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Research Article

Dichotomy Condition and Periodic Solutions for Two Nonlinear Neutral Systems

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In this article, we consider two nonlinear neutral systems with multiple delays. Our main tool here is to use dichotomy theory to construct an implicit solution for these two systems. Utilizing Krasnoselskii's fixed point theorem, we obtain sufficient criteria for the existence of periodic solutions, as well as for the uniqueness of solutions. The main results expand and generalize certain previously published findings.

1. Introduction

Periodic solutions of equations are solutions that describe regularly repeated processes. The periodic solutions of systems of differential equations occupy special importance in branches of science such as the theory of oscillations, dynamical systems, and celestial mechanics, and the analysis of these systems in depth opens up new possibilities and horizons in these sciences. Such a study aids in understanding the geometric behavior of solutions eventually (see [1–4]).

In recent years, several investigators have tried the stability and existence of periodic solutions by using the technique of fixed point, in particular Burton, Furumochi, Zhang, and others (see [5–13]).

By Krasnoselskii's fixed point theorem, Luo et al. [14] investigate the existence of positive periodic solutions for two neutral functional differential equations

$$(y(\zeta) - cy(\zeta - \tau(\zeta)))' = -a(\zeta)y(\zeta) + f(\zeta, y(\zeta - \tau(\zeta))), \quad (1)$$

$$\frac{d}{d\zeta}\left[y(\zeta) - c\int_{-\infty}^{0} Q(\omega)y(\zeta+\omega)d\omega\right] = -a(\zeta)y(\zeta) + \int_{-\infty}^{0} Q(\omega)f(\zeta, y(\zeta+\omega))d\omega, \quad (2)$$

in which $y : \mathbb{R} \longrightarrow \mathbb{R}$; $a(\zeta) \in C(\mathbb{R}, (0,\infty))$; $f \in C(\mathbb{R}\times\mathbb{R}, \mathbb{R})$; $\tau(\zeta) \in C(\mathbb{R}, \mathbb{R})$; $a(\zeta)$, $b(\zeta)$, $\tau(\zeta)$, and $f(\zeta, y)$ are T-periodic functions; $T > 0$ and $|c| < 1$ are constants; $Q(\omega) \in C((-\infty,0], [0,\infty))$; and $\int_{-\infty}^{0} Q(\omega)d\omega = 1$.

The above functional differential equations ((1) and (2)) cover many mathematical ecological and population models, for example, hematopoiesis models (see [15, 16]), Nicholson's blowflies models (see [17, 18]), and blood cell production (see [19]).

Sa Ngiamsunthorn [11] considered the differential system

$$(y(\zeta)-cy(\zeta-\tau))' = A(\zeta)y(\zeta)+f(\zeta, y(\zeta-\sigma_1(\zeta)), \cdots, y(\zeta-\sigma_m(\zeta))), \tag{3}$$

with dichotomy condition (3) periodic coefficients. Similar system of (3) has been studied in [20].

Motivated by the works mentioned above, we are concerned with the existence of periodic solutions for two nonlinear neutral systems of differential equations

$$(y(\zeta)-q(\zeta, y(\zeta-\tau(\zeta))))' = A(\zeta)y(\zeta)+f(\zeta, y(\zeta-\sigma_1(\zeta)), \cdots, y(\zeta-\sigma_m(\zeta))), \tag{4}$$

$$\begin{aligned}&\left(y(\zeta)-\int_{-\infty}^{0} Q(\omega)q(\zeta, y(\zeta+\tau(\omega)))d\omega\right)' \\ &\quad = A(\zeta)y(\zeta)+\int_{-\infty}^{0} Q(\omega)f(\zeta, y(\zeta+\sigma_1(\omega)), \cdots, y(\zeta+\sigma_m(\omega)))d\omega,\end{aligned} \tag{5}$$

in which $y:\mathbb{R}\longrightarrow\mathbb{R}^n$, $\tau(\zeta)$, and $\sigma_i(\zeta)$, $i=1,\cdots,m$, are real continuous T-periodic functions on $\mathbb{R}$, $T>0$. $A(\zeta)$ is a $n\times n$ real continuous matrix T-periodic function defined on $\mathbb{R}$. $Q(\omega)$ is a $n\times n$ real continuous matrix periodic function defined on $(-\infty, 0]$ with $\int_{-\infty}^{0} Q(\omega)d\omega = I$. The functions $q(\zeta, u)$ and $f(\zeta, u_1, \cdots, u_m)$ are real continuous vector functions defined on $\mathbb{R}\times\mathbb{R}^n$ and $\mathbb{R}\times(\mathbb{R}^n)^m$, respectively, such that

$$\begin{aligned} f(\zeta+T, u_1, u_2, \cdots, u_m) &= f(\zeta, u_1, u_2, \cdots, u_m), \\ q(\zeta+T, u) &= q(\zeta, u). \end{aligned} \tag{6}$$

Note that the functional $y(\zeta-\tau(\zeta))$ and function $y(\zeta)$ are in different spaces because $y(\zeta-\tau(\zeta))$ is in the phase space, but their norms are equivalent (for more details on space theory, we refer the reader to the following papers) [21, 22].

