1) Simpson's 1/3 rule:

$$\int_{\theta_1}^{\theta_2} \mathfrak{G}(\varkappa)d\varkappa \approx \frac{\theta_2-\theta_1}{6}\left[\mathfrak{G}(\theta_1)+\mathfrak{G}\left(\frac{\theta_1+\theta_2}{2}\right)+\mathfrak{G}(\theta_2)\right] \tag{1}$$

(2) Simpson's 3/8 rule (Newton rule):

$$\int_{\theta_1}^{\theta_2} \mathfrak{G}(\varkappa)d\varkappa \approx \frac{\theta_2-\theta_1}{8}\left[\mathfrak{G}(\theta_1)+3\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)+3\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)+\mathfrak{G}(\theta_2)\right] \tag{2}$$

Researchers have used fractional calculus to develop different fractional integral inequalities that are beneficial in approximation theory due to their importance. Inequalities like Hermite-Hadamard, Simpson's, midpoint, Ostrowski's, and trapezoidal inequalities are examples of inequalities that may be used to find the boundaries of numerical integration formulas. The bounds of trapezoidal formula and inequality of Hermite-Hadamard type using the Riemann-Liouville fractional integrals were established in [1]. Set [2] used differentiable convexity and established fractional Ostrowski's type inequalities. İscan and Wu [3] proved some bounds of numerical integration and inequality of the Hermite-Hadamard type for reciprocal convex functions via Riemann-Liouville fractional integrals. The bounds of midpoint and a new version of fractional inequality of Hermite-Hadamard type were established by Sarikaya and Yildrim in [4]. The bounds for Simpson's 1/3 formula were obtained by Sarikaya et al. [5] using the general convexity and Riemann-Liouville fractional integral operators. In [6], the authors found some new bounds for Simpson's 1/3 formula using the Riemann-Liouville fractional integrals. The authors of [7] used s-convexity and found some bounds for Simpson's 1/3 formula. In 2020, Sarikaya and Ertugral [8] gave a new class of fractional integrals called generalized fractional integrals and established Hermite-Hadamard-type inequalities connected to the newly defined class of integrals. The main advantage of the newly defined class of fractional integral operators is that it can be converted into the classical integral, Riemann-Liouville fractional integrals, k-fractional integrals, Hadamard fractional integrals, etc. In [9], Zhao et al. obtained some bounds for a trapezoidal formula using the reciprocal convex functions and generalized fractional integral operators. Budak et al. [10] established some bounds for Simpson's 1/3 formula for differentiable convex functions using the generalized fractional integrals. Some bounds for the q-Simpson's and Newton's type inequalities were proved by Budak et al. in [11]. Siricharuanun et al. proved some inequalities of Simpson and Newton type by using quantum numbers in [12]. Until recent years, Newton-type inequalities for fractional integrals had not been proven. Recently, Sitthiwirattham et al. [13] used the Riemann-Liouville fractional integrals operators and obtained some bounds for Newton formula.

Motivated by the ongoing studies, we obtain some new bounds/inequalities for Newton formula using the convexity and generalized fractional integrals. The main edge of newly established inequalities is that these can be converted into classical Newton inequalities, Riemann-Liouville fractional Newton inequalities and new Newton inequalities for k-fractional integrals without establishing one by one. These results can be helpful in finding the error bounds of Newton formulas in fractional calculus, which is the main motivation of this paper. Moreover, the main difference between the results proved in [11–13] and the results of this paper is that while the papers [11, 12] are derived on Newton type inequalities for quantum integrals and the paper [13] focus on Newton type inequalities for Riemann-Liouville fractional integrals operators, we prove some inequalities of Newton type by using the generalized fractional integrals. These inequalities generalize the results of the paper [13] and give some new inequalities for k-fractional integrals, Hadamard fractional integrals, conformable fractional integrals, etc.

On the other hand, there are many other papers related to our topic. One can consult [14–25] and references therein for more inequalities via fractional integrals. Moreover, several papers focused on the functions of bounded variation to prove some important inequalities such as the Ostrowski type [26], Simpson type [27, 28], trapezoid type [29, 30], and midpoint type [31]. For more applications of fractional calculus in other areas of mathematical sciences, one can consult [32–41].

A description of the paper is as follows: In Section 2, the fundamentals of fractional calculus, as well as other pertinent research in this field, are briefly discussed. In Section 3, we develop an essential identity that is vital in identifying the key outcomes of the paper. In Section 4, we use generalized fractional integrals to derive some new Newton's type inequalities for differentiable convex functions. For functions of bounded variation, Section 5 contains certain fractional Newton-type inequalities. Section 6 concludes with some future study ideas.

2. Fractional Integrals and Related Inequalities

Several fundamental fractional integral notations and concepts are reviewed in this section. Different fractional integrals are also used to recall various inequalities.

Definition 1. A function $\mathfrak{G} : I \longrightarrow \mathbb{R}$, where I is an interval in $\mathbb{R}$, is called convex, if it satisfies the inequality

$$\mathfrak{G}(t\varkappa + (1-t)y) \le t\mathfrak{G}(\varkappa) + (1-t)\mathfrak{G}(y), \tag{3}$$

where $\varkappa, y \in I$ and $t \in [0,1]$.

Definition 2 ([42, 43]). Let $\mathfrak{G} \in L_1[\theta_1, \theta_2]$. The Riemann-Liouville fractional integrals (RLFIs) $J^{\alpha}_{\theta_1+}\mathfrak{G}$ and $J^{\alpha}_{\theta_2-}\mathfrak{G}$ of order $\alpha > 0$ with $\theta_1 \ge 0$ are defined as follows:

$$\begin{aligned} J^{\alpha}_{\theta_1+}\mathfrak{G}(\varkappa) &= \frac{1}{\Gamma(\theta_1)}\int_{\theta_1}^{\varkappa}(\varkappa - t)^{\alpha-1}\mathfrak{G}(t)dt, \varkappa > \theta_1, \\ J^{\alpha}_{\theta_2-}\mathfrak{G}(\varkappa) &= \frac{1}{\Gamma(\alpha)}\int_{\varkappa}^{\theta_2}(t-\varkappa)^{\alpha-1}\mathfrak{G}(t)dt, \varkappa < \theta_2, \end{aligned} \tag{4}$$

respectively, where the well-known Gamma function is represented by Γ.

Definition 3 ([44]). Let $\mathfrak{G} \in L_1[\theta_1, \theta_2]$. The k-Riemann-Liouville fractional integrals (KRLFIs) $\mathcal{J}^{\alpha,k}_{\theta_1+}\mathfrak{G}$ and $\mathcal{J}^{\alpha,k}_{\theta_2-,k}\mathfrak{G}$ of order $\alpha, k > 0$ with $\theta_1 \ge 0$ are defined as follows:

$$\begin{aligned} \mathcal{J}^{\alpha,k}_{\theta_1+}\mathfrak{G}(\varkappa) &= \frac{1}{k\Gamma_k(\alpha)}\int_{\theta_1}^{\varkappa}(\varkappa - t)^{(\alpha/k)-1}\mathfrak{G}(t)dt, \varkappa > \theta_1, \\ \mathcal{J}^{\alpha,k}_{\theta_2-,k}\mathfrak{G}(\varkappa) &= \frac{1}{k\Gamma_k(\alpha)}\int_{\varkappa}^{\theta_2}(t-\varkappa)^{(\alpha/k)-1}\mathfrak{G}(t)dt, \varkappa < \theta_2, \end{aligned} \tag{5}$$

respectively, where Γ_k is the well-known k-Gamma function.

