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Research Article

A Novel Numerical Technique for Fractional Ordinary Differential Equations with Proportional Delay

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Some researchers have combined two powerful techniques to establish a new method for solving fractional-order differential equations. In this study, we used a new combined technique, known as the Elzaki residual power series method (ERPSM), to offer approximate and exact solutions for fractional multipantograph systems (FMPS) and pantograph differential equations (PDEs). In Caputo logic, the fractional-order derivative operator is measured. The Elzaki transform method and the residual power series method (RPSM) are combined in this novel technique. The suggested technique is based on a new version of Taylor's series that generates a convergent series as a solution. Establishing the coefficients for a series, like the RPSM, necessitates computing the fractional derivatives each time. As ERPSM just requires the concept of a zero limit, we simply need a few computations to get the coefficients. The novel technique solves nonlinear problems without the need for He's and Adomian polynomials, which is an advantage over the other combined methods based on homotopy perturbation and Adomian decomposition methods. The relative, recurrence, and absolute errors of the problems are analyzed to evaluate the efficiency and consistency of the presented method. Graphical significances are also identified for various values of fractional-order derivatives. As a result, the procedure is quick, precise, and easy to implement, and it yields outstanding results.

1. Introduction

Many differential equations (DEs) that arise in applications are sufficiently complicated that closed-form solutions are not always feasible. Numerical methods offered a powerful substitute means for solving the DEs under the given initial conditions. Numerous methods have been developed in recent years to solve fractional-order differential equations (FODEs), including the homotopy perturbation method [1], the differential transform method [2], the operational matrix method [3], the conformable Shehu transform decomposition method [4], the variational iteration method [5], the Jacobi collocation method [6], the conformable Shehu transform iterative method [7], the spectral tau method [8], the Legendre wavelet method [9], the fractional natural decomposition method [10], the power series method with the conformable operator [11], and the Chebyshev polynomial method [12].

Integral equations (IEs), DEs, and delay differential equations (DDEs) are all solved by employing integral transforms [13–17], which are among the most valuable techniques in mathematics. The conversion of DEs and IEs into terms of a simple algebraic equation is enabled by the appropriate selection of integral transform. The origins of integral transforms can be traced back to Laplace's work in the 1780s and Fourier's work in 1822 [18]. In the beginning, ordinary and partial DEs were solved using the Laplace transform and the Fourier transform, which are two well-known transforms. These transforms were then applied to FODEs [19–24]. In recent years, researchers have proposed lots of new different transformations to solve a variety of mathematical problems. FODEs are solved using the

Aboodh transform [25], fractional complex transform [26], travelling wave transform [27], Sumudu transform [28], and ZZ transform [29]. These transformations are paired with additional analytical, numerical, or homotopy-based techniques to handle FODEs [30–35]. Numerous mathematicians have recently become interested in a transformation known as the Elzaki transform (ET) [36–41]. The ET was introduced by Elzaki to facilitate the process of solving ordinary and partial DEs in the time domain [42]. The ET is derived from the classical Fourier integral transform.

We examine the functions in set H, which are described as

$$H = \left\{ \Theta(\tau) | \exists M, \Upsilon_1, \Upsilon_2 > 0, |\Theta(\tau) < M e^{|\tau|/\Upsilon_j} \quad \text{if } \tau \in (-1)^j \times [0, \infty) \right\}. \tag{1}$$

The formula for E-T is as follows:

$$E[\Theta(\tau)] = \wp(\upsilon) = \upsilon \int_0^\infty \Theta(\tau) e^{-(\tau/\upsilon)} d\tau, \quad \Upsilon_1 \leq \upsilon \leq \Upsilon_2. \tag{2}$$

The following are the key advantages of the ET [36–45]:

(i) The ET can easily be applied to the initial value problems with less computational work

(ii) The ET has unit-preserving properties and may be used to solve problems without resorting to the frequency domain

(iii) Numerous nonlinear DEs with variable coefficients, namely, the time-fractional wavelike equations, can be solved with it

(iv) It may handle a variety of difficult problems in engineering, physics, fluid mechanics, chemistry, and dynamics, such as Maxwell's equations and fluid flow problems

The Jordanian mathematician, Arqub, created the RPSM in 2013 [46]. The RPSM is a semianalytical method; it is a combination of Taylor's series and the residual error function. It provides series solutions of linear and nonlinear DEs in the form of convergence series. In 2013, RPSM was implemented for the first time to find solutions to fuzzy DEs. Furthermore, this method has been successfully used to solve a wide range of FODEs, including time-fractional KdV-Burgers equations [47], time-fractional Schrödinger equations [48], the SIR epidemic model of fractional order [49], conformable-type Coudrey–Dodd–Gibbon–Sawada–Kotera [50], time-fractional Swift–Hohenberg problems [51], time-fractional Phi-4 equation [52], and the Zakharov–Kuznetsov equation [53].

Researchers combined two powerful methods to develop a new method for solving FODEs. Some of these groups are described as a combination of the Adomian decomposition method and the Sumudu transform [54], as well as the homotopy analysis method and the natural transform [55] and the Laplace transformation with homotopy perturbation approach [56]. In this study, we applied the novel combined technique, known as the ERPSM, to provide approximate and exact solutions for FMPS and PDEs. To assess the efficiency and consistency of the proposed method, the relative, recurrence, and absolute errors of the problems are examined. Graphical significance is also found for various values of fractional-order derivatives. As a result, the technique is rapid, precise, and simple to use, and it produces excellent results. The set of rules for this new technique depends on transforming the given equation into the ET space, in the second step; establishing a series solution by using the new form of the Taylor series; and then acquiring the solution in the real space of the equation by applying the inverse ET.

This novel technique can be used to construct power series expansion solutions for linear and nonlinear FODEs without perturbation, linearization, or discretization. Unlike the classical power series method, this method does not need to match the coefficients of the corresponding terms, and a recursion relation is not needed. The new method handles nonlinear problems without the need for He's and Adomian polynomials, which is an advantage over existing combination methods based on homotopy perturbation and Adomian decomposition methods. This technique finds the coefficients of the series, relying on the limit concept but not the fractional derivatives as in the RPSM. Thus, only a few calculations are required to determine the coefficients related to RPSM. The closed-form and approximate solutions can be obtained by the proposed method through a quick convergence series.

A PDE is a special kind of delay differential equation (DDE) with a proportional delay. In 1851, the first-time device named "pantograph" was used in the construction of an electric locomotive, which is where this name originated from that time. British Railways decided to make a new kind of electric locomotive in 1960. The target was to construct a new kind of electric locomotive that moves trains faster. The pantograph was a prominent part of the new fast-speed electric locomotive. Pantographs take current from an overhead wire, which is necessary for the locomotive to move. Therefore, Ockendon and Taylor observed the mechanism of the pantograph. As a result, they built a special kind of DDE form:

$$\frac{d\Theta}{d\tau} = \Omega \, \Theta(\tau) + \sigma \, \Theta(\lambda\tau), \quad \tau \geq 0, \tag{3}$$

where Ω, σ are real constants and $0 < k < 1$, $\lambda \in R$. This article was first published in 1971 [57]. Then, such a type of DDE was named PDE. Various studies on PDEs have recently been published in the scientific literature [58–60]. PDEs are widely used in probability theory, nonlinear dynamical systems, astrophysics, quantum mechanics, electrodynamics, and cell growth [61–64]. In this study, we consider the following FMPS:

$$D_\tau^\omega \Theta_1(\tau) = \Omega_1 \Theta_1(\tau) + \Lambda_1(\tau, \Theta_1(\tau), \Theta_2(\tau), \Theta_1(\varpi_1\tau), \Theta_2(\varpi_2\tau)), \tag{4}$$

$$D_\tau^\omega \Theta_2(\tau) = \Omega_2\Theta_2(\tau) + \Lambda_2(\tau, \Theta_1(\tau), \Theta_2(\tau), \Theta_1(\varpi_1\tau), \Theta_2(\varpi_2\tau)), \tag{5}$$

subject to the initial conditions

$$\begin{aligned} \Theta_1(0) &= \mathfrak{J}_1, \\ \Theta_2(0) &= \mathfrak{J}_2, \end{aligned} \tag{6}$$

where $0 < \varpi_1, \varpi_2 \leq 1, \mathfrak{J}_1, \mathfrak{J}_2$ are finite constants; Λ_1, Λ_2 are analytical functions; and D_τ^ω is the Caputo fractional derivative (CFD). The FMPS is a type of DDE that arises in a number of physics and engineering applications, including quantum mechanics, dynamical systems, electronic systems, and population dynamics [65, 66].

The following linear PDE is given as

$$D_\tau^\omega \Theta(\tau) = l\Theta(\tau) + \sum_{j=1}^{s} m_j \Theta\left(u_j\tau\right) + \sum_{j=1}^{t} p_j D_\tau^\omega \Theta\left(q_j\tau\right) + g(\tau), \tag{7}$$

where $0 < \omega \leq 1$, $\tau \geq 0$, $l, m_j, p_j \in R$, $0 < \omega \leq 1$, $u_j > 0$, and $q_j < 1$. D_τ^ω is the CFD of order ω, with the following initial condition:

$$\Theta(0) = \phi. \tag{8}$$

The nonlinear PDE is as follows:

$$D_\tau^\omega \Theta(\tau) = \xi(\tau^\omega, \Theta(\tau), \Theta(\lambda\tau), D_\tau^\omega \Theta(\lambda\tau)), \tag{9}$$

with the initial condition

$$\Theta(0) = \mathfrak{J}. \tag{10}$$

Definition 1 (see [67]) (a novel fractional Taylor series formula in E-T). Assume that $\Theta(\tau)$ is a piecewise continuous and exponential order and that the E-T of $\Theta(\tau)$, $E[\Phi(\tau)] = \wp(\upsilon)$ is provided by a fractional Taylor series.

$$\wp(\upsilon) = \sum_{\nu=0}^{\infty} \hbar_\nu \upsilon^{\nu\omega+2}, \tag{11}$$

where $\hbar_\nu$ is the νth coefficient of the novel fractional Taylor series formula in E-T.

Lemma 2. *Assume that $\Theta_1(\tau)$ and $\Theta_2(\tau)$ are piecewise continuous and of exponential-order functions, and $E[\Theta_1(\tau)] = \wp_1(\upsilon)$, $E[\Theta_2(\tau)] = \wp_2(\upsilon)$, and λ_1, λ_2 are constants. Then, the following axioms hold [67]:*

(i) $\lim_{\upsilon \longrightarrow 0} (1/\upsilon^2)\wp(\upsilon) = \hbar_0$

(ii) $E[D_\tau^\omega \Theta(\tau)] = (\wp(\upsilon)/\upsilon^\omega) - \sum_{r=0}^{\nu-1} \upsilon^{r-\omega+2}\Theta^{(r)}(0), \nu - 1 < \omega \leq \nu$

(iii) $E[D_\tau^{\nu\omega}\Theta(\tau)] = \wp(\upsilon)/\upsilon^{\nu\omega} - \sum_{r=0}^{\nu-1} \upsilon^{(r-\nu)\omega+2}\Theta^{(r)}(0), 0 < \omega \leq 1$

The framework of this study is as follows. The fundamental recommendation beyond the ERPSM with convergence and absolute error analysis for FMPS is demonstrated in Section 2. We also demonstrated numerical examples of FMPS to exemplify the competency, potential, and straightforwardness of the new combined technique. The linear and nonlinear PDEs are discussed in Sections 3 and 4, respectively. Finally, our findings are summarized in Section 5.

2. The ERPSM to Demonstrate the FMPS

In this section, we use ERPSM to construct the solutions of the FMPS as in Equations (4) and (5). The main algorithm of this method for solving the FMPS can be summarized by the following steps: applies the E-T to Equation (4). As a result, we get an algebraic form in E-T space. In the second step, using the novel fractional Taylor's series formula in E-T, we represent the solution in the E-T space of the algebraic equation obtained in the first step. The coefficients of this expansion are determined with the help of residual function and limit concept. As a result, we have found a solution to the problem in its original space by taking the inverse E-T.

In the next subsection, we derive the main algorithms of the ERPSM for the FMPS.

2.1. The Algorithm of ERPSM for Solving FMPS. We use the following algorithm to create the solution with the help of ERPSM for the FMPS as shown in Equations (4) and (5):

Step 1. Rewrite Equations (4) and (5). We have

$$D_\tau^\omega \Theta_1(\tau) - \Omega_1\Theta_1(\tau) - \Lambda_1(\tau, \Theta_1(\tau), \Theta_2(\tau), \Theta_1(\varpi_1\tau), \Theta_2(\varpi_2\tau)) = 0, \tag{12}$$

$$D_\tau^\omega \Theta_2(\tau) - \Omega_2\Theta_2(\tau) - \Lambda_2(\tau, \Theta_1(\tau), \Theta_2(\tau), \Theta_1(\varpi_1\tau), \Theta_2(\varpi_2\tau)) = 0. \tag{13}$$

Step 2. We get the following result by implementing the E-T on both sides of Equations (12) and (13).

$$\begin{aligned} &E[D_\tau^\omega \Theta_1(\tau)] - E[\Omega_1\Theta_1(\tau)] \\ &\quad - E[\Lambda_1(\tau, \Theta_1(\tau), \Theta_2(\tau), \Theta_1(\varpi_1\tau), \Theta_2(\varpi_2\tau))] = 0, \end{aligned} \tag{14}$$

$$\begin{aligned} &E[D_\tau^\omega \Theta_2(\tau)] - E[\Omega_2\Theta_2(\tau)] \\ &\quad - E[\Lambda_2(\tau, \Theta_1(\tau), \Theta_2(\tau), \Theta_1(\varpi_1\tau), \Theta_2(\varpi_2\tau))] = 0. \end{aligned} \tag{15}$$

By utilizing the second part of Lemma 2, Equations (14) and (15) become as follows:

$$\wp_1(\upsilon) = \upsilon^2\Theta(0) + \upsilon^\omega \Omega_1 \wp_1(\upsilon) + \upsilon^\omega \Lambda_1^l(\upsilon), \tag{16}$$

$$\wp_2(\upsilon) = \upsilon^2\Theta(0) + \upsilon^\omega \Omega_2 \wp_2(\upsilon) + \upsilon^\omega \Lambda_2^l(\upsilon), \tag{17}$$

where

$$\begin{aligned}
E[\Lambda_1(\tau,\Theta_1(\tau),\Theta_2(\tau),\Theta_1(\varpi_1\tau),\Theta_2(\varpi_2\tau))] &= \Lambda_1^l(\upsilon),\\
E[\Lambda_2(\tau,\Theta_1(\tau),\Theta_2(\tau),\Theta_1(\varpi_1\tau),\Theta_2(\varpi_2\tau))] &= \Lambda_2^l(\upsilon),\\
E[\Theta_1(\tau)] &= \wp_1(\upsilon),\\
E[\Theta_2(\tau)] &= \wp_2(\upsilon).
\end{aligned} \tag{18}$$