This paper is arranged as follows: after this introduction, we list a set of definitions and previous results related to integrable dichotomies and fixed point theorems in Section 2. Sections 3 and 4 deal with the existence and uniqueness of periodic solutions of systems (4) and (5), respectively, and are followed by a conclusion.

2. Preliminaries

In this section, we outline some results and definitions of integrable dichotomy that will be crucial in the proof of our results (see [23, 24]). Consider the following linear differential system:

$$z'(\zeta) = A(\zeta)z(\zeta), \tag{7}$$

in which $A(\zeta)$ is a continuous $n\times n$ matrix function. Let $\Psi(\zeta)$ be the fundamental matrix solution of system (7) with $\Psi(0)=I$. Assume P is a projection matrix. We let a green matrix $G := G_P$ be associated with P by

$$G(\zeta, \omega) = \begin{cases} \Psi(\zeta)P\Psi^{-1}(\zeta), & \text{for} \zeta \ge \omega, \\ -\Psi(\zeta)(I-P)\Psi^{-1}(\zeta), & \text{for} \zeta < \omega. \end{cases} \tag{8}$$

Definition 1 (see [23]). If a projection matrix P and a positive constant μ exist for which the associated Green matrix $G = G_P$ satisfies

$$\sup_{\zeta\in\mathbb{R}} \int_{-\infty}^{\infty} \|G(\zeta, \omega)\| d\omega = \mu, \tag{9}$$

the linear differential system (7) has an integrable dichotomy.

Proposition 2 (see [23]). *Assume that system (7) has an integrable dichotomy. Then, $z(\zeta)=0$ is the only bounded solution to (7).*

Now, the set of bounded and continuous functions is designated as $BC(\mathbb{R}, \mathbb{R}^n)$. If we consider the nonhomogeneous linear system

$$z'(\zeta) = A(\zeta)z(\zeta) + f(\zeta), \tag{10}$$

under an integrable dichotomy condition, we take the following theorem from [23].

Theorem 3. *Assume that system (7) has an integrable dichotomy. If $f \in BC(\mathbb{R}, \mathbb{R}^n)$, then system (10) has a unique bounded solution $z \in BC(\mathbb{R}, \mathbb{R}^n)$. Furthermore,*

$$z(\zeta) = \int_{-\infty}^{\infty} G(\zeta, \omega)f(\omega)d\omega. \tag{11}$$

Theorem 4 (see [23]). *Assume that the homogeneous system (7) has an integrable dichotomy for which $\Psi(\zeta)P\Psi^{-1}(\zeta)$ is bounded. If A is T-periodic, then $\Psi(\zeta)P\Psi^{-1}(\zeta)$ is also T-periodic. In addition, if $f \in BC(\mathbb{R}, \mathbb{R}^n)$ is T-periodic, then (10) has a unique periodic solution satisfying (11).*

We present the fixed point theorems that we utilize to demonstrate the existence and uniqueness of periodic solutions to system (4) (see [5, 25]).

Theorem 5 (Banach). *Assume that (Y, ρ) is a complete metric space and $\Gamma : Y \longrightarrow Y$. If there is a constant $\gamma < 1$ such that for $u, v \in Y$,*

$$\rho(\Gamma u, \Gamma v) \le \gamma\rho(u, v), \tag{12}$$

then there is one and only one point $z \in Y$ with $\Gamma z = z$.

Smart [25] established a hybrid result by combining Banach's theorem and Schauder's theorem as follows:

Theorem 6 (Krasnoselskii). *Let Ω be a closed bounded convex nonempty subset of a Banach space Y. Assume that Γ_1 and Γ_2 map Ω into Y such that*

(i) Γ_1 is a contraction mapping on Ω

(ii) Γ_2 is completely continuous on Ω

(iii) $u, v \in \Omega$ implies $\Gamma_1 u + \Gamma_2 v \in \Omega$

Then, there exists $z \in \Omega$ with $z = \Gamma_1 z + \Gamma_2 z$.

