Research Article

K frames for Banach Spaces

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1. Introduction

In 1952, Duffin and Schaeffer [1] introduced the notion of frames in nonharmonic Fourier analysis. Regrettably, their work was not continued until 1986 when Daubechies et al. [2] applied the theory of frames to wavelet and Gabor transform. Subsequently, the theory of frames began to be deeply studied and widely applied to signal processing, image processing, data compressing, sampling theory, and so on.

In Hilbert spaces, some different concepts of frames, such as ordinary frames, frames of subspaces [3], and G frames [4], have been proposed one after another in the recent 20 years. In 2012, Gavruta [5] introduced the concept of K-frames for Hilbert spaces, which generalized the concept of ordinary frames for Hilbert spaces, to reconstruct elements from the range of a linear bounded operator K on a Hilbert space. In [6], Xiao et al. gave some methods to construct K-frames and discussed the stability of K-frames for Hilbert spaces. In [7], Neyshaburi and Arefijamaal focused on the reconstruction formulation to characterize all K -duals of a given K-frame and so gave several approaches for constructing K-frames for Hilbert spaces. In 2019, Jia and Zhu [8] studied the stabilities of K-frames under the operator perturbation. Furthermore, in [9], Shamsabadi and Arefijamaal studied some properties of K-frames, introduced the K-frame multipliers, and focused on representing elements from the range of K by K-frame multipliers. For more works about K-frames, please refer to Refs. [10–14].

The theory of frames for Banach spaces originated from the works of Grochenig [15]. In [15], Grochenig introduced the notion of Banach frames and discussed the atomic decompositions versus frames. Subsequently, Aldroubi et al. [16] proposed p-frames for Banach spaces. In 2005, Casazza et al. [17] generalized the concept of p-frames and introduced Xp-frames for Banach spaces, where Xp was a BK-space. In 2008, Cao et al. [18] introduced the (p, Y)-operator frames for Banach spaces, where Y was a Banach space. (p, Y)-operator frames cover p-frames and Xp-frames. More works about the theory of frames for Banach spaces can be seen in Refs. [19–21].

Inspired by the works in [5, 17] and in order to reconstruct elements from the range of a bounded linear operator K on a separable Banach space, we will introduce the concepts of K-frames and K*-atomic systems for Banach spaces. The concept of K-frames for Banach spaces is not only a generalization of Xp-frames but also a generalization of K-frames for Hilbert spaces. In this paper, we will discuss the relationship between K-frames and K*-atomic systems for Banach spaces and study the sufficient condition for a Banach space to have a K-frame. In addition, in order to study the duality of K-frames for Banach spaces, we will introduce the concepts of the pair of (K, K*)-dual Bessel...
sequences and the pair of approximate \((K, K^*)\)-dual Bessel sequences for Banach spaces and discuss their important properties.

The rest of the paper is organized as follows. In Section 2, we review some existing concepts, such as BK-spaces, \(X_d\)-frames, and \(X_d^*\)-Riesz bases, and give several important lemmas, which will be used in proving the following main results. In Section 3, we introduce \(K\)-frames and \(K^*\)-atomic systems for separable Banach spaces, discuss the relationship between \(K\)-frames and \(K^*\)-atomic systems, and study some properties of \(K\)-frames. In Section 4, we introduce the pair of \((K, K^*)\)-dual Bessel sequences and the pair of approximate \((K, K^*)\)-dual Bessel sequences and study their properties.

Throughout this paper, let \(F\) be the field \(\mathbb{C}\) of all complex numbers or the field \(\mathbb{R}\) of all real numbers, \(X, Y\) be separable Banach spaces, \(X^*\) be the dual space of \(X\), and \(X^{**}\) be the dual of \(X^*\). Assume that \(I_X\) is the natural map of \(X\) into its second dual:

\[
I_X : X \longrightarrow X^{**}, \quad J_X x = \tilde{x} \in X^{**}, \quad \langle x, \tilde{x} \rangle = \langle x, x^* \rangle, \quad \forall x^* \in X^*.
\]

where \(\langle x, x^* \rangle = x^*(x)\) and \(\langle x^*, \tilde{x} \rangle = \tilde{x}(x^*)\). Then, \(X\) can be seen as a subspace of \(X^{**}\), and the map \(I_X\) is a linear isometry with \(\|I_X x\| = \|x\|\). In this paper, we use \(L(X, Y)\) to denote the space of all bounded linear operators from \(X\) into \(Y\). If \(X = Y\), \(L(X, Y)\) is written as \(L(X)\). For given \(T \in L(X, Y)\), we denote the range of \(T\) by \(\text{ran}(T)\).

2. Preliminaries

In this section, some important concepts and propositions that will be used in this paper are reviewed.

A sequence space \(X_d\) is called a BK-space (see [17]), if it is a Banach space and the coordinate functionals are continuous on \(X_d\); i.e., the relations \(x_n = \{a_i^{(n)}\}_{n=1}^{\infty}, x = \{a_i\}_{n=1}^{\infty} \in X_d, \lim_{n \to \infty} x_n = x \) imply \(\lim_{n \to \infty} a_i^{(n)} = a_i, \forall i \in \mathbb{N}\).

Example 1. Let \(C\) be the set of all complex numbers. For \(1 \leq p < \infty\), we denote

\[
\begin{align*}
\ell^p &= \{c_i^{(p)} : c_i \in \mathbb{C}, i = 1, 2, \ldots, \sum_{i=1}^{\infty} |c_i|^p < \infty\}, \\
\ell^\infty &= \{c_i^{(\infty)} : c_i \in \mathbb{C}, i = 1, 2, \ldots, \sup_i |c_i| < \infty\}.
\end{align*}
\]

Then, all spaces \(\ell^p\) are BK-spaces.

Let \(X_d\) be a BK-space and the canonical unit vectors \(\{e_i\}_{i=1}^{\infty}\) form a Schauder basis for \(X_d\); then, the following space

\[
Y_d = \{\{c_i e_i^*\}_{i=1}^{\infty} : c_i \in X_d^*\}
\]

with the norm \(\|\{c_i e_i^*\}_{i=1}^{\infty}\|_{X_d^*} = \|c^*\|_{X_d^*}\) is also a BK-space.

Moreover, Lemma 3.1 in [17] implies that \((Y_d, \|\cdot\|)\) is isometrically isomorphic to \(X_d^*\). In this paper, we always assume that the canonical unit vectors \(\{e_i\}_{i=1}^{\infty}\) form a Schauder basis for \(X_d\) and no longer distinguish between \(Y_d\) and \(X_d^*\).

In [17], Casazza et al. introduced the concept of \(X_d\)-frames for Banach spaces. Let \(X_d\) be a BK-space. A family \(\mathcal{F} = \{f_i\}_{i=1}^{\infty} \subset X^*\) is called an \(X_d\)-frame for \(X\) if it satisfies

\[
\langle \{x, f_i\}_{i=1}^{\infty} \rangle_{i=1}^{\infty} \in X_d, \forall x \in X, \text{ and there exist constants } A, B > 0 \text{ such that}
\]

\[
A \|x\| \leq \|\langle \{x, f_i\}_{i=1}^{\infty} \rangle_{i=1}^{\infty}\|_{X_d} \leq B \|x\|, \quad \forall x \in X,
\]

where \(A, B\) are, respectively, called a lower frame bound and an upper frame bound for \(\mathcal{F}\).

Specifically, if \(X_d = \ell^p, 1 < p < \infty\), we obtain the concept of \(p\)-frame for Banach spaces (see [16, 20]).

In (4), if only the upper condition is satisfied, the family \(\{g_i\}_{i=1}^{\infty}\) is called an \(X_d^*\)-Bessel sequence for \(X\). Let \(\mathcal{F} = \{g_i\}_{i=1}^{\infty}\) be an \(X_d^*\)-Bessel sequence for \(X\) and the operator

\[
R_{\mathcal{F}} : X \longrightarrow X_d
\]

be defined by

\[
R_{\mathcal{F}}(x) = \{\langle x, g_i \rangle\}_{i=1}^{\infty}, \quad \forall x \in X.
\]

It is clear that \(R_{\mathcal{F}}\) is a linear bounded operator. \(R_{\mathcal{F}}\) usually called the analysis operator of \(\mathcal{F}\). Furthermore, another important operator \(T_{\mathcal{F}}\) is defined as follows:

\[
T_{\mathcal{F}} : X_d^* \longrightarrow X^*, \quad T_{\mathcal{F}}(\{c_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} c_i g_i, \quad \forall \{c_i\}_{i=1}^{\infty} \in X_d^*.
\]

If \(\mathcal{F}\) is an \(X_d^*\)-Bessel sequence, it is easy to prove that \(T_{\mathcal{F}}\) is well defined and linear bounded. Here, \(T_{\mathcal{F}}\) is called the synthesis operator of \(\mathcal{F}\).