Definition 4 ([8]). Let $\mathfrak{G} \in L_1[\theta_1, \theta_2]$. The generalized fractional integrals (GRLFIs) ${}_{\theta_1+}I_{\varphi}\mathfrak{G}$ and ${}_{\theta_2-}I_{\varphi}\mathfrak{G}$ with $\theta_1 \ge 0$ are defined as follows:

$$\begin{aligned} {}_{\theta_1+}I_{\varphi}\mathfrak{G}(\varkappa) &= \int_{\theta_1}^{\varkappa}\frac{\varphi(\varkappa - t)}{\varkappa - t}\mathfrak{G}(t)dt, \varkappa > \theta_1, \\ {}_{\theta_2-}I_{\varphi}\mathfrak{G}(\varkappa) &= \int_{\varkappa}^{\theta_2}\frac{\varphi(t-\varkappa)}{t-\varkappa}\mathfrak{G}(t)dt, \varkappa < \theta_2, \end{aligned} \tag{6}$$

respectively, where the mapping is $\varphi : [0,\infty) \longrightarrow [0,\infty)$. One can consult [8] for further information of function φ.

Remark 5. The GRLFIs are significant because they can be converted into classical Riemann integrals, RLFIs, and KFIs for $\varphi(\lambda) = \lambda, \varphi(\lambda) = \lambda^{\alpha}/\Gamma(\alpha)$ and $\varphi(\lambda) = \lambda^{\alpha/k}/k\Gamma_k(\alpha)$, respectively. For more choices of the function φ, one can recapture the different fractional integrals like Katugampola fractional operators, conformable fractional integrals, Hadamard fractional operators, and fractional operators with the exponential kernel (see [8]).

In [45], Ertuğral and Sarikaya used GRLFIs and proved the following Simpson's type inequalities for differentiable convex functions.

Theorem 6. *Let $\mathfrak{G} : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function over I° and $\mathfrak{G}' \in L_1[\theta_1, \theta_2]$. If $|\mathfrak{G}'|$ is convex over $[\theta_1, \theta_2]$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{1}{6}\left[\mathfrak{G}(\theta_1) + 4\mathfrak{G}\left(\frac{\theta_1+\theta_2}{2}\right) + \mathfrak{G}(\theta_2)\right] \right. \\ &\left. - \frac{1}{2\Theta(1)}\left[{}_{\theta_1+}I_{\varphi}\mathfrak{G}\left(\frac{\theta_1+\theta_2}{2}\right) + {}_{\theta_2-}I_{\varphi}\mathfrak{G}\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right| \\ &\le \frac{\theta_2-\theta_1}{2\Theta(1)}\Omega(t)\left[|\mathfrak{G}'(\theta_1)| + |\mathfrak{G}'(\theta_2)|\right], \end{aligned} \tag{7}$$

where

$$\begin{aligned} \Omega(t) &= \int_0^1 \left|\frac{\Theta(t)}{2} - \frac{\Theta(1)}{3}\right| dt, \\ \Theta(\varkappa) &= \int_0^{\varkappa} \frac{\varphi((\theta_2-\theta_1)/2t)}{t} dt. \end{aligned} \tag{8}$$

It is worth mentioning here that the inequality (7) can be turned into classical Simpson's inequality, RLFIs Simpson's inequality, and KRLFIs inequality as follows:

(i) For $\varphi(t) = t$, the following Simpson's inequality for classical Riemann-integral holds (see [5]):

$$\begin{aligned} &\left|\frac{1}{6}\left[\mathfrak{G}(\theta_1) + 4\mathfrak{G}\left(\frac{\theta_1+\theta_2}{2}\right) + \mathfrak{G}(\theta_2)\right] - \frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2}\mathfrak{G}(\varkappa)d\varkappa\right| \\ &\le \frac{5(\theta_2-\theta_1)}{72}\left[|\mathfrak{G}'(\theta_1)| + |\mathfrak{G}'(\theta_2)|\right] \end{aligned} \tag{9}$$

(ii) For $\varphi(t) = t^{\alpha}/\Gamma(\alpha)$, the following Simpson's inequality for RLFIs holds (see [45]):

$$\begin{aligned} &\left|\frac{1}{6}\left[\mathfrak{G}(\theta_1) + 4\mathfrak{G}\left(\frac{\theta_1+\theta_2}{2}\right) + \mathfrak{G}(\theta_2)\right] \right. \\ &\left. - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(\theta_2-\theta_1)^{\alpha}}\left[J^{\alpha}_{\theta_1+}\mathfrak{G}\left(\frac{\theta_1+\theta_2}{2}\right) + J^{\alpha}_{\theta_2-}\mathfrak{G}\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right| \\ &\le \frac{\theta_2-\theta_1}{2}F(\alpha)\left[|\mathfrak{G}'(\theta_1)| + |\mathfrak{G}'(\theta_2)|\right], \end{aligned} \tag{10}$$

where

$$F(\alpha) = \left(\frac{2}{3}\right)^{(1/\alpha)+1}\left(\frac{\alpha}{\alpha+1}\right) + \frac{1}{2(\alpha+1)} - \frac{1}{3} \tag{11}$$

(iii) For $\varphi(t) = t^{\alpha/k}/k\Gamma_k(\alpha)$, the following Simpson's inequality for KRLFIs holds (see [45]):

$$\begin{aligned} &\left|\frac{1}{6}\left[\mathfrak{G}(\theta_1) + 4\mathfrak{G}\left(\frac{\theta_1+\theta_2}{2}\right) + \mathfrak{G}(\theta_2)\right] \right. \\ &\left. - \frac{\Gamma_k(\alpha+k)}{2^{1-\alpha/k}(\theta_2-\theta_1)^{\alpha/k}}\left[\mathcal{J}^{\alpha,k}_{\theta_1+}\mathfrak{G}\left(\frac{\theta_1+\theta_2}{2}\right) + \mathcal{J}^{\alpha,k}_{\theta_2-}\mathfrak{G}\left(\frac{\theta_1+\theta_2}{2}\right)\right]\right| \\ &\le \frac{\theta_2-\theta_1}{2}F(\alpha,k)\left[|\mathfrak{G}'(\theta_1)| + |\mathfrak{G}'(\theta_2)|\right], \end{aligned} \tag{12}$$

where

$$F(\alpha,k) = \left(\frac{2}{3}\right)^{k/\alpha+1}\left(\frac{\alpha}{\alpha+k}\right) + \frac{k}{2(\alpha+k)} - \frac{1}{3} \tag{13}$$

Remark 7. If we set $\alpha = k = 1$ in (10) and (12), then we obtain the classical Simpson's inequality (9).

3. An Identity

In this section, we prove an integral equality in order to demonstrate the primary findings of the paper. For brevity, we shall use the following notation throughout the paper:

$$Y(\varkappa) = \int_0^{\varkappa} \frac{\varphi(((\theta_2-\theta_1)/3)u)}{u} du < +\infty. \tag{14}$$

Lemma 8. *If $\mathfrak{G} : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is a function such that $\mathfrak{G}$ is differentiable over I° and $\mathfrak{G}' \in L_1[\theta_1, \theta_2]$, then the following identity holds for GRLFIs:*

$$\begin{aligned} &\frac{1}{3Y(1)}\left[{}_{\theta_1+}I_{\varphi}\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right) + {}_{(2\theta_1+\theta_2)/3+}I_{\varphi}\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)\right. \\ &\left. + {}_{(\theta_1+2\theta_2)/3+}I_{\varphi}\mathfrak{G}(\theta_2)\right] - \frac{1}{8}\left[\mathfrak{G}(\theta_1) + 3\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)\right. \\ &\left. + 3\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right) + \mathfrak{G}(\theta_2)\right] = \frac{\theta_2-\theta_1}{9Y(1)}[I_1+I_2+I_3], \end{aligned} \tag{15}$$

where

$$\begin{aligned} I_1 &= \int_0^1 \left(Y(t) - \frac{5Y(1)}{8}\right)\mathfrak{G}'\left(t\theta_1 + (1-t)\frac{2\theta_1+\theta_2}{3}\right)dt, \\ I_2 &= \int_0^1 \left(Y(t) - \frac{Y(1)}{2}\right)\mathfrak{G}'\left(t\frac{2\theta_1+\theta_2}{3} + (1-t)\frac{\theta_1+2\theta_2}{3}\right)dt, \end{aligned}$$

$$I_3 = \int_0^1 \left(Y(t) - \frac{3Y(1)}{8} \right) \mathfrak{G}' \left(t \frac{\theta_1 + 2\theta_2}{3} + (1-t)\theta_2 \right) dt. \quad (16)$$

