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Retraction

Retracted: Generalized Intuitionistic Fuzzy Normalized Weighted Optimized Geometric Bonferroni Mean and Their Application to MADM

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope

(2) Discrepancies in the description of the research reported

(3) Discrepancies between the availability of data and the research described

(4) Inappropriate citations

(5) Incoherent, meaningless and/or irrelevant content included in the article

(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] J.-B. Zhang, T.-L. Sun, and M.-H. Shi, "Generalized Intuitionistic Fuzzy Normalized Weighted Optimized Geometric Bonferroni Mean and Their Application to MADM," *Journal of Function Spaces*, vol. 2022, Article ID 6375994, 17 pages, 2022.

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Research Article

Generalized Intuitionistic Fuzzy Normalized Weighted Optimized Geometric Bonferroni Mean and Their Application to MADM

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In an information age, people often need to face a lot of decision-making information when making decisions. Some indicators are on the high side and others are on the low side which is a common phenomenon in decision-making. So, it is difficult to make a correct and rational judgment. Long-term research has proved that information aggregation operator is an effective tool to solve this kind of problem. Bonferroni mean (BM) is an important information aggregation tool which has the main feature of capturing the interrelationships among aggregated arguments. Because the existing geometric Bonferroni mean (GBM) cannot reflect the two-layer average calculation and the weighted GBM do not feature reducibility, this paper develops the intuitionistic fuzzy normalized weighted optimized GBM (IFNWOGBM) and the generalized intuitionistic fuzzy normalized weighted optimized GBM (GIFNWOGBM) and also studies their desirable properties and special cases. In the end, based on the IFNWOGBM and GIFNWOGBM, a method to multiple attribute decision-making (MADM) problem is proposed. In order to verify the effectiveness of the method, it is used to select the location of the library.

1. Introduction

As our society develops, situations human encounter when making decisions is becoming more and more complex, and data and information we rely upon in these situations are highly vague and uncertain. Under these new circumstances, in order to better utilize decision-making information in modeling, scholars have extended the fuzzy set to many other forms, such as triangular fuzzy set, vague set, and intuitionistic fuzzy set. The flexibility and efficacy that the intuitionistic fuzzy set demonstrate during actual decision-makings have won public attention with related theories which further enriched, developed, and applied extensively in intelligent algorithm, graphics and image processing, and other fields [1–4].

Aggregation operator, as an important tool for information aggregation, has always been an academic focus of decision-making. To aggregate the intuitionistic fuzzy decision-making information, a large number of operators have been introduced. By analyzing the shortcomings of the existing weighted averaging (WA) operator, Kumar et al. proposed some improved WA operators and demonstrated their advantages in the field of intuitionistic fuzzy decision-making with a large number of decision-making cases [5]. In order to solve the problem of investment target selection represented by intuitionistic fuzzy information, Zou et al. improved the weighted geometric (WG) operator and proposed an effective method to solve this kind of problem [6]. Combined with the advantages of Choquet integral and arithmetic aggregation operator, Jia et al. proposed some novel operators to solve the intuitionistic fuzzy supplier selection decision-making problem [7].

The operators in the above literature gather information from the perspective of independent evaluation indexes, but in reality, most decision indexes have certain relevance. In order to overcome this shortcoming, Yanger presents a new fuzzy information aggregation operator, known as Bonferroni mean (BM), which can increase reliability of decisions made when data and information is highly correlated. Some scholars seek to replace the arithmetic average with geometric average in the BM so as to generate geometric Bonferroni mean (GBM). More commonly seen nowadays are the GBM and the weighted GBM defined by Xia et al. [8]. Mahmoodi et al. introduced some GBMs to aggregate linguistic Z-number decision-making information [9]. Devaraj and Broumi defined several neutrosophic cubic fuzzy GBMs and proposed a method to solve the financial risk decision-making problem [10]. Huang et al. presented some GBMs to solve a hesitant fuzzy uncertain linguistic MADM problem [11]. Park et al. further propose the optimal weighted GBM and generalized optimal weighted GBM [12]. However, this kind of geometric operators cannot reflect the two-layer average calculation which is the key feature of BM. Moreover, the weighted GBMs mentioned above fail to bear a common feature of classic weighted operators, reducibility. This means when weights are equal, they cannot degenerate back to geometric Bonferroni mean.

In order to get rid of the above shortcomings, based on the full analysis of the construction of geometric operators and BM operators, this paper proposes some improved GBM operators to deal with intuitionistic fuzzy MADM problems. This paper is organized as follows. In Section 2, we review some necessary concepts and operators. In Section 3, we defined IFONWGBM and GIFONWGBM and discussed their properties and special cases. In Section 4, an example about location of the library is used to demonstrate the application of IFONWGBM and GIFONWGBM in MADM. In Section 5, we present the comparative analysis with other MADM methods. Finally, Section 6 gives the summary of the operators and methods proposed in this paper.

2. Preliminaries

2.1. Some Intuitionistic Fuzzy Concepts

Definition 1 (see [13, 14]). Let X be a fixed set. Then, an intuitionistic fuzzy set on X can be defined as

$$A = \{\langle x, \mu_A(x), v_A(x)\rangle x| \in X\}, \tag{1}$$

where $\mu_A(x)$: $X \longrightarrow [0,1]$ and $v_A(x)$: $X \longrightarrow [0,1]$ satisfy the condition $0 \le \mu_A(x) + v_A(x) \le 1$, $\forall x \in X$, and $\mu_A(x)$ and $v_A(x)$ represent the membership degree and the non-membership degree of x to A.

In order to facilitate discussion, Xu calls the pair $\alpha = (\mu_\alpha, v_\alpha)$ an intuitionistic fuzzy number (IFN) with the conditions:

$$\begin{aligned} &\mu_\alpha \in [0,1],\\ &v_\alpha \in [0,1],\\ &\mu_\alpha + v_\alpha \in [0,1]. \end{aligned} \tag{2}$$

Let $\alpha_i = (\mu_{\alpha_i}, v_{\alpha_i})(i = 1,2)$ and $\alpha = (\mu_\alpha, v_\alpha)$ be three IFNs. Xu et al. defined the following operation laws [4]:

(1) $\alpha_1 \oplus \alpha_2 = (\mu_{\alpha_1} + \mu_{\alpha_2} - \mu_{\alpha_1}\mu_{\alpha_2}, v_{\alpha_1}v_{\alpha_2})$
(2) $\alpha_1 \otimes \alpha_2 = (\mu_{\alpha_1}\mu_{\alpha_2}, v_{\alpha_1} + v_{\alpha_2} - v_{\alpha_1}v_{\alpha_2})$
(3) $\lambda\alpha = (1 - (1 - \mu_\alpha)^\lambda, v_\alpha^\lambda), \lambda > 0$
(4) $\alpha^\lambda = (\mu_\alpha^\lambda, 1 - (1 - v_\alpha)^\lambda), \lambda > 0$

The IFNs comparison method used in most literature is given by Xu et al. as follows.

Definition 2 (see [4]). Let $\alpha_i = (\mu_{\alpha_i}, v_{\alpha_i})(i = 1,2)$ and $\alpha = (\mu_\alpha, v_\alpha)$ be three IFNs. $s_\alpha = \mu_\alpha - v_\alpha$ is called the score function of α, and $h_\alpha = \mu_\alpha + v_\alpha$ is called the accuracy degree function of α.

(i) If $s_{\alpha_1} > s_{\alpha_2}$, then $\alpha_1 > \alpha_2$
(ii) If $s_{\alpha_1} = s_{\alpha_2}$, then
  (i) If $h_{\alpha_1} = h_{\alpha_2}$, then $\alpha_1 = \alpha_2$
  (ii) If $h_{\alpha_1} > h_{\alpha_2}$, then $\alpha_1 > \alpha_2$
  (iii) If $h_{\alpha_1} < h_{\alpha_2}$, then $\alpha_1 < \alpha_2$

2.2. Optimized Geometric Bonferroni Mean

Definition 3 (see [8]). Let p, q, and $a_i(i = 1,2,\ldots,n)$ be nonnegative real numbers. If $B^{p,q}(a_1, a_2, \ldots, a_n) = ((1/n(n-1))\sum_{i,j=1,i\neq j}^n a_i^p a_j^q)^{(1/p+q)}$ and $\mathrm{GB}^{p,q}(a_1, a_2, \ldots, a_n) = (1/p+q)\prod_{i,j=1,i\neq j}^n (pa_1 + qa_j)^{(1/n(n-1))}$, then $B^{p,q}$ is called Bonferroni mean (BM), and $\mathrm{GB}^{p,q}$ is called geometric Bonferroni mean (GBM).

Due to the excellent nature of BM and GBM, their weighted forms have been studied in different fuzzy environments by many scholars. However these weighted functions do not have the reducibility. To solve this problem, Zhou introduced normalized weighted Bonferroni mean as follows.

Definition 4 (see [15]). Let p, q, and $a_i(i = 1,2,\ldots,n)$ be nonnegative real numbers. If NWB

$${}^{p,q}(a_1, a_2, \ldots, a_n) = \left(\sum_{\substack{i,j=1\\ i\neq j}}^{n} \frac{w_i w_j}{1 - w_i} a_i^p a_j^q\right)^{(1/p+q)}, \tag{3}$$

then $\mathrm{NWB}^{p,q}$ is called normalized weighted Bonferroni mean (NWB).

