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Research Article

Qualitative Analyses of Fractional Integrodifferential Equations with a Variable Order under the Mittag-Leffler Power Law

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This research paper intends to study some qualitative analyses for a nonlinear fractional integrodifferential equation with a variable order in the frame of a Mittag-Leffler power law. At first, we convert the considered problem of variable order into an equivalent standard problem of constant order using generalized intervals and piecewise constant functions. Next, we prove the existence and uniqueness of analytic results by application of Krasnoselskii's and Banach's fixed point theorems. Besides, the guarantee of the existence of solutions is shown by different types of Ulam-Hyer's stability. Then, we investigate sufficient conditions of positive solutions for the proposed problem. In the end, we discuss an example to illustrate the applicability of our obtained results.

1. Introduction

Fractional calculus and its applications [1, 2] has recently gained in popularity due to their applicability in modeling many complex phenomena in a wide range of science and engineering disciplines. Several biological models [3] and optimal control problems [4] have been presented in the literature through the development of fractional calculus. In order to describe the dynamics of real-world problems, new methods and techniques have been discovered. In particular, Caputo and Fabrizio in [5] investigated a new type of fractional derivatives (FDs) in the exponential kernel. There are some interesting properties of Caputo and Fabrizio that were discussed by Losada and Nieto in [6]. In [7], Atangana and Baleanu, investigated a new type and interesting FD with a Mittag-Leffler (ML) kernel. Atangana-Baleanu (AB) FD was extended to higher arbitrary order by Abdeljawad in [8], along with their associated integral operators. For some theoretical works on AB fractional operator, we refer the reader to a series of papers [9–13].

Variable order fractional operators can be seen as a natural analytical extension of constant order fractional operators. In recent years, variable order fractional operators have been designed and formalized mathematically only. After that, the applications of this effect rolled rapidly. In this regard, Lorenzo and Hartley [14] studied the behaviors of a fractional diffusion problem with fractional operators in the variable order. Afterward, different applications of variable order spaces of a fractional kind have shown up in striking and fascinating points of interest, see [15–20]. For instance, Sun et al. [21] introduced a comparative study on constant and variable order models to describe the memory identification of certain systems. The authors in [22] have formulated a nonlinear model of alcoholism in the frame of FDEs with variable order and discussed the solutions of such a model numerically and analytically. The authors in [23] analyzed a variable order mathematical fuzzy model through a computational approach for nonlinear fuzzy partial FDEs.

The probability of formulating evolutionary control equations has led to the effective application of these operators to model complex real-world problems going from mechanics to transition and control processes to theory and biology. For available applications of variable order fractional operators in the overall area of engineering and scientific modeling, see [24, 25]. Such broad and various applications promptly require a progression of systematic studies on the qualitative analyses of solutions of FDEs with variable order such as existence, uniqueness, and stability.

Recently, Li et al. [26], by a new numerical approach, have studied the following fractional problem

$$\begin{cases} {}^{ML}\mathbf{D}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa) + a(\varkappa)\vartheta(\varkappa) = \mathscr{K}(\varkappa, \vartheta(\varkappa)), & \varkappa \in [0, 1], \\ B(\vartheta) = 0, \end{cases} \tag{1}$$

where ${}^{ML}\mathbf{D}_{0^+}^{\varrho(\varkappa)}$ is the Atangana-Baleanu FD with a variable order $\varrho(\varkappa)$ and $B(\vartheta)$ is the linear boundary condition.

Bouazza et al. [27] established the existence and stability results for a multiterm fractional BVP with a variable order of the form:

$$\begin{cases} {}^{C}\mathbf{D}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa) = \mathscr{K}\left(\varkappa, \vartheta(\varkappa), \mathbf{I}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa)\right), \varkappa \in [0, b], \\ \vartheta(0) = 0, \vartheta(b) = 0, \end{cases} \tag{2}$$