A family \(\{g_i\}_{i=1}^{\infty} \subset X^*\) is said to be independent, if the following condition is satisfied:

\[
\sum_{i=1}^{\infty} c_i g_i = 0, \quad \{c_i\}_{i=1}^{\infty} \in X_d \Rightarrow c_i = 0, \quad i = 1, 2, \ldots.
\]

Definition 2. Let \(X\) be a Banach space and \(X_d\) be a BK-space. A family \(\mathcal{G} = \{g_i\}_{i=1}^{\infty} \subset X^*\) is called an \(X_d^*\)-Riesz basis for \(X\) if the following conditions are satisfied:

(i) \(\overline{\text{span}}\{g_i\} = X^*\)

(ii) There exist constants \(C, D > 0\) such that

\[
C \|\{c_i\}_{i=1}^{\infty}\|_{X_d^*} \leq \left\|\sum_{i=1}^{\infty} c_i g_i\right\| \leq D \|\{c_i\}_{i=1}^{\infty}\|_{X_d^*}, \quad \forall \{c_i\}_{i=1}^{\infty} \in X_d^*.
\]
Specifically, when $X_d = \ell^q$, $1 < p, q < \infty$, $(1/p) + (1/q) = 1$, we call $\mathcal{S}$ a $q$-Riesz basis for $X^*$. Corollary 2.5 in [20] implied that a $q$-Riesz basis $\mathcal{S}$ with bounds $C, D$ is always a $p$-frame for $X$ and $C, D$ are still its frame bounds.

**Lemma 3.** Assume that $T_1 \in L(X_1, X)$, $T_2 \in L(X_2, X)$ are two bounded operators and $T_1$ is injective. If ran $(T_1) \subset$ ran $(T_2)$, then there exists a bounded operator $T \in L(X_1, X_2)$ such that $T_1 = T_2 T$.

**Proof.** Suppose that ran $(T_1) \subset$ ran $(T_2)$, then for every $f \in X_1$, we have $T_1 f \in$ ran $(T_1) \subset$ ran $(T_2)$. Since $T_2$ is injective, there exists a unique element $h \in X_2$ such that $T_2 h = T_1 f$. Set $T f = h$. Then, $T_1 = T_2 T$. Now, we will prove that $T$ is bounded.

In fact, let $\{ (f_i, h_i) \}$ be a sequence of elements in the graph of $T$ such that $\lim_{i \to \infty} (f_i, h_i) = (f, h)$. Since $\lim_{i \to \infty} T_1 f_i = T_1 f$ and $\lim_{i \to \infty} T_2 h_i = T_2 h$, we have $T_1 f = T_2 h$. Thus, $T f = h$. This shows that $T$ is bounded on $X_1$. □

**Lemma 4** (von Neumann). Let $X$ be a Banach space, $T \in L(X)$, and $I$ be the identity operator on $X$. If $\| I - T \| < 1$, then $T$ is invertible and $T^{-1} = I + \sum_{i} (I - T)^i$.

**3. K-frames and $K^*$-atomic Systems for Banach Spaces**

In this section, we will introduce $K$-frames and $K^*$-atomic systems for Banach spaces and study the properties of $K$-frames and $K^*$-atomic systems.

**Definition 5.** Let $X_d$ be a BK-space and $X$ be a Banach space, $K \in L(X)$. A family $\mathcal{S} = \{ g_i \}_{i=1}^\infty \subset X^*$ is called a $K$-frame with respect to $X_d$ for $X$ if the following conditions are satisfied:

(i) For each $x \in X$, the sequence $\{ \langle x, g_i \rangle \}_{i=1}^\infty \in X_d$

(ii) There exist constants $A, B > 0$ such that

$$A \| K x \| \leq \| \{ \langle x, g_i \rangle \}_{i=1}^\infty \|_{X_d} \leq B \| x \|, \quad \forall x \in X,$$  

where $A, B$ are, respectively, called a lower frame bound and an upper frame bound of the $K$-frame $\mathcal{S}$. In particular, $\mathcal{S}$ is called a tight $K$-frame if the following condition is satisfied:

$$A \| K x \| = \| \{ \langle x, g_i \rangle \}_{i=1}^\infty \|_{X_d}, \quad \forall x \in X.$$  

**Example 6.** Let $X = X_d = \ell^1$. Then, the canonical unit vectors sequence $\{ e_i \}_{i=1}^\infty \subset X^* = \ell^\infty$. For every $x = \{ c_i \}_{i=1}^\infty \in X$, we define

$$K x = K\{ c_i \}_{i=1}^\infty = (c_2, c_3, \ldots, c_{n+1}, \ldots).$$  

Clearly, $K$ is a bound linear operator. Thus, we have

$$\| \{ \langle x, e_i \rangle \}_{i=1}^\infty \|_{X_d} \leq \| \{ c_i \}_{i=1}^\infty \| = \sum_{i=1}^\infty |c_i| = \| x \|,$$

$$\| K x \| = \| (c_2, c_3, \ldots, c_{n+1}, \ldots) \| \leq \sum_{i=1}^\infty |c_i| = \| x \| = \| \{ \langle x, e_i \rangle \}_{i=1}^\infty \|_{X_d}.\]$$

This shows that $\{ e_i \}_{i=1}^\infty$ is a $K$-frame with respect to $X_d$ for $X$.

**Remark 7.** Since an $X_d$-frame $\mathcal{S} = \{ g_i \}_{i=1}^\infty \subset X^*$ for $X$ must be an $I$-frame for $X$, where $I$ is the identity operator on $X$, the concept of $K$-frames for Banach spaces is a generalization of the concept of $X_d$-frames.

Let $\{ e_i \}_{i=1}^\infty$ be a Schauder basis for $X$. We define the space as follows:

$$\mathcal{Y} = \{ \{ c_i \}_{i=1}^\infty : c_i \in \mathbb{F}, i = 1, 2, \ldots, \sum_{i=1}^\infty c_i e_i \text{ converges in } X \},$$

$$\| \{ c_i \}_{i=1}^\infty \|_{\mathcal{Y}} = \sup_{N} \| \sum_{i=1}^N c_i e_i \|,$$  

and the operator

$$S : \mathcal{Y} \to X, S(\{ c_i \}_{i=1}^\infty) = \sum_{i=1}^\infty c_i e_i, \quad \forall \{ c_i \}_{i=1}^\infty \in \mathcal{Y}.\]$$

Then, $(\mathcal{Y}, \| \cdot \|_{\mathcal{Y}})$ is a Banach space and $S$ is a topological isomorphic mapping. Moreover, we can prove that $\mathcal{Y}$ is a BK-space. Indeed, let $P_n$ be the $n$-th coordinate functional on $\mathcal{Y}$, i.e., $P_n(\{ c_i \}_{i=1}^\infty) = c_n, n = 1, 2, \ldots$. Suppose that $\{ c_i \}_{i=1}^\infty \|_{Y} \to 0$. Then, for every $N$, we have $\| \sum_{i=1}^N c_i e_i \| \to 0$. When $N = 1$, we get $\| c_1 e_1 \| = |c_1| \| e_1 \| \to 0 (c_1 \neq 0)$, so $c_1 \to 0$. If $N = 2$, we have $\| \sum_{i=1}^2 c_i e_i \| \to 0$. Since

$$\| c_2 e_2 \| = \| c_1 e_1 + c_2 e_2 - c_1 e_1 \| \leq \sum_{i=1}^2 |c_i e_i| + \| c_1 e_1 \| \to 0,$$  

we get $c_2 \to 0$. And so on, $c_n \to 0, \forall n$. This shows that every coordinate functional $P_n(n = 1, 2, \ldots)$ is continuous on $\mathcal{Y}$. Hence, $\mathcal{Y}$ is a BK-space.

**Theorem 8.** Let $X$ be a separable Banach space with a Schauder basis and $K \in L(X)$. Then, $X$ has a $K$-frame.
Proof. Let \( \{\varepsilon_i\}_{i=1}^\infty \) be a Schauder basis for \( X \). Then, for all \( x \in X \), we have

\[
\begin{aligned}
x &= \sum_{i=1}^{\infty} a_i(x) \varepsilon_i = \sum_{i=1}^{\infty} \langle x, a_i \rangle \varepsilon_i,
\end{aligned}
\]

where \( \{a_i\} \) are the coefficient functionals and \( \{\varepsilon_i\}_{i=1}^\infty \) are biorthogonal.