Step 3. Assume that algebraic equations (16) and (17) have the solution in the expansion form as

$$\begin{aligned}
\wp_1(\upsilon) &= \sum_{\nu=0}^{\infty} \hbar_{1\nu}\upsilon^{\nu\omega+2},\\
\wp_2(\upsilon) &= \sum_{\nu=0}^{\infty} \hbar_{2\nu}\upsilon^{\nu\omega+2}.
\end{aligned} \tag{19}$$

The κth-truncated series of $\wp_1(\upsilon)$ and $\wp_2(\upsilon)$ are as

$$\begin{aligned}
\wp_{1\kappa}(\upsilon) &= \sum_{\nu=0}^{\kappa} \hbar_{1\nu}\upsilon^{\nu\omega+2},\\
\wp_{2\kappa}(\upsilon) &= \sum_{\nu=0}^{\kappa} \hbar_{2\nu}\upsilon^{\nu\omega+2}.
\end{aligned} \tag{20}$$

Step 4. By utilizing the following lemma,

$$\begin{aligned}
\hbar_{1,0} &= \lim_{\upsilon\longrightarrow 0}\frac{1}{\upsilon^2}\Theta_1(\tau) = \Theta_1(0),\\
\hbar_{2,0} &= \lim_{\upsilon\longrightarrow 0}\frac{1}{\upsilon^2}\Theta_2(\tau) = \Theta_2(0).
\end{aligned} \tag{21}$$

The κth-truncated series becomes

$$\wp_{\kappa,1}(\upsilon) = \Theta_1(0)\upsilon^2 + \sum_{\nu=1}^{\kappa} \hbar_{1\nu}\upsilon^{\nu\omega+2}, \tag{22}$$

$$\wp_{\kappa,2}(\upsilon) = \Theta_2(0)\upsilon^2 + \sum_{\nu=1}^{\kappa} \hbar_{2,\nu}\upsilon^{\nu\omega+2}. \tag{23}$$

Step 5. Consider the Elzaki residual function (ERF) of Equations (22) and (23) separately, as well as the κth-truncated Elzaki residual functions (ERFs), so that

$$E\mathrm{Res}_1(\upsilon) = \wp_1(\upsilon) - \upsilon^2\Theta(0) - \upsilon^{\omega}\Omega_1\wp_1(\upsilon) - \upsilon^{\omega}\Lambda_1^l(\upsilon), \tag{24}$$

$$E\mathrm{Res}_2(\upsilon) = \wp_2(\upsilon) - \upsilon^2\Theta(0) - \upsilon^{\omega}\Omega_2\wp_2(\upsilon) - \upsilon^{\omega}\Lambda_2^l(\upsilon), \tag{25}$$

$$E\mathrm{Res}_{1,\kappa}(\upsilon) = \wp_{1,\kappa}(\upsilon) - \upsilon^2\Theta(0) - \upsilon^{\omega}\Omega_1\wp_{1,\kappa}(\upsilon) - \upsilon^{\omega}\Lambda_1^l(\upsilon), \tag{26}$$

$$E\mathrm{Res}_{2,\kappa}(\upsilon) = \wp_{2,\kappa}(\upsilon) - \upsilon^2\Theta(0) - \upsilon^{\omega}\Omega_2\wp_{2,\kappa}(\upsilon) - \upsilon^{\omega}\Lambda_2^l(\upsilon). \tag{27}$$

Step 6. Replace the succession arrangement of $\wp_{1,\kappa}(\upsilon)$ and $\wp_{2,\kappa}(\upsilon)$ in Equations (26) and (27).

Step 7. To highlight important facts, we extend some features that arise in the RPSM [67–69].

That is obvious.

$$E\ \mathrm{Re}\ s(\upsilon) = 0. \tag{28}$$

Therefore,

$$\lim_{\kappa\longrightarrow\infty} E\mathrm{Res}_k(\upsilon) = E\mathrm{Res}(\upsilon). \tag{29}$$

Since, $\lim_{\upsilon\longrightarrow 0}(1/\upsilon^2)E\ \mathrm{Re}\ s(\upsilon) = 0$. It is clear that $\lim_{\upsilon\longrightarrow 0}(1/\upsilon^2)$ $E\mathrm{Res}_k(\upsilon) = 0$. As a result, we get the following:

$$\begin{aligned}
\lim_{\upsilon\longrightarrow 0}\left(\frac{1}{\upsilon^{k\omega+2}}E\mathrm{Res}_1(\upsilon)\right) &= \lim_{\upsilon\longrightarrow 0}\left(\frac{1}{\upsilon^{k\omega+2}}E\mathrm{Res}_{1,\kappa}(\upsilon)\right) = 0, \quad k = 1, 2, 3, \cdots,\\
\lim_{\upsilon\longrightarrow 0}\left(\frac{1}{\upsilon^{k\omega+2}}E\mathrm{Res}_2(\upsilon)\right) &= \lim_{\upsilon\longrightarrow 0}\left(\frac{1}{\upsilon^{k\omega+2}}E\mathrm{Res}_{2,\kappa}(\upsilon)\right) = 0, \quad k = 1, 2, 3, \cdots.
\end{aligned} \tag{30}$$

Step 8. Replace the κth attained values of $\hbar_{1,\kappa}$ and $\hbar_{2,\kappa}$ into the κth-truncated series of $\wp_{1,\kappa}(\upsilon)$ and $\wp_{2,\kappa}(\upsilon)$ to become the κth-approximate explanation of Equations (16) and (17).

Step 9. Use the inverse E-T on $\wp_{1,\kappa}(\upsilon)$ and $\wp_{2,\kappa}(\upsilon)$ to obtain the κth-approximate solution $\Theta_{1,\kappa}(\tau)$ and $\Theta_{2,\kappa}(\tau)$ in the original space.

The next theorem clarifies and establishes the conditions for the series solutions to converge.

Theorem 3. *Let Z be a Banach space denoted with a suitable norm $\|.\|$ over which the sequence of partial sums $\sum_{\kappa=0}^{\infty}\wp_\kappa(\upsilon)$ is defined. Assume that the initial guess $\wp_0$ remains inside the ball $B_r(\wp)$ of the solution $\wp(\upsilon)$. Then, the series solution $\sum_{\kappa=0}^{\infty}\wp_\kappa(\upsilon)$ converges if $\exists\theta > 0$ such that $\|\wp_{\nu+1}(\upsilon)\| \leq \theta\|\wp_\nu(\upsilon)\|$.*

Proof. A sequence of partial sums is defined as

$$\begin{aligned}
Y_0 &= \wp_0(\upsilon),\\
Y_1 &= \wp_0(\upsilon) + \wp_1(\upsilon),\\
Y_2 &= \wp_0(\upsilon) + \wp_1(\upsilon) + \wp_2(\upsilon),\\
Y_3 &= \wp_0(\upsilon) + \wp_1(\upsilon) + \wp_2(\upsilon) + \wp_3(\upsilon),\\
&\vdots\ \vdots\ \vdots\\
Y_\nu &= \wp_0(\upsilon) + \wp_1(\upsilon) + \wp_2(\upsilon) + \wp_3(\upsilon) + \cdots + \wp_\nu(\upsilon).
\end{aligned} \tag{31}$$

Next is that we would have to show that $\{Y_\nu\}_{\nu=0}^{\infty}$ is a Cauchy sequence in Z. To demonstrate this, consider the following relationship:

$$\begin{aligned}\|Y_{\nu+1}-Y_\nu\| &= \|\wp_{\nu+1}(\upsilon)\| \le \theta\|\wp_\nu(\upsilon)\| \le \theta^2\|\wp_{\nu-1}(\upsilon)\| \\ &\le \theta^3\|\wp_{\nu-2}(\upsilon)\| \le \theta^4\|\wp_{\nu-3}(\upsilon)\| \cdots \le \theta^{\nu+1}\|\wp_0(\upsilon)\|,\end{aligned} \tag{32}$$

where $\nu = 0, 1, 2, \cdots$.

For every $\nu, \kappa \in N$, $\nu \ge \kappa$, we have

$$\begin{aligned}\|Y_\nu - Y_\kappa\| = \|&(Y_\nu - Y_{\nu-1}) + (Y_{\nu-1} - Y_{\nu-2}) + (Y_{\nu-2} - Y_{\nu-3}) \\ &+ (Y_{\nu-3} - Y_{\nu-4}) \cdots (Y_{\kappa+1} - Y_\kappa)\|.\end{aligned} \tag{33}$$

From triangle inequality, we have

$$\begin{aligned}&\|(Y_\nu - Y_{\nu-1}) + (Y_{\nu-1} - Y_{\nu-2}) + (Y_{\nu-2} - Y_{\nu-3}) \\ &\quad + (Y_{\nu-3} - Y_{\nu-4}) + \cdots (Y_{\kappa+1} - Y_\kappa)\| \le \|(Y_\nu - Y_{\nu-1})\| \\ &\quad + \|(Y_{\nu-1} - Y_{\nu-2})\| + \|(Y_{\nu-2} - Y_{\nu-3})\| + \|(Y_{\nu-3} - Y_{\nu-4})\| \\ &\quad + \cdots + \|(Y_{\kappa+1} - Y_\kappa)\|,\end{aligned} \tag{34}$$

and

$$\begin{aligned}&\|(Y_\nu - Y_{\nu-1})\| + \|(Y_{\nu-1} - Y_{\nu-2})\| + \|(Y_{\nu-2} - Y_{\nu-3})\| \\ &\quad + \|(Y_{\nu-3} - Y_{\nu-4})\| + \cdots + \|(Y_{\kappa+1} - Y_\kappa)\| \le \theta^\nu\|\wp_0(\upsilon)\| \\ &\quad + \theta^{\nu-1}\|\wp_0(\upsilon)\| + \theta^{\nu-2}\|\wp_0(\upsilon)\| + \cdots + \theta^{\kappa+1}\|\wp_0(\upsilon)\| \\ &= \frac{1-\theta^{\nu-\kappa}}{1-\theta}\theta^{\kappa+1}\|\wp_0(\upsilon)\|.\end{aligned} \tag{35}$$

Therefore,

$$\|Y_\nu - Y_\kappa\| = \frac{1-\theta^{\nu-\kappa}}{1-\theta}\theta^{\kappa+1}\|\wp_0(\upsilon)\|. \tag{36}$$

Showing that the sequence is bounded, we can obtain for $0 < \theta < 1$ that

$$\lim_{\nu,\kappa\longrightarrow\infty}\|Y_\nu - Y_\kappa\| = 0. \tag{37}$$

□

This proves that the sequence of partial sums generated by ERPSM is Cauchy and hence convergent.

In the next theorem, we determine the maximum truncation error.

Theorem 4. *Let $\wp(\upsilon)$ be the approximate solution of the truncated finite series $\sum_{\nu=0}^{\infty}\wp_\nu(\upsilon)$. Assume it is attainable to acquire a real number $\theta \in (0, 1)$, in order that $\|\wp_{\nu+1}(\upsilon)\| \le \theta\|\wp_\nu(\upsilon)\|, \forall \nu \in N$; furthermore, the utmost absolute error is*

$$\left\|\wp(\upsilon) - \sum_{\nu=0}^{\kappa}\wp_\nu(\upsilon)\right\| \le \frac{\theta^{\kappa+1}}{1-\kappa}\|\wp_0(\upsilon)\|. \tag{38}$$

Proof. Let the series $\sum_{\nu=0}^{\kappa}\wp_\nu(\upsilon)$ be finite, then

$$\begin{aligned}\left\|\wp(\upsilon) - \sum_{\nu=0}^{\kappa}\wp_\nu(\upsilon)\right\| &= \left\|\sum_{\kappa+1}^{\infty}\wp_\nu(\upsilon)\right\|, \le \sum_{\kappa+1}^{\infty}\|\wp_\nu(\upsilon)\|, \\ &\le \sum_{\kappa+1}^{\infty}\theta^\kappa\|\wp_0(\upsilon)\|, \\ &\le \theta^{\kappa+1}\left(1+\theta+\theta^2+\theta^3+\cdots\right)\|\wp_0(\upsilon)\|, \\ &\le \left(\frac{\theta^{\kappa+1}}{1-\theta}\right)\|\wp_0(\upsilon)\|.\end{aligned} \tag{39}$$

This proof is complete. □

In the next subsection, two problems of FMPS are established to illustrate the performance and appropriateness of the proposed method.

2.2. Numerical Examples. To demonstrate the execution and capability of ERPSM, we investigated two interesting and important problems for FMPS:

Problem 5. Consider the following FMPS:

$$D_\tau^\omega\Theta_1(\tau) - \Theta_1(\tau) + \Theta_2(\tau) - \Theta_1\left(\frac{\tau}{2}\right) + e^{\tau^\omega/2} - e^{-\tau^\omega} = 0, \tag{40}$$

$$D_\tau^\omega\Theta_2(\tau) + \Theta_1(\tau) + \Theta_2(\tau) + \Theta_2\left(\frac{\tau}{2}\right) - e^{-(\tau^\omega/2)} - e^{\tau^\omega} = 0, \tag{41}$$

with the initial conditions $\Theta_1(0) = 1$ and $\Theta_2(0) = 1$.