where ${}^{C}\mathbf{D}_{0^+}^{\varrho(t)}, \mathbf{I}_{0^+}^{\varrho(t)}$ are Caputo's and Riemann-Liouville's operators of variable order $\varrho(t)$. The existence and Ulam-Hyers stability results of a Caputo-type problem (2) have been obtained by Benkerrouche et al. [28]. Kaabar et al. [29] investigated some qualitative analyses of solutions for the following implicit FDE with variable order

$$\begin{cases} {}^{C}\mathbf{D}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa) = \mathscr{K}\left(\varkappa, \vartheta(\varkappa), {}^{C}\mathbf{D}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa)\right), \varkappa \in [0, b], \\ \vartheta(0) = 0, \vartheta(b) = 0, \end{cases} \tag{3}$$

where ${}^{C}\mathbf{D}_{0^+}^{\varrho(\varkappa)}$ is the Caputo FD of variable order $\varrho(\varkappa)$.

Cauchy's type of nonlocal problems can be used to explain differential laws in the development of systems, which is remarkable. Nonnegative quantities, such as the concentration of a species or the distribution of mass or temperature, are often described using these types of equations. In this regard, the first question to ask before analyzing any system or model of a real-world phenomenon is whether or not the problem actually exists. The answer to this question is given by the fixed point theory. We refer here to some results that dealt with the stability approach in the concept of Ulam-Hyers and others related to fixed point techniques, see [30–35].

Motivated by the argumentations above, we intend to analyze and investigate the sufficient conditions of solution for ML-type nonlinear fractional integrodifferential equations with a variable order of the form

$$\begin{cases} {}^{ML}\mathbf{D}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa) = \mathscr{K}\left(\varkappa, \vartheta(\varkappa), {}^{ML}\mathbf{I}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa), {}^{ML}\mathbf{D}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa)\right), \varkappa \in \mathscr{E} \coloneqq (0, b], \\ \vartheta(0) = 0, \vartheta(b) = \sum_{j=1}^{n} \tau_j \vartheta(\kappa_j), \kappa_j \in (0, b), \end{cases} \tag{4}$$

where ${}^{ML}\mathbf{D}_{0^+}^{\varrho(\varkappa)}$ and ${}^{ML}\mathbf{I}_{0^+}^{\varrho(\varkappa)}$ are the ML-type derivative and the ML-type integral of fractional variable order $\varrho(\varkappa) > 0$,, respectively, $\tau_j \in \mathbb{R}$, $\kappa_j, j = 1, 2, \cdots, n$ are prefixed points satisfying $0 < \kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_i < b$ and $\mathscr{K} : \mathscr{E} \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a continuous function fulfilling some later-specified assumptions.

Let $C(\mathscr{E}, \mathbb{R})$ be a Banach space of continuous functions $\vartheta : \mathscr{E} \longrightarrow \mathbb{R}$ equipped with the norm $\|\vartheta\| = \sup \{|\vartheta(\varkappa)|: \varkappa \in \mathscr{E}\}$.

Definition 1 (see [36]). Let $\varrho(\varkappa) \in (n-1, n]$,$\vartheta \in H^1(\mathscr{E})$. Then, the ML-type FD of a variable order $\varrho(\varkappa)$ for a function ϑ with the lower limit zero is defined by

$${}^{ML}\mathbf{D}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa) = \frac{\Upsilon(\varrho\varkappa)}{1-\varrho(\varkappa)} \int_0^{\varkappa} E_{\varrho(\varkappa)}\left(\frac{\varrho(\varkappa)}{\varrho(\varkappa)-1}(\varkappa-\theta)^{\varrho}\right)\vartheta'(\theta)d\theta, \varkappa > 0, \tag{5}$$

respectively. The normalization function $\Upsilon(\varrho(\varkappa))$ satisfies $\Upsilon(0) = \Upsilon(1) = 1$, where $E_{\varrho(\varkappa)}$ is the ML function defined by

$$\mathbf{E}_{\varrho(\varkappa)}(\vartheta) = \sum_{i=0}^{\infty} \frac{\vartheta^i}{\Gamma(i\varrho(\varkappa)+1)}, \operatorname{Re}(\varrho(\varkappa)) > 0, \vartheta \in \mathbb{C}. \tag{6}$$