Put \( g_i = K^* a_i, i = 1, 2, \ldots \); then, for every \( x \in X \), we have

\[
\sum_{i=1}^{\infty} g_i(x) \varepsilon_i = \sum_{i=1}^{\infty} \langle x, g_i \rangle \varepsilon_i = \sum_{i=1}^{\infty} \langle x, K^* a_i \rangle \varepsilon_i = \sum_{i=1}^{\infty} \langle Kx, a_i \rangle \varepsilon_i = \sum_{i=1}^{\infty} a_i(Kx) \varepsilon_i = Kx.
\]

(17)

This shows that \( \sum_{i=1}^{\infty} g_i(x) \varepsilon_i \) converges in \( X \) and \( \{g_i(x)\}_{i=1}^\infty \in \mathcal{Y} \), where \( \mathcal{Y} \) is the following space:

\[
\mathcal{Y} = \left\{ \{c_i\}_{i=1}^\infty : c_i \in F, i = 1, 2, \ldots \right\}
\]

(19)

Define the operator

\[
S(\{c_i\}_{i=1}^\infty) = \sum_{i=1}^{\infty} c_i \varepsilon_i.
\]

(20)

Then, for every \( x \in X \), we have

\[
\begin{aligned}
\left\| \{\langle x, g_i \rangle\}_{i=1}^\infty \right\|_{\mathcal{Y}} &= \sup_{x} \left\| \sum_{i=1}^{N} \langle x, g_i \rangle \varepsilon_i \right\| \\
&= \sup_{x} \left\| \sum_{i=1}^{N} \langle x, K^* a_i \rangle \varepsilon_i \right\| \\
&= \sup_{x} \left\| \sum_{i=1}^{N} a_i(Kx) \varepsilon_i \right\| \\
&= \left\| \sum_{i=1}^{N} a_i(Kx) \varepsilon_i \right\| \\
&\leq \left\| S^{-1}(Kx) \right\| \cdot \| K \| \cdot \| x \|.
\end{aligned}
\]

(21)

Moreover, we get

\[
\begin{aligned}
\left\| Kx \right\| &= \left\| \sum_{i=1}^{\infty} a_i(Kx) \varepsilon_i \right\| \\
&= \left\| \sum_{i=1}^{\infty} \langle x, a_i \rangle \varepsilon_i \right\| \\
&= \left\| \sum_{i=1}^{\infty} \langle x, K^* a_i \rangle \varepsilon_i \right\| \\
&= \left\| \sum_{i=1}^{\infty} a_i(Kx) \varepsilon_i \right\| \\
&\leq \sup_{x} \left\| \sum_{i=1}^{N} a_i(Kx) \varepsilon_i \right\| \\
&\leq \left\| S^{-1}(Kx) \right\| \cdot \| K \| \cdot \| x \|.
\end{aligned}
\]

(22)

The above two inequalities yield that

\[
\| Kx \| \leq \left\| \{g_i(x)\}_{i=1}^\infty \right\|_{\mathcal{Y}} \leq B \| x \|, \quad \forall x \in X,
\]

(23)

where \( B = \| S^{-1} \| \cdot \| K \| \). Hence, \( \{g_i\}_{i=1}^\infty \) is a K-frame with respect to \( \mathcal{Y} \) for \( X \).

**Definition 9.** Let \( X \) be a Banach space and \( K \in L(X) \). A family \( \{g_i\}_{i=1}^\infty \subset X^* \) is called a K*-atomic system with respect to \( X^*_d \) for \( X^* \) if the following conditions are satisfied:

(i) For every \( \{c_i\}_{i=1}^\infty \in X^*_d \), \( \sum_{i=1}^{\infty} c_i g_i \) converges

(ii) For every \( x^* \in X^* \), there exist a sequence \( c_x^* = \{c_i\}_{i=1}^\infty \in X^*_d \) and a constant \( C > 0 \) such that \( \|\{c_i\}_{i=1}^\infty\| \leq C \|x^*\| \) and \( K^* x^* = \sum_{i=1}^{\infty} c_i g_i \).

Now, we will discuss the relationship between K-frames and K*-atomic systems for Banach spaces.

**Theorem 10.** Let \( X_d \) be a BK-space and \( \{g_i\}_{i=1}^\infty \subset X^* \) be a K*-atomic system with respect to \( X^*_d \) for \( X^* \); then, it is a K-frame with respect to \( X_d \) for \( X \).

Proof. Suppose that \( \{g_i\}_{i=1}^\infty \) is a K*-atomic system with respect to \( X^*_d \) for \( X^* \). Then,

\[
\| Kx \| = \sup_{x^*} \|\langle Kx, x^* \rangle\| = \sup_{x^*} \|\langle x, K^* x^* \rangle\|.
\]

(24)

For the same definition of the K*-atomic system, it follows that there exist a sequence \( c_x^* = \{c_i\}_{i=1}^\infty \in X^*_d \) and a constant \( C > 0 \) such that \( \|\{c_i\}_{i=1}^\infty\| \leq C \|x^*\| \) and \( K^* x^* = \sum_{i=1}^{\infty} c_i g_i \). Hence, we have

\[
\| Kx \| = \sup_{x^*} \|\langle Kx, x^* \rangle\| = \sup_{x^*} \|\langle x, K^* x^* \rangle\|
\]

\[
= \sup_{x^*} \left| \sum_{i=1}^{\infty} c_i g_i \right| = \sup_{x^*} \left| \sum_{i=1}^{\infty} \langle x, g_i \rangle \right|
\]

\[
\leq \sup_{x^*} \sum_{i=1}^{\infty} \| c_i g_i \| \cdot \| x \| \leq C \|x^*\| \| \{g_i\}_{i=1}^\infty \|_{X^*_d}
\]

(25)

On the other hand, define the operator:

\[
T : X^*_d \rightarrow X^* \ni T(\{c_i\}_{i=1}^\infty) = \sum_{i=1}^{\infty} c_i g_i.
\]

(26)

Then, \( \sum_{i=1}^{\infty} c_i g_i \) converges in \( X^* \). By the Banach-Steinhaus theorem, we can conclude that \( T \) is bounded. Denote the adjoint of the operator \( T \) by \( T^* \); then, for every \( \{c_i\}_{i=1}^\infty \in X^*_d \) and \( x \in X \), the operator \( T^* \) can be determined by the following calculation:
\[ \langle x, T(\{c_i\}_{i=1}^{\infty}) \rangle = \left\langle x, \sum_{i=1}^{\infty} c_i g_i \right\rangle = \sum_{i=1}^{\infty} \langle x, c_i g_i \rangle = \langle x, \{c_i\}_{i=1}^{\infty} \rangle = \langle T^* x, \{c_i\}_{i=1}^{\infty} \rangle, \]
\[
\text{for } i=1, 2, \ldots.
\]

Thus, we have
\[ \left\| \{x, g_i\}_{i=1}^{\infty} \right\|_X = \|T^* x\| \leq \|T^*\| \cdot \|x\|, \quad \forall x \in X. \quad (29) \]

Therefore, \( \{g_i\}_{i=1}^{\infty} \) is a K-frame with respect to \( X_d \) for \( X \).

\[ \\Box \]

**Theorem 11.** Let \( 1 < p < \infty \) and \( \mathcal{F} = \{f_i\}_{i=1}^{\infty} \) be a K-frame with respect to \( \ell^p \) for \( X \). If \( \mathcal{F} \) is independent and the operators \( K, T_{\mathcal{F}} \) satisfy \( \text{ran}(K^*) \subset \text{ran}(T_{\mathcal{F}}) \), then \( \mathcal{F} \) is a \( K^* \)-atomic system with respect to \( \ell^p \) for \( X^* \), where \( T_{\mathcal{F}} \) is the synthesis operator of \( \mathcal{F} \).

**Proof.** Suppose that \( \{g_i\}_{i=1}^{\infty} \) is a K-frame with respect to \( \ell^p \) for \( X \) with frame bounds \( A, B > 0 \) and \( T_{\mathcal{F}} \) is the synthesis operator of \( \mathcal{F} \); then, we have

\[ T_{\mathcal{F}}: \ell^p \longrightarrow X^*, \quad T_{\mathcal{F}}(\{c_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} c_i g_i. \quad (30) \]

Since the canonical unit vectors \( \{e_i\}_{i=1}^{\infty} \) form a Schauder basis for \( \ell^p \), we can get

\[ g_i = T_{\mathcal{F}} e_i, \quad i = 1, 2, \ldots. \quad (31) \]

Assume that \( \mathcal{F} \) is independent and ran \( (K^*) \subset \text{ran}(T_{\mathcal{F}}) \); then, the operator \( T_{\mathcal{F}} : \ell^p \to X^* \) is injective and there exists a bounded operator \( C \in L(X^*, \ell^p) \) such that \( K^* = T_{\mathcal{F}} C \) by using Lemma 3. For every \( x^* \in X^* \), since \( C x^* \in \ell^q \), we denote the \( i \)-th element of \( C x^* \) by \( c_i x^* \), \( i = 1, 2, \ldots; \)

\[ \left\| \{c_i\}_{i=1}^{\infty} \right\| = \|C x^*\| \leq \|C\| \cdot \|x^*\|. \quad (32) \]

and we have
\[ K^* x^* = T_{\mathcal{F}} C x^* = \sum_{i=1}^{\infty} c_i g_i x^* \quad \forall x^* \in X^*. \quad (33) \]

This shows that \( \{g_i\}_{i=1}^{\infty} \) is a \( K^* \)-atomic system with respect to \( \ell^q \) for \( X^* \).