Obviously, if $w_1 = w_2 = \cdots = w_n = (1/n)$, then $\mathrm{NWB}^{p,q}(a_1, a_2, \ldots, a_n) = B^{p,q}(a_1, a_2, \ldots, a_n)$. In particular, by rearranging the terms in $\mathrm{NWB}^{p,q}$, the NWB is expressed as follows:

$$\begin{aligned} &\mathrm{NWB}^{p,q}(a_1, a_2, \ldots, a_n)\\ &= \left(\sum_{i=1}^{n} w_i a_i^p \left(\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{w_j}{1 - w_i} a_j^q\right)\right)^{(1/p+q)}. \end{aligned} \tag{4}$$

Then, it is easy to see that NWB contains two weighted means. $\sum_{j=1,j\neq i}^n (w_j/1 - w_c)a_j^q$ is the weighted average of $a_j^q(j \neq i)$; $\sum_{i=1}^n w_i a_i^p(\sum_{j=1,j\neq i}^n (w_j/1 - w_i)a_j^q)$ is the weighted average of a_i^p and $\sum_{j=1,j\neq i}^n (w_j/1 - w_i)a_j^q$, that is, the distinguishing characteristic of the BM [9]. However, the above definition of GBM and its weighted forms cannot reflect two geometric means such as BM or NWB. Furthermore, to our knowledge, there has been no report concerning the reducible weighted geometric Bonferroni mean previously. So, based on the work of Zhou [15] and Xia et al. [8], we introduce a new GBM and its weighted forms.

Definition 5. Let p, q, and $a_i\,(i = 1, 2, \ldots, n)$ are nonnegative real numbers. If

$$\mathrm{OGBM}^{p,q}(a_1, a_2, \ldots, a_n) = \frac{1}{p+q}\prod_{i=1}^{n}\left(pa_i + \prod_{\substack{j=1\\ j\neq i}}^{n}\left(qa_j\right)^{(1/n-1)}\right)^{1/n}, \tag{5}$$

then $\mathrm{OGBM}^{p,q}$ is called the optimized GBM (OGBM).

Obviously, the OGBM have the following properties:

(1) $\mathrm{OGBM}^{p,q}(0, 0, \ldots, 0) = 0$

(2) If $a_i = a\,(i = 1, 2, \ldots, n)$, then $\mathrm{OGBM}^{p,q}(a, a, \ldots, a) = a$

(3) If $a_i \geq d_i\,(i = 1, 2, \ldots, n)$, then $\mathrm{OGBM}^{p,q}(a_1, a_2, \ldots, a_n) \geq \mathrm{OGBM}^{p,q}(d_1, d_2, \ldots, d_n)$

(4) $\min_{1\leq i\leq n}\{a_i\} \leq \mathrm{GBM}^{p,q}(a_1, a_2, \ldots, a_n) \leq \max_{1\leq i\leq n}\{a_i\}$

(5) If $q = 0$, then $\mathrm{OGBM}^{p,q}(a_1, a_2, \ldots, a_n) = \prod_{i=1}^{n}(a_i)^{1/n}$

Definition 6. Let $p \geq 0$, $q \geq 0$, and $a_i\,(i = 1, 2, \ldots, n)$ be nonnegative real numbers with the weight $w_i\,(i = 1, 2, \ldots, n)$, $w_i \in [0, 1]$, $\sum_{i=1}^{n} w_i = 1$. If

$$\mathrm{NWOGBM}^{p,q}(a_1, a_2, \ldots, a_n) = \frac{1}{p+q}\prod_{i=1}^{n}\left(pa_i + \prod_{\substack{j=1\\ j\neq i}}^{n}\left(qa_j\right)^{\left(w_j/1-w_i\right)}\right)^{w_i}, \tag{6}$$

then $\mathrm{NWOGBM}^{p,q}$ is called the normalized weighted OGBM (NWOGBM).

Definition 7. Let p, q, r, and $a_i\,(i = 1, 2, \ldots, n)$ be nonnegative real numbers. If

$$\mathrm{GOGBM}^{p,q,r}(a_1, a_2, \ldots, a_n) = \frac{1}{p+q+r}\prod_{i=1}^{n}\left(pa_1 + \prod_{\substack{j=1\\ j\neq i}}^{n}\left(qa_j + \prod_{\substack{k=1\\ k\neq j\neq i}}^{n}\left(ra_k\right)^{(1/n-2)}\right)^{(1/n-1)}\right)^{1/n}. \tag{7}$$

then $\mathrm{GOGBM}^{p,q,r}$ is called the generalized OGBM(GOGBM).

Definition 8. Let $p \geq 0$, $q \geq 0$, $r \geq 0$, and $a_i\,(i = 1, 2, \ldots, n)$ be nonnegative real numbers with the weight $w_i\,(i = 1, 2, \ldots, n)$, $w_i \in [0, 1]$, and $\sum_{i=1}^{n} w_i = 1$. If

$$\mathrm{GNWOGBM}^{p,q,r}(a_1, a_2, \ldots, a_n) = \frac{1}{p+q+r}\prod_{i=1}^{n}\left(pa_i + \prod_{\substack{j=1\\ j\neq i}}^{n}\left(qa_j + \prod_{\substack{k=1\\ k\neq j\neq i}}^{n}\left(ra_k\right)^{\left(w_k/1-w_i-w_j\right)}\right)^{\left(w_j/1-w_i\right)}\right)^{w_i}, \tag{8}$$

then $\mathrm{GNWOGBM}^{p,q,r}$ is called the generalized normalized weighted OGBM (GNWOGBM).

It is obvious that GNOGBM reduces to NOGBM if $r = 0$. When the weights are the same,

$$\begin{aligned}\mathrm{GNWOGBM}^{p,q,r}(a_1, a_2, \ldots, a_n) &= \mathrm{GOGBM}^{p,q,r}(a_1, a_2, \ldots, a_n),\\ \mathrm{NWOGBM}^{p,q}(a_1, a_2, \ldots, a_n) &= \mathrm{OGBM}^{p,q}(a_1, a_2, \ldots, a_n),\end{aligned} \tag{9}$$

which reflect GNWOGBM and NWOGBM have the reducibility.

3. Intuitionistic Fuzzy Weighted OGBM

Definition 9. Let $p \geq 0$, $q \geq 0$, and $\alpha_i\,(i = 1, 2, \ldots, n)$ be intuitionistic fuzzy numbers with the weight $w_i\,(i = 1, 2, \ldots, n)$, $w_i \in [0, 1]$, and $\sum_{i=1}^{n} w_i = 1$. If

$$\text{IFNWOGBM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{1}{p+q} \otimes_{i=1}^{n} \left(p\alpha_i \oplus \otimes_{j=1, j\neq i}^{n} \left(q\alpha_j\right)^{\left(w_j/1-w_i\right)} \right)^{w_i}, \tag{10}$$

then $\text{IFNWOGBM}^{p,q}$ is called the intuitionistic fuzzy normalized weighted OGBM (IFNWOGBM).

If $w_i = (1/n)\,(i = 1, 2, \ldots, n)$, then

$$\text{IFNWOGBM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{1}{p+q} \otimes_{i=1}^{n} \left(p\alpha_i \oplus \otimes_{j=1, j\neq i}^{n} \left(q\alpha_j\right)^{(1/n-1)} \right)^{n}, \tag{11}$$

which we call the intuitionistic fuzzy OGBM (IFOGBM).

Theorem 1. *Let $p \geq 0$, $q \geq 0$, and $\alpha_i = (\mu_{\alpha_i}, v_{\alpha_i})$ $(i = 1, 2, \ldots, n)$ be intuitionistic fuzzy numbers with the weight $w_i\,(i = 1, 2, \ldots, n)$, $w_i \in [0, 1]$, and $\sum_{i=1}^{n} w_i = 1$; then,*

$$\text{IFNWOGBM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(\begin{array}{c} \left(1 - \left(1 - \prod_{i=1}^{n} \left(1 - \left(1 - \mu_{\alpha_i}\right)^{p} \left(1 - \prod_{j=1, j\neq i}^{n} \left(1 - \left(1 - \mu_{\alpha_j}\right)^{q}\right)^{\left(w_j/1-w_i\right)}\right)\right)\right)^{w_i}\right)^{(1/p+q)}, \\ \left(1 - \prod_{i=1}^{n} \left(1 - v_{a_i}^{p} \left(1 - \prod_{j=1, j\neq i}^{n} \left(1 - v_{a_j}^{q}\right)^{\left(w_j/1-w_i\right)}\right)\right)\right)^{w_i \, (1/p+q)} \end{array} \right), \tag{12}$$

and $\text{IFNWOGBM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is also an IFN.

Proof. Since $p\alpha_i = (1 - (1 - \mu_{\alpha_i})^p, v_{a_i}^{p})$ and $q\alpha_j = (1 - (1 - \mu_{\alpha_j})^q, v_{a_j}^{q})$, then

$$\begin{aligned} \otimes_{j=1, j\neq i}^{N} \left(q\alpha_j\right)^{\left(w_j/1-w_i\right)} &= \otimes_{j=1, j\neq i}^{N} \left(\left(1 - \left(1 - \mu_{\alpha_j}\right)^{q}\right)^{\left(w_j/1-w_i\right)}, 1 - \left(1 - v_{a_j}^{q}\right)^{\left(w_j/1-w_i\right)} \right) \\ &= \left(\prod_{j=1, j\neq i}^{n} \left(1 - \left(1 - \mu_{\alpha_j}\right)^{q}\right)^{\left(w_j/1-w_i\right)}, 1 - \prod_{j=1, j\neq i}^{n} \left(1 - \left(1 - \left(1 - v_{a_j}^{q}\right)^{\left(w_j/1-w_i\right)}\right)\right) \right) \\ &= \left(\prod_{j=1, j\neq i}^{n} \left(1 - \left(1 - \mu_{\alpha_j}\right)^{q}\right)^{\left(w_j/1-w_i\right)}, 1 - \prod_{j=1, j\neq i}^{n} \left(1 - v_{a_j}^{q}\right)^{\left(w_j/1-w_i\right)} \right). \end{aligned} \tag{13}$$