Assume $M > 0$ be a constant. Denote

$$\Omega = \{u \in BC(\mathbb{R}, \mathbb{R}^n): \|u\| \le M \text{ and } u(\zeta + T) = u(\zeta) \text{ for all } \zeta \in \mathbb{R}\}. \tag{13}$$

Clearly, the set Ω is a bounded nonempty closed and convex subset of $BC(\mathbb{R}, \mathbb{R}^n)$.

Assume that, for $u, v \in \Omega$, there exists $L_1 \in (0, 1)$ such that

$$|q(\zeta, u) - q(\zeta, v)| \le L_1 |u - v|, \text{ for all } \zeta \in \mathbb{R}, \tag{14}$$

and for $u_1, u_2, \cdots u_m, v_1, v_2, \cdots, v_m \in \Omega$, there exists $L_2 > 0$ such that

$$\begin{aligned} &|f(\zeta, u_1, u_2, \cdots, u_m) - f(\zeta, v_1, v_2, \cdots, v_m)| \\ &\quad \le L_2(|u_1 - v_1| + \cdots + |u_m - v_m|), \text{ for all } \zeta \in \mathbb{R}. \end{aligned} \tag{15}$$

Denote $\sup_{\zeta \in [0,T]} |q(\zeta, 0)| = \alpha$, $\sup_{\zeta \in [0,T]} |f(\zeta, 0, \cdots, 0)| = \beta$, and $\sup_{\zeta \in [0,T]} \|A(\zeta)\| = \lambda$, and we assume also

$$L_1 M + \alpha + \mu[\lambda(L_1 M + \alpha) + (L_2 m M + \beta)] \le M. \tag{16}$$

3. Existence of Periodic Solutions for (4)

In this section, we show the existence and the uniqueness of solution of (4) under the conditions stated in the previous section. So, let

$$z(\zeta) = y(\zeta) - q(\zeta, y(\zeta - \tau(\zeta))). \tag{17}$$

Hence,

$$z'(\zeta) = A(\zeta)z(\zeta) + A(\zeta)q(\zeta, y(\zeta - \tau(\zeta))) + f(\zeta, y(\zeta - \sigma_1(\zeta)), \cdots, y(\zeta - \sigma_m(\zeta))). \tag{18}$$

By Theorem 3, system (4) holds the integral equation

$$\begin{aligned} z(\zeta) = \int_{-\infty}^{\infty} &G(\zeta, \omega)[A(\omega)q(\omega, y(\omega - \tau(\omega))) \\ &+ f(\omega, y(\omega - \sigma_1(\omega)), \cdots, y(\omega - \sigma_m(\omega)))]d\omega. \end{aligned} \tag{19}$$

The above equation is equivalent to

$$\begin{aligned} y(\zeta) = q(\zeta, y(\zeta - \tau(\zeta))) + \int_{-\infty}^{\infty} &G(\zeta, \omega)[A(\omega)q(\omega, y(\omega - \tau(\zeta))) \\ &+ f(\omega, y(\omega - \sigma_1(\omega)), \cdots, y(\omega - \sigma_m(\omega)))]d\omega. \end{aligned} \tag{20}$$

Define the operators Γ_1 and Γ_2 by

$$(\Gamma_1 u)(\zeta) := q(\zeta, u(\zeta - \tau(\zeta))) \quad \text{for} \quad u \in BC(\mathbb{R}, \mathbb{R}^n), \tag{21}$$

$$\begin{aligned} (\Gamma_2 u)(\zeta) := &\int_{-\infty}^{\infty} G(\zeta, \omega)A(\omega)q(\omega, u(\omega - \tau(\omega)))d\omega \\ &+ \int_{-\infty}^{\infty} G(\zeta, \omega)f(\omega, u(\omega - \sigma_1(\omega)), \cdots, u(\omega - \sigma_m(\omega)))d\omega. \end{aligned} \tag{22}$$

Note that if the operator $\Gamma_1 + \Gamma_2$ has a fixed point, then this fixed point is a periodic solution of (4).