We give a sufficient condition for a p-Bessel sequence to become a K-frame by choosing a linear bounded operator \( K \).

\[ \Box \]

**Theorem 12.** Let \( 1 < p, q < \infty, (1/p) + (1/q) = 1 \) and \( \mathcal{F} = \{g_i\}_{i=1}^{\infty} \subset X^* \) be a p-Bessel sequence for \( X \). If \( X \) has a p-Riesz basis, then there exists a linear bounded operator \( K \in L(X) \) such that \( \mathcal{F} \) is a K-frame with respect to \( \ell^p \) for \( X \).

**Proof.** Let \( \mathcal{F} = \{f_i\}_{i=1}^{\infty} \subset X \) be a p-Riesz basis for \( X \) with bounds \( A, B \). By using Corollary 2.5 in [20], \( \mathcal{F} \) is a q-frame for \( X^* \) with bounds \( A, B \). In addition, from Theorem 2.8 in [20], it follows that there exists a unique \( q \)-Riesz basis \( \{h_i\}_{i=1}^{\infty} \subset X^* \) for \( X^* \) such that

\[ x = \sum_{i=1}^{\infty} \langle x, h_i \rangle f_i, \quad \forall x \in X, \quad x^* = \sum_{i=1}^{\infty} \langle f_i, x^* \rangle h_i, \quad \forall x^* \in X^*. \quad (34) \]

Set \( U_h = g_i, i = 1, 2, \ldots; \) then, \( U \) is a bounded operator from \( X^* \) to \( X^* \). Indeed, for every \( x^* \in X^* \), we have

\[ U \left( \sum_{i=1}^{N} \langle f_i, x^* \rangle h_i \right) = \sum_{i=1}^{N} \langle f_i, x^* \rangle g_i = U x^*. \quad (36) \]

This shows that \( U \) is well defined. Let \( K \in L(X^*) \) be the analysis operator of \( \mathcal{F} \) \( T_{\mathcal{F}} \) be the synthesis operator of \( \mathcal{F} \). The above equation yields that \( U = T_{\mathcal{F}} K \). Hence, \( U \) is bounded.

Put \( K = U^* \); then, for every \( x \in X \), we have

\[ \|K x\| = \sup_{\|x^*\| \leq 1} \left| \langle K x, x^* \rangle \right| = \sup_{\|x^*\| \leq 1} \left( \sum_{i=1}^{\infty} \langle K x, h_i \rangle f_i, x^* \right) \leq \sup_{\|x^*\| \leq 1} \left( \sum_{i=1}^{\infty} \langle x, K^* h_i \rangle f_i, x^* \right)^{1/p} \]
\[ \leq \sup_{\|x^*\| \leq 1} \left( \sum_{i=1}^{\infty} \langle x, U h_i \rangle f_i, x^* \right)^{1/p} \leq B \left( \sum_{i=1}^{\infty} \|\langle x, g_i \rangle\|^q \right)^{1/q}. \quad (37) \]

Since \( \{g_i\}_{i=1}^{\infty} \subset X^* \) is a Bessel sequence, the above inequality yields that it is a K-frame with respect to \( \ell^q \) for \( X^* \).

\[ \Box \]

**4. The Duality of K-frames**

In this section, we will study the duality of K-frames for Banach spaces and reconstruct elements from the range of a linear bounded operator \( K \) on a separable Banach space.
Definition 13. Let $K \in L(X, \mathcal{F} = \{g_i\}_{i=1}^{\infty} \subset X^*$ be an $X_d$-Bessel sequence for $X$, and $\mathcal{F} = \{f_j\}_{j=1}^{\infty}$ be a $X'_d$-Bessel sequence for $X^*$. $(\mathcal{F}, \mathcal{G})$ is called a pair of $(K, K^*)$-dual Bessel sequences if the following conditions are satisfied:

$$Kx = \sum_{i=1}^{\infty} \langle x, g_i \rangle f_j, \quad \forall x \in X,$$

(38)

$$K^* x^* = \sum_{i=1}^{\infty} (f_j, x^*) g_i, \quad \forall x^* \in X^*.$$  

(39)

In fact, if $(\mathcal{F}, \mathcal{G})$ form a pair of $(K, K^*)$-dual Bessel sequences, they are a $K^*$-frame and a $K$-frame, respectively.

Proposition 14. Let $K \in L(X, \mathcal{G} = \{g_i\}_{i=1}^{\infty} \subset X^*$ be an $X_d$-Bessel sequence, and $\mathcal{F} = \{f_j\}_{j=1}^{\infty} \subset X$ be an $X'_d$-Bessel sequence. If $(\mathcal{F}, \mathcal{G})$ is a pair of $(K, K^*)$-dual Bessel sequences, then $\mathcal{G}$ is a $K$-frame with respect to $X_d$ for $X$ and $\mathcal{F}$ is a $K^*$-frame with respect to $X_d$ for $X^*$.

Proof. Suppose that $\mathcal{F} = \{f_j\}_{j=1}^{\infty}$ has a Bessel bound $B_1$; then,

$$\| (f_j, x^*) \|_{\mathcal{G}} \leq B_1 \| x^* \|, \quad \forall x^* \in X^*.$$  

(40)

Since $(\mathcal{F}, \mathcal{G})$ is a pair of $(K, K^*)$-dual Bessel sequences, for all $x^* \in X^*$, we have

$$K^* x^* = \sum_{i=1}^{\infty} (f_j, x^*) g_i.$$  

(41)

So

$$\| K^* x^* \| = \sup_{\| x^* \| \leq 1} \| x^* \| \sum_{i=1}^{\infty} \langle x^*, g_i \rangle f_j \| = \sup_{\| x^* \| \leq 1} \| \langle x^*, g_i \rangle \|_{\mathcal{G}} = \sup_{\| x^* \| \leq 1} \| (f_j, x^*) \|_{\mathcal{G}} \leq B_2 \| x^* \|,$$

(42)

where $T_g$ is the analysis operator of the Bessel sequence $\mathcal{G}$ and $B_2$ is its Bessel bound. Combining the above inequality and (40), we obtain

$$B_2^{-1} \| K^* x^* \| \leq \| (f_j, x^*) \| \| x^* \| \leq B_1 \| x^* \|, \quad \forall x^* \in X^*.$$  

(43)

Therefore, $\mathcal{F}$ is a $K^*$-frame with respect to $X'_d$ for $X^*$. Similarly, we can prove that $\mathcal{G}$ is a $K$-frame with respect to $X_d$ for $X$.

From the above proposition, when (38) and (39) are satisfied, $(\mathcal{F}, \mathcal{G}) = \{f_j\}_{j=1}^{\infty}$ and $\mathcal{G} = \{g_i\}_{i=1}^{\infty}$ are, respectively, a $K^*$-frame and a $K$-frame. Therefore, $(\mathcal{F}, \mathcal{G})$ is also called a pair of $(K, K^*)$-dual frames. Moreover, $\mathcal{F}$ is called a $K^*$-dual of $\mathcal{G}$ and $\mathcal{G}$ is called a $K$-dual of $\mathcal{F}$.

Theorem 15. Let $1 < p < \infty$ and $\mathcal{G} = \{g_i\}_{i=1}^{\infty} \subset X^*$ be a $K$-frame with respect to $\mathcal{F}$ for $X$. If its analysis operator $R_{\mathcal{F}}$ is invertible, then $\mathcal{G}$ must have a $K^*$-dual.