Applying the E-T to Equations (40) and (41), we get

$$E\left[D_\tau^\omega\Theta_1(\tau) - \Theta_1(\tau) + \Theta_2(\tau) - \Theta_1\left(\frac{\tau}{2}\right) + e^{\tau^\omega/2} - e^{-\tau^\omega}\right] = 0, \tag{42}$$

$$E\left[D_\tau^\omega\Theta_2(\tau) + \Theta_1(\tau) + \Theta_2(\tau) + \Theta_2\left(\frac{\tau}{2}\right) - e^{-(\tau^\omega/2)} - e^{\tau^\omega}\right] = 0. \tag{43}$$

We have the following results from Equations (42) and (43) using the procedure mentioned in Subsection 2.1:

$$\begin{aligned}\wp_1(\upsilon) = \upsilon^2\Theta_1(0) &+ \upsilon^\omega\wp_1(\upsilon) - \upsilon^\omega\wp_2(\upsilon) + 4\upsilon^\omega\wp_1\left(\frac{\upsilon}{2}\right) \\ &- 2\frac{\upsilon^{\omega+2}}{2-\upsilon} + \frac{\upsilon^{\omega+2}}{1+\upsilon},\end{aligned} \tag{44}$$

$$\wp_2(\upsilon) = \upsilon^2\Theta_2(0) - \upsilon^\omega\wp_1(\upsilon) - 4\upsilon^\omega\wp_2\left(\frac{\upsilon}{2}\right) + 2\frac{\upsilon^{\omega+2}}{2+\upsilon} + \frac{\upsilon^{\omega+2}}{1-\upsilon}. \tag{45}$$

Assume that Equations (44) and (45) have a series solution in the following form:

$$\wp_1(\upsilon) = \sum_{\nu=0}^{\infty} \hbar_{1,\nu}\upsilon^{\nu\omega+2},$$
$$\wp_2(\upsilon) = \sum_{\nu=0}^{\infty} \hbar_{2,\nu}\upsilon^{\nu\omega+2}. \tag{46}$$

The κth-truncated expansion is as follows:

$$\wp_{1,\kappa}(\upsilon) = \sum_{\nu=0}^{\kappa} \hbar_{1,\nu}\upsilon^{\nu\omega+2},$$
$$\wp_{2,\kappa}(\upsilon) = \sum_{\nu=0}^{\kappa} \hbar_{2,\nu}\upsilon^{\nu\omega+2}. \tag{47}$$

By using the first part of Lemma 2, the κth-truncated series becomes

$$\wp_{1,\kappa}(\upsilon) = \upsilon^2 + \sum_{\nu=1}^{\kappa} \hbar_{1,\nu}\upsilon^{\nu\omega+2},$$
$$\wp_{2,\kappa}(\upsilon) = \upsilon^2 + \sum_{\nu=1}^{\kappa} \hbar_{2,\nu}\upsilon^{\nu\omega+2}. \tag{48}$$

The ERFs are formulated as

$$E\mathrm{Res}_1(\upsilon) = \wp_1(\upsilon) - \upsilon^2\Theta_1(0) - \upsilon^\omega\wp_1(\upsilon) + \upsilon^\omega\wp_2(\upsilon) - 4\upsilon^\omega\wp_1\left(\frac{\upsilon}{2}\right) + 2\frac{\upsilon^{\omega+2}}{2-\upsilon} - \frac{\upsilon^{\omega+2}}{1+\upsilon},$$

$$E\mathrm{Res}_2(\upsilon) = \wp_2(\upsilon) - \upsilon^2\Theta_2(0) + \upsilon^\omega\wp_1(\upsilon) + 4\upsilon^\omega\wp_2\left(\frac{\upsilon}{2}\right) - 2\frac{\upsilon^{\omega+2}}{2+\upsilon} - \frac{\upsilon^{\omega+2}}{1-\upsilon}. \tag{49}$$

The κth-truncated ERF takes the following form:

$$E\mathrm{Res}_{1,\kappa}(\upsilon) = \wp_{1,\kappa}(\upsilon) - \upsilon^2\Theta_1(0) - \upsilon^\omega\wp_{1,\kappa}(\upsilon) + \upsilon^\omega\wp_{2,\kappa}(\upsilon) - 4\upsilon^\omega\wp_{1,\kappa}\left(\frac{\upsilon}{2}\right) + 2\frac{\upsilon^{\omega+2}}{2-\upsilon} - \frac{\upsilon^{\omega+2}}{1+\upsilon},$$

$$E\mathrm{Res}_{2,\kappa}(\upsilon) = \wp_{2,\kappa}(\upsilon) - \upsilon^2\Theta_2(0) + \upsilon^\omega\wp_{1,\kappa}(\upsilon) + 4\upsilon^\omega\wp_{2,\kappa}\left(\frac{\upsilon}{2}\right) - 2\frac{\upsilon^{\omega+2}}{2+\upsilon} - \frac{\upsilon^{\omega+2}}{1-\upsilon}. \tag{50}$$

Substitute $\kappa = 1, 2, 3, 4, 5$ into Equations (65) and (66) and solve the following expression to find the unknown coefficients.

$$\lim_{\upsilon\longrightarrow 0}\left(\frac{1}{\upsilon^{k\omega+2}}E\mathrm{Res}_{1,\kappa}(\upsilon)\right) = 0,$$
$$\lim_{\upsilon\longrightarrow 0}\left(\frac{1}{\upsilon^{k\omega+2}}E\mathrm{Res}_{2,\kappa}(\upsilon)\right) = 0. \tag{51}$$

Thus, we have

$$\hbar_{1,1} = 1,$$
$$\hbar_{2,1} = -1,$$
$$\hbar_{1,2} = 2 + \frac{1}{(2)^\omega} - \frac{3\Gamma(\omega+1)}{2},$$
$$\hbar_{2,2} = \frac{1}{(2)^\omega} + \frac{\Gamma(\omega+1)}{2},$$
$$\hbar_{1,3} = \frac{1}{8^{1+\omega}}\left(8 + 2^{\omega+4} + 2^{3\omega+4} - 2^{\omega+2}\left(3+4^{\omega+1}\right)(\Gamma(\omega+1) + 38^\omega\Gamma(2\omega+1)\right),$$
$$\hbar_{2,3} = -2 - 2^{1-\omega} - 2^{-3\omega} + \left(1 - 2^{-2\omega-1}\right)\Gamma(\omega+1) + \frac{5}{8}\Gamma(2\omega+1),$$
$$\hbar_{1,4} = -\frac{1}{4^{3\omega+2}}\left(-16 - 2^{\omega+5} - 2^{4\omega+5} - 2^{3\omega+6} - 2^{5\omega+5} - 2^{6\omega+6} + 2^{\omega+3}\left(3 + 2^{3\omega+1} + 3(2)^{5\omega+1} + 2^{2\omega+2}\right) \times \Gamma(\omega+1) + 2^{3\omega+1}\left(2^{3\omega+1} - 3\right)\Gamma(2\omega+1) + 3(2)^{6\omega}\Gamma(3\omega+1)\right),$$
$$\hbar_{2,4} = \frac{1}{4^{3\omega+2}}\left(16 + 2^{2\omega+5} + 2^{3\omega+5} - 2^{4\omega+5} + 2^{5\omega+5} + 2^{\omega+3} \cdot \left(1 - 2^{2\omega+1} + 2^{3\omega+2} + 2^{5\omega+1}\right)\Gamma(\omega+1) - 2^{3\omega+1}\left(8^{\omega+1}+5\right) \cdot \Gamma(2\omega+1) + \frac{7}{3}\Gamma(3\omega+1)2^{6\omega}\right),$$
$$\begin{aligned}\hbar_{1,5} = {} & 4 + 2^{-9\omega+1} + 2^{-7\omega+2} + 2^{-6\omega+1} + 2^{-5\omega+2} + 2^{-4\omega+1} + 2^{-3\omega+1} \\ & + 4^{-\omega+1} + 2^{-10\omega} - \frac{1}{3}\Gamma(3\omega+1) - 3\times 2^{-4(\omega+1)}\Gamma(3\omega+1) \\ & + 5\times 2^7\Gamma(4\omega+1) - 4\Gamma(\omega+1) - \frac{1}{3}\Gamma(3\omega+1) \\ & - 3\times 2^{-4(\omega+1)}\Gamma(3\omega+1) + 5\times 2^7\Gamma(4\omega+1) - 4\Gamma(\omega+1) \\ & - (3)(2)^{-9\omega-1}\Gamma(\omega+1) - 2^{-7\omega+1}\Gamma(\omega+1) \\ & - 2^{-5\omega+1}\Gamma(\omega+1) - 2^{-6\omega}\Gamma(\omega+1) - (3)(2)^{-4\omega}\Gamma(\omega+1) \\ & - (3)(4)^{-\omega}\Gamma(\omega+1) - (8)^{-\omega}\Gamma(\omega+1) + \frac{3}{4}\Gamma(2\omega+1) \\ & +-(32)^{-4\omega}\Gamma(\omega+1) - (34)^{-\omega}\Gamma(\omega+1) - (8)^{-\omega}\Gamma(\omega+1),\end{aligned}$$
$$\begin{aligned}\hbar_{2,5} = {} & -4 - 2^{-8\omega+1} - 2^{-7\omega+1} - 2^{-5\omega+2} - 2^{-4\omega+1} - 3\times 2^{-3\omega+1} \\ & - 2^{-\omega+2} - 2^{-10\omega} + \frac{(4^\omega+1)\Gamma(\omega+1)}{2^{7\omega}} - \frac{7\Gamma(3\omega+1)}{3\times 2^{(4\omega+4)}} \\ & + 2\Gamma(\omega+1) - \frac{\Gamma(\omega+1)}{2^{9\omega+1}} - \frac{\Gamma(\omega+1)}{2^{6\omega-1}} - 2^{-4\omega}\Gamma(\omega+1) \\ & - 4^{-\omega}\Gamma(\omega+1) + (3)(8)^{-\omega}\Gamma(\omega+1) + \frac{5}{4}\Gamma(2\omega+1) \\ & + 5\times\frac{\Gamma(2\omega+1)}{2^{7\omega+3}} + \frac{\Gamma(2\omega+1)}{2^{4\omega}} + \frac{\Gamma(3\omega+1)}{24} \\ & + \frac{17\Gamma(4\omega+1)}{384}.\end{aligned} \tag{52}$$

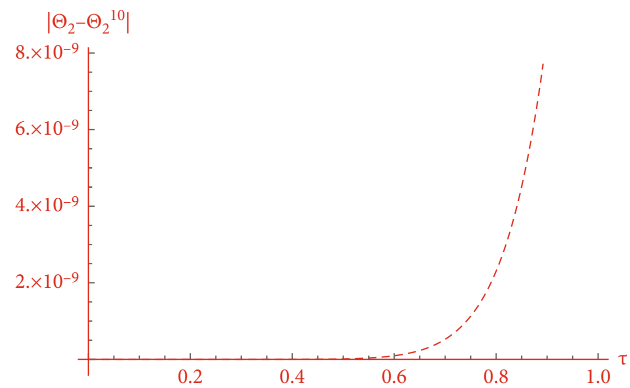

Figure 1

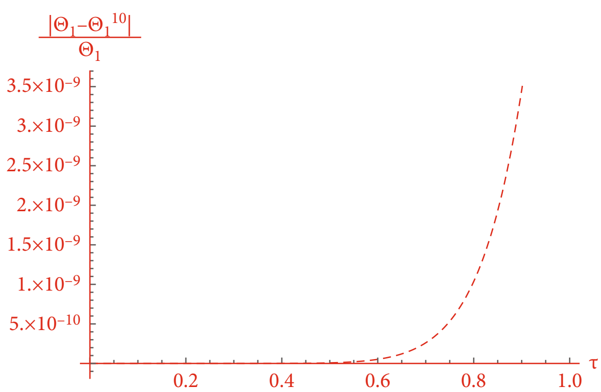

Figure 2

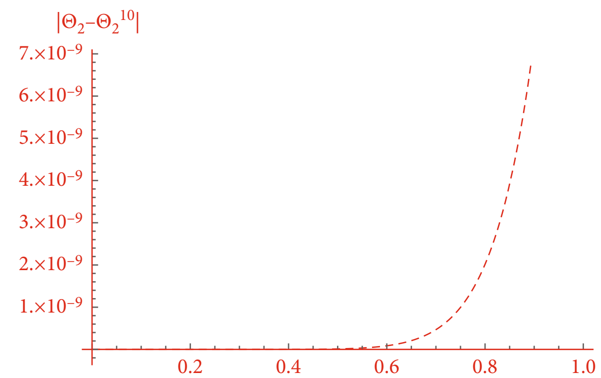

Figure 3

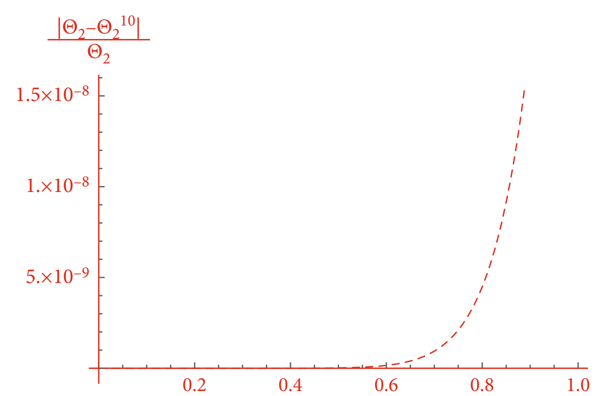

Figure 4

We have the following 5th-order approximate solution in the original space for Equations (40) and (41) when $\omega = 1$ uses the procedure mentioned in Subsection 2.1:

$$\Theta_{1,5}(\tau) = 1 + \tau + \frac{\tau^2}{2!} + \frac{\tau^3}{3!} + \frac{\tau^4}{4!} + \frac{\tau^5}{5!}, \tag{53}$$

$$\Theta_{2,5}(\tau) = 1 - \tau + \frac{\tau^2}{2!} - \frac{\tau^3}{3!} + \frac{\tau^4}{4!} - \frac{\tau^5}{5!}. \tag{54}$$

Equations (53) and (54) represent the first five terms of e^{τ} and $e^{-\tau}$, respectively, and therefore the exact solutions of Equations (40) and (41).

The following 2-D graphs show the absolute and relative error for Example 1.

Figures 1 and 2 demonstrate the 2-D graphs of absolute and relative errors in the intervals $\tau \in [0, 1]$ over the ten-step approximate and exact solutions of Equation (40) at $\omega = 1$, respectively. According to the figures, the approximate solution is extremely close to the precise solution. Figures 3 and 4 are graphs of absolute and relative errors in the intervals $\tau \in [0, 1]$ over the ten-step approximate and exact solutions of Equation (41) at $\omega = 1$, respectively. One can perceive the equivalent verdicts depicted for Equation (40).

Error functions are presented to observe the exactness and capability of the numerical method. To prove the exactness and capability of ERPSM, we selected three kinds of error functions, such as absolute, residual, and error functions.

Table 1 shows the absolute and relative errors at reasonable nominated grid points in the interval $\tau \in [0, 1]$ among the five-step approximate and exact solutions of Equations (40) and (41) at $\omega = 1$ attained using ERPSM. Table 1 shows that the approximate and exact solutions are quite close to each other, confirming the effectiveness of the recommended strategy.

Problem 6. Consider the following FMPS:

$$D_{\tau}^{\omega}\Theta_1(\tau) + \Theta_1(\tau) + e^{-\tau^{\omega}} \cos\left(\frac{\tau^{\omega}}{2}\right)\Theta_2\left(\frac{\tau}{2}\right) + 2e^{-(3/4)\tau^{\omega}} \cos \cdot \left(\frac{\tau^{\omega}}{2}\right) \sin\left(\frac{\tau^{\omega}}{4}\right)\Theta_1\left(\frac{\tau}{4}\right) = 0, \tag{55}$$

$$D_{\tau}^{\omega}\Theta_2(\tau) - e^{\tau^{\omega}}\Theta_1^2\left(\frac{\tau}{2}\right) + \Theta_2^2\left(\frac{\tau}{2}\right) = 0, \tag{56}$$

with the initial conditions $\Theta_1(0) = 1, \Theta_2(0) = 0$.

Table 1: The absolute and relative error of Example 1.