Lemma 7. *If (14) and (15) hold, then the operators Γ_1 and Γ_2 are defined by (21) and (22), respectively, from Ω into $BC(\mathbb{R}, \mathbb{R}^n)$, that is, $\Gamma_1, \Gamma_2 : \Omega \longrightarrow BC(\mathbb{R}, \mathbb{R}^n)$.*

Proof. Let $u \in \Omega$, by (14). Therefore,

$$\begin{aligned} |(\Gamma_1 u)(\zeta)| = |q(\zeta, u(\zeta - \tau(\zeta)))| &\le L_1 |u(\zeta - \tau(\zeta))| + |q(\zeta, 0)| \\ &\le L_1 \|u\| + \sup_{\zeta \in [0,\zeta]} |q(\zeta, 0)| \le L_1 M + \alpha. \end{aligned} \tag{23}$$

Secondly, for $u \in \Omega$, by (14) and (15), we get

$$\begin{aligned} |(\Gamma_2 u)(\zeta)| = &\int_{-\infty}^{\infty} \|G(\zeta, \omega)\| \|A(\omega)\| |q(\omega, u(\omega - \tau(\omega)))| d\omega \\ &+ \int_{-\infty}^{\infty} \|G(\zeta, \omega)\| |f(\omega, u(\omega - \sigma_1(\omega)), \cdots, u(\omega - \sigma_m(\omega)))| d\omega \\ &\le [\lambda(L_1 M + \alpha) + (L_2 m M + \beta)] \int_{-\infty}^{\infty} \|G(\zeta, \omega)\| d\omega \\ &\le \mu[\lambda(L_1 M + \alpha) + (L_2 m M + \beta)]. \end{aligned} \tag{24}$$

Since all quantities in Γ_1 and Γ_2 are periodic, then Γ_1, $\Gamma_2 : \Omega \longrightarrow BC(\mathbb{R}, \mathbb{R}^n)$. □

Lemma 8. *If (14) holds, then the operator $\Gamma_1 : \Omega \longrightarrow BC(\mathbb{R}, \mathbb{R}^n)$ defined by (21) is a contraction.*

Proof. Let $u, v \in \Omega$. By using (14), we get

$$\begin{aligned} |(\Gamma_1 u)(\zeta) - (\Gamma_1 v)(\zeta)| &= |q(\zeta, u(\zeta - \tau(\zeta))) - q(\zeta, v(\zeta - \tau(\zeta)))| \\ &\le L_1 |u(\zeta - \tau(\zeta)) - v(\zeta - \tau(\zeta))| \le L_1 \|u - v\|. \end{aligned} \tag{25}$$

Then,

$$\|\Gamma_1 u - \Gamma_1 v\| \le L_1 \|u - v\|. \quad (26)$$

Therefore, Γ_1 is a contraction because $L_1 \in (0, 1)$. □

Lemma 9. *If (14) and (15) hold, then the operator $\Gamma_2 : \Omega \longrightarrow BC(\mathbb{R}, \mathbb{R}^n)$ defined by (22) is completely continuous.*

Proof. To prove the operator $\Gamma_2 : \Omega \longrightarrow BC(\mathbb{R}, \mathbb{R}^n)$ completely continuous, we must prove that Γ_2 is continuous and $\Gamma_2(\Omega)$ is contained in a compact set; for this purpose, let $u_n \in \Omega$ where n is a positive integer such that $u_n \longrightarrow u$ as $n \longrightarrow \infty$. Then,

$$\begin{aligned} |(\Gamma_2 u_n)(\zeta) - (\Gamma_2 u)(\zeta)| \le & \int_{-\infty}^{\infty} \|G(\zeta,\omega)\| \|A(\omega)\| |q(\omega, u_n(\omega - \tau(\omega))) \\ & - q(\omega, u(\omega - \tau(\omega)))| d\omega + \int_{-\infty}^{\infty} \|G(\zeta,\omega)\| \\ & \times |f(\omega, u_n(\omega - \sigma_1(\omega)), \cdots, u_n(\omega - \sigma_m(\omega))) \\ & - f(\omega, u(\omega - \sigma_1(\omega)), \cdots, u(\omega - \sigma_m(\omega)))| d\omega \\ \le & \mu(\lambda L_1 + L_2 m) \|u_n - u\|. \end{aligned} \quad (27)$$

So, the dominated convergence theorem implies

$$\lim_{n \longrightarrow \infty} |(\Gamma_2 u_n)(\zeta) - (\Gamma_2 u)(\zeta)| = 0, \quad (28)$$

which implies that Γ_2 is continuous. Next, we show that the image of Γ_2 is contained in a compact set. Let $u_n \in \Omega$, and by (24), we have

$$\|\Gamma_2 u_n\| \le \mu[\lambda(L_1 M + \alpha) + (L_2 m M + \beta)]. \quad (29)$$