Therefore,

$$p\alpha_i \oplus \otimes_{j=1, j\neq i}^{N} \left(q\alpha_j\right)^{\left(w_j/1-w_i\right)} = \left(\begin{array}{c} 1 - \left(1 - \mu_{\alpha_i}\right)^{p} + \prod_{j=1, j\neq i}^{n} \left(1 - \left(1 - \mu_{\alpha_j}\right)^{q}\right)^{\left(w_j/1-w_i\right)} \\ -\left(1 - \left(1 - \mu_{\alpha_i}\right)^{p}\right) \prod_{j=1, j\neq i}^{n} \left(1 - \left(1 - \mu_{\alpha_j}\right)^{q}\right)^{\left(w_j/1-w_i\right)}, v_{a_i}^{p} \left(1 - \prod_{j=1, j\neq i}^{n} \left(1 - v_{a_j}^{q}\right)^{\left(w_j/1-w_i\right)}\right) \end{array} \right)$$

$$= \left(1 - \left(1-\mu_{\alpha_i}\right)^p + \left(1-\mu_{\alpha_i}\right)^p \prod_{j=1, j\neq i}^{n} \left(1 - \left(1-\mu_{\alpha_j}\right)^q\right)^{\left(w_j/1-w_i\right)}, v_{a_i}^p \left(1 - \prod_{j=1, j\neq i}^{n} \left(1 - v_{a_j}^q\right)^{\left(w_j/1-w_i\right)}\right)\right)$$

$$= \left(1 - \left(1-\mu_{\alpha_i}\right)^p \left(1 - \prod_{j=1, j\neq i}^{n} \left(1 - \left(1-\mu_{\alpha_j}\right)^q\right)^{\left(w_j/1-w_i\right)}\right), v_{a_i}^p \left(1 - \prod_{j=1, j\neq i}^{n} \left(1 - v_{a_j}^q\right)^{\left(w_j/1-w_i\right)}\right)\right) \tag{14}$$

$$\left(p\alpha_i \oplus \otimes_{j=1, j\neq i}^{n} \left(q\alpha_j\right)^{\left(w_j/1-w_i\right)}\right)^{w_i}$$

$$= \left(\left(1 - \left(1-\mu_{\alpha_i}\right)^p \left(1 - \prod_{j=1, j\neq i}^{n} \left(1 - \left(1-\mu_{\alpha_j}\right)^q\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}, 1 - \left(1 - v_{a_i}^p \left(1 - \prod_{j=1, j\neq i}^{n} \left(1 - v_{a_j}^q\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}\right).$$

Hence, we obtain

$$\text{IFNWOGBM}^{p,q}\left(\alpha_1, \alpha_2, \ldots, \alpha_n\right) = \frac{1}{p+q} \otimes_{i=1}^{n} \left(p\alpha_i \oplus \otimes_{j=1, j\neq i}^{n} \left(q\alpha_j\right)^{\left(w_j/1-w_i\right)}\right)^{w_i}$$

$$= \frac{1}{p+q} \otimes_{i=1}^{n} \left(\begin{array}{c} \left(1 - \left(1-\mu_{\alpha_i}\right)^p \left(1 - \prod_{j=1, j\neq i}^{n} \left(1 - \left(1-\mu_{\alpha_j}\right)^q\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}, \\ 1 - \left(1 - v_{a_i}^p \left(1 - \prod_{j=1, j\neq i}^{n} \left(1 - v_{a_j}^q\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i} \end{array} \right)$$

$$= \frac{1}{p+q} \left(\begin{array}{c} \prod_{i=1}^{n} \left(1 - \left(1-\mu_{\alpha_i}\right)^p \left(1 - \prod_{j=1, j\neq i}^{n} \left(1 - \left(1-\mu_{\alpha_j}\right)^q\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}, \\ 1 - \prod_{i=1}^{n} \left(1 - v_{a_i}^p \left(1 - \prod_{j=1, j\neq i}^{n} \left(1 - v_{a_j}^q\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i} \end{array} \right)$$

$$= \left(\begin{array}{c} 1 - \left(1 - \prod_{i=1}^{n} \left(1 - \left(1-\mu_{\alpha_i}\right)^p \left(1 - \prod_{j=1, j\neq i}^{n} \left(1 - \left(1-\mu_{\alpha_j}\right)^q\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}\right)^{(1/p+q)}, \\ \left(1 - \prod_{i=1}^{n} \left(1 - v_{a_i}^p \left(1 - \prod_{j=1, j\neq i}^{n} \left(1 - v_{a_j}^q\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}\right)^{(1/p+q)} \end{array} \right). \tag{15}$$

By $\mu_{\alpha_i} \in [0,1]$, $v_{\alpha_i} \in [0,1]$, and $\mu_{\alpha_i} + v_{\alpha_i} \in [0,1]$, we have

$$0 \leq 1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\mu_{\alpha_i}\right)^{p}\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-\mu_{\alpha_j}\right)^{q}\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}\right)^{(1/p+q)} \leq 1,$$
$$0 \leq \left(1-\prod_{i=1}^{n}\left(1-\upsilon_{a_i}^{p}\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\upsilon_{a_j}^{q}\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}\right)^{(1/p+q)} \leq 1. \tag{16}$$

In addition, since $\upsilon_{\alpha_j} \leq 1-\mu_{\alpha_j} \Rightarrow 1-\upsilon_{a_j}^{q} \geq 1-(1-\mu_{\alpha_j})^{q}$,

$$\Rightarrow\left(1-\upsilon_{a_j}^{q}\right)^{\left(w_j/1-w_i\right)} \geq \left(1-\left(1-\mu_{\alpha_j}\right)^{q}\right)^{\left(w_j/1-w_i\right)}$$
$$\Rightarrow \prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\upsilon_{a_j}^{q}\right)^{\left(w_j/1-w_i\right)} \geq \prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-\mu_{\alpha_j}\right)^{q}\right)^{\left(w_j/1-w_i\right)}$$
$$\Rightarrow 1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\upsilon_{a_j}^{q}\right)^{\left(w_j/1-w_i\right)} \leq 1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-\mu_{\alpha_j}\right)^{q}\right)^{\left(w_j/1-w_i\right)}$$
$$\Rightarrow \upsilon_{a_i}^{p}\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\upsilon_{a_j}^{q}\right)^{\left(w_j/1-w_i\right)}\right) \leq \left(1-\mu_{\alpha_i}\right)^{p}\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-\mu_{\alpha_j}\right)^{q}\right)^{\left(w_j/1-w_i\right)}\right)$$
$$\Rightarrow \prod_{i=1}^{n}\left(1-\upsilon_{a_i}^{p}\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\upsilon_{a_j}^{q}\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i} \geq \prod_{i=1}^{n}\left(1-\left(1-\mu_{\alpha_i}\right)^{p}\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-\mu_{\alpha_j}\right)^{q}\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}$$
$$\Rightarrow \left(1-\prod_{i=1}^{n}\left(1-\upsilon_{a_i}^{p}\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\upsilon_{a_j}^{q}\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}\right)^{(1/p+q)}$$
$$\leq \left(1-\prod_{i=1}^{n}\left(1-\left(1-\mu_{\alpha_i}\right)^{p}\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-\mu_{\alpha_j}\right)^{q}\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}\right)^{(1/p+q)}$$
$$\Rightarrow 1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\mu_{\alpha_i}\right)^{p}\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-\mu_{\alpha_j}\right)^{q}\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}\right)^{(1/p+q)}$$
$$+\left(1-\prod_{i=1}^{n}\left(1-\upsilon_{a_i}^{p}\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\upsilon_{a_j}^{q}\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}\right)^{(1/p+q)} \leq 1. \tag{17}$$

This completes the proof. □

Property 1.

(1) Let $\alpha_i\,(i = 1, 2, \ldots, n)$ be a collection of IFNs; if $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$, then

$$\text{IFNWOGBM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha. \tag{18}$$

(2) Let $\alpha_i\,(i = 1, 2, \ldots, n)$ be a collection of IFNs; if $\alpha_1', \alpha_2', \ldots, \alpha_N'$ is any permutation of $\alpha_1, \alpha_2, \ldots, \alpha_n$, then

$$\begin{aligned}&\text{IFNWOGBM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n)\\ &= \text{IFNWOGBM}^{p,q}(\alpha_1', \alpha_2', \ldots, \alpha_n').\end{aligned} \tag{19}$$

(3) Let $\alpha_i = (\mu_{\alpha_i}, v_{\alpha_i})$ and $\beta_i = (\mu_{\beta_i}, v_{\beta_i})$ $(i = 1, 2, \ldots, n)$ be two collection of IFNs; if $\mu_{\alpha_i} \le \mu_{\beta_i}$ and $v_{\alpha_i} \ge v_{\beta_i}\,(i = 1, 2, \ldots, n)$, then

$$\begin{aligned}&\text{IFNWOGBM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n)\\ &\le \text{IFNWOGBM}^{p,q}(\beta_1, \beta_2, \ldots, \beta_n).\end{aligned} \tag{20}$$

(4) Let $\alpha_i = (\mu_{\alpha_i}, v_{\alpha_i})$ $(i = 1, 2, \ldots, n)$ be a collection of IFNs, and

$$\begin{aligned}\alpha^- &= \left(\min_{1\le i\le n}\{\mu_{\alpha_i}\}, \max_{1\le i\le n}\{v_{\alpha_i}\}\right),\\ \alpha^+ &= \left(\max_{1\le i\le n}\{\mu_{\alpha_i}\}, \min_{1\le i\le n}\{v_{\alpha_i}\}\right),\end{aligned} \tag{21}$$

then

$$\alpha^- \le \text{IFNWOGBM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) \le \alpha^+. \tag{22}$$