Proof. Using the laws of integration by parts and variables change, we have

$$\begin{aligned} I_1 &= \int_0^1 \left(Y(t) - \frac{5Y(1)}{8} \right) \mathfrak{G}' \left(t\theta_1 + (1-t)\frac{2\theta_1 + \theta_2}{3} \right) dt \\ &= \frac{3}{(\theta_2 - \theta_1)} \int_0^1 \frac{\varphi(((\theta_2 - \theta_1)/3)t)}{t} \mathfrak{G} \left(t\theta_1 + (1-t)\frac{2\theta_1 + \theta_2}{3} \right) dt \\ &\quad - \frac{Y(1)}{\theta_2 - \theta_1} \left[\frac{15}{8} \mathfrak{G} \left(\frac{2\theta_1 + \theta_2}{3} \right) + \frac{9}{8} \mathfrak{G}(\theta_1) \right] \\ &= \frac{3}{\theta_2 - \theta_1} {}_{\theta_1 +}I_\varphi \mathfrak{G} \left(\frac{2\theta_1 + \theta_2}{3} \right) - \frac{Y(1)}{\theta_2 - \theta_1} \left[\frac{15}{8} \mathfrak{G} \left(\frac{2\theta_1 + \theta_2}{3} \right) + \frac{9}{8} \mathfrak{G}(\theta_1) \right]. \end{aligned} \quad (17)$$

Also, we have

$$\begin{aligned} I_2 &= \int_0^1 \left(Y(t) - \frac{Y(1)}{2} \right) \mathfrak{G}' \left(t\frac{2\theta_1 + \theta_2}{3} + (1-t)\frac{\theta_1 + 2\theta_2}{3} \right) dt \\ &= \frac{3}{\theta_2 - \theta_1} {}_{(2\theta 1 + \theta 2)/3 +}I_\varphi \mathfrak{G} \left(\frac{\theta_1 + 2\theta_2}{3} \right) \\ &\quad - \frac{Y(1)}{\theta_2 - \theta_1} \left[\frac{3}{2} \mathfrak{G} \left(\frac{\theta_1 + 2\theta_2}{3} \right) + \frac{3}{2} \mathfrak{G} \left(\frac{2\theta_1 + \theta_2}{3} \right) \right], \end{aligned} \quad (18)$$

$$\begin{aligned} I_3 &= \int_0^1 \left(Y(t) - \frac{3Y(1)}{8} \right) \mathfrak{G}' \left(t\frac{\theta_1 + 2\theta_2}{3} + (1-t)\theta_2 \right) dt \\ &= \frac{3}{\theta_2 - \theta_1} {}_{(\theta 1 + 2\theta 2)/3 +}I_\varphi \mathfrak{G}(\theta_2) - \frac{Y(1)}{\theta_2 - \theta_1} \left[\frac{9}{8} \mathfrak{G}(\theta_2) \right. \\ &\quad \left. + \frac{15}{8} \mathfrak{G} \left(\frac{\theta_1 + 2\theta_2}{3} \right) \right]. \end{aligned} \quad (19)$$

As a consequence, we may get the resultant equality by adding (17)–(19) and multiplying the resultant one by $(\theta_2 - \theta_1)/9Y(1)$. □

4. Newton's Inequalities for Convex Functions

We will utilize GRLFIs to demonstrate some new Newton's inequalities for differentiable convex functions in this section. We use the following notations for sake of brevity:

$$\begin{aligned} A_1(\alpha) &= \int_0^1 t \left| Y(t) - \frac{3Y(1)}{8} \right| dt, \\ A_2(\alpha) &= \int_0^1 \left| Y(t) - \frac{3Y(1)}{8} \right| dt, \\ A_3(\alpha) &= \int_0^1 t \left| Y(t) - \frac{Y(1)}{2} \right| dt, \\ A_4(\alpha) &= \int_0^1 \left| Y(t) - \frac{Y(1)}{2} \right| dt, \\ A_5(\alpha) &= \int_0^1 t \left| Y(t) - \frac{5Y(1)}{8} \right| dt, \\ A_6(\alpha) &= \int_0^1 \left| Y(t) - \frac{5Y(1)}{8} \right| dt. \end{aligned} \quad (20)$$

Theorem 9. *If $|\mathfrak{G}'|$ is a convex function and assumptions of Lemma 8 hold, then we obtain the following Newton's type inequality:*

$$\begin{aligned} &\left| \frac{1}{3Y(1)} \left[{}_{\theta_1 +}I_\varphi \mathfrak{G} \left(\frac{2\theta_1 + \theta_2}{3} \right) + {}_{(2\theta 1 + \theta 2)/3 +}I_\varphi \mathfrak{G} \left(\frac{\theta_1 + 2\theta_2}{3} \right) \right. \right. \\ &\quad \left. + {}_{(\theta 1 + 2\theta 2)/3 +}I_\varphi \mathfrak{G}(\theta_2) \right] - \frac{1}{8} \left[\mathfrak{G}(\theta_1) + 3\mathfrak{G} \left(\frac{2\theta_1 + \theta_2}{3} \right) \right. \\ &\quad \left. \left. + 3\mathfrak{G} \left(\frac{\theta_1 + 2\theta_2}{3} \right) + \mathfrak{G}(\theta_2) \right] \right| \\ &\leq \frac{\theta_2 - \theta_1}{27Y(1)} \left[|\mathfrak{G}'(\theta_2)| (3A_2(\alpha) - A_1(\alpha) + 2A_4(\alpha) \right. \\ &\quad - A_3(\alpha) + A_6(\alpha) - A_5(\alpha)) + |\mathfrak{G}'(\theta_1)| (A_1(\alpha) \\ &\quad \left. + A_4(\alpha) + A_3(\alpha) + 2A_6(\alpha) + A_5(\alpha)) \right]. \end{aligned} \quad (21)$$

Proof. Using the convexity of $|\mathfrak{G}'|$ and the modulus in (15), we get

$$\begin{aligned} &\left| \frac{1}{3Y(1)} \left[{}_{\theta_1 +}I_\varphi \mathfrak{G} \left(\frac{2\theta_1 + \theta_2}{3} \right) + {}_{(2\theta 1 + \theta 2)/3 +}I_\varphi \mathfrak{G} \left(\frac{\theta_1 + 2\theta_2}{3} \right) \right. \right. \\ &\quad \left. + {}_{(\theta 1 + 2\theta 2)/3 +}I_\varphi \mathfrak{G}(\theta_2) \right] - \frac{1}{8} \left[\mathfrak{G}(\theta_1) + 3\mathfrak{G} \left(\frac{2\theta_1 + \theta_2}{3} \right) \right. \\ &\quad \left. \left. + 3\mathfrak{G} \left(\frac{\theta_1 + 2\theta_2}{3} \right) + \mathfrak{G}(\theta_2) \right] \right| \\ &\leq \frac{\theta_2 - \theta_1}{9} \left[\int_0^1 \left| Y(t) - \frac{3Y(1)}{8} \right| \left| \mathfrak{G}' \left(t\frac{\theta_1 + 2\theta_2}{3} + (1-t)\theta_2 \right) \right| dt \right. \\ &\quad + \int_0^1 \left| Y(t) - \frac{Y(1)}{2} \right| \left| \mathfrak{G}' \left(t\frac{2\theta_1 + \theta_2}{3} + (1-t)\frac{\theta_1 + 2\theta_2}{3} \right) \right| dt \\ &\quad \left. + \int_0^1 \left| Y(t) - \frac{5Y(1)}{8} \right| \left| \mathfrak{G}' \left(t\theta_1 + (1-t)\frac{2\theta_1 + \theta_2}{3} \right) \right| dt \right] \\ &= \frac{\theta_2 - \theta_1}{9} \left[\int_0^1 \left| Y(t) - \frac{3Y(1)}{8} \right| \left| \mathfrak{G}' \left(\frac{3-t}{3}\theta_2 + \frac{t}{3}\theta_1 \right) \right| dt \right. \\ &\quad + \int_0^1 \left| Y(t) - \frac{Y(1)}{2} \right| \left| \mathfrak{G}' \left(\frac{2-t}{3}\theta_2 + \frac{1+t}{3}\theta_1 \right) \right| dt \\ &\quad \left. + \int_0^1 \left| Y(t) - \frac{5Y(1)}{8} \right| \left| \mathfrak{G}' \left(\frac{1-t}{3}\theta_2 + \frac{2+t}{3}\theta_1 \right) \right| dt \right] \\ &\leq \frac{\theta_2 - \theta_1}{9} \left[|\mathfrak{G}'(\theta_2)| \int_0^1 \frac{3-t}{3} \left| Y(t) - \frac{3Y(1)}{8} \right| dt \right. \\ &\quad + |\mathfrak{G}'(\theta_1)| \int_0^1 \frac{t}{3} \left| Y(t) - \frac{3Y(1)}{8} \right| dt + |\mathfrak{G}'(\theta_2)| \int_0^1 \frac{2-t}{3} |Y(t) \end{aligned}$$