Second, we calculate $(\Gamma_2 u_n)'(\zeta)$ and show that it is uniformly bounded.

$$\begin{aligned} (\Gamma_2 u_n)'(\zeta) &= \left(\int_{-\infty}^{\infty} G(\zeta,\omega)[A(\omega) q(\omega, u_n(\omega - \tau(\omega))) \right. \\ & \quad \left. + f(\omega, u_n(\omega - \sigma_1(\omega)), \cdots, u_n(\omega - \sigma_m(\omega)))] d\omega \right)' \\ &= u_n'(\zeta) - q'(\zeta, u_n(\zeta - \tau(\zeta))) \\ &= A(\zeta) u(\zeta) + f(\zeta, u_n(\zeta - \sigma_1(\zeta)), \cdots, u_n(\zeta - \sigma_m(\zeta))) \\ &= A(\zeta)(\Gamma_1 u_n + \Gamma_2 u_n) + f(\zeta, u_n(\zeta - \sigma_1(\zeta)), \cdots, u_n(\zeta - \sigma_m(\zeta))). \end{aligned} \quad (30)$$

Then,

$$\|(\Gamma_2 u_n)'\| \le (\lambda + L_2 m) M + \beta. \quad (31)$$

Thus, the sequence $(\Gamma_2 u_n)$ is uniformly bounded and equicontinuous. As a result, by Ascoli-Arzela's theorem $\Gamma_2(\Omega)$ is relatively compact. □

We next prove for any $u, v \in \Omega$ that $\Gamma_1 u + \Gamma_2 v \in \Omega$.

Lemma 10. *If (14)–(16) hold, then for any $u, v \in \Omega$, we have $\Gamma_1 u + \Gamma_2 v \in \Omega$.*

Proof. Let $u, v \in \Omega$. Then, $\|u\|, \|v\| \le M$. By (16), we have

$$\begin{aligned} |(\Gamma_1 u)(\zeta) + (\Gamma_2 v)(\zeta)| \le & |q(\zeta, u(\zeta - \tau(\zeta)))| \\ & + \int_{-\infty}^{\infty} \|G(\zeta,\omega)\| \|A(\omega)\| |q(\omega, v(\omega - \tau(\omega)))| d\omega \\ & + \int_{-\infty}^{\infty} \|G(\zeta,\omega)\| |f(\omega, v(\omega - \sigma_1(\omega)), \cdots, v(\omega - \sigma_m(\omega)))| d\omega \\ \le & L_1 M + \alpha + [\lambda(L_1 M + \alpha) + (L_2 m M + \beta)] \int_{-\infty}^{\infty} \|G(\zeta,\omega)\| d\omega \\ \le & L_1 M + \alpha + \mu[\lambda(L_1 M + \alpha) + (L_2 m M + \beta)] \le M. \end{aligned} \quad (32)$$

It follows that

$$\|\Gamma_1 u + \Gamma_2 v\| \le M, \quad (33)$$

for all $u, v \in \Omega$. Hence, $\Gamma_1 u + \Gamma_2 v \in \Omega$. □

Theorem 11. *Assume that system (7) has an integrable dichotomy. If conditions (14)–(16) hold, then system (4) has at least one T-periodic solution.*

Proof. Clearly, by Lemmas 7–10, all the requirements of the Krasnoselskii's theorem are satisfied. Thus, there exists a fixed point $z \in \Omega$ such that $z = \Gamma_1 z + \Gamma_2 z$; this fixed point is a solution of (4). Hence, (4) has a T-periodic solution. □

Theorem 12. *Assume that system (7) has an integrable dichotomy. If conditions (14) and (15) and*

$$L_1 + \mu(\lambda L_1 + L_2 m) < 1, \quad (34)$$

hold, then system (4) has a unique T-periodic solution.

Proof. Let the mapping Γ be presented by

$$\begin{aligned} (\Gamma u)(\zeta) = & q(\zeta, u(\zeta - \tau(\zeta))) + \int_{-\infty}^{\infty} G(\zeta,\omega)[A(\omega) q(\omega, u(\omega - \tau(\omega))) \\ & + f(\omega, u(\omega - \sigma_1(\omega)), \cdots, u(\omega - \sigma_m(\omega)))] d\omega. \end{aligned} \quad (35)$$