Proof. Since the analysis operator $R_{\mathcal{F}}$ is invertible, for every $x \in X$, we have $R_{\mathcal{F}}^{-1} R_{\mathcal{F}} x = x$. Since the canonical unit vectors $\{e_i\}_{i=1}^{\infty}$ form a Schauder basis for the BK-space $\ell^p$, we can set $f_i = KR_{\mathcal{F}}^{-1} e_i, i = 1, 2, \ldots$; then,

$$\{ (x^*, f_j) \}_{j=1}^{\infty} = \{ (x^*, KR_{\mathcal{F}}^{-1} e_i) \}_{i=1}^{\infty} = \{ (R_{\mathcal{F}}^{-1} K^* x^*, e_i) \}_{i=1}^{\infty} = (R_{\mathcal{F}}^{-1})^* K^* x^* \in \ell^p, \quad \forall x^* \in X^*.$$  

(44)

So we obtain

$$\| ( (x^*, f_j) \|_{\mathcal{G}} = \| (R_{\mathcal{F}}^{-1})^* K^* x^* \|_{\mathcal{G}} \leq \| (R_{\mathcal{F}}^{-1})^* \| \| K^* \| \| x^* \|.$$  

This shows that $\{f_j\}_{j=1}^{\infty}$ is a $q$-Bessel sequence ($(1/p) + (1/q) = 1$). In addition, for every $x \in X$, we have

$$Kx = KR_{\mathcal{F}}^{-1} R_{\mathcal{F}} x = KR_{\mathcal{F}}^{-1} \sum_{i=1}^{\infty} \langle x, g_i \rangle e_i = \sum_{i=1}^{\infty} \langle x, g_i \rangle f_i,$$

(45)

i.e.,

$$Kx = \sum_{i=1}^{\infty} \langle x, g_i \rangle f_i, \quad \forall x \in X.$$  

(46)

Since $\{ (f_j, x^*) \}_{j=1}^{\infty} \in \ell^p(\forall x^* \in X^*)$, for any $\varepsilon > 0$, there exists a positive integer $N$ such that $\forall n, m > N, \| \sum_{i=n}^{m} (f_j, x^*) e_i \| < \varepsilon$. Here, $B$ is the upper frame bound. Hence, we have

$$\| \sum_{i=n}^{m} (f_j, x^*) g_i \| \leq \sup_{\| x^* \| \leq 1} \| (\sum_{i=n}^{m} \langle f_j, x^* \rangle g_i) \| \leq \sup_{\| x^* \| \leq 1} \left( \sum_{i=n}^{m} |\langle x^*, g_i \rangle|^p \right)^{1/p} \leq B \cdot \left( \sum_{i=n}^{m} |\langle x^*, g_i \rangle|^p \right)^{1/p} \leq B \cdot \left( \sum_{i=n}^{m} |\langle x^*, g_i \rangle|^p \right)^{1/p} = B \cdot \| x^* \| e_i \| < \varepsilon.$$  

(47)

This shows that $\sum_{i=1}^{\infty} (f_j, x^*) g_i$ converges in $X^*$. Here, we get
\[ \langle x, K^* x^* \rangle = \langle K x, x^* \rangle = \left\langle \sum_{i=1}^{\infty} \langle x, g_i \rangle f_i, x^* \right\rangle \]
\[ = \sum_{i=1}^{\infty} \langle x, g_i \rangle \langle f_i, x^* \rangle = \left\langle x, \sum_{i=1}^{\infty} \langle f_i, x^* \rangle g_i \right\rangle. \]  

Thus,
\[ K^* x^* = \sum_{i=1}^{\infty} \langle f_i, x^* \rangle g_i, \quad \forall x^* \in X^*. \]  

From (46) and (49), it follows that \( \{ f_i \}_{i=1}^{\infty} \) is the \( K^* \)-dual of \( \mathcal{F} \).

Although the dual of a frame for Hilbert spaces always exists, to find it is very complicated. In [22], Christensen and Laugesen introduced the concept of an approximate dual to solve such problem. For Banach spaces, the duality of all kinds of frames such as \( p \)-frames, \( X_d \)-frames, \( (p, Y) \)-operator frames, or \( K \)-frames, does not necessarily exist. Therefore, for a \( K \)-frame \( \mathcal{G} = \{ g_i \}_{i=1}^{\infty} \subset X^* \), we aim to identify a sequence \( \{ f_i \}_{i=1}^{\infty} \subset X \), following the approximate series expansion of \( K x \) and \( K^* x^* \):
\[ K x = \sum_{i=1}^{\infty} \langle x, g_i \rangle f_i, \quad \forall x \in X, \]  
\[ K^* x^* = \sum_{i=1}^{\infty} \langle f_i, x^* \rangle g_i, \quad \forall x^* \in X^*. \]

To realize the approximate reconstruction of elements from the range of the linear bounded operator on a Banach space, the concept of approximate \( (K, K^*) \)-dual Bessel sequences for Banach spaces will be introduced.

**Definition 16.** Let \( K \in L(X) \), \( \mathcal{F} = \{ f_i \}_{i=1}^{\infty} \subset X^* \) be an \( X_d \)-Bessel sequence for \( X \), and \( \mathcal{G} = \{ g_i \}_{i=1}^{\infty} \subset X^* \) be an \( X_d^* \)-Bessel sequence for \( X^* \). \( (\mathcal{F}, \mathcal{G}) \) is called a pair of approximate \( (K, K^*) \)-dual Bessel sequences if the following conditions are satisfied:
\[ \left| \left| K x - \sum_{i=1}^{\infty} \langle x, g_i \rangle f_i \right| \right| \leq \varepsilon \| x \|, \quad \forall x \in X, \]  
\[ \left| \left| K^* x^* - \sum_{i=1}^{\infty} \langle f_i, x^* \rangle g_i \right| \right| \leq \varepsilon \| x^* \|, \quad \forall x^* \in X^*, \]  
where \( \varepsilon \) is a small positive number.

**Remark 17.** To ensure that (50) and (51) are satisfied, the constant \( \varepsilon \) in (52) and (53) is usually set to a very small value. Assume that \( \varepsilon < 1 \); we may define the concept of approximate \( (K, K^*) \)-dual Bessel sequences by the analysis operator and the synthesis operator of Bessel sequences.

**Definition 18.** Let \( K \in L(X) \), \( \mathcal{F} = \{ f_i \}_{i=1}^{\infty} \subset X^* \) be an \( X_d \)-Bessel sequence for \( X \), and \( \mathcal{G} = \{ g_i \}_{i=1}^{\infty} \subset X^* \) be an \( X_d^* \)-Bessel sequence for \( X^* \). \( (\mathcal{F}, \mathcal{G}) \) is called a pair of approximate \( (K, K^*) \)-dual Bessel sequences if \( \| K - T_{\mathcal{F}} R_{\mathcal{G}} \| < 1 \) and \( \| K^* - T_{\mathcal{G}} R_{\mathcal{F}} \| < 1 \).

If \( K \in L(X) \) is injective, then for every \( y \in \text{ran} \( K \) \), there exists a unique \( x_y \in X \) such that \( K x_y = y \). Thus, we can define an operator \( \mathcal{H} \) as follows:
\[ \mathcal{H} : \text{ran} \( K \) \subset X \longrightarrow X, \mathcal{H} y = x_y, \quad \forall y \in \text{ran} \( K \). \]

It is obvious that \( \mathcal{H} \) is invertible on \( \text{ran} \( K \) \) and satisfies the following equations:
\[ \mathcal{H} K x = x, \quad \forall x \in X, \quad K \mathcal{H} y = y, \quad \forall y \in \text{ran} \( K \). \]

Furthermore, if the operator \( K \in L(X) \) has a generalized inverse \( K^\dagger \), then it satisfies that
\[ KK^\dagger K = K, \quad K^\dagger KK^\dagger = K^\dagger. \]

The concept of the generalized inverse of a linear bounded operator in Banach spaces can be seen Ref. [23].

**Lemma 19.** Let \( K \in L(X) \), \( \| K \| > 1 \), \( \mathcal{G} = \{ g_i \}_{i=1}^{\infty} \subset X^* \) be an \( X_d^* \)-Bessel sequence, \( \mathcal{F} = \{ f_i \}_{i=1}^{\infty} \subset X \) be an \( X_d \)-Bessel sequence, and \( (\mathcal{F}, \mathcal{G}) \) be a pair of approximate \( (K, K^*) \)-dual Bessel sequences. If \( K \) is injective and has a generalized inverse \( K^\dagger \), then the operator \( K^\dagger T_{\mathcal{F}} R_{\mathcal{G}} \) is invertible.

**Proof.** Suppose that \( (\mathcal{F}, \mathcal{G}) \) is a pair of approximate \( (K, K^*) \)-dual Bessel sequences, then \( \| K - T_{\mathcal{F}} R_{\mathcal{G}} \| < 1 \). Since \( \| K \| \geq 1 \) and it is injective and has a generalized inverse \( K^\dagger \), for every \( x \in X \), we have
\[ \| I - K^\dagger T_{\mathcal{F}} R_{\mathcal{G}} \| = \sup_{\| x \| \leq 1} \| x - K^\dagger T_{\mathcal{F}} R_{\mathcal{G}} x \| \]
\[ = \sup_{\| x \| \leq 1} \left\| \mathcal{H} K x - \mathcal{H} K^\dagger K x \right\| \]
\[ = \sup_{\| x \| \leq 1} \left\| \mathcal{H} (K K^\dagger) x - \mathcal{H} K^\dagger K x \right\| \]
\[ \leq \left\| \mathcal{H} K^\dagger K \right\| \cdot \sup_{\| x \| \leq 1} \| K x - T_{\mathcal{F}} R_{\mathcal{G}} x \| \]
\[ \leq \| K^\dagger \| \cdot \| K - T_{\mathcal{F}} R_{\mathcal{G}} \| < 1. \]

From Lemma 4, it follows that \( K^\dagger T_{\mathcal{F}} R_{\mathcal{G}} \) is invertible on \( X \).