Proof

(1)

$$\begin{aligned}\text{IFNWOGBM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) &= \frac{1}{p+q} \otimes_{i=1}^{n} \left(p\alpha_i \oplus \otimes_{j=1, j\ne i}^{n} \left(q\alpha_j\right)^{\left(w_j/1-w_i\right)}\right)^{w_i}\\ &= \frac{1}{p+q} \otimes_{i=1}^{n} \left(p\alpha \oplus \otimes_{j=1, j\ne i}^{n} (q\alpha)^{\left(w_j/1-w_i\right)}\right)^{w_i}\\ &= \frac{1}{p+q} \otimes_{i=1}^{n} (p\alpha \oplus q\alpha)^{w_i}\\ &= \frac{1}{p+q} \otimes_{i=1}^{n} ((p+q)\alpha)^{w_i} = \alpha.\end{aligned} \tag{23}$$

(2)

$$\begin{aligned}\text{IFNWOGBM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) &= \frac{1}{p+q} \otimes_{i=1}^{n} \left(p\alpha_i \oplus \otimes_{j=1, j\ne i}^{n} \left(q\alpha_j\right)^{\left(w_j/1-w_i\right)}\right)^{w_i}\\ &= \frac{1}{p+q} \otimes_{i=1}^{n} \left(p\alpha_i' \oplus \otimes_{j=1, j\ne i}^{n} \left(q\alpha_j'\right)^{\left(w_j/1-w_i\right)}\right)^{w_i}\\ &= \text{IFNWOGBM}^{p,q}(\alpha_1', \alpha_2', \ldots, \alpha_n').\end{aligned} \tag{24}$$

(3) According to $\mu_{\alpha_i} \le \mu_{\beta_i}$ and $v_{\alpha_i} \ge v_{\beta_i}$ $(i = 1, 2, \ldots, n)$, we can obtain

$$\prod_{i=1}^{n}\left(1-\left(1-\mu_{\alpha_i}\right)^{p}\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-\mu_{\alpha_j}\right)^{q}\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i} \leq \prod_{i=1}^{n}\left(1-\left(1-\mu_{\beta_i}\right)^{p}\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-\mu_{\beta_j}\right)^{q}\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}. \tag{25}$$

Thus,

$$\begin{aligned}&1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\mu_{\alpha_i}\right)^{p}\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-\mu_{\alpha_j}\right)^{q}\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}\right)^{(1/p+q)}\\ &\leq 1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\mu_{\beta_i}\right)^{p}\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-\mu_{\beta_j}\right)^{q}\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}\right)^{(1/p+q)}.\end{aligned} \tag{26}$$

Similarly, we can obtain

$$\left(1-\prod_{i=1}^{n}\left(1-\upsilon_{a_i}^{p}\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\upsilon_{a_j}^{q}\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}\right)^{(1/p+q)} \geq \left(1-\prod_{i=1}^{n}\left(1-\upsilon_{\beta_i}^{p}\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\upsilon_{\beta_j}^{q}\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}\right)^{(1/p+q)}. \tag{27}$$

Then,

$$\begin{aligned}&1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\mu_{\beta_i}\right)^{p}\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-\mu_{\beta_j}\right)^{q}\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}\right)^{(1/p+q)}\\ &-\left(1-\prod_{i=1}^{n}\left(1-\upsilon_{\beta_i}^{p}\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\upsilon_{\beta_j}^{q}\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}\right)^{(1/p+q)}\\ &\geq 1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\mu_{\alpha_i}\right)^{p}\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-\mu_{\alpha_j}\right)^{q}\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}\right)^{(1/p+q)}\\ &-\left(1-\prod_{i=1}^{n}\left(1-\upsilon_{a_i}^{p}\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\upsilon_{a_j}^{q}\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}\right)^{(1/p+q)} * .\end{aligned} \tag{28}$$

Let the score and accuracy degree values of $\text{IFNWOGBM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\text{IFNWOGBM}^{p,q}(\beta_1, \beta_2, \ldots, \beta_n)$ be s_α, s_β and h_α, h_β, respectively. Then, equation $*$ can be denoted as $s_\beta \geq s_\alpha$.

(1) If $s_\beta > s_\alpha$, then, by Definition 2, we have $\text{IFNWOGBM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) < \text{IFNWOGBM}^{p,q}(\beta_1, \beta_2, \ldots, \beta_n)$.

(2) If $s_\alpha = s_\beta$, then, by $\mu_{\alpha_i} \leq \mu_{\beta_i}$ and $v_{\alpha_i} \geq v_{\beta_i}$ $(i = 1, 2, \ldots, n)$, we have

$$\begin{aligned}
&1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\mu_{\alpha_i}\right)^{p}\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-\mu_{\alpha_j}\right)^{q}\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}\right)^{(1/p+q)}\\
&=1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\mu_{\beta_i}\right)^{p}\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-\mu_{\beta_j}\right)^{q}\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}\right)^{(1/p+q)}\\
&\left(1-\prod_{i=1}^{n}\left(1-v_{a_i}^{p}\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-v_{a_j}^{q}\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}\right)^{(1/p+q)}\\
&=\left(1-\prod_{i=1}^{n}\left(1-v_{\beta_i}^{p}\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-v_{\beta_j}^{q}\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}\right)^{(1/p+q)}.
\end{aligned} \tag{29}$$

Thus, $h_\alpha = h_\beta$ and

$$\begin{aligned}
&\text{IFNWOGBM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n)\\
&= \text{IFNWOGBM}^{p,q}(\beta_1, \beta_2, \ldots, \beta_n).
\end{aligned} \tag{30}$$

Therefore, (1) and (2) indicate that

$$\begin{aligned}
&\text{IFNWOGBM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n)\\
&\leq \text{IFNWOGBM}^{p,q}(\beta_1, \beta_2, \ldots, \beta_n).
\end{aligned} \tag{31}$$

(4) Based on (1) and (3), we have

$$\begin{aligned}
&\text{IFNWOGBM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \text{IFNWOGBM}^{p,q}(\alpha^+, \alpha^+, \ldots, \alpha^+) = \alpha^+,\\
&\alpha^- = \text{IFNWOGBM}^{p,q}(\alpha^-, \alpha^-, \ldots, \alpha^-) \leq \text{IFNWOGBM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n).
\end{aligned} \tag{32}$$

Then, $\alpha^- \leq \text{IFNWOGBM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \alpha^+$.

This completes the property.

Let us now further consider some specials of IFNWOGBM with respect to the parameters p and q.

Case 1: if $q \longrightarrow 0$, then IFNWOGBM can be converted to GIFWGBM [16]:

$$\begin{aligned}
\text{IFNWOGBM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) &= \frac{1}{p} \otimes_{i=1}^{n} (p\alpha_i)^{w_i}\\
&= \frac{1}{p} \otimes_{i=1}^{n} \left(1-\left(1-\mu_{\alpha_i}\right)^{p}, v_{a_i}^{p}\right)^{w_i}\\
&= \frac{1}{p} \otimes_{i=1}^{n} \left(\left(1-\left(1-\mu_{\alpha_i}\right)^{p}\right)^{w_i}, 1-\left(1-v_{a_i}^{p}\right)^{w_i}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{p}\left(\prod_{i=1}^{n}\left(1-\left(1-\mu_{\alpha_i}\right)^{p}\right)^{w_i}, 1-\prod_{i=1}^{n}\left(1-v_{a_i}^{p}\right)^{w_i}\right) \\
&= \left(1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\mu_{\alpha_i}\right)^{p}\right)^{w_i}\right)^{1/p}, \left(1-\prod_{i=1}^{n}\left(1-v_{a_i}^{p}\right)^{w_i}\right)^{1/p}\right).
\end{aligned}
\tag{33}
$$

Case 2: if $p = 1$ and $q \longrightarrow 0$, then IFNWOGBM can be converted to IFWGM [16]:

$$
\begin{aligned}
&\text{IFNWOGBM}^{p,q}\left(\alpha_1, \alpha_2, \ldots, \alpha_n\right) = \otimes_{i=1}^{N} \alpha_i^{w_i} \\
&= \left(\prod_{i=1}^{n} \mu_{\alpha_i}^{w_i} 1-\prod_{i=1}^{n}\left(1-v_{a_i}\right)^{w_i}\right).
\end{aligned}
\tag{34}
$$

Case 3: if $p = 2$ and $q \longrightarrow 0$, then IFNWOGBM can be converted to IFWSGM [3]:

$$
\text{IFNWOGBM}^{p,q}\left(\alpha_1, \alpha_2, \ldots, \alpha_n\right) = \left(1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\mu_{\alpha_i}\right)^{2}\right)^{w_i}\right)^{1/2}, \left(1-\prod_{i=1}^{n}\left(1-v_{a_i}^{2}\right)^{w_i}\right)^{1/2}\right). \tag{35}
$$

Case 4: if $p = 1$ and $q = 1$, then IFNWOGBM can be converted to intuitionistic fuzzy normalized weighted optimized square GBM (IFNWOSGBM):

$$
\begin{aligned}
&\text{IFNWOGBM}^{p,q}\left(\alpha_1, \alpha_2, \ldots, \alpha_n\right) \\
&= \left(1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\mu_{\alpha_i}\right)\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n} \mu_{\alpha_j}^{\left(w_j/1-w_i\right)}\right)\right)\right)^{w_i}\right)^{1/2}, \\
&\left(1-\prod_{i=1}^{n}\left(1-v_{a_i}\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-v_{a_j}\right)^{\left(w_j/1-w_i\right)}\right)\right)\right)^{w_i}\right)^{1/2}\right)
\end{aligned}
\tag{36}
$$

4. Generalized Intuitionistic Fuzzy Weighted OGBM

Definition 10. Let $p \geq 0$, $q \geq 0$, $r \geq 0$, and $\alpha_i\,(i = 1, 2, \ldots, n)$ be intuitionistic fuzzy numbers with the weight $w_i\,(i = 1, 2, \ldots, n)$, $w_i \in [0, 1]$, and $\sum_{i=1}^{n} w_i = 1$. If

$$
\text{GIFNWOGBM}^{p,q,r}\left(\alpha_1, \alpha_2, \ldots, \alpha_n\right) = \frac{1}{p+q+r} \otimes_{i=1}^{N}\left(p\alpha_i \oplus \otimes_{j=1, j\neq i}^{n}\left(q\alpha_j \oplus \otimes_{k=1, k\neq j\neq i}^{n}\left(r\alpha_k\right)^{\left(w_k/1-w_i-w_j\right)}\right)^{\left(w_j/1-w_i\right)}\right)^{w_i}, \tag{37}
$$

then $\mathrm{GIFNWOGBM}^{p,q}$ is called the generalized intuitionistic fuzzy normalized weighted OGBM (GIFNWOGBM).