τ	Θ_1 Abs.error	Θ_1 Rel.error	Θ_2 Abs.error	Θ_2 Rel.error
0.2	$9.149350321813188 \times 10^{-8}$	$7.490854479152386 \times 10^{-8}$	$8.641131510334077 \times 10^{-8}$	$1.055430186034679 \times 10^{-7}$
0.4	$6.030974603721262 \times 10^{-6}$	$4.042683174006208 \times 10^{-6}$	$5.379368972713294 \times 10^{-6}$	$8.025075491218842 \times 10^{-6}$
0.6	$7.080039050877396 \times 10^{-5}$	$3.885607815121622 \times 10^{-5}$	$5.963609402637182 \times 10^{-5}$	$1.086640481073082 \times 10^{-4}$
0.8	$4.102618258010615 \times 10^{-4}$	$1.843425212040309 \times 10^{-4}$	$3.262974505549021 \times 10^{-4}$	$7.261883310726820 \times 10^{-4}$
1.0	$1.615161333333059 \times 10^{-3}$	$5.941846488086295 \times 10^{-4}$	$1.212774566900720 \times 10^{-3}$	$3.296663066666799 \times 10^{-3}$

Applying the E-T to Equations (55) and (56), we get

$$E\left[D_\tau^\omega \Theta_1(\tau) + \Theta_1(\tau) + e^{-\tau^\omega} \cos\left(\frac{\tau^\omega}{2}\right)\Theta_2\left(\frac{\tau}{2}\right) + 2e^{-(3/4)\tau^\omega} \cos \cdot \left(\frac{\tau^\omega}{2}\right) \sin\left(\frac{\tau^\omega}{4}\right)\Theta_1\left(\frac{\tau}{4}\right)\right] = 0, \tag{57}$$

$$E\left[D_\tau^\omega \Theta_2(\tau) - e^{\tau^\omega}\Theta_1^2\left(\frac{\tau}{2}\right) + \Theta_2^2\left(\frac{\tau}{2}\right)\right] = 0. \tag{58}$$

Using the approach mentioned in Subsection 2.1, we obtain the following results from Equations (57) and (58) as

$$\begin{aligned}\wp_1(\upsilon) = {} & \upsilon^2\Theta_1(0) - \upsilon^\omega \wp_1(\upsilon) \\ & - \upsilon^\omega E\left[E^{-1}\left[\frac{\upsilon^2}{1+\upsilon}\right]E^{-1}\left[\frac{4\upsilon^2}{4+\upsilon^2}\right]E^{-1}\left[4\wp_2\left(\frac{\upsilon}{2}\right)\right]\right] \\ & - 2\upsilon^\omega E\left[E^{-1}\left[\frac{4\upsilon^2}{4+3\upsilon}\right]E^{-1}\left[\frac{4\upsilon^2}{4+\upsilon^2}\right]E^{-1}\left[\frac{4\upsilon^3}{16+\upsilon^2}\right] \cdot E^{-1}\left[16\wp_1\left(\frac{\upsilon}{4}\right)\right]\right],\end{aligned} \tag{59}$$

$$\begin{aligned}\wp_2(\upsilon) = {} & \upsilon^2\Theta_2(0) + \upsilon^\omega E\left[E^{-1}\left[\frac{\upsilon^2}{1-\upsilon}\right]\left(E^{-1}\left[4\wp_1\left(\frac{\upsilon}{2}\right)\right]\right)^2 \right. \\ & \left. - \left(E^{-1}\left[4\wp_2\left(\frac{\upsilon}{2}\right)\right]\right)^2\right].\end{aligned} \tag{60}$$

Assume that Equations (59) and (60) have the series solution in the following form:

$$\begin{aligned}\wp_1(\upsilon) &= \sum_{\nu=0}^{\infty} \hbar_{1,\nu}\upsilon^{\nu\omega+2}, \\ \wp_2(\upsilon) &= \sum_{\nu=0}^{\infty} \hbar_{2,\nu}\upsilon^{\nu\omega+2}.\end{aligned} \tag{61}$$

The κth-truncated expansions are as

$$\begin{aligned}\wp_{1,\kappa}(\upsilon) &= \sum_{\nu=0}^{\kappa} \hbar_{1,\nu}\upsilon^{\nu\omega+2}, \\ \wp_{2,\kappa}(\upsilon) &= \sum_{\nu=0}^{\kappa} \hbar_{2,\nu}\upsilon^{\nu\omega+2}.\end{aligned} \tag{62}$$

By using the first part of Lemma 2, the κth-truncated expansions become

$$\begin{aligned}\wp_{1,\kappa}(\upsilon) &= \upsilon^2 + \sum_{\nu=1}^{\kappa} \hbar_{1,\nu}\upsilon^{\nu\omega+2}, \\ \wp_{2,\kappa}(\upsilon) &= \upsilon^2 + \sum_{\nu=1}^{\kappa} \hbar_{2,\nu}\upsilon^{\nu\omega+2}.\end{aligned} \tag{63}$$

The ERFs are formulated as

$$\begin{aligned}E\mathrm{Res}_1(\upsilon) = {} & \wp_1(\upsilon) - \upsilon^2 + \upsilon^\omega \wp_1(\upsilon) \\ & + \upsilon^\omega E\left[E^{-1}\left[\frac{\upsilon^2}{1+\upsilon}\right]E^{-1}\left[\frac{4\upsilon^2}{4+\upsilon^2}E^{-1}\left[4\wp_2\left(\frac{\upsilon}{2}\right)\right]\right]\right. \\ & + 2\upsilon^\omega E\left[E^{-1}\left[\frac{4\upsilon^2}{4+3\upsilon}\right]E^{-1}\left[\frac{4\upsilon^2}{4+\upsilon^2}\right]E^{-1}\left[\frac{4\upsilon^3}{16+\upsilon^2}\right] \cdot E^{-1}\left[16\wp_1\left(\frac{\upsilon}{4}\right)\right]\right], \\ E\mathrm{Res}_2(\upsilon) = {} & \wp_2(\upsilon) - \upsilon^\omega E\left[E^{-1}\left[\frac{\upsilon^2}{1-\upsilon}\right]\left(E^{-1}\left[4\wp_1\left(\frac{\upsilon}{2}\right)\right]\right)^2 \right. \\ & \left. + \left(E^{-1}\left[4\wp_2\left(\frac{\upsilon}{2}\right)\right]\right)^2\right].\end{aligned} \tag{64}$$

The κth-truncated ERF takes the following form:

$$\begin{aligned}E\mathrm{Res}_{1,\kappa}(\upsilon) = {} & \wp_{1,\kappa}(\upsilon) - \upsilon^2 + \upsilon^\omega \wp_{1,\kappa}(\upsilon) \\ & + \upsilon^\omega E\left[E^{-1}\left[\frac{\upsilon^2}{1+\upsilon}\right]E^{-1}\left[\frac{4\upsilon^2}{4+\upsilon^2}\right]E^{-1}\left[4\wp_{2,\kappa}\left(\frac{\upsilon}{2}\right)\right]\right] \\ & + 2\upsilon^\omega E\left[E^{-1}\left[\frac{4\upsilon^2}{4+3\upsilon}\right]E^{-1}\left[\frac{4\upsilon^2}{4+\upsilon^2}\right]E^{-1}\left[\frac{4\upsilon^3}{16+\upsilon^2}\right] \cdot E^{-1}\left[16\wp_{1,\kappa}\left(\frac{\upsilon}{4}\right)\right]\right],\end{aligned} \tag{65}$$

$$E\mathrm{Res}_{2,\kappa}(\upsilon)=\wp_2,\kappa(\upsilon)-\upsilon^{\omega}E\left[\mathrm{E}^{-1}\left[\frac{\upsilon^2}{1-\upsilon}\right]\left(E^{-1}\left[4\wp_{1,\kappa}\left(\frac{\upsilon}{2}\right)\right]\right)^2\right]+\left(E^{-1}\left[4\wp_{2,\kappa}\left(\frac{\upsilon}{2}\right)\right]\right)^2. \tag{66}$$

Substitute $\kappa=1,2,3,4,5$ into Equations (65) and (66) and solve the following expression to find the unknown coefficients.

$$\begin{aligned}&\lim_{\upsilon\longrightarrow 0}\left(\frac{1}{\upsilon^{k\omega+2}}E\ \mathrm{Re}\ s_{1,\kappa}(\upsilon)\right)=0,\\&\lim_{\upsilon\longrightarrow 0}\left(\frac{1}{\upsilon^{k\omega+2}}E\ \mathrm{Re}\ s_{2,\kappa}(\upsilon)\right)=0.\end{aligned} \tag{67}$$

Thus, we have

$$\hbar_{1,1}=-1,$$

$$\hbar_{2,1}=1,$$

$$\hbar_{1,2}=1-\frac{1}{(2)^{\omega}}-\frac{\Gamma(\omega+1)}{2},$$

$$\hbar_{2,2}=-\frac{1}{(2)^{\omega-1}}+\Gamma(\omega+1),$$

$$\begin{aligned}\hbar_{1,3}=&\frac{3}{8}\Gamma(2\omega+1)+\frac{2^{-1}\left(2^{\omega+1}+1\right)\Gamma((1/2)+\omega)}{\sqrt{\pi}}\\&+\Gamma(\omega+1)\left(\frac{1}{2}-\frac{1}{2^{2\omega}}\right)-1+2^{1-3\omega}+2^{-\omega},\end{aligned}$$

$$\begin{aligned}\hbar_{2,3}=&\frac{1}{8^{\omega}}\left(-2+2^{\omega+1}-\frac{2^{4\omega+1}\Gamma((1/2)+\omega)}{\sqrt{\pi}}-2^{\omega}\Gamma(\omega+1)\right)\\&+\frac{1}{2}\Gamma(2\omega+1),\end{aligned}$$

$$\begin{aligned}\hbar_{1,4}=&1+2^{-6\omega+1}-2^{-5\omega+1}-2^{-3\omega+1}-2^{-\omega}\\&-\frac{1}{2}\Gamma(\omega+1)-\frac{1}{2\times 8^{\omega}}\Gamma(2\omega+1)\\&-\frac{2^{-2\omega-1}\left(4^{\omega}+2^{3\omega+1}-4\right)\Gamma((1/2)+\omega)}{\sqrt{\pi}}-\frac{3}{8}\Gamma(2\omega+1)\\&-\frac{7}{96}\Gamma(3\omega+1)+\frac{\sqrt{\pi}\Gamma(3\omega+1)}{4^{3\omega+1}\Gamma(1/2+\omega)}\\&-\frac{3\Gamma(3\omega+1)(2^{\omega}+1)}{2^{2\omega+3}\Gamma(\omega+1)}-\frac{\Gamma(3\omega+1)\left(4^{\omega+1}+2^{\omega}-1\right)}{2^{5\omega+1}\Gamma(2\omega+1)}\\&+\frac{\Gamma(\omega+1)((8^{\omega}+1)\Gamma(2\omega+1)+8^{\omega}\Gamma(3\omega+1))}{2^{5\omega}\Gamma(2\omega+1)},\end{aligned}$$

$$\begin{aligned}\hbar_{2,4}=&-\frac{\Gamma(3\omega+1)(2^{\omega}-3)}{2^{4\omega-1}\Gamma(2\omega+1)\Gamma(\omega+1)}\\&-\frac{\Gamma(3\omega+1)\left(-2^{\omega+1}+3+2^{\omega}\Gamma(\omega+1)\right)}{8^{\omega}\Gamma(2\omega+1)}\\&+\frac{\Gamma(3\omega+1)}{4^{\omega}\Gamma(\omega+1)\Gamma(\omega+1)}+\frac{\Gamma(3\omega+1)}{6}+\frac{3\Gamma(2\omega+1)}{2^{3\omega+2}}\\&+\frac{\Gamma(\omega+1)}{8^{\omega}}-\frac{\Gamma(3\omega+1)}{2^{\omega}\Gamma(\omega+1)}+\frac{\left(2^{\omega+1}+1\right)\Gamma((1/2)+\omega)}{8^{\omega}\sqrt{\pi}}\\&-\frac{\Gamma(\omega+1)}{2^{5\omega-1}}+\frac{\left(2+4^{\omega}-8^{\omega}\right)}{2^{6\omega-1}}.\end{aligned} \tag{68}$$

We have the following 5th-order approximate solution in the original space for Equations (55) and (56) when $\omega=1$ using the procedure mentioned in Subsection 2.1:

$$\Theta_{1,5}(\tau)=1-\tau+\frac{\tau^3}{3}-\frac{\tau^4}{6}+\frac{\tau^5}{30}, \tag{69}$$

$$\Theta_{2,5}(\tau)=\tau-\frac{\tau^3}{6}+\frac{\tau^5}{120}. \tag{70}$$

Equations (69) and (70) represent the first five terms of $e^{-\tau}\cos\tau$ and $\sin\tau$, respectively, and therefore the exact solutions of Equations (55) and (56).

In the next section, we will use our new ERPSM to find approximate and exact solutions to linear PDE.

3. The ERPSM to Demonstrate the Linear PDE

In this section, we use our new ERPSM to construct the solutions of the linear PDE as in Equation (7). In the next subsection, we derive the main algorithms of the ERPSM for the linear PDE.

3.1. The Algorithm of ERPSM for Solving Linear PDE. In this subsection, we exploit ERPSM to create the solution to the linear PDE. We start by taking E-T on Equation (7). We get the following:

$$\begin{aligned}E[D_{\tau}^{\omega}\Theta(\tau)]=&lE[\Theta(\tau)]+\sum_{j=1}^{s}m_jE\left[\Theta\left(u_j\tau\right)\right]\\&+\sum_{j=1}^{t}p_jE\left[D_{\tau}^{\omega}\Theta\left(q_j\tau\right)\right]+E[g(\tau)].\end{aligned} \tag{71}$$

By using the second part of Lemma 2, Equation (71) becomes

$$\begin{aligned}\wp(\upsilon)=&\upsilon^2\phi+\upsilon^{\omega}l\wp(\upsilon)+\upsilon^{\omega}\sum_{j=1}^{s}m_i\frac{\wp\left(u_j\upsilon\right)}{u_i^{\,2}}\\&+\sum_{j=1}^{t}p_j\left(\frac{\wp\left(q_j\upsilon\right)}{q_j^{\,2}}-\upsilon^2\phi\right)+\upsilon^{\omega}G(\upsilon).\end{aligned} \tag{72}$$

Assuming that algebraic equation (72) has the solution in the expansion form as

$$\wp(\upsilon) = \sum_{\nu=0}^{\infty} \hbar_{\nu} \upsilon^{\nu\omega+2}. \tag{73}$$

The κth-truncated series of $\wp(\upsilon)$ is as follows:

$$\wp_{\kappa}(\upsilon) = \sum_{\nu=0}^{\kappa} \hbar_{\nu} \upsilon^{\nu\omega+2}. \tag{74}$$

By using the first part of Lemma 2, we have the following:

$$\hbar_0 = \phi. \tag{75}$$

The κth-truncated series becomes

$$\wp_k(\upsilon) = \phi\upsilon^2 + \sum_{\nu=1}^{k} \hbar_{\nu} \upsilon^{\nu\omega+2}. \tag{76}$$

Now, define the ERF in the following form:

$$\begin{aligned} E\text{Res}(\upsilon) &= \wp(\upsilon) - \upsilon^2\phi - \upsilon^{\omega} l\wp(\upsilon) - \upsilon^{\omega} \sum_{j=1}^{s} m_j \frac{\wp(u_j\upsilon)}{u_j^{\,2}} \\ &\quad - \sum_{j=1}^{t} p_j \left(\frac{\wp\left(q_j\upsilon\right)}{q_j^{\,2}} - \upsilon^2\phi \right) - \upsilon^{\omega} G(\upsilon). \end{aligned} \tag{77}$$

The κth-ERF is as follows:

$$\begin{aligned} E\text{Res}_k(\upsilon) &= \wp_k(\upsilon) - \upsilon^2\phi - \upsilon^{\varpi} l\wp_k(\upsilon) - \upsilon^{\varpi} \sum_{j=1}^{s} m_i \frac{\wp_k(u_j\upsilon)}{u_j^{\,2}} \\ &\quad - \sum_{j=1}^{t} p_j \left(\frac{\wp_k\left(q_j\upsilon\right)}{q_j^{\,2}} - \upsilon^2\phi \right) - \upsilon^{\varpi} G(\upsilon). \end{aligned} \tag{78}$$

To highlight basic points, we generalize certain features that arise in the RPSM [67–69].