$$
\begin{aligned}
&-\frac{Y(1)}{2}|dt+|\mathfrak{G}'(\theta_1)|\int_0^1\frac{1+t}{3}\left|Y(t)-\frac{Y(1)}{2}\right|dt\\
&+|\mathfrak{G}'(\theta_2)|\int_0^1\frac{1-t}{3}\left|Y(t)-\frac{5Y(1)}{8}\right|dt\\
&+|\mathfrak{G}'(\theta_1)|\int_0^1\frac{2+t}{3}\left|Y(t)-\frac{5Y(1)}{8}\right|dt]\\
&=\frac{\theta_2-\theta_1}{27Y(1)}\left[|\mathfrak{G}'(\theta_2)|(3A_2(\alpha)-A_1(\alpha)+2A_4(\alpha)\right.\\
&-A_3(\alpha)+A_6(\alpha)-A_5(\alpha))+|\mathfrak{G}'(\theta_1)|(A_1(\alpha)\\
&\left.+A_4(\alpha)+A_3(\alpha)+2A_6(\alpha)+A_5(\alpha))\right].
\end{aligned}
\tag{22}
$$

The proof is now completed. □

Remark 10. In Theorem 9, we have the following:

(i) By setting $\varphi(t)=t$, we reclaim the inequality established in ([13], Remark 3)

(ii) By setting $\varphi(t)=t^{\alpha}/\Gamma(\alpha)$, we reclaim the inequality established in ([13], Theorem 4)

Corollary 11. *By setting $\varphi(t)=t^{\alpha/k}/k\Gamma_k(\alpha)$ in Theorem 9, we get the following new Newton's inequality for KRLFIs:*

$$
\begin{aligned}
&\left|\frac{3^{\alpha/k-1}\Gamma_k(\alpha+1)}{(\theta_2-\theta_1)^{\alpha/k}}\left[\mathcal{J}^{\alpha,k}_{\theta_1+}\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)+\mathcal{J}^{\alpha,k}_{(2\theta_1+\theta_2)/3+}\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)\right.\right.\\
&\left.+\mathcal{J}^{\alpha,k}_{(\theta_1+2\theta_2)/3+}\mathfrak{G}(\theta_2)\right]-\frac{1}{8}\left[\mathfrak{G}(\theta_1)+3\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)\right.\\
&\left.\left.+3\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)+\mathfrak{G}(\theta_2)\right]\right|\\
&\le\frac{\theta_2-\theta_1}{27}\left[|\mathfrak{G}'(\theta_2)|(3A_2(\alpha,k)-A_1(\alpha,k)+2A_4(\alpha,k)\right.\\
&-A_3(\alpha,k)+A_6(\alpha,k)-A_5(\alpha,k))+|\mathfrak{G}'(\theta_1)|(A_1(\alpha,k)\\
&\left.+A_4(\alpha,k)+A_3(\alpha,k)+2A_6(\alpha,k)+A_5(\alpha,k))\right],
\end{aligned}
\tag{23}
$$

where

$$
\begin{aligned}
A_1(\alpha,k)&=\int_0^1 t\left|t^{\alpha/k}-\frac{3}{8}\right|dt=\frac{\alpha}{\alpha+2k}\left(\frac{3}{8}\right)^{(\alpha+2k)/\alpha}+\frac{k}{\alpha+2k}-\frac{3}{16},\\
A_2(\alpha,k)&=\int_0^1\left|t^{\alpha/k}-\frac{3}{8}\right|dt=\frac{2\alpha}{\alpha+k}\left(\frac{3}{8}\right)^{(\alpha+k)/\alpha}+\frac{k}{\alpha+k}-\frac{3}{8},\\
A_3(\alpha,k)&=\int_0^1 t\left|t^{\alpha/k}-\frac{1}{2}\right|dt=\frac{\alpha}{\alpha+2k}\left(\frac{1}{2}\right)^{(\alpha+2k)/\alpha}+\frac{k}{\alpha+2k}-\frac{1}{4},\\
A_4(\alpha,k)&=\int_0^1\left|t^{\alpha/k}-\frac{1}{2}\right|dt=\frac{2\alpha}{\alpha+k}\left(\frac{1}{2}\right)^{(\alpha+k)/\alpha}+\frac{1}{\alpha+k}-\frac{1}{2},\\
A_5(\alpha,k)&=\int_0^1 t\left|t^{\alpha/k}-\frac{5}{8}\right|dt=\frac{\alpha}{\alpha+2k}\left(\frac{5}{8}\right)^{(\alpha+2k)/a}+\frac{k}{\alpha+2k}-\frac{5}{16},\\
A_6(\alpha,k)&=\int_0^1\left|t^{\alpha/k}-\frac{5}{8}\right|dt=\frac{2\alpha}{\alpha+k}\left(\frac{5}{8}\right)^{(\alpha+k)/a}+\frac{k}{\alpha+k}-\frac{5}{8}.
\end{aligned}
\tag{24}
$$

Theorem 12. *If $|\mathfrak{G}'|^q$, $q\ge 1$ is a convex function and assumptions of Lemma 8 hold, then we get the following Newton's type inequality:*

$$
\begin{aligned}
&\left|\frac{1}{3Y(1)}\left[{}_{\theta_1+}I_{\varphi}\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)+{}_{(2\theta1+\theta2)/3+}I_{\varphi}\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)\right.\right.\\
&\left.+{}_{(\theta1+2\theta2)/3+}I_{\varphi}\mathfrak{G}(\theta_2)\right]-\frac{1}{8}\left[\mathfrak{G}(\theta_1)+3\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)\right.\\
&\left.\left.+3\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)+\mathfrak{G}(\theta_2)\right]\right|\\
&\le\frac{\theta_2-\theta_1}{9Y(1)}\left[A_2^{1-(1/q)}(\alpha)\left(|\mathfrak{G}'(\theta_2)|^q\frac{3A_2(\alpha)-A_1(\alpha)}{3}\right.\right.\\
&\left.+|\mathfrak{G}'(\theta_1)|^q\frac{A_1(\alpha)}{3}\right)^{1/q}+A_4^{1-(1/q)}(\alpha)\\
&\cdot\left(|\mathfrak{G}'(\theta_2)|^q\frac{2A_4(\alpha)-A_3(\alpha)}{3}+|\mathfrak{G}'(\theta_1)|^q\frac{A_4(\alpha)+A_3(\alpha)}{3}\right)^{1/q}\\
&+A_6^{1-(1/q)}(\alpha)\left(|\mathfrak{G}'(\theta_2)|\frac{A_6(\alpha)-A_5(\alpha)}{3}\right.\\
&\left.\left.+|\mathfrak{G}'(\theta_1)|^q\frac{2A_6(\alpha)+A_5(\alpha)}{3}\right)^{1/q}\right].
\end{aligned}
\tag{25}
$$