If $w_i = (1/n)(i = 1, 2, \ldots, n)$, then

$$\mathrm{GIFNWOGBM}^{p,q,r}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{1}{p+q+r} \otimes_{i=1}^{n} \left(p\alpha_i \oplus \otimes_{j=1, j\neq i}^{n} \left(q\alpha_j \oplus \otimes_{k=1, k\neq j\neq i}^{n} (r\alpha_k)^{(1/n-2)} \right)^{(1/n-1)} \right)^{1/n}, \tag{38}$$

which we call the generalized intuitionistic fuzzy OGBM (GIFOGBM).

Theorem 2. *Let $p \geq 0$, $q \geq 0$, $r \geq 0$, and $\alpha_i = (\mu_{\alpha_i}, v_{\alpha_i})$ $(i = 1, 2, \ldots, n)$ are intuitionistic fuzzy numbers with the weight $w_i (i = 1, 2, \ldots, n)$, $w_i \in [0, 1]$, and $\sum_{i=1}^{n} w_i = 1$; then,*

$$\mathrm{GIFNWOGBM}^{p,q,r}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(\begin{array}{c} 1 - \left(1 - \prod_{i=1}^{n} \left(1 - \left(1 - \mu_{\alpha_i}\right)^{p} \left(1 - \prod_{\substack{j=1 \\ j\neq i}}^{n} \left(1 - \left(1 - \mu_{\alpha_j}\right)^{q} \left(1 - \prod_{\substack{k=1 \\ k\neq j\neq i}}^{n} \left(1 - \left(1 - \mu_{\alpha_k}\right)^{r}\right)^{\left(w_k/1-w_i-w_j\right)} \right) \right)^{\left(w_j/1-w_i\right)} \right) \right)^{w_i} \right)^{(1/p+q+r)}, \\ \left(1 - \prod_{i=1}^{n} \left(1 - v_{a_i}^{p} \left(1 - \prod_{\substack{j=1 \\ j\neq i}}^{n} \left(1 - v_{a_j}^{q} \left(1 - \prod_{\substack{k=1 \\ k\neq j\neq i}}^{n} \left(1 - v_{\alpha_k}^{r}\right)^{\left(w_k/1-w_i-w_j\right)} \right) \right)^{\left(w_j/1-w_i\right)} \right) \right)^{w_i} \right)^{(1/p+q+r)} \end{array} \right), \tag{39}$$

and $\mathrm{GIFNWOGBM}^{p,q,r}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is also an IFN.

Proof. Since $p\alpha_i = (1 - (1 - \mu_{\alpha_i})^{p}, v_{a_i}^{p})$, $q\alpha_j = (1 - (1 - \mu_{\alpha_j})^{q}, v_{a_j}^{q})$, and $r\alpha_k = (1 - (1 - \mu_{\alpha_k})^{r}, v_{a_k}^{r})$, then

$$\begin{aligned} \otimes_{k=1, k\neq j\neq i}^{n} (r\alpha_k)^{\left(w_k/1-w_i-w_j\right)} &= \otimes_{k=1, k\neq j\neq i}^{n} \left(\left(1 - \left(1 - \mu_{\alpha_k}\right)^{r}\right)^{\left(w_k/1-w_i-w_j\right)}, 1 - \left(1 - v_{\alpha_k}^{r}\right)^{\left(w_k/1-w_i-w_j\right)} \right) \\ &= \left(\prod_{k=1, k\neq j\neq i}^{n} \left(1 - \left(1 - \mu_{\alpha_k}\right)^{r}\right)^{\left(w_k/1-w_i-w_j\right)}, 1 - \prod_{k=1, k\neq j\neq i}^{n} \left(1 - v_{\alpha_k}^{r}\right)^{\left(w_k/1-w_i-w_j\right)} \right) \end{aligned}$$

$$\begin{aligned} & q\alpha_j \oplus \otimes_{k=1, k\neq j\neq i}^{n} (r\alpha_k)^{\left(w_k/1-w_i-w_j\right)} \\ &= \left(\begin{array}{c} 1 - \left(1 - \mu_{\alpha_j}\right)^{q} + \prod_{k=1, k\neq j\neq i}^{n} \left(1 - \left(1 - \mu_{\alpha_k}\right)^{r}\right)^{\left(w_k/1-w_i-w_j\right)} \\ - \left(1 - \left(1 - \mu_{\alpha_j}\right)^{q}\right) \prod_{k=1, k\neq j\neq i}^{n} \left(1 - \left(1 - \mu_{\alpha_k}\right)^{r}\right)^{\left(w_k/1-w_i-w_j\right)}, v_{a_j}^{q} \left(1 - \prod_{k=1, k\neq j\neq i}^{n} \left(1 - v_{\alpha_k}^{r}\right)^{\left(w_k/1-w_i-w_j\right)} \right) \end{array} \right) \\ &= \left(1 - \left(1 - \mu_{\alpha_j}\right)^{q} \left(1 - \prod_{k=1, k\neq j\neq i}^{n} \left(1 - \left(1 - \mu_{\alpha_k}\right)^{r}\right)^{\left(w_k/1-w_i-w_j\right)} \right), v_{a_j}^{q} \left(1 - \prod_{k=1, k\neq j\neq i}^{n} \left(1 - v_{\alpha_k}^{r}\right)^{\left(w_k/1-w_i-w_j\right)} \right) \right) \end{aligned}$$

$$\begin{aligned} & \otimes_{j=1, j\neq i}^{n} \left(q\alpha_j \oplus \otimes_{k=1, k\neq j\neq i}^{n} (r\alpha_k)^{\left(w_k/1-w_i-w_j\right)} \right)^{\left(w_j/1-w_i\right)} \\ &= \otimes_{j=1, j\neq i}^{n} \left(\begin{array}{c} \left(1 - \left(1 - \mu_{\alpha_j}\right)^{q} \left(1 - \prod_{k=1, k\neq j\neq i}^{n} \left(1 - \left(1 - \mu_{\alpha_k}\right)^{r}\right)^{\left(w_k/1-w_i-w_j\right)} \right) \right)^{\left(w_j/1-w_i\right)}, \\ 1 - \left(1 - v_{a_j}^{q} \left(1 - \prod_{k=1, k\neq j\neq i}^{n} \left(1 - v_{\alpha_k}^{r}\right)^{\left(w_k/1-w_i-w_j\right)} \right) \right)^{\left(w_j/1-w_i\right)} \end{array} \right) \end{aligned}$$

$$
\begin{aligned}
&= \left(\prod_{j=1, j\neq i}^{n} \left(1 - \left(1 - \mu_{\alpha_j}\right)^{q} \left(1 - \prod_{k=1, k\neq j\neq i}^{n} \left(1 - \left(1 - \mu_{\alpha_k}\right)^{r} \right)^{\left(w_k/1-w_i-w_j\right)} \right) \right)^{\left(w_j/1-w_i\right)}, \right. \\
&\qquad \left. 1 - \prod_{j=1, j\neq i}^{n} \left(1 - v_{a_j}^{q} \left(1 - \prod_{k=1, k\neq j\neq i}^{n} \left(1 - v_{\alpha_k}^{r} \right)^{\left(w_k/1-w_i-w_j\right)} \right) \right)^{\left(w_j/1-w_i\right)} \right) \\
&p\alpha_i \oplus \otimes_{j=1, j\neq i}^{n} \left(q\alpha_j \oplus \otimes_{k=1, k\neq j\neq i}^{n} \left(r\alpha_k \right)^{\left(w_k/1-w_i-w_j\right)} \right)^{\left(w_j/1-w_i\right)} \\
&= \left(1 - \left(1 - \mu_{\alpha_i}\right)^{p} \left(1 - \prod_{j=1, j\neq i}^{n} \left(1 - \left(1 - \mu_{\alpha_j}\right)^{q} \left(1 - \prod_{k=1, k\neq j\neq i}^{n} \left(1 - \left(1 - \mu_{\alpha_k}\right)^{r} \right)^{\left(w_k/1-w_i-w_j\right)} \right) \right)^{\left(w_j/1-w_i\right)} \right), \right. \\
&\qquad \left. v_{a_i}^{p} \left(1 - \prod_{j=1, j\neq i}^{n} \left(1 - v_{a_j}^{q} \left(1 - \prod_{k=1, k\neq j\neq i}^{n} \left(1 - v_{\alpha_k}^{r} \right)^{\left(w_k/1-w_i-w_j\right)} \right) \right)^{\left(w_j/1-w_i\right)} \right) \right).
\end{aligned} \tag{40}
$$