For $u_1, u_2 \in BC(\mathbb{R}, \mathbb{R}^n)$, we obtain

$$\begin{aligned} |(\Gamma u_1)(\zeta) - (\Gamma u_2)(\zeta)| \le & |q(\zeta, u_1(\zeta - \tau(\zeta))) - q(\zeta, u_2(\zeta - \tau(\zeta)))| \\ & + \int_{-\infty}^{\infty} \|G(\zeta,\omega)\| \|A(\omega)\| |q(\omega, u_1(\omega - \tau(\omega))) \\ & - q(\omega, u_2(\omega - \tau(\omega)))| d\omega + \int_{-\infty}^{\infty} \|G(\zeta,\omega)\| \\ & \times |f(\omega, u_1(\omega - \sigma_1(\omega)), \cdots, u_1(\omega - \sigma_m(\omega))) \\ & - f(\omega, u_2(\omega - \sigma_1(\omega)), \cdots, u_2(\omega - \sigma_m(\omega)))| d\omega \\ = & (L_1 + \mu(\lambda L_1 + L_2 m)) \|u_1 - u_2\|. \end{aligned} \quad (36)$$

Since (34) hold, the contraction mapping completes the proof. □

Example 1. Consider system (4) with $n = 2$, $m = 2$, $T = 2\pi$, and $y = (y_1, y_2)^t$, and

$$y(\zeta) = \begin{pmatrix} y_1(\zeta) \\ y_2(\zeta) \end{pmatrix}, q(\zeta, y(\zeta - \tau(\zeta))) = 10^{-4} \sin(\zeta) \begin{pmatrix} y_2(\zeta - \cos(\zeta)) \\ y_1(\zeta - \cos(\zeta)) \end{pmatrix},$$

$$A(\zeta) = \begin{pmatrix} 10^{-2} \sin(t) & -0.99 \\ 0.99 & 10^{-3} \sin(t) \end{pmatrix},$$

$$\begin{aligned} &f(\zeta, y(\zeta - \sigma_1(\zeta)), y(\zeta - \sigma_2(\zeta))) \\ &\quad = 10^{-5} \cos(\zeta) \begin{pmatrix} y_1(\zeta - 10^{-2}) + y_2(\zeta - \sin(\zeta)) \\ y_2(\zeta - 10^{-2}) + y_1(\zeta - \sin(\zeta)) \end{pmatrix}. \end{aligned} \tag{37}$$

Let the set

$$\Omega = \{u \in BC([0, 2\pi], \mathbb{R}^n): \|u\| \le M \text{ and } u(\zeta + 2\pi) = u(\zeta) \text{ for all } \zeta \in \mathbb{R}\}. \tag{38}$$

Clearly, the set Ω is a bounded nonempty closed and convex subset of $BC([0, 2\pi], \mathbb{R}^n)$ for any positive constant M.

Note that $L_1 = 10^{-4}$, $L_2 = 10^{-5}$, $\alpha = 0$, and $\beta = 0$, and we use $\|A\| = \max_{1\le j\le 2} \sum_{i=1}^{2} a_{ij}$ to get

$$\begin{aligned} \|A(\zeta)\| &= \left\| \begin{pmatrix} 10^{-2} \sin(\zeta) & -0.99 \\ 0.99 & 10^{-3} \sin(\zeta) \end{pmatrix} \right\| \\ &= \max\left\{ \left|10^{-2} \sin(\zeta)\right| + 0.99, \left|10^{-3} \sin(\zeta)\right| + 0.99 \right\} \\ &= \left|10^{-2} \sin(\zeta)\right| + 0.99. \end{aligned} \tag{39}$$

Then, $\lambda = 10^{-1}$.

We can see that conditions (14) and (15) hold.

We substitute all quantities in the inequality (16), and we have

$$10^{-4} + \mu\left[10^{-5} + 2 \times 10^{-5}\right] \le 1. \tag{40}$$

Now, since the matrix A is continuous and periodic, then system (4) has an integrable dichotomy, and we have two cases: if $\mu \le ((1 - 10^{-4})/(3 \times 10^{-5}))$, then (16) holds for any positive constant M, and by Theorem 11, system (4) has at least one 2π-periodic solution.

If $\mu < ((1 - 10^{-4})/(3 \times 10^{-5}))$, then condition (34) holds, and by Theorem 11, system (4) has a unique 2π-periodic solution.