Proof. Applying power mean inequality in (15) after taking the modulus, we have

$$
\begin{aligned}
&\left|\frac{1}{3Y(1)}\left[{}_{\theta_1+}I_{\varphi}\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)+{}_{(2\theta1+\theta2)/3+}I_{\varphi}\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)\right.\right.\\
&\left.+{}_{(\theta1+2\theta2)+}I_{\varphi}\mathfrak{G}(\theta_2)\right]-\frac{1}{8}\left[\mathfrak{G}(\theta_1)+3\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)\right.\\
&\left.\left.+3\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)+\mathfrak{G}(\theta_2)\right]\right|\frac{\theta_2-\theta_1}{9Y(1)}\left[\int_0^1\left|Y(t)-\frac{3Y(1)}{8}\right||\mathfrak{G}'\right.\\
&\cdot\left(\frac{3-t}{3}\theta_2+\frac{t}{3}\theta_1\right)\Big|dt+\int_0^1\left|Y(t)-\frac{Y(1)}{2}\right||\mathfrak{G}'\\
&\cdot\left(\frac{2-t}{3}\theta_2+\frac{1+t}{3}\theta_1\right)\Big|dt+\int_0^1\left|Y(t)-\frac{5Y(1)}{8}\right||\mathfrak{G}'\\
&\left.\cdot\left(\frac{1-t}{3}\theta_2+\frac{2+t}{3}\theta_1\right)\Big|dt\right]\\
&\le\frac{\theta_2-\theta_1}{9Y(1)}\left[\left(\int_0^1\left|Y(t)-\frac{3Y(1)}{8}\right|dt\right)^{1-(1/q)}\left(\int_0^1\left|Y(t)-\frac{3Y(1)}{8}\right||\mathfrak{G}'\right.\right.\\
&\left.\cdot\left(\frac{3-t}{3}\theta_2+\frac{t}{3}\theta_1\right)\Big|^q dt\right)^{(1/q)}+\left(\int_0^1\left|Y(t)-\frac{Y(1)}{2}\right|dt\right)^{1-(1/q)}\\
&\cdot\left(\int_0^1\left|Y(t)-\frac{Y(1)}{2}\right|\left|\mathfrak{G}'\left(\frac{2-t}{3}\theta_2+\frac{1+t}{3}\theta_1\right)\right|^q dt\right)^{(1/q)}\\
&+\left(\int_0^1\left|Y(t)-\frac{5Y(1)}{8}\right|dt\right)^{1-(1/q)}\left(\int_0^1\left|Y(t)-\frac{5Y(1)}{8}\right||\mathfrak{G}'\right.\\
&\left.\left.\cdot\left(\frac{1-t}{3}\theta_2+\frac{2+t}{3}\theta_1\right)\Big|^q dt\right)^{1/q}\right].
\end{aligned}
\tag{26}
$$

Using the convexity of $|\mathfrak{G}'|^q$, we have

$$\begin{aligned}
&\left|\frac{\theta_2-\theta_1}{3Y(1)}\left[{}_{\theta_1+}I_{\varphi}\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)+{}_{(2\theta_1+\theta_2)/3+}I_{\varphi}\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)\right.\right.\\
&\quad\left.+{}_{(\theta_1+2\theta_2)/3+}I_{\varphi}\mathfrak{G}(\theta_2)\right]-\frac{1}{8}\left[\mathfrak{G}(\theta_1)+3\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)\right.\\
&\quad\left.\left.+3\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)+\mathfrak{G}(\theta_2)\right]\right|\\
&\leq\frac{\theta_2-\theta_1}{9Y(1)}\left[\left(\int_0^1\left|Y(t)-\frac{3Y(1)}{8}\right|dt\right)^{1-(1/q)}\right.\\
&\quad\times\left(\left|\mathfrak{G}'(\theta_2)\right|^q\int_0^1\frac{3-t}{3}\left|Y(t)-\frac{3Y(1)}{8}\right|dt+\left|\mathfrak{G}'(\theta_1)\right|^q\right.\\
&\quad\left.\cdot\int_0^1\frac{t}{3}\left|Y(t)-\frac{3Y(1)}{8}\right|dt\right)^{(1/q)}+\left(\int_0^1\left|Y(t)-\frac{Y(1)}{2}\right|dt\right)^{1-(1/q)}\\
&\quad\times\left(\left|\mathfrak{G}'(\theta_2)\right|^q\int_0^1\frac{2-t}{3}\left|Y(t)-\frac{Y(1)}{2}\right|dt+\left|\mathfrak{G}'(\theta_1)\right|^q\right.\\
&\quad\left.\cdot\int_0^1\frac{1+t}{3}\left|Y(t)-\frac{Y(1)}{2}\right|dt\right)^{(1/q)}+\left(\int_0^1\left|Y(t)-\frac{5Y(1)}{8}\right|dt\right)^{1-(1/q)}\\
&\quad\times\left(\left|\mathfrak{G}'(\theta_2)\right|^q\int_0^1\frac{1-t}{3}\left|Y(t)-\frac{5Y(1)}{8}\right|dt\right.\\
&\quad\left.\left.+\left|\mathfrak{G}'(\theta_1)\right|^q\int_0^1\frac{2+t}{3}\left|Y(t)-\frac{5Y(1)}{8}\right|dt\right)^{(1/q)}\right]\\
&=\frac{\theta_2-\theta_1}{9Y(1)}\left[A_2^{1-(1/q)}(\alpha)\left(\left|\mathfrak{G}'(\theta_2)\right|^q\frac{3A_2(\alpha)-A_1(\alpha)}{3}\right.\right.\\
&\quad\left.+\left|\mathfrak{G}'(\theta_1)\right|^q\frac{A_1(\alpha)}{3}\right)^{(1/q)}+A_4^{1-(1/q)}(\alpha)\\
&\left(\left|\mathfrak{G}'(\theta_2)\right|^q\frac{2A_4(\alpha)-A_3(\alpha)}{3}+\left|\mathfrak{G}'(\theta_1)\right|^q\frac{A_4(\alpha)+A_3(\alpha)}{3}\right)^{1/q}\\
&\quad+A_6^{1-(1/q)}(\alpha)\left(\left|\mathfrak{G}'(\theta_2)\right|\frac{A_6(\alpha)-A_5(\alpha)}{3}\right.\\
&\quad\left.\left.+\left|\mathfrak{G}'(\theta_1)\right|^q\frac{2A_6(\alpha)+A_5(\alpha)}{3}\right)^{(1/q)}\right].
\end{aligned}\tag{27}$$

Thus, the proof is completed. □

Remark 13. In Theorem 12, we have the following:

(i) By setting $\varphi(t)=t$, we reclaim the inequality established in ([13], Remark 4)

(ii) By setting $\varphi(t)=t^{\alpha}/\Gamma(\alpha)$, we reclaim the inequality established in ([13], Theorem 5)

Corollary 14. *By setting $\varphi(t)=t^{\alpha/k}/k\Gamma_k(\alpha)$ in Theorem 12, we obtain the following new Newton's inequality for KRLFIs:*

$$\begin{aligned}
&\left|\frac{3^{(\alpha/k)-1}\Gamma_k(\alpha+1)}{(\theta_2-\theta_1)^{\alpha/k}}\left[\mathcal{J}^{\alpha,k}_{\theta_1+}\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)+\mathcal{J}^{\alpha,k}_{(2\theta_1+\theta_2)/3+}\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)\right.\right.\\
&\quad\left.+\mathcal{J}^{\alpha,k}_{(\theta_1+2\theta_2)/3+}\mathfrak{G}(\theta_2)\right]-\frac{1}{8}[\mathfrak{G}(\theta_1)\\
&\quad\left.+3\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)+3\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)+\mathfrak{G}(\theta_2)]\right|\\
&\leq\frac{\theta_2-\theta_1}{9}\left[A_2^{1-(1/q)}(\alpha,k)\left(\left|\mathfrak{G}'(\theta_2)\right|^q\frac{3A_2(\alpha,k)-A_1(\alpha,k)}{3}\right.\right.\\
&\quad\left.+\left|\mathfrak{G}'(\theta_1)\right|^q\frac{A_1(\alpha,k)}{3}\right)^{(1/q)}+A_4^{1-(1/q)}(\alpha,k)\\
&\quad\cdot\left(\left|\mathfrak{G}'(\theta_2)\right|^q\frac{2A_4(\alpha,k)-A_3(\alpha,k)}{3}+\left|\mathfrak{G}'(\theta_1)\right|^q\frac{A_4(\alpha,k)+A_3(\alpha,k)}{3}\right)^{(1/q)}\\
&\quad+A_6^{1-(1/q)}(\alpha,k)\left(\left|\mathfrak{G}'(\theta_2)\right|\frac{A_6(\alpha,k)-A_5(\alpha,k)}{3}\right.\\
&\quad\left.\left.+\left|\mathfrak{G}'(\theta_1)\right|^q\frac{2A_6(\alpha,k)+A_5(\alpha,k)}{3}\right)^{(1/q)}\right].
\end{aligned}\tag{28}$$