Therefore,

$$
\begin{aligned}
&\text{GIFNWOGBM}^{p,q}\left(\alpha_1, \alpha_2, \ldots, \alpha_n\right) = \frac{1}{p+q+r} \otimes_{i=1}^{n} \left(p\alpha_i \oplus \otimes_{j=1, j\neq i}^{n} \left(q\alpha_j \oplus \otimes_{k=1, k\neq j\neq i}^{n} \left(r\alpha_k \right)^{\left(w_k/1-w_i-w_j\right)} \right)^{\left(w_j/1-w_i\right)} \right)^{w_i} \\
&= \frac{1}{p+q+r} \otimes_{i=1}^{n} \left(\left(1 - \left(1 - \mu_{\alpha_i}\right)^{p} \left(1 - \prod_{j=1, j\neq i}^{n} \left(1 - \left(1 - \mu_{\alpha_j}\right)^{q} \left(1 - \prod_{k=1, k\neq j\neq i}^{n} \left(1 - \left(1 - \mu_{\alpha_k}\right)^{r} \right)^{\left(w_k/1-w_i-w_j\right)} \right) \right)^{\left(w_j/1-w_i\right)} \right) \right)^{w_i}, \right. \\
&\qquad \left. 1 - \left(1 - v_{a_i}^{p} \left(1 - \prod_{j=1, j\neq i}^{n} \left(1 - v_{a_j}^{q} \left(1 - \prod_{k=1, k\neq j\neq i}^{n} \left(1 - v_{\alpha_k}^{r} \right)^{\left(w_k/1-w_i-w_j\right)} \right) \right)^{\left(w_j/1-w_i\right)} \right) \right)^{w_i} \right) \\
&= \frac{1}{p+q+r} \left(\prod_{i=1}^{n} \left(1 - \left(1 - \mu_{\alpha_i}\right)^{p} \left(1 - \prod_{j=1, j\neq i}^{n} \left(1 - \left(1 - \mu_{\alpha_j}\right)^{q} \left(1 - \prod_{k=1, k\neq j\neq i}^{n} \left(1 - \left(1 - \mu_{\alpha_k}\right)^{r} \right)^{\left(w_k/1-w_i-w_j\right)} \right) \right)^{\left(w_j/1-w_i\right)} \right) \right)^{w_i}, \right. \\
&\qquad \left. 1 - \prod_{i=1}^{n} \left(1 - v_{a_i}^{p} \left(1 - \prod_{j=1, j\neq i}^{n} \left(1 - v_{a_j}^{q} \left(1 - \prod_{\substack{k=1 \\ k\neq j\neq i}}^{n} \left(1 - v_{\alpha_k}^{r} \right)^{\left(w_k/1-w_i-w_j\right)} \right) \right)^{\left(w_j/1-w_i\right)} \right) \right)^{w_i} \right) \\
&= \left(1 - \left(1 - \prod_{i=1}^{n} \left(1 - \left(1 - \mu_{\alpha_i}\right)^{p} \left(1 - \prod_{j=1, j\neq i}^{n} \left(1 - \left(1 - \mu_{\alpha_j}\right)^{q} \left(1 - \prod_{k=1, k\neq j\neq i}^{n} \left(1 - \left(1 - \mu_{\alpha_k}\right)^{r} \right)^{\left(w_k/1-w_i-w_j\right)} \right) \right)^{\left(w_j/1-w_i\right)} \right) \right)^{w_i} \right)^{(1/p+q+r)}, \right. \\
&\qquad \left. \left(1 - \prod_{i=1}^{n} \left(1 - v_{a_i}^{p} \left(1 - \prod_{j=1, j\neq i}^{n} \left(1 - v_{a_j}^{q} \left(1 - \prod_{k=1, k\neq j\neq i}^{n} \left(1 - v_{\alpha_k}^{r} \right)^{\left(w_k/1-w_i-w_j\right)} \right) \right)^{\left(w_j/1-w_i\right)} \right) \right)^{w_i} \right)^{(1/p+q+r)} \right).
\end{aligned} \tag{41}
$$

Since $\mu_{\alpha_i} \in [0,1]$, $v_{\alpha_i} \in [0,1]$, and $\mu_{\alpha_i} + v_{\alpha_i} \in [0,1]$, then

$$0 \le 1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\mu_{\alpha_i}\right)^{p}\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-\mu_{\alpha_j}\right)^{q}\left(1-\prod_{\substack{k=1\\ k\neq j\neq i}}^{n}\left(1-\left(1-\mu_{\alpha_k}\right)^{r}\right)^{\left(w_k/1-w_i-w_j\right)}\right)\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}\right)^{(1/p+q+r)} \le 1,$$

$$0 \le \left(1-\prod_{i=1}^{n}\left(1-v_{a_i}^{p}\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-v_{a_j}^{q}\left(1-\prod_{\substack{k=1\\ k\neq j\neq i}}^{n}\left(1-v_{\alpha_k}^{r}\right)^{\left(w_k/1-w_i-w_j\right)}\right)\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}\right)^{(1/p+q+r)} \le 1. \tag{42}$$

In addition, since $v_{\alpha_i} \le 1-\mu_{\alpha_i}$, $v_{\alpha_j} \le 1-\mu_{\alpha_j}$, and $v_{\alpha_k} \le 1-\mu_{\alpha_k}$, we obtain

$$\left(1-\prod_{i=1}^{n}\left(1-v_{a_i}^{p}\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-v_{a_j}^{q}\left(1-\prod_{\substack{k=1\\ k\neq j\neq i}}^{n}\left(1-v_{\alpha_k}^{r}\right)^{\left(w_k/1-w_i-w_j\right)}\right)\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}\right)^{(1/p+q+r)}$$
$$\le \left(1-\prod_{i=1}^{n}\left(1-\left(1-\mu_{\alpha_i}\right)^{p}\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-\mu_{\alpha_j}\right)^{q}\left(1-\prod_{\substack{k=1\\ k\neq j\neq i}}^{n}\left(1-\left(1-\mu_{\alpha_k}\right)^{r}\right)^{\left(w_k/1-w_i-w_j\right)}\right)\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}\right)^{(1/p+q+r)}, \tag{43}$$

and thus,

$$1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\mu_{\alpha_i}\right)^{p}\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-\mu_{\alpha_j}\right)^{q}\left(1-\prod_{\substack{k=1\\ k\neq j\neq i}}^{n}\left(1-\left(1-\mu_{\alpha_k}\right)^{r}\right)^{\left(w_k/1-w_i-w_j\right)}\right)\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}\right)^{(1/p+q+r)}$$
$$+\left(1-\prod_{i=1}^{n}\left(1-v_{a_i}^{p}\left(1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-v_{a_j}^{q}\left(1-\prod_{\substack{k=1\\ k\neq j\neq i}}^{n}\left(1-v_{\alpha_k}^{r}\right)^{\left(w_k/1-w_i-w_j\right)}\right)\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}\right)^{(1/p+q+r)} \le 1. \tag{44}$$

This completes the proof. □

Property 2

(1) Let $\alpha_i\,(i = 1, 2, \ldots, n)$ be a collection of IFNs; if $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$, then

$$\mathrm{GIFNWOGBM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha. \tag{45}$$

(2) Let $\alpha_i\,(i = 1, 2, \ldots, n)$ be a collection of IFNs; if $\alpha_1', \alpha_2', \ldots, \alpha_n'$ is any permutation of $\alpha_1, \alpha_2, \ldots, \alpha_n$, then

$$\begin{aligned}&\mathrm{GIFNWOGBM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n)\\ &= \mathrm{GIFNWOGBM}^{p,q}(\alpha_1', \alpha_2', \ldots, \alpha_n').\end{aligned} \tag{46}$$

(3) Let $\alpha_i = (\mu_{\alpha_i}, v_{\alpha_i})$ and $\beta_i = (\mu_{\beta_i}, v_{\beta_i})\,(i = 1, 2, \ldots, n)$ be two collection of IFNs; if $\mu_{\alpha_i} \le \mu_{\beta_i}$ and $v_{\alpha_i} \ge v_{\beta_i}\,(i = 1, 2, \ldots, n)$, then

$$\begin{aligned}&\mathrm{GIFNWOGBM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n)\\ &\le \mathrm{GIFNWOGBM}^{p,q}(\beta_1, \beta_2, \ldots, \beta_n).\end{aligned} \tag{47}$$

(4) Let $\alpha_i = (\mu_{\alpha_i}, v_{\alpha_i})\,(i = 1, 2, \ldots, n)$ be a collection of IFNs, and

$$\begin{aligned}\alpha^- &= \left(\min_{1\le i\le n}\{\mu_{\alpha_i}\}, \max_{1\le i\le n}\{v_{\alpha_i}\}\right),\\ \alpha^+ &= \left(\max_{1\le i\le n}\{\mu_{\alpha_i}\}, \min_{1\le i\le n}\{v_{\alpha_i}\}\right),\end{aligned} \tag{48}$$

then, $\alpha^- \le \mathrm{GIFNWOGBM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) \le \alpha^+$

The proof of Property 2 is similar to Property 1 and is not displayed.

Let us now further consider some specials of GIFNWOGBM.