4. Existence of Periodic Solutions for (5)

In this section, we show the existence and the uniqueness of the solution of (5) under the conditions stated in the previous section. So, let

$$z(\zeta) = y(\zeta) - \int_{-\infty}^{0} Q(r) q(\zeta, y(\zeta + \tau(r))) dr. \tag{41}$$

Then,

$$\begin{aligned} z'(\zeta) &= A(\zeta) z(\zeta) + A(\zeta) \int_{-\infty}^{0} Q(r) q(\zeta, y(\zeta + \tau(r))) dr \\ &\quad + \int_{-\infty}^{0} Q(r) f(\zeta, y(\zeta + \sigma_1(r)), \cdots, y(\zeta + \sigma_m(r))) dr. \end{aligned} \tag{42}$$

By Theorem 3, system (5) holds the integral equation

$$\begin{aligned} z(\zeta) &= \int_{-\infty}^{\infty} G(\zeta,\omega) A(\omega) \int_{-\infty}^{0} Q(r) q(\omega, y(\omega + \tau(r))) dr d\omega \\ &\quad + \int_{-\infty}^{\infty} G(\zeta,\omega) \int_{-\infty}^{0} Q(r) f(\omega, y(\omega + \sigma_1(r)), \cdots, y(\omega + \sigma_m(r))) dr d\omega. \end{aligned} \tag{43}$$

The above equation is equivalent to

$$\begin{aligned} y(\zeta) &= \int_{-\infty}^{0} Q(r) q(\zeta, y(\zeta + \tau(r))) dr \\ &\quad + \int_{-\infty}^{\infty} G(\zeta,\omega) A(\omega) \int_{-\infty}^{0} Q(r) q(\omega, y(\omega + \tau(r))) dr d\omega \\ &\quad + \int_{-\infty}^{\infty} G(\zeta,\omega) \int_{-\infty}^{0} Q(r) f(\omega, y(\omega + \sigma_1(r)), \cdots, y(\omega + \sigma_m(r))) dr d\omega. \end{aligned} \tag{44}$$

We define, for $u \in BC(\mathbb{R}, \mathbb{R}^n)$, the operators Y_1 and Y_2 by

$$(Y_1 u)(\zeta) = \int_{-\infty}^{0} Q(r) q(\zeta, u(\zeta + \tau(r))) dr, \tag{45}$$

$$\begin{aligned} (Y_2 u)(\zeta) &= \int_{-\infty}^{\infty} G(\zeta,\omega) A(\omega) \int_{-\infty}^{0} Q(r) q(\omega, u(\omega + \tau(r))) dr d\omega \\ &\quad + \int_{-\infty}^{\infty} G(\zeta,\omega) \int_{-\infty}^{0} Q(r) f(\omega, u(\omega + \sigma_1(r)), \cdots, u(\omega + \sigma_m(r))) dr d\omega. \end{aligned} \tag{46}$$

Lemma 13. *If (14) and (15) hold, then the operators Y_1 and Y_2 defined above are operators from Ω into $BC(\mathbb{R}, \mathbb{R}^n)$, that is, $Y_1, Y_2 : \Omega \longrightarrow BC(\mathbb{R}, \mathbb{R}^n)$.*

Proof. Let $u \in \Omega$; by (14), we get

$$\begin{aligned} |(Y_1 u)(\zeta)| &= \left| \int_{-\infty}^{0} Q(r) q(\zeta, u(\zeta + \tau(r))) dr \right| \\ &\le (L_1 |u(\zeta - \tau(\zeta))| + |q(\zeta, 0)|) \left| \int_{-\infty}^{0} Q(r) dr \right| \\ &\le \left(L_1 \|u\| + \sup_{\zeta \in [0,\zeta]} |q(\zeta, 0)| \right) \|I\| \le L_1 M + \alpha. \end{aligned} \tag{47}$$

Secondly, for $u \in \Omega$, by (14) and (15), we get

$$\begin{aligned} |(Y_2 u)(\zeta)| &= \int_{-\infty}^{\infty} \|G(\zeta,\omega)\| \|A(\omega)\| \left| \int_{-\infty}^{0} Q(r) q(\omega, u(\omega + \tau(r))) dr \right| d\omega \\ &\quad + \int_{-\infty}^{\infty} \|G(\zeta,\omega)\| \left| \int_{-\infty}^{0} Q(r) f(\omega, u(\omega + \sigma_1(r)), \cdots, u(\omega + \sigma_m(r))) dr \right| d\omega \\ &\le [\lambda(L_1 M + \alpha) + (L_2 m M + \beta)] \int_{-\infty}^{\infty} \|G(\zeta,\omega)\| \left| \int_{-\infty}^{0} Q(r) dr \right| d\omega \\ &\le \mu[\lambda(L_1 M + \alpha) + (L_2 m M + \beta)]. \end{aligned} \tag{48}$$

Since all quantities in Y_1 and Y_2 are periodic, then Y_1, $Y_2 : \Omega \longrightarrow BC(\mathbb{R}, \mathbb{R}^n)$.