**Proposition 20.** Let \( K \in L(X) \), \( \| K \| \geq 1 \), \( \mathcal{G} = \{ g_i \}_{i=1}^{\infty} \subset X^* \) be an \( X_d^* \)-Bessel sequence, \( \mathcal{F} = \{ f_i \}_{i=1}^{\infty} \subset X \) be an \( X_d \)-Bessel sequence, and \( (\mathcal{F}, \mathcal{G}) \) be a pair of approximate \( (K, K^*) \)-dual Bessel sequences. If \( K \) is injective and has a generalized inverse \( K^\dagger \), then \( \mathcal{G} \) is a \( K \)-frame with respect to \( X_d \) for \( X \).

**Proof.** Since \( (\mathcal{F}, \mathcal{G}) \) is a pair of approximate \( (K, K^*) \)-dual Bessel sequences, from Lemma 19, it follows that \( K^\dagger T_{\mathcal{F}} R_{\mathcal{G}} \) is invertible. For every \( x \in X \), we have
Suppose that the Bessel bound of \( G \) is \( B \); then, we have

\[
\|Kx\| = \sup_{\|x\| \leq 1} \|Kx, x^*\| = \sup_{\|x\| \leq 1} \|\langle x, K^* x^* \rangle\|
\]

\[
= \sup_{\|x\| \leq 1} \left| \sum_{i=1}^{\infty} \langle x, g_i \rangle \langle f_i, K^* x^* \rangle \right|
\]

\[
= \sup_{\|x\| \leq 1} \left| \sum_{i=1}^{\infty} \langle x, g_i \rangle \left( \langle f_i, (K^T R_g)^{-1} K^* x^* \rangle \right) \right|
\]

\[
= \sup_{\|x\| \leq 1} \left| \sum_{i=1}^{\infty} \langle x, g_i \rangle \left( \langle f_i, (K^T R_g)^{-1} K^* x^* \rangle \right) \right|
\]

\[
\leq \sup_{\|x\| \leq 1} \left| \sum_{i=1}^{\infty} \langle x, g_i \rangle \left( \langle f_i, (K^T R_g)^{-1} K^* x^* \rangle \right) \right|
\]

\[
\leq B \sup_{\|x\| \leq 1} \left| \sum_{i=1}^{\infty} \langle x, g_i \rangle \left( \langle f_i, (K^T R_g)^{-1} K^* x^* \rangle \right) \right|
\]

\[
\leq B \sup_{\|x\| \leq 1} \left| \sum_{i=1}^{\infty} \langle x, g_i \rangle \left( \langle f_i, (K^T R_g)^{-1} K^* x^* \rangle \right) \right|
\]

(59)

where \( \beta = B \sup_{\|x\| \leq 1} \left| \sum_{i=1}^{\infty} \langle x, g_i \rangle \left( \langle f_i, (K^T R_g)^{-1} K^* x^* \rangle \right) \right| \). Thus, we obtain

\[
\frac{1}{\beta} \|Kx\| \leq \|\langle x, g_i \rangle\|_{X^*_d}, \quad \forall x \in X.
\]

(60)

Since \( \mathcal{G} \) is an \( X_d \)-Bessel sequence, it is a \( K \)-frame with respect to \( X_d \) for \( X^* \).

Remark 21. Similarly, in the above proposition, under some conditions, \( \mathcal{F} \) can become a \( K^* \)-frame with respect to \( X_d^* \) for \( X^* \).

Theorem 22. Let \( K \in L(X), \|K\| \geq 1, \mathcal{F} = \{g_i\}_{i=1}^{\infty} \subset X^* \) be an \( X_d \)-Bessel sequence, \( \mathcal{F} = \{f_i\}_{i=1}^{\infty} \subset X \) be an \( X_d^* \)-Bessel sequence, and \( (\mathcal{F}, \mathcal{G}) \) be a pair of approximate \( (K, K^*) \)-dual Bessel sequences. If \( K \) is injective and has a generalized inverse \( K^d \), then

(i) The sequence \( \{K(K^T R_g)^{-1} K^* f_i\}_{i=1}^{\infty} \) is a \( K^* \)-dual of \( \mathcal{G} \), and the operator \( K(K^T R_g)^{-1} K^* \) can be written as

\[
K(K^T R_g)^{-1} K^* f_i = KK^d f_i + \sum_{k=1}^{\infty} K(I - K^T R_g)^k K^d f_i,
\]

\[
i = 1, 2, \ldots.
\]

(61)

(ii) For every fixed natural number \( n \), we put

\[
Y^n = KK^d f_i + \sum_{k=1}^{n} K(I - K^T R_g)^k K^d f_i, i = 1, 2, \ldots.
\]

(62)

Then, \( (Y^n, \mathcal{G}) \) is a pair of approximate \( (K, K^*) \)-dual Bessel sequences, where \( Y^n = \{Y^n_i\}_{i=1}^{\infty} \).

Proof.

(i) Suppose that \( (\mathcal{F}, \mathcal{G}) \) is a pair of approximate \( (K, K^*) \)-dual Bessel sequences; from Lemmas 4 and 19, it follows that \( (K^T R_g)^{-1} \) is invertible and

\[
(K^T R_g)^{-1} = I + \sum_{k=1}^{\infty} (I - K^T R_g)^k.
\]

(63)

Thus, for every \( f_i, i = 1, 2, \ldots \), we have

\[
K(K^T R_g)^{-1} K^* f_i = KK^d f_i + \sum_{k=1}^{\infty} K(I - K^T R_g)^k K^d f_i.
\]

(64)

From the proof of Proposition 20, it follows that

\[
Kx = K \left( \sum_{i=1}^{\infty} \langle x, g_i \rangle (K^T R_g)^{-1} K^d f_i \right)
\]

(65)

On the other hand, for every \( x \in X, x^* \in X^* \), we have

\[
\langle x, K^* x^* \rangle = \langle Kx, x^* \rangle = \left( \sum_{i=1}^{\infty} \langle x, g_i \rangle (K^T R_g)^{-1} K^d f_i, x^* \right)
\]

(66)
Then, for every \( x^* \in X^* \), we have

\[
\begin{align*}
K^* x^* &= \sum_{i=1}^{\infty} \left( K(K^* T_{\mathcal{F}} R_{\mathcal{G}})^{-1} K^* f_i, x^* \right) g_i, \quad \forall x^* \in X^*. \end{align*}
\] 

(67)

Therefore, \( \{K(K^* T_{\mathcal{F}} R_{\mathcal{G}})^{-1} K^* f_i\}_{i=1}^{\infty} \) is a \( K^* \)-dual of \( \mathcal{F} \).

(ii) Given a fixed natural number \( n \), set

\[
\gamma_i^n = K K^* f_i + \sum_{k=1}^{n} K (I - K^* T_{\mathcal{F}} R_{\mathcal{G}})^k K^* f_i, i = 1, 2, \ldots. \tag{68}
\]

Then, for every \( x^* \in X^* \), we have

\[
\begin{align*}
\| (\gamma_i^n, x^*)_{i=1}^{\infty} \|_{\mathcal{F}^*} &= \left\{ \left( \langle K K^* f_i, x^* \rangle + \sum_{k=1}^{n} \langle K (I - K^* T_{\mathcal{F}} R_{\mathcal{G}})^k K^* f_i, x^* \rangle \right)_{i=1}^{\infty} \right\}_{i=1}^{\infty} \\
&\leq \| K K^* f_i \| + \left\{ \left( \sum_{k=1}^{n} \langle K (I - K^* T_{\mathcal{F}} R_{\mathcal{G}})^k K^* f_i, x^* \rangle \right)_{i=1}^{\infty} \right\}_{i=1}^{\infty} \\
&\leq B \| (K K^*) f_i \| + \left\{ \left( \sum_{k=1}^{n} \langle K (I - K^* T_{\mathcal{F}} R_{\mathcal{G}})^k K^* f_i, x^* \rangle \right)_{i=1}^{\infty} \right\}_{i=1}^{\infty} \\
&= B \| f_i \| + B \left\{ \left( \sum_{k=1}^{n} \langle K (I - K^* T_{\mathcal{F}} R_{\mathcal{G}})^k K^* f_i, x^* \rangle \right)_{i=1}^{\infty} \right\}_{i=1}^{\infty} \\
&= B \| f_i \| + B \left\{ \left( \sum_{k=1}^{n} \langle K (I - K^* T_{\mathcal{F}} R_{\mathcal{G}})^k K^* f_i, x^* \rangle \right)_{i=1}^{\infty} \right\}_{i=1}^{\infty} \\
&= \| f_i \| + M \| x^* \|. \tag{69}
\end{align*}
\]

where \( B \) is the Bessel bound of \( \mathcal{F} \) and \( M = B(1 + \| \sum_{k=1}^{n} K (I - K^* T_{\mathcal{F}} R_{\mathcal{G}})^k K^* \| ) \). This shows that \( Y^n = \{ \gamma_i^n \}_{i=1}^{\infty} \) is an \( X^*_d \)-Bessel sequence.