Case 1: if $r \longrightarrow 0$, then GIFNWOGBM can be converted to intuitionistic fuzzy normalized weighted OGBM (IFNWOGBM):

$$\begin{aligned}&\mathrm{GIFNWOGBM}^{p,q,r}(\alpha_1, \alpha_2, \ldots, \alpha_n)\\ &= \left(\begin{array}{c} 1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\mu_{\alpha_i}\right)^p\left(1-\prod_{j=1, j\ne i}^{n}\left(1-\left(1-\mu_{\alpha_i}\right)^q\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}\right)^{(1/p+q)}\\ ,\left(1-\prod_{i=1}^{n}\left(1-v_{a_1}^p\left(1-\prod_{j=1, j\ne i}^{n}\left(1-v_{a_1}^p\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}\right)^{1/p+q}\end{array}\right).\end{aligned} \tag{49}$$

Case 2: if $p = 1$, $q \longrightarrow 0$, and $r \longrightarrow 0$, then GIFNWOGBM can be converted to IFWGM [16]:

$$\mathrm{GIFNWOGBM}^{p,q,r}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \otimes_{i=1}^{N}\alpha_i^{w_i} = \left(\prod_{i=1}^{n}\mu_{\alpha_i}^{w_i}, 1-\prod_{i=1}^{n}\left(1-v_{a_i}\right)^{w_i}\right). \tag{50}$$

Case 3: if $p = 2$, $q \longrightarrow 0$, and $r \longrightarrow 0$, then GIFNWOGBM can be converted to IFWSGM [3]:

$$\mathrm{GIFNWOGBM}^{p,q,r}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\mu_{\alpha_i}\right)^2\right)^{w_i}\right)^{1/2}, \left(1-\prod_{i=1}^{n}\left(1-v_{a_i}^2\right)^{w_i}\right)^{1/2}\right). \tag{51}$$

Case 4: if $p = 1$, $q = 1$, and $r \longrightarrow 0$, then GIFNWOGBM can be converted to generalized intuitionistic fuzzy normalized weighted optimized triple GBM (GIFNWOTGBM):

$$\begin{aligned}&\mathrm{GIFNWOGBM}^{p,q,r}\left(\alpha_1,\alpha_2,\ldots,\alpha_n\right)\\&=\left(\begin{array}{c}1-\left(1-\prod_{i=1}^{N}\left(1-\left(1-\mu_{\alpha_i}\right)\left(1-\prod_{j=1,j\neq i}^{N}\left(1-\left(1-\mu_{\alpha_j}\right)\left(1-\prod_{k=1,k\neq j\neq i}^{n}\left(1-\left(1-\mu_{\alpha_k}\right)\right)^{\left(w_k/1-w_i-w_j\right)}\right)\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}\right)^{1/3},\\ \left(1-\prod_{i=1}^{n}\left(1-v_{a_i}\left(1-\prod_{j=1,j\neq i}^{n}\left(1-v_{a_j}\left(1-\prod_{k=1,k\neq j\neq i}^{n}\left(1-v_{a_k}\right)^{\left(w_k/1-w_i-w_j\right)}\right)\right)^{\left(w_j/1-w_i\right)}\right)\right)^{w_i}\right)^{1/3}\end{array}\right).\end{aligned}\tag{52}$$

5. A Method of Intuitionistic Fuzzy MADM

Based on the IFNWOGBM and GIFNWOGBM below, we present a method to aggregate multicriteria information under intuitionistic fuzzy environment.

Let $X=\{x_1,x_2,\ldots,x_n\}$ be a set of n alternatives and $G=\{g_1,g_2,\ldots,g_l\}$ be a set of attributes with the weight vector $w_i\,(i=1,2,\ldots,n)^T$, $w_i\in[0,1]$, and $\sum_{i=1}^{n}w_i=1$. Suppose that the decision maker provides the intuitionistic fuzzy evaluated values under the attribute $g_j\in G$ for the alternative $x_i\in X$, denoted by a IFN $\alpha_{ij}=(\mu_{\alpha_{ij}},v_{\alpha_{ij}})$, and constructs the intuitionistic fuzzy decision matrix $H=(\alpha_{ij})_{n\times l}$.

Step 1: transform matrix $H=(\alpha_{ij})_{n\times l}$ into the positive matrix $\overline{H}=(\overline{\alpha}_{ij})_{n\times l}$, where $\overline{\alpha}_{ij}=\alpha_{ij}$, for positive index (the bigger the number, the better the evaluation) g_j; $\overline{\alpha}_{ij}=N(\alpha_{ij})=(v_{\alpha_{ij}},\mu_{\alpha_{ij}})$ for negative index (the smaller the number, the better the evaluation) g_j, $i=1,2,\ldots,n$ and $j=1,2,\ldots,l$.

Step 2: utilize the IFNWOGBM (or GIFNWOGBM) to aggregate the ith line of $\overline{H}=(\overline{\alpha}_{ij})_{n\times l}$:

$$\begin{aligned}\alpha_i&=\mathrm{IFNWOGBM}\left(\overline{\alpha}_{i1},\overline{\alpha}_{i2},\ldots,\overline{\alpha}_{il}\right),\quad i=1,2,\ldots,n,\\ \text{or }\alpha_i&=\mathrm{GIFNWOGBM}\left(\overline{\alpha}_{i1},\overline{\alpha}_{i2},\ldots,\overline{\alpha}_{il}\right),\quad i=1,2,\ldots,n,\end{aligned}\tag{53}$$

and get the comprehensive evaluation value α_i of alternative x_i.

Step 3: rank all the alternatives $x_i\,(i=1,2,\ldots,n)$ according to $\alpha_i\,(i=1,2,\ldots,n)$ in descending order by the binary relation described in Definition 2.

Example 1 (see [17]). In human history, library is the product of the development of human civilization to a certain stage. The development of culture, science, and technology has led to the emergence of books, the record carrier of knowledge and information, and the increase of books has produced an early library whose main function is to preserve books. It was the preservation function of the early library that preserved the excellent cultural and scientific achievements of mankind. In modern times, the function of the library has changed from book collection to borrowing, and the range of readers has expanded from senior intellectuals to ordinary people. Library plays an irreplaceable role in the continuous progress of mankind and the sustainable development of society. Libraries have gradually become the basis of sustainable social development. At present, more and more local governments are establishing public libraries.

Suppose a city wants to build a large, comprehensive, and modern public library. The city administrators need to decide what brand of air conditioning products used in library. Three aspects of alternative (whose weighting vector is $w=(0.3,0.5,0.2)^T$) are evaluated by experts, which are as follows:

g_1: economic

g_2: functional

g_3: operational

The five feasible alternatives $x_i\,(i=1,2,\ldots,5)$ are to be evaluated using IFNs under the above three criteria, and construct the following matrix, see Table 1:

Step 1: in this example, all criteria $g_i\,(i=1,2,3)$ are benefit-type criteria and do not need normalization

Step 2: utilize the IFNWOGBM (or GIFNWOGBM) to obtain the comprehensive evaluation value α_i of alternative x_i (see Tables 2 and 3).

Step 3: rank all the alternatives $x_i\,(i=1,2,\ldots,5)$ according to $\alpha_i\,(i=1,2,\ldots,5)$ in the descending order by the binary relation described in Definition 2 (see Table 4)

From Table 4, we can see that the decision-making results derived by the IFNWOGBM or GIFNWOGBM depend on the choice of parameters (p,q,r). Moreover, the optimal alternative given by the IFNWOGBM is different from that of the GIFNWOGBM. This is principally because the IFNWOGBM captures the interrelationship between any two aggregated arguments, but the GIFNWOGBM captures the interrelationship of any three aggregated arguments. Therefore, the GIFNWOGBM is more focused on the overall performance of aggregated arguments. As can be seen from Tables 2 and 3, the intuitionistic fuzzy comprehensive evaluation values derived by the IFNWOGBM or GIFNWOGBM are influenced by the input parameters p, q, and r. With larger or smaller parameters, it may lead to the degradation of recognition ability in decision-making process. For this reason, we recommend taking $p=q=r=1$, which not only makes the calculations easier but also interrelationship of the aggregated arguments can be fully take into account.

Table 1: Decision matrix.

	g_1	g_2	g_3
x_1	$\alpha_{11} = (0.3, 0.4)$	$\alpha_{12} = (0.7, 0.2)$	$\alpha_{13} = (0.5, 0.3)$
x_2	$\alpha_{21} = (0.5, 0.2)$	$\alpha_{22} = (0.4, 0.1)$	$\alpha_{23} = (0.7, 0.1)$
x_3	$\alpha_{31} = (0.4, 0.5)$	$\alpha_{32} = (0.7, 0.2)$	$\alpha_{33} = (0.4, 0.4)$
x_4	$\alpha_{41} = (0.2, 0.6)$	$\alpha_{42} = (0.8, 0.1)$	$\alpha_{43} = (0.8, 0.2)$
x_5	$\alpha_{51} = (0.9, 0.1)$	$\alpha_{52} = (0.6, 0.3)$	$\alpha_{53} = (0.2, 0.5)$

Table 2: The comprehensive evaluation value by IFNWOGBM.

p, q	x_1	x_2	x_3	x_4	x_5
$p = 10,\ q = 10$	$\alpha_1 = (0.459, 0.317)$	$\alpha_2 = (0.470, 0.000)$	$\alpha_3 = (0.452, 0.408)$	$\alpha_4 = (0.608, 0.316)$	$\alpha_5 = (0.465, 0.366)$
$p = 6,\ q = 6$	$\alpha_1 = (0.485, 0.304)$	$\alpha_2 = (0.480, 0.137)$	$\alpha_3 = (0.481, 0.386)$	$\alpha_4 = (0.612, 0.299)$	$\alpha_5 = (0.483, 0.353)$
$p = 2,\ q = 2$	$\alpha_1 = (0.522, 0.288)$	$\alpha_2 = (0.501, 0.132)$	$\alpha_3 = (0.527, 0.344)$	$\alpha_4 = (0.625, 0.259)$	$\alpha_5 = (0.547, 0.305)$
$p = 1,\ q = 1$	$\alpha_1 = (0.530, 0.284)$	$\alpha_2 = (0.507, 0.130)$	$\alpha_3 = (0.536, 0.335)$	$\alpha_4 = (0.633, 0.256)$	$\alpha_5 = (0.588, 0.279)$
$p = 1,\ q = 0.001$	$\alpha_1 = (0.508, 0.286)$	$\alpha_2 = (0.478, 0.131)$	$\alpha_3 = (0.529, 0.344)$	$\alpha_4 = (0.529, 0.310)$	$\alpha_5 = (0.544, 0.294)$
$H_s^1,\ q = 1$	$\alpha_1 = (0.463, 0.309)$	$\alpha_2 = (0.510, 0.139)$	$\alpha_3 = (0.484, 0.382)$	$\alpha_4 = (0.476, 0.358)$	$\alpha_5 = (0.511, 0.301)$

Table 3: The comprehensive evaluation value by GIFNWOGBM.