It is understandable that

$$E\text{Res}(\upsilon) = 0. \tag{79}$$

Therefore,

$$\lim_{\kappa \longrightarrow \infty} E\text{Res}_k(\upsilon) = E \text{ Re } s(\upsilon). \tag{80}$$

Since, $\lim_{\upsilon \longrightarrow 0} (1/\upsilon^2) E\text{Res}(\upsilon) = 0$. It is obvious that $\lim_{\upsilon \longrightarrow 0} (1/\upsilon^2) E\text{Res}_k(\upsilon) = 0$. As a result, we get the following:

$$\begin{aligned} &\lim_{\upsilon \longrightarrow 0} \left(\frac{1}{\upsilon^{k\omega+2}} E \text{ Re } s(\upsilon) \right) \\ &\quad = \lim_{\upsilon \longrightarrow 0} \left(\frac{1}{\upsilon^{k\omega+2}} E \text{ Re } s_k(\upsilon) \right) = 0, \quad k = 1, 2, 3, \cdots. \end{aligned} \tag{81}$$

To determine the first undefined coefficient $\hbar_1$, substitute $k = 1$, in Equations (76) and (78). As a result, we obtain as following:

$$\begin{aligned} E\text{Res}_1(\upsilon) &= \wp_1(\upsilon) - \upsilon^2\phi - \upsilon^{\omega} l\wp_1(\upsilon) - \upsilon^{\omega} \sum_{j=1}^{s} m_j \frac{\wp_1(u_j\upsilon)}{u_j^{\,2}} \\ &\quad - \sum_{j=1}^{s} p_j \left(\frac{\wp_1\left(q_j\upsilon\right)}{q_j^{\,2}} - \upsilon^2\phi \right) - \upsilon^{\omega} G(\upsilon), \end{aligned} \tag{82}$$

$$\wp_1(\upsilon) = \phi\upsilon^2 + \hbar_1 \upsilon^{2+\omega}, \tag{83}$$

$$\wp_1(u_j\upsilon) = \phi u_j^2 \upsilon^2 + \hbar_1 u_j^{\,\omega+2} \upsilon^{\omega+2}, \tag{84}$$

$$\wp_1\left(q_j\upsilon\right) = \phi q_j^2 \upsilon^2 + \hbar_1 q_j^{\,\omega+2} \upsilon^{\omega+2}. \tag{85}$$

Using Equations (83), (84), and (85) in (82), we get the following:

$$\begin{aligned} E \text{ Re } s_1(\upsilon) &= \left(\phi\upsilon^2 + \hbar_1\upsilon^{\omega+2}\right) - \upsilon^2\phi - \upsilon^{\omega} l\left(\phi\upsilon^2 + \hbar_1\upsilon^{\omega+2}\right) \\ &\quad - \upsilon^{\omega} \sum_{j=1}^{s} m_i \frac{\phi u_j^2 \upsilon^2 + \hbar_1 u_j^{\,\omega+2} \upsilon^{\omega+2}}{u_j^{\,2}} \\ &\quad - \sum_{j=1}^{t} p_j \left(\frac{\phi q_j^2 \upsilon^2 + \hbar_1 q_j^{\,\omega+2} \upsilon^{\omega+2}}{q_j^{\,2}} - \upsilon^2\phi \right) \\ &\quad - \upsilon^{\omega} G(\upsilon), \end{aligned} \tag{86}$$

$$\begin{aligned} E\text{Res}_1(\upsilon) &= \hbar_1\upsilon^{\omega+2} - \upsilon^{\omega} l\left(\phi\upsilon^2 + \hbar_1\upsilon^{\omega+2}\right) \\ &\quad - \upsilon^{\omega} \sum_{j=1}^{s} m_j \left(\phi\upsilon^2 + \hbar_1 u_j^{\,\omega} \upsilon^{\omega+2}\right) \\ &\quad - \sum_{j=1}^{t} p_j \left(\hbar_1 q_j^{\,\omega} \upsilon^{\omega+2}\right) - \upsilon^{\omega} G(\upsilon). \end{aligned} \tag{87}$$

Dividing $\upsilon^{2+\omega}$ by Equation (87), we get

$$\begin{aligned} \frac{1}{\upsilon^{2+\omega}} E\text{Res}_1(\upsilon) &= \hbar_1 \frac{1}{\upsilon^{2+\omega}} \upsilon^{\omega+2} - \frac{1}{\upsilon^{2+\omega}} \upsilon^{\omega} l\left(\phi\upsilon^2 + \hbar_1\upsilon^{\omega+2}\right) \\ &\quad - \frac{1}{\upsilon^{2+\omega}} \upsilon^{\omega} \sum_{j=1}^{s} m_j \left(\phi\upsilon^2 + \hbar_1 u_j^{\,\omega} \upsilon^{\omega+2}\right) \\ &\quad - \frac{1}{\upsilon^{2+\omega}} \sum_{j=1}^{t} p_j \left(\hbar_1 q_j^{\,\omega} \upsilon^{\omega+2}\right) - \frac{1}{\upsilon^{2+\omega}} \upsilon^{\omega} G(\upsilon). \end{aligned} \tag{88}$$

By using $\lim_{\upsilon\longrightarrow 0}$ to the Equation (88), we get

$$\lim_{\upsilon\longrightarrow 0}\frac{1}{\upsilon^{2+\omega}}E\mathrm{Res}_1(\upsilon)=\hbar_1-l\left(\phi+\hbar_1\lim_{\upsilon\longrightarrow 0}\upsilon^2\right)-\sum_{j=1}^{s}m_j\left(\phi+\hbar_1 u_j^{\omega}\lim_{\upsilon\longrightarrow 0}\upsilon^2\right)-\sum_{j=1}^{t}p_j\left(\hbar_1 q_j^{\omega}\right)-\lim_{\upsilon\longrightarrow 0}\frac{1}{\upsilon^2}G(\upsilon). \tag{89}$$

When we put $\lim_{\upsilon\longrightarrow 0}((1/\upsilon^{2+\omega})E\mathrm{Res}_1(\upsilon))=0$ into Equation (89), we get

$$\hbar_1=\frac{l\phi+\sum_{j=1}^{s}m_j\phi+g(0)}{1-\sum_{j=1}^{t}p_jq_j^{\omega}}. \tag{90}$$

In the same manner, to determine the second undefined coefficient, put $k=2$ in Equations (76) and (78); we get the following:

$$\wp_k(\upsilon)=\phi\upsilon^2+\hbar_1\upsilon^{\omega+2}+\hbar_2\upsilon^{2\omega+2}, \tag{91}$$

$$\wp_k\left(u_j\upsilon\right)=\phi u_j^2\upsilon^2+\hbar_1 u_j^{\omega+2}\upsilon^{\omega+2}+\hbar_2 u_j^{2\omega+2}\upsilon^{2\omega+2}, \tag{92}$$

$$\wp_k\left(q_j\upsilon\right)=\phi q_j^2\upsilon^2+\hbar_1 q_j^{\omega+2}\upsilon^{\omega+2}+\hbar_2 q_j^{2\omega+2}\upsilon^{2\omega+2}, \tag{93}$$

$$E\mathrm{Res}_2(\upsilon)=\wp_2(\upsilon)-\upsilon^2\phi-\upsilon^{\varpi}l\wp_2(\upsilon)-\upsilon^{\varpi}\sum_{j=1}^{s}m_i\frac{\wp_2\left(u_j\upsilon\right)}{u_j^{2}}-\sum_{j=1}^{t}p_j\left(\frac{\wp_2\left(q_j\upsilon\right)}{q_j^{2}}-\upsilon^2\phi\right)-\upsilon^{\varpi}G(\upsilon). \tag{94}$$

Using Equations (91), (92), and (93) in (94), we get the following:

$$\begin{aligned}E\mathrm{Res}_2(\upsilon)&=\hbar_1\upsilon^{\omega+2}+\hbar_2\upsilon^{2\omega+2}-\left(l\phi\upsilon^{2+\omega}+l\hbar_1\upsilon^{2\omega+2}+l\hbar_2\upsilon^{3\omega+2}\right)\\&\quad-\upsilon^{\omega}\sum_{j=1}^{s}m_j\frac{\phi u_j^{2}\upsilon^2+\hbar_1 u_j^{\varpi+2}\upsilon^{\varpi+2}+\hbar_2 u_j^{2\omega+2}\upsilon^{2\omega+2}}{\rho_i^{2}}\\&\quad-\sum_{j=1}^{t}p_j\left(\frac{\hbar_1 q_j^{\omega+2}\upsilon^{\omega+2}+\hbar_2 q_j^{2\omega+2}\upsilon^{2\omega+2}}{q_j^{2}}\right)\\&\quad-\upsilon^{\omega}G(\upsilon).\end{aligned} \tag{95}$$

Dividing $\upsilon^{2+2\omega}$ on Eq. (95), we have the following results:

$$\frac{1}{\upsilon^{2\omega+2}}E\mathrm{Res}_2(\upsilon)=\hbar_2-l\hbar_1-l\hbar_2\upsilon^{\omega}+\sum_{j=1}^{s}m_j\omega_1u_j^{\omega}+\sum_{j=1}^{s}m_j\hbar_2u_j^{2\omega}\upsilon^{\omega}+\sum_{j=1}^{t}p_j\hbar_2q_j^{2\omega}-\frac{1}{\upsilon^2}\left(\frac{1}{\upsilon^{\omega}}G(\upsilon)-g(0)\right). \tag{96}$$

Apply $\lim_{\upsilon\longrightarrow 0}$ to Equation (96), we get

$$\lim_{\upsilon\longrightarrow 0}\frac{1}{\upsilon^{2\omega+2}}E\mathrm{Res}_2(\upsilon)=\hbar_2-l\hbar_1-l\hbar_2\upsilon^{\omega}+\sum_{j=1}^{s}m_j\hbar_1u_j^{\omega}+\sum_{j=1}^{s}m_j\phi\hbar_2u_j^{2\omega}\lim_{\upsilon\longrightarrow 0}\upsilon^{\omega}+\sum_{j=1}^{t}p_j\hbar_2q_j^{2\omega}-\lim_{\upsilon\longrightarrow 0}\frac{1}{\upsilon^2}E[D_{\tau}^{\omega}g(\tau)]. \tag{97}$$

By using the second part of Lemma 2, we have

$$\frac{1}{\upsilon^{\omega}}G(\upsilon)-g(0)=E[D_{\tau}^{\omega}g(\tau)]. \tag{98}$$

Put $\lim_{\upsilon\longrightarrow 0}(1/\upsilon^{2\omega+2})E\mathrm{Res}_2(\upsilon)=0$ in Equation (97), and we have the second coefficient in the following form:

$$\hbar_2=\frac{l\hbar_1+\sum_{j=1}^{s}m_j\hbar_1u_j^{\omega}+(D_{\tau}^{\omega}g(\tau))(0)}{1-\sum_{j=1}^{t}p_jq_j^{2\omega}}. \tag{99}$$

Now, to define the third coefficient, put $k=3$ in Equations (76) and (78), and repeat the same steps. We get the following:

$$\begin{aligned}E\mathrm{Res}_3(\upsilon)&=\hbar_1\upsilon^{\omega+2}\left(1-\sum_{j=1}^{t}p_jq_j^{\omega}\right)+\hbar_2\upsilon^{2\omega+2}\left(1-\sum_{j=1}^{t}r_jq_j^{2\omega}\right)\\&\quad+\hbar_3\upsilon^{3\omega+2}-l\phi\upsilon^{2+\omega}-l\hbar_1\upsilon^{2\omega+2}-l\hbar_2\upsilon^{3\omega+2}\\&\quad-l\hbar_3\upsilon^{4\omega+2}-\phi\upsilon^{\omega+2}\sum_{j=1}^{s}m_j-\hbar_1\upsilon^{2\omega+2}\sum_{j=1}^{s}m_ju_j^{\omega}\\&\quad-\hbar_2\upsilon^{3\omega+2}\sum_{j=1}^{s}m_ju_j^{2\omega}-\hbar_3\upsilon^{4\omega+2}\sum_{j=1}^{s}m_ju_j^{3\omega}\\&\quad-\hbar_3\upsilon^{3\omega+2}\sum_{j=1}^{t}p_jq_j^{3\omega}-\upsilon^{\omega}G(\upsilon).\end{aligned} \tag{100}$$

Divide $\upsilon^{2+3\omega}$ by Equation (100).

$$\frac{1}{\upsilon^{3\omega+2}}\mathrm{ERes}_3(\upsilon)=\hbar_3-l\hbar_2-l\hbar_3\upsilon^{\omega}-\hbar_2\sum_{j=1}^{s}m_ju_j^{2\omega}-\hbar_3\upsilon^{\omega}\sum_{j=1}^{s}m_ju_j^{3\omega}-\hbar_3\sum_{j=1}^{t}p_jq_j^{3\omega}-\frac{1}{\upsilon^2}\cdot\left(\frac{1}{\upsilon^{2\omega}}G(\upsilon)-\frac{1}{\upsilon^{2\omega-2}}g(0)-\frac{1}{\upsilon^{\omega-2}}(D_\tau^{\omega}g(\tau))(0)\right), \tag{101}$$

By using $\lim_{\upsilon\longrightarrow 0}$ to Equation (101), we get the third part of Lemma 2 in the following form:

$$(D_\tau^{2\omega}g(\tau))(0)=\lim_{\upsilon\longrightarrow 0}\frac{1}{\upsilon^2}\left(\frac{1}{\upsilon^{2\omega}}G(\upsilon)-\frac{1}{\upsilon^{2\omega-2}}g(0)-\frac{1}{\upsilon^{\omega-2}}(D_\tau^{\omega}g(\tau))(0)\right). \tag{102}$$