By the same technique proofs in Lemmas 8–10, we state the following lemmas without proofs. □

Lemma 14. *If (14) holds, then the operator $Y_1 : \Omega \longrightarrow BC(\mathbb{R}, \mathbb{R}^n)$ defined by (45) is a contraction.*

Lemma 15. *If (14) and (15) hold, then the operator $Y_2 : \Omega \longrightarrow BC(\mathbb{R}, \mathbb{R}^n)$ defined by (46) is completely continuous.*

Lemma 16. *If (14)–(16) hold, then for any $u, v \in \Omega$, we have $Y_1 u + Y_2 v \in \Omega$.*

Theorem 17. *Assume that system (7) has an integrable dichotomy. If conditions (14)–(16) hold, then system (5) has at least one T-periodic solution.*

Proof. By Lemmas 13–16, all the requirements of the Krasnoselskii's theorem are satisfied. Thus, there exists a fixed point $z \in \Omega$ such that $z = Y_1 z + Y_2 z$; this fixed point is a solution of (5). Hence, (5) has a ζ-periodic solution. □

Theorem 18. *Assume that system (7) has an integrable dichotomy. If conditions (14), (15), and (34) hold, then system (5) has a unique T-periodic solution.*

Proof. Let the mapping Y be presented by

$$\begin{aligned} (Yu)(\zeta) &= \int_{-\infty}^{0} Q(r) q(\zeta, u(\zeta + \tau(r))) dr \\ &\quad + \int_{-\infty}^{\infty} G(\zeta,\omega) A(\omega) \int_{-\infty}^{0} Q(r) q(\omega, u(\omega + \tau(r))) dr d\omega \\ &\quad + \int_{-\infty}^{\infty} G(\zeta,\omega) \int_{-\infty}^{0} Q(r) f(\omega, u(\omega + \sigma_1(r)), \cdots, u(\omega + \sigma_m(r))) dr d\omega. \end{aligned} \tag{49}$$

For $u_1, u_2 \in BC(\mathbb{R}, \mathbb{R}^n)$, we have

$$\begin{aligned} |(Yu_1)(\zeta) - (Yu_2)(\zeta)| &\le \left| \int_{-\infty}^{0} Q(r) q(\zeta, u_1(\zeta + \tau(r))) dr - \int_{-\infty}^{0} Q(r) q(\zeta, u_2(\zeta + \tau(r))) dr \right| \\ &\quad + \int_{-\infty}^{\infty} \|G(\zeta,\omega)\| \|A(\omega)\| \left| \int_{-\infty}^{0} Q(r) (q(\omega, u_1(\omega + \tau(r))) \right. \\ &\quad \left. - q(\omega, u_2(\omega + \tau(r)))) dr \right| d\omega \\ &\quad + \int_{-\infty}^{\infty} \|G(\zeta,\omega)\| \left| \int_{-\infty}^{0} Q(r) | f(\omega, u_1(\omega + \sigma_1(r)), \cdots, u_1(\omega + \sigma_m(r))) \right. \\ &\quad \left. - f(\omega, u_2(\omega + \sigma_1(r)), \cdots, u_2(\omega + \sigma_m(r))) | dr \right| d\omega \\ &= (L_1 + \mu(\lambda L_1 + L_2 m)) \|u_1 - u_2\|. \end{aligned} \tag{50}$$

Since (34) holds, the contraction mapping completes the proof. □

5. Conclusion

In this paper, we dealt with the study of types of neutral equations more generally, represented in nonlinear systems with several delays under the dichotomy condition, where the fixed point theorems were used to prove existence and uniqueness.

The benefit of this paper is to generalize several well-known researches such as [11, 14]. So that if $q(\zeta, y(\zeta - \tau(\zeta))) = cy(\zeta - \tau)$, then our results will apply to systems (3) of [11]. Also, the periodicity of (1) and (2) in [14] is generalized by our systems (4) and (5) in n-dimentional case.

Data Availability

The numerical data used to support the findings of this study are included in the article.

Conflicts of Interest

The authors declare that they have no competing interests. There are no any nonfinancial competing interests (political, personal, religious, ideological, academic, intellectual, commercial, or any other) to declare in relation to this manuscript.

Authors' Contributions

Mesmouli directed the study and helped inspection. All the authors carried out the main results of this article and drafted the manuscript and read and approved the final manuscript.

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