p, q, r	x_1	x_2	x_3	x_4	x_5
$p = 10,\ q = 10,\ r = 10$	$\alpha_1 = (0.460, 0.560)$	$\alpha_2 = (0.393, 0.503)$	$\alpha_3 = (0.482, 0.573)$	$\alpha_4 = (0.508, 0.697)$	$\alpha_5 = (0.683, 0.570)$
$p = 1,\ q = 5,\ r = 10$	$\alpha_1 = (0.318, 0.560)$	$\alpha_2 = (0.368, 0.503)$	$\alpha_3 = (0.356, 0.573)$	$\alpha_4 = (0.297, 0.697)$	$\alpha_5 = (0.486, 0.570)$
$p = 1,\ q = 1,\ r = 1$	$\alpha_1 = (0.476, 0.413)$	$\alpha_2 = (0.454, 0.355)$	$\alpha_3 = (0.489, 0.427)$	$\alpha_4 = (0.556, 0.434)$	$\alpha_5 = (0.683, 0.443)$
$p = 1,\ q = 0.5,\ r = 1$	$\alpha_1 = (0.483, 0.386)$	$\alpha_2 = (0.454, 0.308)$	$\alpha_3 = (0.490, 0.412)$	$\alpha_4 = (0.584, 0.407)$	$\alpha_5 = (0.662, 0.409)$
$p = 0.5,\ q = 1,\ r = 1$	$\alpha_1 = (0.440, 0.445)$	$\alpha_2 = (0.454, 0.429)$	$\alpha_3 = (0.462, 0.446)$	$\alpha_4 = (0.486, 0.449)$	$\alpha_5 = (0.669, 0.481)$
$p = 1,\ q = 1,\ r = 0.5$	$\alpha_1 = (0.493, 0.397)$	$\alpha_2 = (0.457, 0.314)$	$\alpha_3 = (0.504, 0.415)$	$\alpha_4 = (0.586, 0.401)$	$\alpha_5 = (0.674, 0.419)$

Table 4: Decision-making results.

IFNWOGBM operator		GIFNWOGBM operator	
p, q	Result	$p\ q, r$	Result
$p = 10,\ q = 10$	$x_2 \succ x_4 \succ x_1 \succ x_5 \succ x_3$	$p = 10\ q = 10,\ r = 10$	$x_5 \succ x_3 \succ x_1 \succ x_2 \succ x_4$
$p = 6,\ q = 6$	$x_2 \succ x_4 \succ x_1 \succ x_5 \succ x_3$	$p = 1,\ q = 1,\ r = 1$	$x_5 \succ x_2 \succ x_3 \succ x_1 \succ x_4$
$p = 2,\ q = 2$	$x_2 \succ x_4 \succ x_5 \succ x_1 \succ x_3$	$p = 1,\ q = 0.001,\ r = 0.001$	$x_5 \succ x_4 \succ x_2 \succ x_1 \succ x_3$
$p = 1,\ q = 1$	$x_2 \succ x_4 \succ x_5 \succ x_1 \succ x_3$	$p = 0.001,\ q = 1,\ r = 1$	$x_5 \succ x_4 \succ x_2 \succ x_1 \succ x_3$
$p = 1,\ q = 0.001$	$x_2 \succ x_5 \succ x_1 \succ x_4 \succ x_3$	$p = 1,\ q = 0.001,\ r = 1$	$x_5 \succ x_4 \succ x_2 \succ x_1 \succ x_3$
$H_s^1,\ q = 1$	$x_2 \succ x_5 \succ x_1 \succ x_4 \succ x_3$	$p = 0.5,\ q = 0.5,\ r = 0.5$	$x_5 \succ x_4 \succ x_2 \succ x_1 \succ x_3$

Table 5: Comparison with other methods.

	[17]	[8]	[12]	Proposed method
Information	Intuitionistic fuzzy number	Intuitionistic fuzzy number	Intuitionistic fuzzy number	Intuitionistic fuzzy number
Aggregation	IFWBM	IFWGBM	IFOWGBM	IFNWOGBM/ GIFNWOGBM
Conditions	Intuitionistic fuzzy number, attribute weights are known	Intuitionistic fuzzy number, attribute weights are known	Intuitionistic fuzzy number, attribute weights are known	Intuitionistic fuzzy number, attribute weights are known
Result	Ranking of alternatives	Ranking of alternatives	Ranking of alternatives	Ranking of alternatives
Example 1 $p = q = 1$	$x_2 \succ x_5 \succ x_1 \succ x_4 \succ x_3$	$x_2 \succ x_4 \succ x_5 \succ x_1 \succ x_3$	$x_2 \succ x_4 \succ x_5 \succ x_1 \succ x_3$	$x_2 \succ x_4 \succ x_1 \succ x_5 \succ x_3$

6. Comparative Analysis with Other Methods

To verify the effectiveness of the proposed method, it is compared with the classical MADM method based on the aggregation operator. Comparative analysis results are shown in Table 5.

As can be seen from Table 5, the primary difference in the above four methods is in the aggregation operators. Xu and Yager [17] use IFWBM based on average mean, but the operators of other three kinds of methods based on geometric mean. The IFWBM and IFWGBM do not mine the interrelationship between attributes in the process of information aggregation, while the IFOWGBM and IFNWOGBM (GIFNWOGBM) take this into account. Moreover, unlike the other three kinds of operators, the IFNWOGBM and GIFNWOGBM have the reducibility. However, any kind of operators' flaws and certain restrictions would exist in the aggregation of decision information. Therefore, the decision makers should choose the proper operators according to practical circumstances for further improving the effect of decision-making.

7. Conclusions

Based on the normalized weighted BM and geometric mean, this paper introduces the OGBM, the NWOGBM, and the GNWOGBM. The three kinds of operators designed in this paper have strong operability and can effectively capture the correlation between decision evaluation indexes.

To deal with the MADM problem that the criteria evaluation information is intuitionistic fuzzy numbers, an approach has been proposed on the basis of the IFNWOGBM and GIFNWOGBM. Finally, the practicability and validity of the approach are verified with a case of library location.

Hopefully, we will use the proposed operators to solve the problems of criminal identification, optimal financial scheme selection, optimal driving path selection, and so on and extend the operators proposed in this paper to other fuzzy environments, such as hesitant fuzzy language environment.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] J.-B. Liu, Y. Bao, W. T. Zheng, and S. Hayat, "Network coherence analysis on a family of nested weighted n-polygon networks," *Fractals*, vol. 29, no. 8, p. 2150260, 2021.

[2] Z. Xu and H. Liao, "Intuitionistic fuzzy analytic hierarchy process," *IEEE Transactions on Fuzzy Systems*, vol. 22, no. 4, pp. 749–761, 2014.

[3] Z. S. Xu, *Intuitionistic Preference Modeling and Interactive Decision Making*, Springer, Berlin, Germany, 2014.

[4] Z. S. Xu and X. Q. Cai, *Intuitionistic Fuzzy Information Aggregation*, Springer, Berlin, Germany, 2012.

[5] K. Kumar and S. M. Chen, "Multiattribute decision making based on the improved intuitionistic fuzzy Einstein weighted averaging operator of intuitionistic fuzzy values," *Information Sciences*, vol. 568, pp. 369–383, 2021.

[6] X. Y. Zou, S. M. Chen, and K. Y. Fan, "Multiple attribute decision making using improved intuitionistic fuzzy weighted geometric operators of intuitionistic fuzzy values," *Information Sciences*, vol. 535, pp. 242–253, 2020.

[7] X. Jia and Y. Wang, "Choquet integral-based intuitionistic fuzzy arithmetic aggregation operators in multi-criteria decision-making," *Expert Systems with Applications*, vol. 191, Article ID 116242, 2022.

[8] M. Xia, Z. Xu, and B. Zhu, "Geometric bonferroni means with their application in multi-criteria decision making," *Knowledge-Based Systems*, vol. 40, pp. 88–100, 2013.

[9] A. H. Mahmoodi, S. J. Sadjadi, S. Sadi-Nezhad, R. Soltani, and F. M. Sobhani, "Linguistic Z-number bonferroni mean and linguistic Z-number geometric bonferroni mean operators: their applications in portfolio selection problems," *IEEE Access*, vol. 8, pp. 98742–98760, 2020.

[10] A. Devaraj and S. Broumi, "Aldring an MCDM method under neutrosophic cubic fuzzy sets with geometric bonferroni mean operator," *Neutrosophic Sets and Systems*, vol. 32, pp. 187–202, 2020.

[11] L. Huang, X. Yuan, and Y. Ren, "The q-Rung orthopair hesitant fuzzy uncertain linguistic aggregation operators and their application in multi-attribute decision making," *IEEE Access*, vol. 8, pp. 187084–187113, 2020.

[12] J. H. Park and J. Y. Kim, "Kwun intuitionistic fuzzy optimized weighted geometric bonferroni means and their applications in group decision making," *Fundamenta Informaticae*, vol. 144, no. 3-4, pp. 363–381, 2016.

[13] R. Tao, Z. Liu, R. Cai, and K. H. Cheong, "A dynamic group MCDM model with intuitionistic fuzzy set: perspective of alternative queuing method," *Information Sciences*, vol. 555, pp. 85–103, 2021.

[14] J. Moko and D. Hnar, "On unification of methods in theories of fuzzy sets, hesitant fuzzy set," *Fuzzy Soft Sets and Intuitionistic Fuzzy Sets*, vol. 9, pp. 1–26, 2021.

[15] W. Zhou, "On hesitant fuzzy reducible weighted bonferroni mean and its generalized form for multicriteria aggregation," *Journal of Applied Mathematics*, vol. 2014, Article ID 954520, 10 pages, 2014.

[16] Z. Xu and R. R. Yager, "Some geometric aggregation operators based on intuitionistic fuzzy sets," *International Journal of General Systems*, vol. 35, no. 4, pp. 417–433, 2006.

[17] Z. S. Xu and R. R. Yager, "Intuitionistic fuzzy bonferroni means," *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, vol. 41, no. 2, pp. 568–578, 2011.