Theorem 15. *If $|\mathfrak{G}'|^q$, $q>1$ is a convex function and assumptions of Lemma 8 hold, then we have the following Newton's type inequality:*

$$\begin{aligned}
&\left|\frac{1}{3Y(1)}\left[{}_{\theta_1+}I_{\varphi}\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)+{}_{(2\theta_1+\theta_2)/3+}I_{\varphi}\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)\right.\right.\\
&\quad\left.+{}_{(\theta_1+2\theta_2)/3+}I_{\varphi}\mathfrak{G}(\theta_2)\right]-\frac{1}{8}[\mathfrak{G}(\theta_1)\\
&\quad\left.+3\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)+3\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)+\mathfrak{G}(\theta_2)]\right|\\
&\leq\frac{\theta_2-\theta_1}{9Y(1)}\left[A_7^{(1/p)}(\alpha,p)\left(\frac{5\left|\mathfrak{G}'(\theta_2)\right|^q+\left|\mathfrak{G}'(\theta_1)\right|^q}{6}\right)^{(1/q)}\right.\\
&\quad+A_8^{(1/p)}(\alpha,p)\left(\frac{\left|\mathfrak{G}'(\theta_2)\right|^q+\left|\mathfrak{G}'(\theta_1)\right|^q}{2}\right)^{(1/q)}\\
&\quad\left.+A_9^{(1/p)}(\alpha,p)\left(\frac{\left|\mathfrak{G}'(\theta_2)\right|^q+5\left|\mathfrak{G}'(\theta_1)\right|^q}{6}\right)^{(1/q)}\right],
\end{aligned}\tag{29}$$

where $q^{-1}+p^{-1}=1$ and

$$\begin{aligned}
A_7(\alpha,p)&=\int_0^1\left|Y(t)-\frac{3Y(1)}{8}\right|^p dt,\\
A_8(\alpha,p)&=\int_0^1\left|Y(t)-\frac{Y(1)}{2}\right|^p dt,\\
A_9(\alpha,p)&=\int_0^1\left|Y(t)-\frac{5Y(1)}{8}\right|^p dt.
\end{aligned}\tag{30}$$

Proof. Applying Hölder's inequality in (15) after taking the modulus, we have

$$\begin{aligned}
&\left|\frac{1}{3Y(1)}\left[{}_{\theta_1+}I_{\varphi}\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)+{}_{(2\theta_1+\theta_2)/3+}I_{\varphi}\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)\right.\right.\\
&\quad\left.+{}_{(\theta_1+2\theta_2)/3+}I_{\varphi}\mathfrak{G}(\theta_2)\right]-\frac{1}{8}\left[\mathfrak{G}(\theta_1)+3\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)\right.\\
&\quad\left.+3\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)+\mathfrak{G}(\theta_2)]\right|\\
&=\frac{\theta_2-\theta_1}{9Y(1)}\left[\int_0^1\left|Y(t)-\frac{3Y(1)}{8}\right|\left|\mathfrak{G}'\left(\frac{3-t}{3}\theta_2+\frac{t}{3}\theta_1\right)\right|dt\right.\\
&\quad+\int_0^1\left|Y(t)-\frac{Y(1)}{2}\right|\left|\mathfrak{G}'\left(\frac{2-t}{3}\theta_2+\frac{1+t}{3}\theta_1\right)\right|dt\\
&\quad\left.+\int_0^1\left|Y(t)-\frac{5Y(1)}{8}\right|\left|\mathfrak{G}'\left(\frac{1-t}{3}\theta_2+\frac{2+t}{3}\theta_1\right)\right|dt\right]\\
&\leq\frac{\theta_2-\theta_1}{9Y(1)}\left[\left(\int_0^1\left|Y(t)-\frac{3Y(1)}{8}\right|^p dt\right)^{(1/p)}\right.\\
&\quad\cdot\left(\int_0^1\left|\mathfrak{G}'\left(\frac{3-t}{3}\theta_2+\frac{t}{3}\theta_1\right)\right|^q dt\right)^{(1/q)}+\left(\int_0^1\left|Y(t)-\frac{Y(1)}{2}\right|^p dt\right)^{(1/p)}\\
&\quad\cdot\left(\int_0^1\left|\mathfrak{G}'\left(\frac{2-t}{3}\theta_2+\frac{1+t}{3}\theta_1\right)\right|^q dt\right)^{(1/q)}+\left(\int_0^1\left|Y(t)-\frac{5Y(1)}{8}\right|^p dt\right)^{(1/p)}\\
&\quad\left.\cdot\left(\int_0^1\left|\mathfrak{G}'\left(\frac{1-t}{3}\theta_2+\frac{2+t}{3}\theta_1\right)\right|^q dt\right)^{(1/q)}\right].
\end{aligned}\tag{31}$$

From convexity of $|\mathfrak{G}'|^q$, $q>1$, we obtain

$$\begin{aligned}
&\left|\frac{\theta_2-\theta_1}{3Y(1)}\left[{}_{\theta_1+}I_\varphi\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)+{}_{(2\theta_1+\theta_2)/3+}I_\varphi\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)\right.\right.\\
&\qquad\left.+{}_{(\theta_1+2\theta_2)/3+}I_\varphi\mathfrak{G}(\theta_2)\right]-\frac{1}{8}\left[\mathfrak{G}(\theta_1)+3\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)\right.\\
&\qquad\left.\left.+3\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)+\mathfrak{G}(\theta_2)\right]\right|\\
&\le\frac{\theta_2-\theta_1}{9Y(1)}\left[\left(\int_0^1\left|Y(t)-\frac{3Y(1)}{8}\right|^p dt\right)^{(1/p)}\right.\\
&\quad\cdot\left(|\mathfrak{G}'(\theta_2)|^q\int_0^1\frac{3-t}{3}dt+|\mathfrak{G}'(\theta_1)|^q\int_0^1\frac{t}{3}dt\right)^{(1/q)}\\
&\quad+\left(\int_0^1\left|Y(t)-\frac{Y(1)}{2}\right|^p dt\right)^{(1/p)}\left(|\mathfrak{G}'(\theta_2)|^q\int_0^1\frac{2-t}{3}dt\right.\\
&\quad\left.+|\mathfrak{G}'(\theta_1)|^q\int_0^1\frac{1+t}{3}dt\right)^{(1/q)}+\left(\int_0^1\left|Y(t)-\frac{5Y(1)}{8}\right|^p dt\right)^{(1/p)}\\
&\quad\left.\cdot\left(|\mathfrak{G}'(\theta_2)|^q\int_0^1\frac{1-t}{3}dt+|\mathfrak{G}'(\theta_1)|^q\int_0^1\frac{2+t}{3}dt\right)^{(1/q)}\right]\\
&=\frac{\theta_2-\theta_1}{9Y(1)}\left[A_7^{(1/p)}(\alpha,p)\left(\frac{5|\mathfrak{G}'(\theta_2)|^q+|\mathfrak{G}'(\theta_1)|^q}{6}\right)^{(1/q)}\right.\\
&\quad+A_8^{(1/p)}(\alpha,p)\left(\frac{|\mathfrak{G}'(\theta_2)|^q+|\mathfrak{G}'(\theta_1)|^q}{2}\right)^{(1/q)}\\
&\quad\left.+A_9^{(1/p)}(\alpha,p)\left(\frac{|\mathfrak{G}'(\theta_2)|^q+5|\mathfrak{G}'(\theta_1)|^q}{6}\right)^{(1/q)}\right].
\end{aligned}\tag{32}$$