The correspondent ML fractional integral is given by

$${}^{ML}\mathbf{I}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa) = \frac{1-\varrho(\varkappa)}{\Upsilon(\varrho(\varkappa))}\vartheta(\varkappa) + \frac{\varrho(\varkappa)}{\Upsilon(\varrho(\varkappa))\Gamma(\varrho(\varkappa))}\int_0^{\varkappa}(\varkappa-s)^{\varrho(\varkappa)-1}\vartheta(s)ds. \tag{7}$$

For an integer $n \in \mathbb{N}$ and $\mathscr{B}$ is a partition of the interval $\mathscr{E}$ defined as

$$\mathscr{B} = \{\mathscr{E}_1 = [0, b_1], \mathscr{E}_2 = [b_1, b_2], \mathscr{E}_3 = [b_2, b_3], \ldots \cdots, \mathscr{E}_n = [b_{n-1}, b_n]\}. \tag{8}$$

Let $\varrho(\varkappa)$: $\mathscr{E} \longrightarrow (1, 2]$ be a piecewise constant function with respect to $\mathscr{B}$ such that

$$\varrho(\varkappa)=\sum_{l=1}^{n}\varrho_l Q_l(\varkappa)=\begin{cases}\varrho_1, & \text{if } \varkappa\in\mathscr{E}_1\\ \varrho_2, & \text{if } \varkappa\in\mathscr{E}_2\\ \cdot & \\ \cdot & \\ \cdot & \\ \varrho_n, & \text{if } \varkappa\in\mathscr{E}_n,\end{cases} \tag{9}$$

where $\varrho_l\in(1,2]$ are constants, and Q_l is the indicator of the interval $\mathscr{E}_l:=(b_{l-1},b_l]$, $l=1,2,\cdots,n$ (with $b_0=0$, $b_n=b$) such that

$$Q_l(\varkappa)=\begin{cases}1, & \text{if } \varkappa\in\mathscr{E}_l\\ 0, & \text{for } elsewhere.\end{cases} \tag{10}$$

Let $C(\mathscr{E}_l,\mathbb{R})$, $l\in\{1,2,\cdots,n\}$ be a Banach space of continuous functions $\vartheta:\mathscr{E}_l\longrightarrow\mathbb{R}$ equipped with the norm $\|\vartheta\|=\sup\{|\vartheta(\varkappa)|:\varkappa\in\mathscr{E}_l\}$. Then, for $\varkappa\in\mathscr{E}_l$, $l=1,2,\cdots,n$ the ML-type FD of variable order $\varrho(\varkappa)$ for a function $\vartheta\in C(\mathscr{E},\mathbb{R})$ defined by (5) can be written as a sum of left ML-type FD of constant orders ϱ_l, $l=1,2,\cdots,n$ as follows.

$$\begin{aligned}{}^{ML}\mathbf{D}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa)&=\frac{\Upsilon(\varrho(\varkappa))}{1-\varrho(\varkappa)}\int_0^{b_1}E_{\varrho(\varkappa)}\left(\frac{\varrho(\varkappa)}{\varrho(\varkappa)-1}(\varkappa-\theta)^{\varrho(\varkappa)}\right)\\&\quad\cdot\vartheta'(\theta)d\theta+.\cdots\cdots+\frac{\Upsilon(\varrho(\varkappa))}{1-\varrho(\varkappa)}\int_{b_{l-1}}^{\varkappa}E_{\varrho(\varkappa)}\\&\quad\cdot\left(\frac{\varrho(\varkappa)}{\varrho(\varkappa)-1}(\varkappa-\theta)^{\varrho(\varkappa)}\right)\vartheta'(\theta)d\theta.\end{aligned} \tag{11}$$