Let \( T_{x^*} \) be the synthesis operator of \( Y^n \). For every \( x \in X \), we have

\[
\begin{align*}
T_{x^*} R_{\mathcal{G}} x &= \sum_{k=0}^{n} \langle x, g_k \rangle Y_k = \sum_{k=0}^{n} \langle x, g_k \rangle \left( \sum_{i=1}^{\infty} \langle K (I - K^* T_{\mathcal{F}} R_{\mathcal{G}})^k K^* f_i, x^* \rangle g_i \right) \\
&= \sum_{k=0}^{n} K (I - K^* T_{\mathcal{F}} R_{\mathcal{G}})^k K^* T_{\mathcal{F}} R_{\mathcal{G}} x = \sum_{k=0}^{n} K (I - K^* T_{\mathcal{F}} R_{\mathcal{G}})^k \cdot (K^* T_{\mathcal{F}} R_{\mathcal{G}}) x = \sum_{k=0}^{n} K (I - K^* T_{\mathcal{F}} R_{\mathcal{G}})^k \cdot [I - (I - K^* T_{\mathcal{F}} R_{\mathcal{G}})] x \\
&= \sum_{k=0}^{n} K (I - K^* T_{\mathcal{F}} R_{\mathcal{G}})^k x - \sum_{k=0}^{n} K (I - K^* T_{\mathcal{F}} R_{\mathcal{G}})^k \cdot (I - K^* T_{\mathcal{F}} R_{\mathcal{G}})^k x \\
&= K x - K (I - K^* T_{\mathcal{F}} R_{\mathcal{G}})^n x. \tag{70}
\end{align*}
\]

That is,

\[
T_{x^*} R_{\mathcal{G}} x = K x - K (I - K^* T_{\mathcal{F}} R_{\mathcal{G}})^{n+1} x. \tag{71}
\]

Thus,

\[
\| K x - T_{y^*} R_{\mathcal{G}} x \| = \left\| K (I - K^* T_{\mathcal{F}} R_{\mathcal{G}})^{n+1} x \right\|. \tag{72}
\]

It follows that

\[
\| K - T_{y^*} R_{\mathcal{G}} \| = \sup_{\| x \| \leq 1} \| K x - T_{y^*} R_{\mathcal{G}} x \| \\
\leq \| K \| \cdot \sup_{\| x \| \leq 1} \| (I - K^* T_{\mathcal{F}} R_{\mathcal{G}})^{n+1} x \| \\
= \| K \| \cdot \left\| (I - K^* T_{\mathcal{F}} R_{\mathcal{G}})^{n+1} \right\| \\
\leq \| K \| \cdot \left\| I - K^* T_{\mathcal{F}} R_{\mathcal{G}} \right\|^{n+1} \\
\leq \| K \| \cdot \| K^* \|^{n+1} \cdot \| K - T_{y^*} R_{\mathcal{G}} \|^{n+1}. \tag{73}
\]

Since \( \| K - T_{y^*} R_{\mathcal{G}} \| < 1 \) and \( \| K \| \geq 1 \), we obtain \( \| K - T_{y^*} R_{\mathcal{G}} \| < 1 \). Similarly, we can prove that \( \| K^* - T_{y^*} R_{\mathcal{G}} \| < 1 \). Hence, \( (Y^n, \mathcal{G}) \) is a pair of approximate \( (K, K^*) \)-dual Bessel sequences.

In fact, from part (ii) of Theorem 22, we find that as \( n \) increases, \( Y^n = \{ \gamma_i^n \}_{i=1}^{\infty} \) is closer to a \( K^* \)-dual of \( \mathcal{G} \). Hence, by choosing \( n \) sufficiently large, the reconstruction of \( K x \) is almost perfect.” If we denote \( y_i = \lim_{n \to \infty} \gamma_i^n, i = 1, 2, \ldots, \) then \( \{ y_i \}_{i=1}^{\infty} \) happens to be a \( K^* \)-dual of \( \mathcal{G} \).

\[\Box\]

Theorem 23. Let \( K \in L(X), \mathcal{F} = \{ f_i \}_{i=1}^{\infty} \) be an \( X^*_d \)-Bessel sequence with the bound \( \mathcal{G} = \{ g_i \}_{i=1}^{\infty} \) be an \( X^*_d \)-Bessel sequence for \( X \), and \( (\mathcal{F}, \mathcal{G}) \) be a pair of \( (K, K^*) \)-dual Bessel sequences. For a given sequence \( \mathcal{H} = \{ h_i \}_{i=1}^{\infty} \subset X^* \), if \( \{ x_i, h_i \}_{i=1}^{\infty} \in X_d \) and there exists a constant \( \lambda > 0 \) such that \( \lambda < 1/B \) and

\[
\| \{ \langle x, g_i - h_i \rangle \}_{i=1}^{\infty} \|_{X_d} \leq \lambda \| x \|, \quad \forall x \in X, \tag{74}
\]

then \( (\mathcal{F}, \mathcal{H}) \) is a pair of approximate \( (K, K^*) \)-dual Bessel sequences.

Proof. Let \( T_{x^*} \) be the synthesis operator of \( \mathcal{F} \) and \( R_{\mathcal{H}} \) be the analysis operator of \( \mathcal{H} \). For every \( x \in X \), we have

\[
\| \{ x_i, h_i \}_{i=1}^{\infty} \|_{X_d} \leq \| \{ x, g_i - h_i \}_{i=1}^{\infty} \|_{X_d} + \| \{ x, g_i \}_{i=1}^{\infty} \|_{X_d} \leq \lambda \| x \| + B_1 \| x \| \leq (\lambda + B_1) \| x \|, \tag{75}
\]

where \( B_1 \) is the upper frame bound of \( \mathcal{H} \). Thus, \( \mathcal{H} \) is an \( X^*_d \)-Bessel sequence. Suppose that \( R_{\mathcal{H}} \) is the analysis operator of \( \mathcal{H} \). Then, (74) complies that
\[ \|R_{\mathcal{J}} - R_{\mathcal{K}}\| \leq \lambda. \quad (76) \]

In addition, for all \( \{e_i\}_{i=1}^{\infty} \in X_d^* \), we have
\[
\|(T_{\mathcal{J}} - T_{\mathcal{K}})(\{e_i\}_{i=1}^{\infty})\| = \sup_{\|x\| \leq 1} \left( \|x, (T_{\mathcal{J}} - T_{\mathcal{K}})(\{e_i\}_{i=1}^{\infty})\| \right)
= \sup_{\|x\| \leq 1} \left( \|x, \sum_{i=1}^{\infty} e_i (g_i - h_i)\| \right)
= \sup_{\|x\| \leq 1} \left( \|\{x, g_i - h_i\}_{i=1}^{\infty}, \{e_i\}_{i=1}^{\infty}\| \right)
\leq \sup_{\|x\| \leq 1} \left( \|\{x, g_i - h_i\}_{i=1}^{\infty}\| \right) \cdot \|\{e_i\}_{i=1}^{\infty}\|
\leq \lambda \cdot \|\{e_i\}_{i=1}^{\infty}\|. \quad (77)
\]

That is,
\[ \|T_{\mathcal{J}} - T_{\mathcal{K}}\| \leq \lambda. \quad (78) \]

Since \((\mathcal{F}, \mathcal{G})\) is a pair of \((K, K^*)\)-dual Bessel sequences, for every \( x \in X, x^* \in X^* \), we have
\[ Kx = \sum_{i=1}^{\infty} \langle x, g_i \rangle f_i, K^* x^* = \sum_{i=1}^{\infty} \langle f_i, x^* \rangle g_i, \quad (79) \]
i.e.,
\[ K = T_{\mathcal{G}} R_{\mathcal{G}}, \quad K^* = T_{\mathcal{G}} R_{\mathcal{G}}. \quad (80) \]