Put $\lim_{\upsilon\longrightarrow 0}(1/\upsilon^{3\omega+2})ERes_3(\upsilon)=0$, in Equation (101); as a result, we get

$$\hbar_3=\frac{l\hbar_{(\nu-1)}+\sum_{j=1}^{s}\hbar_{(\nu-1)}m_ju_j^{(\nu-1)\omega}+\left(D_\tau^{(\nu-1)\omega}g(\tau)\right)(0)}{\left(1-\sum_{j=1}^{t}p_jq_j^{\nu\omega}\right)}. \tag{103}$$

As a consequence of the generalization, we acquire the following:

$$\hbar_\nu=\begin{cases}\hbar_0, & \nu=0,\\ \dfrac{l\hbar_{(\nu-1)}+\sum_{i=1}^{l}\hbar_{(\nu-1)}m_ju_j^{(\nu-1)\omega}+\left(D_\Upsilon^{(\nu-1)\omega}g(\tau)\right)(0)}{\left(1-\sum_{j=1}^{t}p_jq_j^{\nu\omega}\right)}, & \nu=1,2,3,\cdots.\end{cases} \tag{104}$$

The κth-approximate solution of Equation (72) is as follows:

$$\begin{aligned}\wp_k(\upsilon)=\hbar_0\upsilon^2&+\left(\frac{l\phi+\sum_{j=1}^{s}\beta m_j\phi+g(0)}{1-\sum_{j=1}^{t}p_jq_j^{\omega}}\right)\upsilon^{2+\omega}\\&+\left(\frac{l\hbar_1+\sum_{j=1}^{s}m_j\hbar_1u_j^{\omega}+(D_\tau^{\omega}g(\tau))(0)}{1-\sum_{j=1}^{t}p_jq_j^{2\omega}}\right)\upsilon^{2+2\omega}\\&+\left(\frac{l\hbar_2+\sum_{j=1}^{s}\hbar_2m_ju_j^{2\omega}+\left(D_\tau^{2\omega}g(\tau)\right)(0)}{\left(1-\sum_{j=1}^{t}p_jq_j^{3\omega}\right)}\right)\upsilon^{2+3\omega}+\cdots\\&+\frac{l\hbar_{(\kappa-1)}+\sum_{i=1}^{l}\hbar_{(\kappa-1)}m_ju_j^{(\kappa-1)\omega}+\left(D_\tau^{(\kappa-1)\omega}g(\tau)\right)(0)}{\left(1-\sum_{j=1}^{t}p_jq_j^{\kappa\omega}\right)}\upsilon^{2+\kappa\omega}.\end{aligned} \tag{105}$$

By taking inverse E-T on Equation (105), we get the κth-step approximate solution in the original space of Equation (7).

$$\begin{aligned}\Theta_\kappa(\tau)=\hbar_0&+\left(\frac{l\phi+\sum_{j=1}^{s}m_j\phi+g(0)}{1-\sum_{j=1}^{t}p_jq_j^{\omega}}\right)\frac{\tau^{\omega}}{\Gamma(\omega+1)}\\&+\left(\frac{l\hbar_1+\sum_{j=1}^{s}m_j\hbar_1u_j^{\omega}+(D_\tau^{\omega}g(\tau))(0)}{1-\sum_{j=1}^{t}p_jq_j^{2\omega}}\right)\frac{\tau^{2\omega}}{\Gamma(2\omega+1)}\\&+\left(\frac{\alpha\hbar_2+\sum_{j=1}^{s}\hbar_2m_ju_j^{2\omega}+\left(D_\tau^{2\omega}g(\tau)\right)(0)}{\left(1-\sum_{j=1}^{t}p_jq_j^{3\omega}\right)}\right)\frac{\tau^{3\omega}}{\Gamma(3\omega+1)}+\cdots\\&+\frac{l\hbar_{(\kappa-1)}+\sum_{i=1}^{l}\hbar_{(\kappa-1)}m_ju_j^{(\kappa-1)\omega}+\left(D_\tau^{(\kappa-1)\omega}g(\tau)\right)(0)}{\left(1-\sum_{j=1}^{t}p_jq_j^{\kappa\omega}\right)}\frac{\tau^{\kappa\omega}}{\Gamma(\kappa\omega+1)}.\end{aligned} \tag{106}$$

In the next subsection, one PDE problem is established to illustrate the performance and appropriateness of the suggested method.

3.2. Numerical Example. In this subsection, we provided a problem to validate the performance and effectiveness of ERPSM for linear PDE.

Example 1. Consider the following linear PDE:

$$D_\tau^{\omega}\Theta(\tau)=-\Theta(\tau)+\frac{1}{10}\Theta\left(\frac{4}{5}\tau\right)+\frac{1}{2}\left(\frac{5}{4}\right)^{\omega}D_\tau^{\omega}\Theta\left(\frac{4\tau}{5}\right)+\left(\frac{8\tau^{\omega}}{25}-\frac{1}{2}\right)e^{(-(4\tau^{\omega}/5))}+e^{-\tau^{\omega}},\quad 0<\omega\leq 1, \tag{107}$$

subject to the initial condition

$$\Theta(0)=0. \tag{108}$$

By comparing Equation (107) with (7), we get

$$\begin{aligned}\hbar_0&=0,\\ l&=-1,\\ g(\tau)&=\left(\frac{8\tau^{\omega}}{25}-\frac{1}{2}\right)e^{(-(4\tau^{\omega}/5))}+e^{-\tau^{\omega}},\\ t&=s=1,\\ m_1&=0.1,\\ u_1&=q_1=0.8,\\ p_1&=\frac{1}{2}\left(\frac{5}{4}\right)^{\omega}.\end{aligned} \tag{109}$$

The recurrence relation that defines the values of the series coefficients as a result of Equation (104) is

$$\hbar_\nu=\begin{cases}0, & \nu=0,\\ \dfrac{\left((1/10)(4/5)^{(\nu-1)\omega}-1\right)\hbar_{(\nu-1)}+\left(D_\tau^{(\nu-1)\omega}g\right)(0)}{1-(1/2)(5/4)^{\omega}(4/5)^{\nu\omega}}, & \nu=1,2,3,\cdots.\end{cases} \tag{110}$$

First of all, we will compute a few of the terms of $(D_\tau^{(\nu-1)\omega} g)(0)$.

Expand the following form in a few terms to accomplish this:

$$g(\tau) = \left(\frac{8\tau^\omega}{25} - \frac{1}{2}\right) e^{(-(4\tau^\omega/5))} + e^{-\tau^\omega},$$

$$\begin{aligned}
g(\tau) &= \frac{1}{2} - \frac{7\tau^\omega}{25} + \frac{21\tau^{2\omega}}{250} - \frac{27\tau^{3\omega}}{1250} + \frac{437\tau^{4\omega}}{75000} - \frac{113\tau^{5\omega}}{75000}, \\
D_\tau^\omega[g(\tau)] &= -\frac{7\Gamma(\omega+1)}{25} + \frac{21\tau^\omega\Gamma(2\omega+1)}{250\Gamma(\omega+1)} - \frac{27\tau^{2\omega}\Gamma(3\omega+1)}{1250\Gamma(2\omega+1)} \\
&\quad + \frac{437\tau^{3\omega}\Gamma(4\omega+1)}{75000\Gamma(3\omega+1)} - \frac{113\tau^{4\omega}\Gamma(5\omega+1)}{75000\Gamma(4\omega+1)}, \\
D_\tau^{2\omega}[g(\tau)] &= \frac{21\Gamma(2\omega+1)}{250} - \frac{27\tau^\omega\Gamma(3\omega+1)}{1250\Gamma(\omega+1)} \\
&\quad + \frac{437\tau^{2\omega}\Gamma(4\omega+1)}{75000\Gamma(2\omega+1)} - \frac{113\tau^{3\omega}\Gamma(5\omega+1)}{75000\Gamma(3\omega+1)}, \\
D_\tau^{3\omega}[g(\tau)] &= -\frac{27\Gamma(3\omega+1)}{1250} + \frac{437\tau^\omega\Gamma(4\omega+1)}{75000\Gamma(\omega+1)} \\
&\quad - \frac{113\tau^{2\omega}\Gamma(5\omega+1)}{75000\Gamma(2\omega+1)}, \\
D_\tau^{4\omega}[g(\tau)] &= \frac{437\Gamma(4\omega+1)}{75000} - \frac{113\tau^\omega\Gamma(5\omega+1)}{75000\Gamma(\omega+1)}, \\
D_\tau^{5\omega}[g(\tau)] &= -\frac{113\Gamma(5\omega+1)}{75000}.
\end{aligned} \tag{111}$$

Returning to Equation (110), we can easily find the values of $\hbar_\nu$, for $\nu = 1, 2, 3, 4, 5$, respectively, as

$$\begin{aligned}
\hbar_0 &= 0, \\
\hbar_1 &= 1, \\
\hbar_2 &= \frac{2(5)^\omega(25 + 7\Gamma(\omega+1)) - 5(4)^\omega}{25(4^\omega - 2(5)^\omega)}, \\
\hbar_3 &= \frac{(16 - 2(5)^{1+2\omega})(2(5)^\omega(5(4)^\omega - 2(5)^\omega(25 + 7\Gamma(\omega+1)))}{125(4^\omega - 2(5)^\omega)(16^\omega - 2(25)^\omega)} \\
&\quad + \frac{21\Gamma(2\omega+1)}{250 - 5^{3-2\omega}16^\omega}, \\
\hbar_4 &= \frac{27\Gamma(3\omega+1)}{625(64)^\omega(125)^{-\omega} - 1250} - \frac{(64^\omega - 10(125)^\omega)}{5(64)^\omega - 10(125)^\omega} \\
&\quad \times \left(\frac{((64)^\varpi - 10(100)^\varpi)(-5(4)^\varpi(5)^{-\varpi} + 50 + 14\Gamma(\varpi+1))}{125(2 - 4^\varpi(5)^{-\varpi})(64^\varpi - 2(100)^\varpi} \right. \\
&\quad \left. + \frac{21\Gamma(2\varpi+1)}{250 - 5^{3-2\varpi}16^\varpi}\right), \\
\hbar_5 &= \frac{-437\Gamma(4\omega+1)}{37500(256)^\omega(625)^{-\omega} - 75000} - \left(\frac{(256)^\omega - 10(625)^\omega}{5(256)^\omega - 10(625)^\omega}\right) \\
&\quad \times \left(\frac{27\Gamma(3\omega+1)}{625(64)^\omega(125)^{-\omega} - 1250} - \frac{(64^\omega - 10(125)^\omega)}{5(64)^\omega - 10(125)^\omega} \right. \\
&\quad \times \left(\frac{((64)^\omega - 10(100)^\omega)(-5(4)^\omega(5)^{-\omega} + 50 + 14\Gamma(\omega+1))}{125(2 - 4^\omega(5)^{-\omega})(64^\omega - 2(100)^\omega} \right. \\
&\quad \left.\left. + \frac{21\Gamma(2\omega+1)}{250 - 5^{3-2\omega}16^\omega}\right)\right).
\end{aligned} \tag{112}$$

For $\omega = 1$, we have the following:

$$\begin{aligned}
\hbar_0 &= 0, \\
\hbar_1 &= 1, \\
\hbar_2 &= -2, \\
\hbar_3 &= 3, \\
\hbar_4 &= -4, \\
\hbar_5 &= 5.
\end{aligned} \tag{113}$$

The 5th-term approximate solution is as follows:

$$\wp_5(\upsilon) = \upsilon^{2+1} - 2\upsilon^{2+2} + 3\upsilon^{2+3} - 4\upsilon^{2+4} + 5\upsilon^{2+5}. \tag{114}$$

By taking inverse E-T on both sides of Equation (114),

$$\Theta_5(\tau) = \tau - \tau^2 + \frac{\tau^3}{2} - \frac{\tau^4}{6} + \frac{\tau^5}{24}. \tag{115}$$

Equation (115) represents first five terms of $\tau e^{-\tau}$; therefore, the exact solution of Equation (107) is $\tau e^{-\tau}$ at $\omega = 1$.

The following 2-D graphs show the absolute and relative errors for Example 1.

Figures 5 and 6 demonstrate the 2-D graphs of absolute and relative errors in the intervals $\tau \in [0, 1]$ over the ten-step approximate and exact solutions of Equation (107) at $\omega = 1$, respectively. According to the figures, the approximate solution is extremely close to the exact solution.

Table 2 shows the absolute and relative errors at reasonable nominated grid points in the interval $\tau \in [0, 1]$ among the five-step approximate and exact solutions of Equation (107) at $\omega = 1$ attained using ERPSM. From Table 2, it can be perceived that the approximate solution is in eminent contract with the exact solution, and this sanctions the efficiency of the recommended method. The convergence of the approximate solution to the exact solution for Equation (107) has been shown numerically as in Table 3. From the obtained results, it is evident that the present technique is an effective and convenient algorithm to solve certain classes of fractional-order DEs with fewer calculations and iteration steps.

In the next section, we will use our new ERPSM to find approximate and exact solutions to nonlinear PDE.

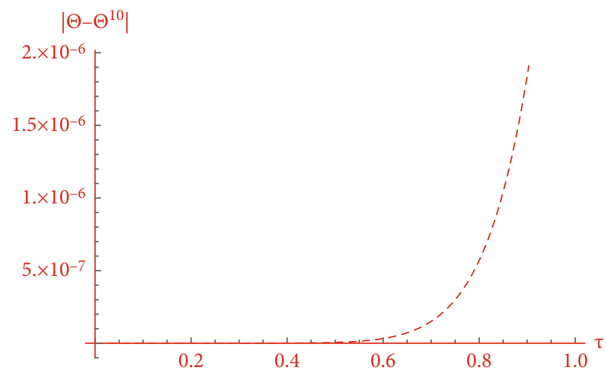

FIGURE 5

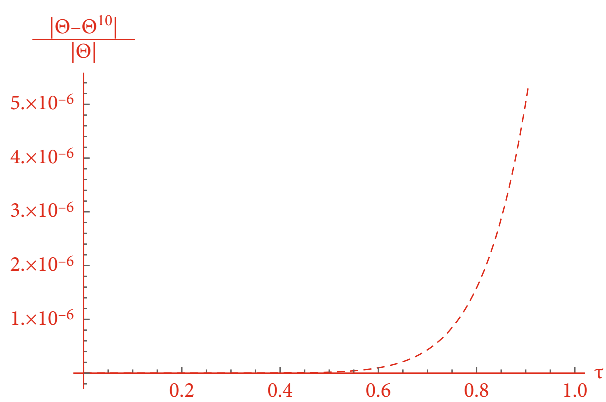

FIGURE 6

TABLE 2: The absolute and relative errors of Example 1.