Thus, the proof is completed. □

Remark 16. In Theorem 15, we have the following:

(i) By setting $\varphi(t)=t$, we reclaim the inequality established in ([13], Remark 5)

(ii) By setting $\varphi(t)=t^\alpha/\Gamma(\alpha)$, we reclaim the inequality established in ([13], Theorem 6)

Corollary 17. *By setting $\varphi(t)=t^{\alpha/k}k\Gamma_k(\alpha)$ in Theorem 15, we obtain the following new Newton's inequality for KRLFIs:*

$$\begin{aligned}
&\left|\frac{3^{(\alpha/k)-1}\Gamma_k(\alpha+1)}{(\theta_2-\theta_1)^{\alpha/k}}\left[\mathcal{J}^{\alpha,k}_{\theta_1+}\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)+\mathcal{J}^{\alpha,k}_{(2\theta_1+\theta_2)/3+}\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)\right.\right.\\
&\qquad\left.+\mathcal{J}^{\alpha,k}_{(\theta_1+2\theta_2)/3+}\mathfrak{G}(\theta_2)\right]-\frac{1}{8}\left[\mathfrak{G}(\theta_1)+3\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)\right.\\
&\qquad\left.\left.+3\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)+\mathfrak{G}(\theta_2)\right]\right|\le\frac{\theta_2-\theta_1}{9}\\
&\quad\cdot\left[A_7^{(1/p)}(\alpha,p,k)\left(\frac{5|\mathfrak{G}'(\theta_2)|^q+|\mathfrak{G}'(\theta_1)|^q}{6}\right)^{(1/q)}\right.\\
&\quad+A_8^{(1/p)}(\alpha,k,p)\left(\frac{|\mathfrak{G}'(\theta_2)|^q+|\mathfrak{G}'(\theta_1)|^q}{2}\right)^{(1/q)}\\
&\quad\left.+A_9^{(1/p)}(\alpha,k,p)\left(\frac{|\mathfrak{G}'(\theta_2)|^q+5|\mathfrak{G}'(\theta_1)|^q}{6}\right)^{(1/q)}\right],
\end{aligned}\tag{33}$$

where $q^{-1}+p^{-1}=1$ and

$$\begin{aligned}
A_7(\alpha,k,p)&=\int_0^1\left|t^{\alpha/k}-\frac{3}{8}\right|^p dt,\\
A_8(\alpha,k,p)&=\int_0^1\left|t^{\alpha/k}-\frac{1}{2}\right|^p dt,\\
A_9(\alpha,k,p)&=\int_0^1\left|t^{\alpha/k}-\frac{5}{8}\right|^p dt.
\end{aligned}\tag{34}$$

5. Fractional Newton-Type Inequality for Functions of Bounded Variation

In this section, we prove a Newton-type inequality for function of bounded variation via generalized fractional integrals.

Theorem 18. *Let $\mathfrak{G}:[\theta_1,\theta_2]\longrightarrow\mathbb{R}$ be a function of bounded variation on $[\theta_1,\theta_2]$. Then we have the following Newton-type inequality for generalized fractional integrals:*

$$\begin{aligned}
&\left|\frac{1}{3Y(1)}\left[{}_{\theta_1+}I_\varphi\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)+{}_{(2\theta_1+\theta_2)/3+}I_\varphi\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)\right.\right.\\
&\qquad\left.+{}_{(\theta_1+2\theta_2)/3+}I_\varphi\mathfrak{G}(\theta_2)\right]-\frac{1}{8}\left[\mathfrak{G}(\theta_1)+3\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)\right.\\
&\qquad\left.\left.+3\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)+\mathfrak{G}(\theta_2)\right]\right|\le\frac{5}{24}\bigvee_{\theta_1}^{\theta_2}(\mathfrak{G}),
\end{aligned}\tag{35}$$

where $\bigvee_c^d(\mathfrak{G})$ denotes the total variation of $\mathfrak{G}$ on $[c,d]$.

Proof. Define the mapping $\Psi_\varphi(\varkappa)$ by

$$\Psi_\varphi(\varkappa)=\begin{cases}Y\left(\dfrac{3}{\theta_2-\theta_1}\left(\dfrac{2\theta_1+\theta_2}{3}-\varkappa\right)\right)-\dfrac{5Y(1)}{8}, & \text{for } \theta_1\le\varkappa\le\dfrac{2\theta_1+\theta_2}{3};\\[2ex] Y\left(\dfrac{3}{\theta_2-\theta_1}\left(\dfrac{\theta_1+2\theta_2}{3}-\varkappa\right)\right)-\dfrac{Y(1)}{2}, & \text{for } \dfrac{2\theta_1+\theta_2}{3}<\varkappa\le\dfrac{\theta_1+2\theta_2}{3};\\[2ex] Y\left(\dfrac{3}{\theta_2-\theta_1}(\theta_2-\varkappa)\right)-\dfrac{3Y(1)}{8}, & \text{for } \dfrac{\theta_1+2\theta_2}{3}<\varkappa\le\theta_2.\end{cases}\tag{36}$$

It follows from that

$$\int_{\theta_1}^{\theta_2}\Psi_\varphi(\varkappa)d\mathfrak{G}(\varkappa)=\int_{\theta_1}^{(2\theta_1+\theta_2)/3}\left(Y\left(\frac{3}{\theta_2-\theta_1}\left(\frac{2\theta_1+\theta_2}{3}-\varkappa\right)\right)-\frac{5Y(1)}{8}\right)d\mathfrak{G}(\varkappa)+\int_{(2\theta_1+\theta_2)/3}^{(\theta_1+2\theta_2)/3}\left(Y\left(\frac{3}{\theta_2-\theta_1}\left(\frac{\theta_1+2\theta_2}{3}-\varkappa\right)\right)-\frac{Y(1)}{2}\right)d\mathfrak{G}(\varkappa)+\int_{(\theta_1 2+\theta_2)/3}^{\theta_2}\left(Y\left(\frac{3}{\theta_2-\theta_1}(\theta_2-\varkappa)\right)-\frac{3Y(1)}{8}\right)d\mathfrak{G}(\varkappa).\tag{37}$$

Integrating by parts, we get

$$\begin{aligned}\int_{\theta_1}^{(2\theta_1+\theta_2)/3}&\left(Y\left(\frac{3}{\theta_2-\theta_1}\left(\frac{2\theta_1+\theta_2}{3}-\varkappa\right)\right)-\frac{5Y(1)}{8}\right)d\mathfrak{G}(\varkappa)\\&=\left(Y\left(\frac{3}{\theta_2-\theta_1}\left(\frac{2\theta_1+\theta_2}{3}-\varkappa\right)\right)-\frac{5Y(1)}{8}\right)\mathfrak{G}(\varkappa)\Big|_{\theta_1}^{(2\theta_1+\theta_2)/3}+\int_{\theta_1}^{(2\theta_1+\theta_2)/3}\frac{\varphi(((2\theta_1+\theta_2)/3)-\varkappa)}{((2\theta_1+\theta_2)/3)-\varkappa}\mathfrak{G}(\varkappa)d\varkappa\\&=-\frac{5Y(1)}{8}\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{2}\right)-\frac{3Y(1)}{8}\mathfrak{G}(\theta_1)+{}_{\theta_1+}I_\varphi\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right).\end{aligned}\tag{38}$$