Hence, we obtain
\[
\|K - T_{\mathcal{G}} R_{\mathcal{G}}\| = \|T_{\mathcal{G}} R_{\mathcal{G}} - T_{\mathcal{G}} R_{\mathcal{G}}\| \leq \|T_{\mathcal{G}}\| \cdot \|R_{\mathcal{G}} - R_{\mathcal{G}}\| \leq B\lambda < 1, \]
\[
\|K^* - T_{\mathcal{G}} R_{\mathcal{G}}\| = \|T_{\mathcal{G}} R_{\mathcal{G}} - T_{\mathcal{G}} R_{\mathcal{G}}\| \leq \|T_{\mathcal{G}} - T_{\mathcal{G}}\| \cdot \|R_{\mathcal{G}}\| \leq B\lambda < 1. \quad (81) \]

It proves that \((\mathcal{F}, \mathcal{G})\) is a pair of approximate \((K, K^*)\)-dual Bessel sequences. \(\square\)

**Theorem 24.** Let \( K \in \mathcal{L}(X), \mathcal{F} = \{f_i\}_{i=1}^{\infty} \) be an \( X_d^*\)-Bessel sequence with bound \( B_1 \), \( \mathcal{G} = \{g_i\}_{i=1}^{\infty} \) be an \( X_d^*\)-Bessel sequence for \( X \) with the bound \( B_2 \), and \( \mathcal{H} = \{h_i\}_{i=1}^{\infty} \subset X^* \) be an \( X_d^*\)-Bessel sequence. If \((\mathcal{F}, \mathcal{G})\) is a pair of \((K, K^*)\)-dual Bessel sequences and there exist constants \( \lambda_1, \mu > 0, 1 > \lambda_2 > 0 \) such that
\[
\frac{B_1 B_2 (1 + \lambda_1) + B_1 \mu}{1 - \lambda_2} < 1 - \lambda_1 \|K\|, \quad (82) \]

\[
\sum_{i=1}^{n} c_i (x_i - h_i) \leq \lambda_1 \sum_{i=1}^{n} c_i g_i + \lambda_2 \sum_{i=1}^{n} c_i h_i + \mu \|\{c_i\}_{i=1}^{\infty}\|, \quad \forall n \in \mathbb{N}, \forall \{c_i\}_{i=1}^{\infty} \in X_d^*. \quad (83) \]

then \((\mathcal{F}, \mathcal{H})\) is a pair of approximate \((K, K^*)\)-dual Bessel sequences.

**Proof.** Let \( T_{\mathcal{G}}, T_{\mathcal{F}}, T_{\mathcal{H}} \) be, respectively, the synthesis operators of \((\mathcal{F}, \mathcal{G}), (\mathcal{F}, \mathcal{H})\) and \( R_{\mathcal{G}}\lambda\) be the analysis operator of \( \mathcal{G}\); then, we have \( \|T_{\mathcal{G}}\| \leq B_1 \) and \( \|R_{\mathcal{G}}\| \leq B_2\). From (83), for every \( \{c_i\}_{i=1}^{\infty} \in X_d^* \), it follows that
\[
\sum_{i=1}^{n} c_i (x_i - h_i) \leq \lambda_1 \sum_{i=1}^{n} c_i g_i + \lambda_2 \sum_{i=1}^{n} c_i h_i + \mu \|\{c_i\}_{i=1}^{\infty}\|. \quad (84) \]

Thus, we have
\[
\sum_{i=1}^{n} c_i (x_i - h_i) \leq \frac{1 + \lambda_1}{1 - \lambda_2} \sum_{i=1}^{n} c_i g_i + \frac{\mu}{1 - \lambda_2} \|\{c_i\}_{i=1}^{\infty}\| \quad (85) \]

So \( \|T_{\mathcal{G}}\| \leq ((1 + \lambda_1) B_2 + \mu)/(1 - \lambda_2) \).

Next, we will prove that \((\mathcal{F}, \mathcal{H})\) is a pair of approximate \((K, K^*)\)-dual Bessel sequences. In fact, for every \( x^* \in X^* \), take \( \{c_i\}_{i=1}^{\infty} = R_{\mathcal{F}} x^* \); by (83), we have
\[
\|T_{\mathcal{G}} R_{\mathcal{F}} x^* - T_{\mathcal{F}} R_{\mathcal{F}} x^*\| \leq \lambda_1 \|T_{\mathcal{G}} R_{\mathcal{F}} x^*\| + \lambda_2 \|T_{\mathcal{F}} R_{\mathcal{F}} x^*\| + \mu \|R_{\mathcal{F}} x^*\|. \quad (86) \]

Since \((\mathcal{F}, \mathcal{G})\) is a pair of \((K, K^*)\)-dual Bessel sequences, for every \( x \in X, x^* \in X^* \), we have
\[ Kx = \sum_{i=1}^{\infty} \langle x, g_i \rangle f_i, \quad K^* x^* = \sum_{i=1}^{\infty} \langle f_i, x^* \rangle g_i, \quad (87) \]
i.e.,
\[ K = T_{\mathcal{F}} R_{\mathcal{F}}, \quad K^* = T_{\mathcal{F}} R_{\mathcal{F}}. \quad (88) \]
Combining (82), (83), and (86), we have
\[
\|K^* x^* - T_{\mathcal{F}} R_{\mathcal{F}} x^*\| \leq \lambda_1 \|K^* x^*\| + \lambda_2 \|T_{\mathcal{F}} R_{\mathcal{F}} x^*\| + \mu \|R_{\mathcal{F}} x^*\|
\]
\[
\leq \lambda_1 \|K^*\| \cdot \|x^*\| + \lambda_2 \|T_{\mathcal{F}}\| \cdot B_1 \cdot \|x^*\|
\]
\[
+ \mu B_1 \cdot \|x^*\| \leq \lambda_1 \|K^*\| \cdot \|x^*\|
\]
\[
+ \lambda_2 \left[\frac{1 + \lambda_1 B_2 + \mu}{1 - \lambda_2}\right] \cdot B_1 \cdot \|x^*\| + \mu B_1 \cdot \|x^*\|
\]
\[
= \left[\lambda_1 \|K^*\| + \lambda_2 (1 + \lambda_1 B_2) + \mu B_1\right] \cdot \|x^*\| \leq \|x^*\|.
\]
(89)

Thus, \(\|K^* - T_{\mathcal{F}} R_{\mathcal{F}}\| < 1\).

On the other hand, for every \(x \in X\), we have
\[
\|K x - T_{\mathcal{F}} R_{\mathcal{F}} x\| = \sup_{\|x^*\|\leq 1} |\langle K x - T_{\mathcal{F}} R_{\mathcal{F}} x, x^* \rangle|
\]
\[
= \sup_{\|x^*\|\leq 1} \left|\langle K x - \sum_{i=1}^{\infty} \langle x, h_i \rangle f_i, x^* \rangle\right|
\]
\[
= \sup_{\|x^*\|\leq 1} \left|\langle K x, x^* \rangle - \sum_{i=1}^{\infty} \langle x, h_i \rangle f_i, x^* \rangle\right|
\]
\[
= \sup_{\|x^*\|\leq 1} \left|\langle K x, x^* \rangle - \sum_{i=1}^{\infty} \langle x, h_i \rangle (f_i, x^*)\right|
\]
\[
= \sup_{\|x^*\|\leq 1} \left|\langle K x, x^* \rangle - \langle x, \sum_{i=1}^{\infty} (f_i, x^*) h_i \rangle\right|
\]
\[
= \sup_{\|x^*\|\leq 1} \left|\langle x, (K^* - T_{\mathcal{F}} R_{\mathcal{F}}) x^* \rangle\right|
\]
\[
\leq \sup_{\|x^*\|\leq 1} \|x\| \cdot \|K^* - T_{\mathcal{F}} R_{\mathcal{F}}\| \cdot \|x^*\| \leq \|K^* - T_{\mathcal{F}} R_{\mathcal{F}}\| \cdot \|x\| < \|x\|.
\]
(90)

Thus, \(\|K - T_{\mathcal{F}} R_{\mathcal{F}}\| < 1\). Therefore, \((\mathcal{F}, \mathcal{R})\) is a pair of approximate \((K, K^*)\)-dual Bessel sequences. \(\square\)

Remark 25. The above two theorems show that if a sequence \(\mathcal{R} = \{h_i\}_{i=1}^{\infty} \in X^*\) is close enough to a \(K\)-dual of the \(X^*_v\)-Bessel sequence \(\mathcal{F}\), it would become an approximate \(K\)-dual of \(\mathcal{F}\).

Data Availability

All data included in this study are available upon request by contact with the corresponding author.

Conflicts of Interest

The authors declare that they have no conflict of interest.

Authors’ Contributions

All the authors contributed equally to the writing of the present article. And they also read and approved the final paper.

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