τ	Abs.error	Rel.error
0.2	$5.1605107029284450 \times 10^{-7}$	$3.1515310030359390 \times 10^{-6}$
0.4	$3.1981585744245145 \times 10^{-5}$	$1.1927729870749216 \times 10^{-4}$
0.6	$3.5301834358408835 \times 10^{-4}$	$1.0720689345454725 \times 10^{-3}$
0.8	$1.9234953728893833 \times 10^{-3}$	$5.3510220976640050 \times 10^{-3}$
1.0	$7.1205587664326390 \times 10^{-3}$	$1.9355685668873270 \times 10^{-2}$

4. The ERPSM to Demonstrate the Nonlinear PDE

In this section, we use our new ERPSM to construct the solutions of the nonlinear PDE as in Equation (9). In the next subsection, we derive the main algorithms of the ERPSM for the nonlinear PDE.

4.1. The Algorithm of ERPSM for Solving Nonlinear PDE. We summarized the method of finding the ERPS solution for the nonlinear PDE with the following algorithm:

Step 1. Rewriting Equation (9), we have

$$D_\tau^\omega \Theta(\tau) - \xi(\tau^\omega, \Theta(\tau), \Theta(\bar{\lambda}\tau), D_\tau^\omega \Theta(\bar{\lambda}\tau)) = 0. \tag{116}$$

Step 2. E-T is used on both sides of Equation (116).

$$E[D_\tau^\omega \Theta(\tau)] - E[\xi(\tau^\omega, \Theta(\tau), \Theta(\bar{\lambda}\tau), D_\tau^\omega \Theta(\bar{\lambda}\tau))] = 0. \tag{117}$$

By using the second part of Lemma 2, we get the following:

$$\wp(\upsilon) - \upsilon^2 \hbar - \upsilon^\omega \wp(\upsilon) = 0, \tag{118}$$

where $E[\Theta(\tau)] = \wp(\upsilon)$ and $E[\xi(\tau^\omega, \Theta(\tau), \Theta(\bar{\lambda}\tau), D_\tau^\omega \Theta(\bar{\lambda}\tau))] = \wp(\upsilon)$.

Step 3. Assume that the solution to Equation (118) has the following extension:

$$\wp(\upsilon) = \sum_{\nu=0}^{\infty} \hbar_\nu \upsilon^{\nu\omega+2}. \tag{119}$$

Introduce the κth-truncated series in the following form:

$$\wp_k(\upsilon) = \sum_{\nu=0}^{k} \hbar_\nu \upsilon^{\nu\omega+2}. \tag{120}$$

Step 4. By using the first part of Lemma 2, we have

$$\hbar_0 = \Theta(0) = \mathfrak{I}. \tag{121}$$

Step 5. The κth-truncated series of $\wp(\upsilon)$ becomes as follows:

$$\wp_k(\upsilon) = \mathfrak{I}\upsilon^2 + \sum_{\nu=1}^{k} \hbar_\nu \upsilon^{\nu\omega+2}. \tag{122}$$

Step 6. Describe the ERF and the κth-ERF of Equation (118), respectively, as follows:

$$E\mathrm{Res}(\upsilon) = \wp(\upsilon) - \upsilon^2 \mathfrak{I} - \upsilon^\omega \wp(\upsilon), \tag{123}$$

$$E\mathrm{Res}_\kappa(\upsilon) = \wp_\kappa(\upsilon) - \upsilon^2 \mathfrak{I} - \upsilon^\omega \wp_\kappa(\upsilon). \tag{124}$$

Step 7. Replace the succession arrangement of $\wp_k(\upsilon)$ as in Equation (120) with (124).

Step 8. Solve the succeeding expression for $\nu = 1, 2, 3, \cdots, \kappa$ step by step to obtain the unknown coefficients

$$\lim_{\upsilon \longrightarrow 0} \left(\frac{1}{\upsilon^{\kappa\omega+2}} E[\mathrm{Res}_\kappa(\upsilon)] \right) = 0, \quad \kappa = 1, 2, 3, \cdots. \tag{125}$$

Step 9. Replace the attained values of $\hbar_\nu$ into the κth-truncated series of $\wp_\kappa(\upsilon)$ to obtain the κth-approximate solution of Equation (118).

Step 10. Use the inverse E-T on $\wp_k(\upsilon)$ to obtain the κth-approximate solution $\Theta_\kappa(\tau)$ in the original space.

In the next subsection, we determine the appropriateness of the recommended method for nonlinear PDFs.

Table 3: The recurrence errors $|\Theta^5(\tau)-\Theta^4(\tau)|$ of the five-step approximate solution with different values of ω for Example 1.

τ	$\omega=0.7$	$\omega=0.8$	$\omega=0.9$	$\omega=1.0$
0.2	$5.5251129381940380\times10^{-7}$	$3.557469322866979\times10^{-8}$	$2.210719076023441\times10^{-9}$	$1.337989787908603\times10^{-10}$
0.4	$6.2509517206696540\times10^{-6}$	$5.691950916587166\times10^{-7}$	$5.002286239854830\times10^{-8}$	$4.281567321307531\times10^{-9}$
0.6	$2.5838396081572233\times10^{-5}$	$2.881550151522252\times10^{-6}$	$3.101552986356802\times10^{-7}$	$3.251315184617906\times10^{-8}$
0.8	$7.0721445608883680\times10^{-5}$	$9.107121466539466\times10^{-6}$	$1.131888166924002\times10^{-7}$	$1.370101542818409\times10^{-7}$
1.0	$1.5443160141456830\times10^{-4}$	$2.223418326791861\times10^{-5}$	$3.089573833214964\times10^{-6}$	$4.181218087214387\times10^{-7}$

4.2. Numerical Examples. In this subsection, we will provide two problems with nonlinear PDEs to validate the performance and effectiveness of ERPSM.

Example 2. Examine the following nonlinear fractional pantograph differential equation:

$$D_\tau^\omega[\Theta(\tau]=1-2\Theta^2\left(\frac{\tau}{2}\right),\quad 1<\omega\le 2, 0\le\tau\le 1, \tag{126}$$

subject to the initial condition

$$\Theta(0)=1. \tag{127}$$

By employing E-T on Equation (126),

$$\wp(\upsilon)-\upsilon^2-\upsilon^{2+\omega}+2\upsilon^\omega E\left[\left(E^{-1}[4\wp(2\upsilon)]\right)^2\right]=0. \tag{128}$$

Assume that algebraic equation (128) has the solution in the expansion form as

$$\wp(\upsilon)=\sum_{\nu=0}^{\infty}\hbar_\nu \upsilon^{\nu\omega+2}. \tag{129}$$

The κth-truncated series of Equation (128) is as follows:

$$\wp_k(\upsilon)=\sum_{\nu=0}^{k}\hbar_\nu \upsilon^{\nu\omega+2}. \tag{130}$$

By using the first part of Lemma 2, we get

$$\hbar_0=\Theta(0)=1. \tag{131}$$

The κth-truncated series becomes as follows:

$$\wp_k(\upsilon)=\upsilon^2+\sum_{\nu=1}^{k}\hbar_\nu \upsilon^{\nu\omega+2}. \tag{132}$$

Now, define the ERF in the following form:

$$E[\mathrm{Res}(\upsilon)]=\wp(\upsilon)-\upsilon^2-\upsilon^{2+\omega}+2\upsilon^\omega E\left[\left(E^{-1}[4\wp(2\upsilon)]\right)^2\right]. \tag{133}$$

The κth-ERF is as follows:

$$E[\mathrm{Res}_\kappa(\upsilon)]=\wp_\kappa(\upsilon)-\upsilon^2-\upsilon^{2+\omega}+2\upsilon^\omega E\left[\left(E^{-1}[4\wp_\kappa(2\upsilon)]\right)^2\right]. \tag{134}$$

Therefore,

$$\lim_{\upsilon\longrightarrow 0}\frac{1}{\upsilon^{\kappa\omega+2}}E[\mathrm{Res}_\kappa(\upsilon)]=0. \tag{135}$$

To determine the undefined coefficients $\hbar_\nu$, substitute $\kappa=1,2,3,4,5,$ in Equations (132) and (134) and then utilize (135); thus, we have

$$\begin{aligned}
\hbar_1&=-1,\\
\hbar_2&=4(2)^{-\omega},\\
\hbar_3&=-\frac{2\Gamma((1/2)+\omega)}{\sqrt{\pi}\Gamma(1+\omega)}-\frac{16}{(8)^\omega},\\
\hbar_4&=\frac{8\sqrt{3}(27)^\omega\Gamma((2/3)+\omega)\Gamma((1/3)+\omega)}{(64)^\omega\sqrt{\pi}\Gamma(1+\omega)\Gamma((1/2)+\omega)}+\frac{64}{(64)^\omega}+\frac{8\Gamma((1/2)+\omega)}{(8)^\omega\sqrt{\pi}\Gamma(1+\omega)},\\
\hbar_5&=-\frac{128\sqrt{3}\Gamma((1/2)+\omega)\Gamma(1/2+2\omega)}{3(54)^\omega\Gamma((1/3)+\omega)\Gamma((2/3)+\omega)\Gamma(1+\omega)}-\frac{32\Gamma((1/2)+2\omega)}{(16)^\omega\Gamma((1/2)+\omega)\Gamma(1+\omega)}\\
&\quad-\frac{16\sqrt{3}(4)^\omega\Gamma((1/2)+\omega)^2\Gamma((1/2)+2\omega)}{3\sqrt{\pi}(27)^\omega\Gamma((1/3)+\omega)\Gamma((2/3)+\omega)\Gamma(1+\omega)^2}-\frac{256}{(1024)^\omega}-\frac{32\sqrt{3}(27)^\omega\Gamma((1/3)+\omega)\Gamma((2/3)+\omega)}{\sqrt{\pi}(1024)^\omega\Gamma((1/2)+\omega)\Gamma(1+\omega)}\\
&\quad-\frac{32\Gamma((1/2)+\omega)}{(128)^\omega\sqrt{\pi}\Gamma(1+\omega)}.
\end{aligned} \tag{136}$$

The five-step approximate solution of Equation (128) is as follows:

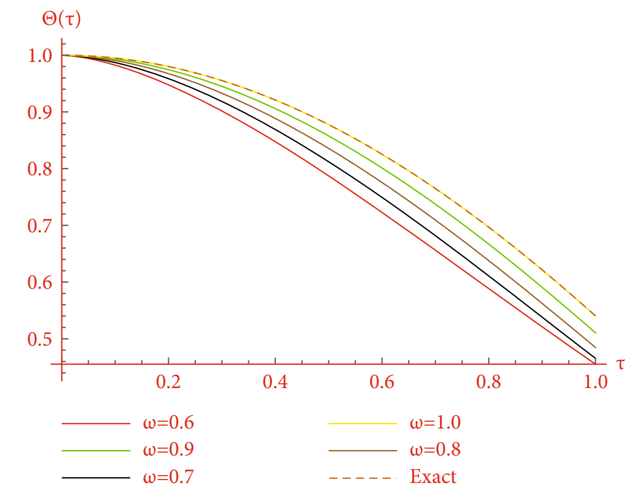

FIGURE 7

$$\wp_5(\upsilon) = \upsilon^2 - \upsilon^{2+\omega} + 4(2)^{-\omega}\upsilon^{2+2\omega} + \hbar_3\upsilon^{2+3\omega} + \hbar_4\upsilon^{2+4\omega} + \hbar_5\upsilon^{2+5\omega}. \tag{137}$$

By taking inverse E-T at both sides of Equation (137),

$$\Theta_5(\tau) = 1 - \frac{\tau^\omega}{\Gamma(1+\omega)} + 4(2)^{-\omega}\frac{\Upsilon^{2\omega}}{\Gamma(1+2\omega)} + \hbar_3\frac{\tau^{3\omega}}{\Gamma(1+3\omega)} + \hbar_4\frac{\tau^{4\omega}}{\Gamma(1+4\omega)} + \hbar_5\frac{\tau^{5\omega}}{\Gamma(1+5\omega)}. \tag{138}$$

For $\omega = 2$, Equation (138) becomes

$$\Theta_5(\tau) = 1 - \frac{\tau^2}{2!} + \frac{\tau^4}{4!} - \frac{\tau^6}{6!} + \frac{\tau^8}{8!} - \frac{\tau^{10}}{10!}. \tag{139}$$

Equation (139) represents the first six terms of $\cos\tau$; therefore, the exact solution of Equation (126) is $\cos\tau$ at $\omega = 2$.

The following 2-D graphs show the behavior of approximate and exact solutions to Example 2.

Figure 7 shows the performance of the five-step approximate solution of Equation (126) for various values of ω and exact solution at $\omega = 2$. Clearly, the findings for various fractional ω values converge to the result in the case of $\omega = 2$. Furthermore, the approximate solution does overlap with the exact solution, which indicates the accuracy and effectiveness of the suggested method.

The following 2-D graphs show the absolute and relative errors for Example 2.

Figures 8 and 9 demonstrate the 2-D graphs of absolute and relative errors in the intervals $\tau \in [0, 1]$ over the five-step approximate and exact solutions of Equation (126) at $\omega = 1$, respectively. The approximate solution is extremely close to the exact solution, as seen in the figures.

Table 4 shows the absolute and relative errors at reasonable nominated grid points in the interval $\tau \in [0, 1]$ among the five-step approximate and exact solutions of Equation (126) at $\omega = 1$ attained using ERPSM. From Table 4, it can be perceived that the approximate and exact solutions are in very good agreement, and this sanctions the efficiency of the recommended method. The convergence of the approximate solution to the exact solution for Equation (126) has been shown numerically as in Table 5. From the obtained results, it is evident that the present technique is an effective and convenient algorithm to solve certain classes of fractional-order DEs with fewer calculations and iteration steps.

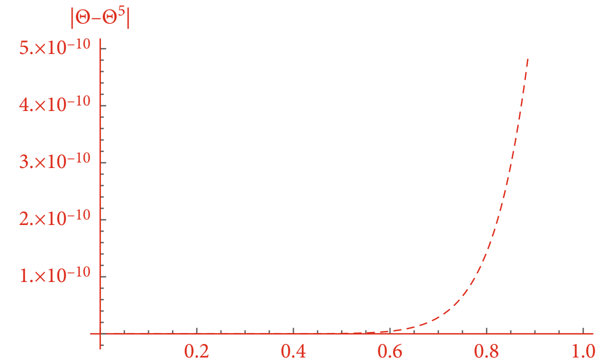

FIGURE 8

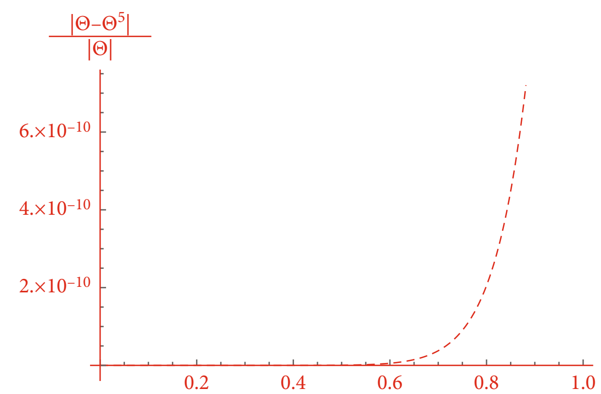

FIGURE 9

TABLE 4: The absolute and relative errors of Example 2.