Similarly, we have

$$\begin{aligned}\int_{(2\theta_1+\theta_2)/3}^{(\theta_1+2\theta_2)/3}&\left(Y\left(\frac{3}{\theta_2-\theta_1}\left(\frac{\theta_1+2\theta_2}{3}-\varkappa\right)\right)-\frac{Y(1)}{2}\right)d\mathfrak{G}(\varkappa)\\&=-\frac{Y(1)}{2}\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{2}\right)-\frac{Y(1)}{2}\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{2}\right)+\frac{2\theta_1+\theta_2}{3}+I_\varphi\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right),\end{aligned}\tag{39}$$

$$\begin{aligned}\int_{(\theta_1+2\theta_2)/3}^{\theta_2}&\left(Y\left(\frac{3}{\theta_2-\theta_1}(\theta_2-\varkappa)\right)-\frac{3Y(1)}{8}\right)d\mathfrak{G}(\varkappa)\\&=-\frac{3Y(1)}{8}\mathfrak{G}(\theta_2)-\frac{5Y(1)}{8}\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{2}\right)+\frac{\theta_1+2\theta_2}{3}+I_\varphi\mathfrak{G}(\theta_2).\end{aligned}\tag{40}$$

By putting the equalities (38)–(40) in (37), we have

$$\begin{aligned}&\left|\frac{1}{3Y(1)}\left[{}_{\theta_1+}I_\varphi\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)+\frac{2\theta_1+\theta_2}{3}+I_\varphi\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)+\frac{\theta_1+2\theta_2}{3}+I_\varphi\mathfrak{G}(\theta_2)\right]-\frac{1}{8}\left[\mathfrak{G}(\theta_1)+3\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)\right.\right.\\&\left.\left.+3\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)+\mathfrak{G}(\theta_2)\right]\right|=\int_{\theta_1}^{\theta_2}\Psi_\varphi(\varkappa)d\mathfrak{G}(\varkappa).\end{aligned}\tag{41}$$

It is well known that if $g,\mathfrak{G}:[\theta_1,\theta_2]\longrightarrow\mathbb{R}$ are such that g is continuous on $[\theta_1,\theta_2]$ and $\mathfrak{G}$ is of bounded variation on $[\theta_1,\theta_2]$, then $\int_{\theta_1}^{\theta_2}g(t)d\mathfrak{G}(t)$ exist and

$$\left|\int_{\theta_1}^{\theta_2}g(t)d\mathfrak{G}(t)\right|\le\sup_{t\in[\theta_1,\theta_2]}|g(t)|\bigvee_{\theta_1}^{\theta_2}(\mathfrak{G}).\tag{42}$$

On the other hand, using (42), we get

$$\begin{aligned}&\left|\frac{1}{3Y(1)}\left[{}_{\theta_1+}I_\varphi\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)+\frac{2\theta_1+\theta_2}{3}+I_\varphi\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)+\frac{\theta_1+2\theta_2}{3}+I_\varphi\mathfrak{G}(\theta_2)\right]-\frac{1}{8}\left[\mathfrak{G}(\theta_1)+3\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)\right.\right.\\&\left.\left.+3\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)+\mathfrak{G}(\theta_2)\right]\right|=\frac{1}{3Y(1)}\left|\int_{\theta_1}^{\theta_2}\Psi_\varphi(\varkappa)d\mathfrak{G}(\varkappa)\right|\\&\le\frac{1}{3Y(1)}\left[\left|\int_{\theta_1}^{(2\theta_1+\theta_2)/3}\left(Y\left(\frac{3}{\theta_2-\theta_1}\left(\frac{2\theta_1+\theta_2}{3}-\varkappa\right)\right)-\frac{5Y(1)}{8}\right)d\mathfrak{G}(\varkappa)\right|+\left|\int_{(2\theta_1+\theta_2)/3}^{(\theta_1+2\theta_2)/3}\cdot\left(Y\left(\frac{3}{\theta_2-\theta_1}\left(\frac{\theta_1+2\theta_2}{3}-\varkappa\right)\right)-\frac{Y(1)}{2}\right)d\mathfrak{G}(\varkappa)\right|\right.\\&\left.+\left|\int_{(\theta_1+2\theta_2)/3}^{\theta_2}\left(Y\left(\frac{3}{\theta_2-\theta_1}(\theta_2-\varkappa)\right)-\frac{3Y(1)}{8}\right)d\mathfrak{G}(\varkappa)\right|\right]\\&\le\frac{1}{3Y(1)}\left[\sup_{\varkappa\in[\theta_1,(2\theta_1+\theta_2)/3]}\left|Y\left(\frac{3}{\theta_2-\theta_1}\left(\frac{2\theta_1+\theta_2}{3}-\varkappa\right)\right)-\frac{5Y(1)}{8}\right|\bigvee_{\theta_1}^{(2\theta_1+\theta_2)/3}(\mathfrak{G})+\sup_{\varkappa\in[(2\theta_1+\theta_2)/3,(\theta_1+2\theta_2)/3]}\left|Y\cdot\left(\frac{3}{\theta_2-\theta_1}\left(\frac{\theta_1+2\theta_2}{3}-\varkappa\right)\right)-\frac{Y(1)}{2}\right|\bigvee_{(2\theta_1+\theta_2)/3}^{(\theta_1+2\theta_2)/3}(\mathfrak{G})\right.\\&\left.+\sup_{\varkappa\in[(\theta_1+2\theta_2)/3,\theta_2]}\left|Y\left(\frac{3}{\theta_2-\theta_1}(\theta_2-\varkappa)\right)-\frac{3Y(1)}{8}\right|\bigvee_{(\theta_1+2\theta_2)/3}^{\theta_2}(\mathfrak{G})\right]=\frac{1}{3Y(1)}\left[\frac{5Y(1)}{8}\bigvee_{\theta_1}^{(2\theta_1+\theta_2)/3}(\mathfrak{G})\right.\\&\left.+\frac{Y(1)}{2}\bigvee_{(2\theta_1+2\theta_2)/3}^{(\theta_1+2\theta_2)/3}(\mathfrak{G})+\frac{5Y(1)}{8}\bigvee_{(\theta_1+2\theta_2)/3}^{\theta_2}(\mathfrak{G})\right]\le\frac{5}{24}\bigvee_{\theta_1}^{\theta_2}(\mathfrak{G}).\end{aligned}\tag{43}$$

This completes the proof. □

Remark 19. In Theorem 18, we have the following:

(i) If we take $\varphi(t)=t$, then we recapture the inequality proved in ([46], Corollary 3)

(ii) If we set $\varphi(t)=t^\alpha/\Gamma(\alpha)$, then we recapture the inequality established in ([13], Theorem 7)

Corollary 20. *If we choose* $\varphi(t) = (t^{(\alpha/k)}/k\Gamma_k(\alpha))$*, then we obtain the following new Newton's inequality for KRLFIs:*

$$\begin{aligned}&\left|\frac{3^{(\alpha/k)-1}\Gamma_k(\alpha+1)}{(\theta_2-\theta_1)^{\alpha/k}}\left[\mathcal{J}_{\theta_1}^{\alpha,k}+\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)+\mathcal{J}_{(2\theta_1+\theta_2)/3}^{\alpha,k}\right.\right.\\&\left.+\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)+\mathcal{J}_{(\theta_1+2\theta_2)/3}^{\alpha,k}+\mathfrak{G}(\theta_2)\right]\\&-\frac{1}{8}\left[\mathfrak{G}(\theta_1)+3\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)\right.\\&\left.\left.+3\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)+\mathfrak{G}(\theta_2)\right]\right|\leq\frac{5}{24}\bigvee_{\theta_1}^{\theta_2}(\mathfrak{G}).\end{aligned}\tag{44}$$

6. Conclusion

We demonstrated some new Simpson's second-type inequalities for differentiable convex functions using Riemann-Liouville fractional integrals. Furthermore, we established fractional Newton-type inequalities for bounded variation functions. It is also shown that the newly established inequalities are an extension of the previously obtained inequalities. It is worth to mentioning here that we can obtain similar inequalities via Katugampola fractional operators, conformable fractional operators, Hadamard fractional operators, and fractional operators with the exponential kernel for different choices of the function φ. In their future work, future researchers can get similar inequalities for various types of convexity and coordinated convexity on fractals, which is an exciting and new problem.

Data Availability

Data sharing is not applicable to this paper as no data sets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no competing interests.

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