Thus, according to (11), the BVP (4) can be written for any $\varkappa\in\mathscr{E}_l$, $l=1,2,\cdots,n$ in the form

$$\begin{aligned}&\frac{\Upsilon(\varrho(\varkappa))}{1-\varrho(\varkappa)}\int_0^{b_1}E_{\varrho(\varkappa)}\left(\frac{\varrho(\varkappa)}{\varrho(\varkappa)-1}(b_1-\theta)^{\varrho(\varkappa)}\right)\vartheta'(\theta)d\theta+.\cdots\cdots\\&\quad+\frac{\Upsilon(\varrho(\varkappa))}{1-\varrho(\varkappa)}\int_{b_{l-1}}^{\varkappa}E_{\varrho(\varkappa)}\left(\frac{\varrho(\varkappa)}{\varrho(\varkappa)-1}(b-\theta)^{\varrho(\varkappa)}\right)\vartheta'(\theta)d\theta\\&=\mathscr{K}\left(\varkappa,\vartheta(\varkappa),{}^{ML}\mathbf{I}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa),{}^{ML}\mathbf{D}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa)\right).\end{aligned} \tag{12}$$

Let the function $\vartheta\in C(\mathscr{E}_l,\mathbb{R})$ be such that $\vartheta(\varkappa)=0$ on $\varkappa\in[0,b_{l-1}]$ and such that it solves the integral equation (12). Then, (12) is reduced to

$$\begin{cases}{}^{ML}\mathbf{D}_{b_{l-1}^+}^{\varrho_1}\vartheta(\varkappa)=\mathscr{K}\left(\varkappa,\vartheta(\varkappa),{}^{ML}\mathbf{I}_{b_{l-1}^+}^{\varrho_1}\vartheta(\varkappa),{}^{ML}\mathbf{D}_{b_{l-1}^+}^{\varrho_1}\vartheta(\varkappa)\right),\varkappa\in\mathscr{E}_1,\\ {}^{ML}\mathbf{D}_{b_{l-1}^+}^{\varrho_2}\vartheta(\varkappa)=\mathscr{K}\left(\varkappa,\vartheta(\varkappa),{}^{ML}\mathbf{I}_{b_{l-1}^+}^{\varrho_2}\vartheta(\varkappa),{}^{ML}\mathbf{D}_{b_{l-1}^+}^{\varrho_2}\vartheta(\varkappa)\right),\varkappa\in\mathscr{E}_2,\\ \cdot\\ \cdot\\ {}^{ML}\mathbf{D}_{b_{l-1}^+}^{\varrho_n}\vartheta(\varkappa)=\mathscr{K}\left(\varkappa,\vartheta(\varkappa),{}^{ML}\mathbf{I}_{b_{l-1}^+}^{\varrho_n}\vartheta(\varkappa),{}^{ML}\mathbf{D}_{b_{l-1}^+}^{\varrho_n}\vartheta(\varkappa)\right),\varkappa\in\mathscr{E}_n.\end{cases} \tag{13}$$

In our forthcoming analysis, we shall deal with the following BVP:

$$\begin{cases}{}^{ML}\mathbf{D}_{b_{l-1}^+}^{\varrho_l}\vartheta(\varkappa)=\mathscr{K}\left(\varkappa,\vartheta(\varkappa),{}^{ML}\mathbf{I}_{b_{l-1}^+}^{\varrho_l}\vartheta(\varkappa),{}^{ML}\mathbf{D}_{b_{l-1}^+}^{\varrho_l}\vartheta(\varkappa)\right),\varkappa\in\mathscr{E}_l\\ \vartheta(b_{l-1})=0,\vartheta(b_l)=\sum_{j=1}^{n}\tau_j\vartheta(\kappa_j),\kappa_j\in(0,b).\end{cases} \tag{14}$$

Observe that, problem (4) is more general of problems (1), (2), and (3). In addition, it is assumed that it should be noted that due to the complexity of the computations and the division of the underlying time interval, not many papers can be found in the literature in which the authors focused on the existence and stability results of fractional variable order integrodifferential equations. To fill this gap, we investigate some qualitative analyses for the fractional variable order problem (14) in the frame of ML-type fractional operators. More precisely, we convert the ML-type fractional variable order problem into an equivalent standard ML-type fractional constant order problem using generalized intervals and piecewise constant functions. Then, we prove the existence and uniqueness of solutions for problem (14) via Krasnoselskii's and Banach's fixed point techniques. Also, we discuss the Ulam-Hyers stability result to the proposed problem. Further, we establish the sufficient conditions of positive solutions for problem (14).