τ	Abs.error	Rel.error
0.2	0.0	0.0
0.4	$3.508304757815494 \times 10^{-14}$	$3.808982011678268 \times 10^{-14}$
0.6	$4.535483100198689 \times 10^{-12}$	$5.495319744192834 \times 10^{-12}$
0.8	$1.429608653680247 \times 10^{-10}$	$2.051951896683681 \times 10^{-10}$
1.0	$7.120558766432639 \times 10^{-9}$	$2.051765518966810 \times 10^{-9}$

Example 3. Consider the following nonlinear PDE:

$$D_\tau^\omega\Theta(\tau) = \frac{1}{2}\Theta(\tau) + \frac{1}{2^{1-\omega}}\Theta\left(\frac{\tau}{2}\right)D_\tau^\omega\Theta\left(\frac{\tau}{2}\right), \quad \tau \geq 0, 0 < \omega \leq 1, \tag{140}$$

Table 5: The recurrence errors $|\Theta^5(\tau) - \Theta^4(\tau)|$ of the five-step approximate solution with different values of ω for Example 2.

τ	$\omega = 0.7$	$\omega = 0.8$	$\omega = 0.9$	$\omega = 1.0$
0.2	$6.296560270772086 \times 10^{-20}$	$1.969389500244771 \times 10^{-21}$	$5.554963006701661 \times 10^{-22}$	$2.821869488536155 \times 10^{-24}$
0.4	$2.279598318393725 \times 10^{-17}$	$1.008327424125322 \times 10^{-18}$	$4.022222859549857 \times 10^{-19}$	$2.889594356261023 \times 10^{-21}$
0.6	$7.155401866274551 \times 10^{-16}$	$3.876349353331781 \times 10^{-17}$	$1.893795546581747 \times 10^{-17}$	$1.666285714285714 \times 10^{-19}$
0.8	$8.253027478106390 \times 10^{-15}$	$5.162636411521651 \times 10^{-16}$	$2.912400444857598 \times 10^{-16}$	$2.958944620811287 \times 10^{-18}$
1.0	$5.499819058543307 \times 10^{-14}$	$3.846463867665569 \times 10^{-15}$	$2.426030253023737 \times 10^{-15}$	$2.755731922398591 \times 10^{-17}$

subject to the initial condition

$$\Theta(0) = 1. \tag{141}$$

By employing the E-T on Equation (140),

$$\wp(\upsilon) = \upsilon^2 + \frac{\upsilon^\omega}{2}\wp(\upsilon) + \frac{\upsilon^\omega}{2^{1-\omega}} E\left[E^{-1}\left[4\wp\left(\frac{\upsilon}{2}\right)\right]E^{-1}\left[\frac{4\wp(\upsilon/2)}{\upsilon^\omega} - \upsilon^{2-\omega}\right]\right]. \tag{142}$$

Assume that algebraic equation (142) has the solution in the expansion form as

$$\wp(\upsilon) = \sum_{\nu=0}^{\infty} \hbar_\nu \upsilon^{\nu\omega+2}. \tag{143}$$

The κth-truncated series of Equation (142) is as follows:

$$\wp_k(\upsilon) = \sum_{\nu=0}^{k} \hbar_\nu \upsilon^{\nu\omega+2}. \tag{144}$$

By employing the first part of Lemma 2, we get

$$\hbar_0 = \Theta(0) = 1. \tag{145}$$

The κth-truncated series of Equation (142) becomes as follows:

$$\wp_k(\upsilon) = \upsilon^2 + \sum_{\nu=1}^{k} \hbar_\nu \upsilon^{\nu\omega+2}. \tag{146}$$

Now, define the ERF in the following form:

$$E\ \mathrm{Re}\ s(\upsilon) = \wp(\upsilon) - \upsilon^2 - \frac{\upsilon^\omega}{2}\wp(\upsilon) - \frac{\upsilon^\omega}{2^{1-\omega}} \cdot E\left[E^{-1}\left[4\wp\left(\frac{\upsilon}{2}\right)\right]E^{-1}\left[\frac{4\wp(\upsilon/2)}{\upsilon^\omega} - \upsilon^{2-\omega}\right]\right]. \tag{147}$$

The κth-ERF is as follows:

$$E\mathrm{Res}_\kappa(\upsilon) = \wp_\kappa(\upsilon) - \upsilon^2 - \frac{\upsilon^\omega}{2}\wp_\kappa(\upsilon) - \frac{\upsilon^\omega}{2^{1-\omega}} \cdot E\left[E^{-1}\left[4\wp_\kappa\left(\frac{\upsilon}{2}\right)\right]E^{-1}\left[\frac{4\wp_\kappa(\upsilon/2)}{\upsilon^\omega} - \upsilon^{2-\omega}\right]\right]. \tag{148}$$

To determine the undefined coefficients $\hbar_\nu$, substitute $\kappa = 1, 2, 3, 4$ in Equations (146) and (148) and then solve the equation $\lim_{\upsilon \longrightarrow 0}(1/\upsilon^{\kappa\omega+2})E[\mathrm{Re}\ s_\kappa(\upsilon)] = 0$. By utilizing this algorithm, we get the following first four coefficients of the series Equation (146):

$$\hbar_1 = 1,$$

$$\hbar_2 = \frac{1 + 2^\omega}{-1 + 2^{\omega+1}},$$

$$\hbar_3 = -\frac{(8^\omega - 16^\omega - 2(32)^\omega)\Gamma((1/2)+\omega) + \sqrt{\pi}(2^\omega - 4^\omega - 8^\omega - 16^\omega - 2(32)^\omega)\Gamma(1+\omega)}{\sqrt{\pi}\Gamma(1+\omega)\left(8(32)^\omega + 2^{2\omega+2} - 2(8^\omega) - 8(16^\omega) - 2^\omega\right)},$$

$$\begin{aligned}\hbar_4 = {} & \frac{\sqrt{\pi}\left(2 - 2^{-\omega} - 2(4)^\omega + 4(2)^{2\omega} + 5(2)^\omega\right)\Gamma(1+3\omega)}{(2(2)^\omega - 1)^2(2(8)^\omega - 1)(2(4)^\omega - 1)\Gamma((1/2)+\omega)\Gamma(1+\omega)^2} \\ & + \frac{4^\omega(16^\omega + 8^\omega + 2^\omega + 1)\Gamma((1/2)+\omega)}{\sqrt{\pi}(2(2)^\omega - 1)(2(8)^\omega - 1)(2(4)^\omega - 1)\Gamma(1+\omega)} \\ & + \frac{(1 + 2(8)^\omega + 4^\omega + 3(2)^\omega + (64)^\omega + (16)^\omega + (2)^\omega))}{(2(2)^\omega - 1)(2(8)^\omega - 1)(2(4)^\omega - 1)} \\ & + \frac{(3 + 3(2)^\omega)\Gamma(3\omega)}{(2(2)^\omega - 1)(2(8)^\omega - 1)(2(4)^\omega - 1)\Gamma(1+\omega)^3}.\end{aligned} \tag{149}$$

The four-step approximate solution of Equation (142) is as follows:

$$\wp_4(\upsilon) = \upsilon^2 + \upsilon^{2+\omega} + \left(\frac{1 + 2^\omega}{1 - 2^{\omega+1}}\right)\upsilon^{2+2\omega} + \hbar_3\upsilon^{2+3\omega} + \hbar_4\upsilon^{2+4\omega}. \tag{150}$$

By employing inverse E-T on both sides of Equation (150),

Table 6: The absolute and relative errors of Example 3.

τ	Abs.error	Rel.error
0.2	$9.149350321813188 \times 10^{-8}$	$7.490854479152386 \times 10^{-8}$
0.4	$6.030974603721262 \times 10^{-6}$	$4.042683174006208 \times 10^{-6}$
0.6	$7.080039050877396 \times 10^{-5}$	$3.885607815121622 \times 10^{-5}$
0.8	$4.102618258010615 \times 10^{-4}$	$1.843425212040309 \times 10^{-4}$
1.0	$1.615161333333059 \times 10^{-3}$	$5.941846488086295 \times 10^{-4}$

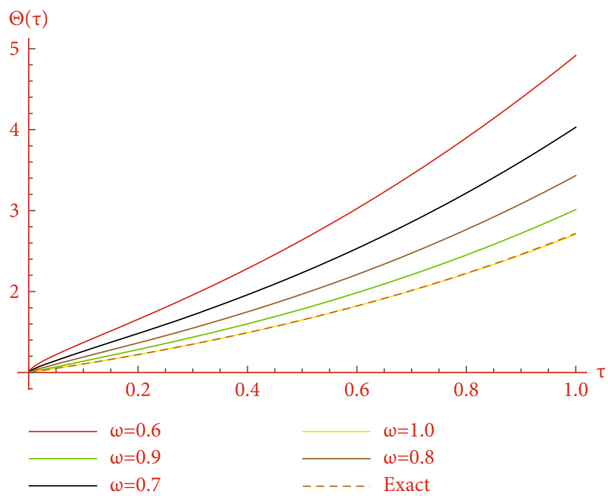

Figure 10

$$\Theta_4(\tau) = 1 + \frac{\tau^{\omega}}{\Gamma(1+\omega)} + \left(\frac{1+2^{\omega}}{1-2^{\omega+1}}\right)\frac{\tau^{2\omega}}{\Gamma(1+2\omega)} + \hbar_3 \frac{\tau^{3\omega}}{\Gamma(1+3\omega)} + \hbar_4 \frac{\tau^{4\omega}}{\Gamma(1+4\omega)}. \quad (151)$$

For$\omega = 1$in Equation (151), it becomes

$$\Theta_4(\tau) = 1 + \tau + \frac{\tau^2}{2!} + \frac{\tau^3}{3!} + \frac{\tau^4}{4!}. \quad (152)$$

Equation (152) represents the first five terms of e^{τ}; therefore, the exact solution of Equation (140) is e^{τ} at $\omega = 1$.

Table 6 shows the absolute and relative errors of the five-step approximate solution of the nonlinear PDE of Example 3 at $\omega = 1$ on various chosen grid points in the interval $\tau \in [0, 1]$. The numerical findings in Table 6 prove that the approximate solution converges to the exact solution quickly. This indicates the accuracy and effectiveness of the suggested scheme.

The following 2-D graphs show the behavior of approximate and exact solutions to Example 3.

Figure 10 exemplifies the performance of the five-step approximate solutions of Equation (140) for some values of ω and exact solution at $\omega = 1$. Apparently, results in cases of fractional values of ω converging result in the case of $\omega = 1$. Furthermore, the approximate result overlaps with the precise result at $\omega = 1$, and this once more agrees with the efficiency and precision of the recommended scheme.

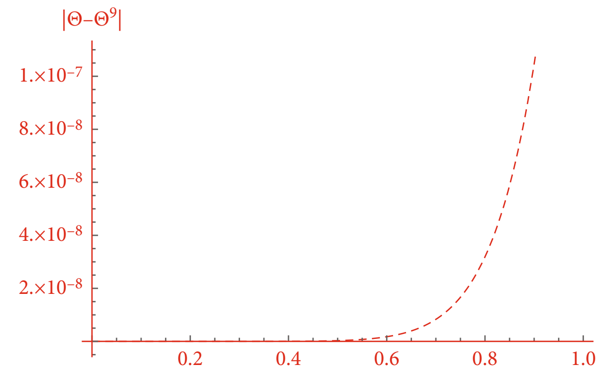

Figure 11

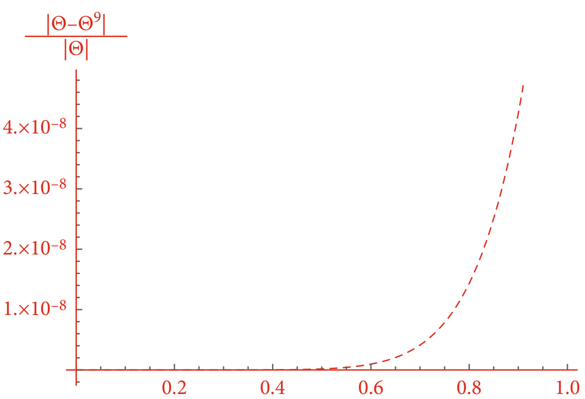

Figure 12

The following 2-D graphs show the absolute and relative errors for Example 3.

Figures 11 and 12 demonstrate the 2-D graphs of absolute and relative errors in the intervals $\tau \in [0, 1]$ over the nine-step approximate and exact solutions of Equation (140) at $\omega = 1$, respectively. The approximate solution is extremely close to the exact solution, as seen in the figures.

5. Conclusion

In this paper, we describe a novel method for solving FMPS and PDEs that combines the RPSM with the E-T. To assess the effectiveness and reliability of ERPSM for FMPS and PDEs, the absolute, relative, and recurrence errors of linear and nonlinear problems are studied graphically and numerically. The interpretation of the 2-D plots and tables for different values of fractional order also validates that the approximate solution is rapidly convergent to the exact solution. The numerical and graphical consequences confirm that the ERPSM is extremely effective and precise.

The ERPSM stands apart from other numerical methods in four major ways. This method has the advantage of requiring no minor or major physical parameter assumptions in the problem. As a result, it applies to both weakly and strongly nonlinear problems, overcoming some of the inherent limits of traditional perturbation approaches. Second, while addressing nonlinear problems, the ERPSM does not require He's polynomials or Adomian polynomials. To solve nonlinear DEs, only a very small number of calculations are needed. As a consequence, it outperforms homotopy analysis and Adomian decomposition methods significantly. Third, the ERPSM provided a simple and rapid way to observe the coefficients of the recommended series as a solution to the problem. In contrast to the traditional RPSM, establishing the coefficients for a series requires computing the fractional derivative every time, while the ERPSM only requires the concept of the limit at zero in establishing the coefficients for the series. Finally, unlike conventional analytic approximation techniques, the ERPSM can create expansion solutions for linear and nonlinear FODEs without the need for perturbation, linearization, or discretization.

Therefore, we conclude that our novel technique is simple to apply, accurate, adaptive, and efficient according to the results. It is significant to consider that implementing the ERPSM to solve other kinds of ordinary and partial DEs of noninteger order is actively attainable. For example, fractional KdV equations, fractional phi-4 equations, and fractional Schrodinger equations.

Data Availability

The numerical data used to support the findings of this study is included within the article.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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