The major contribution of this work is to develop the nonlocal fractional calculus with respect to variable order and learn more properties for the proposed ML-type fractional problems, which makes use of nonsingular kernel derivatives with fractional variable order. Already significant amount of work on constant fractional order for different operators has been done in literature. But to the best of our information, variable order problems have not been well studied so for fractional calculus. There is a waste gap between constant and variable fractional order problems in literature, the first one has got tremendous attention as compared to the second one. Very recently, the area of variable order has started attention to be investigated. In line with these developments, a new approach is used in this work to discuss some qualitative properties of solution for the considered problem. Multiple terms can be solved using this approach. To the best of our understanding, this is the first work dealing with the ML-type derivative with fractional variable order. The results of this work will therefore make a useful contribution to the existing literature on this subject.

The outline of our work is as follows. Some basic notions and axiom results are presented in Section 2. Our main results are obtained in Sections 3, 4, and 5 based on Krasnoselskii's and Banach's fixed point theorems. An illustrative example is fitted in Section 6. Concluding remarks about our results are in the final section.

2. Auxiliary Results

Definition 2. Problem (14) has a solution in $C(\mathscr{E}, \mathbb{R})$, if there are functions $\vartheta_l, l = 1, 2, \cdots, n$, so that $\vartheta_l \in C(\mathscr{E}_l, \mathbb{R})$, satisfying Equation (12), and $\vartheta_l(0) = 0, \vartheta_l(b) = \sum_{j=1}^{n} \tau_j \vartheta(\kappa_j)$.

Lemma 3 (see [8]). *Let $\vartheta(\varkappa)$ be a function defined on $[0, b]$ and $n < \varrho \leq n + 1$, for some $n \in \mathbb{N}_0$, we have*

$$\left({}^{ML}\mathbf{I}_{0^+}^{\varrho\,ML}\mathbf{D}_{0^+}^{\varrho}\vartheta\right)(\varkappa) = \vartheta(\varkappa) - \sum_{i=0}^{n} \frac{\vartheta^{(i)}(0)}{i!}\varkappa^i. \tag{15}$$

Theorem 4 (see [37]). *Let $\mathscr{S}$ be closed subspace from a Banach space $\mathscr{K}$, and let $\Pi : \mathscr{S} \longrightarrow \mathscr{S}$ be a strict contraction such that*

$$\|\Pi(x) - \Pi(y)\| \leq \rho\|x - y\|, \tag{16}$$

for some $0 < \rho < 1$, and for all $x, y \in \mathscr{S}$. Then, Π has a fixed point in $\mathscr{S}$.

Theorem 5 (see [38]). *Let K be a nonempty, closed, convex, and bounded subset of the Banach space $\mathfrak{X}$. If there are two operators Φ^1, Φ^2 such that*

(1) $\Phi^1 u + \Phi^2 v \in \mathfrak{X}$, for all $u, v \in \mathfrak{X}$,

(2) Φ^1 is compact and continuous

(3) Φ^2 is a contraction mapping

Then, there exists a function $z \in K$ such that $z = \Phi^1 z + \Phi^2 z$..

Remark 6. Let $u(\varkappa), v(\varkappa) \in C(\mathscr{E}, \mathbb{R})$ be two functions. We notice that the semigroup property is not valid, meaning that

$${}^{ML}\mathbf{I}_{0^+}^{u(\varkappa)\,ML}\mathbf{I}_{0^+}^{v(\varkappa)}\vartheta(\varkappa) \neq {}^{ML}\mathbf{I}_{0^+}^{u(\varkappa)+v(\varkappa)}\vartheta(\varkappa). \tag{17}$$

Lemma 7 (see [8]). *Let $\varrho \in (1, 2]$ and $\hbar \in C(\mathscr{E}, \mathbb{R})$. Then, the following ML-type linear problem,*

$$\begin{cases} {}^{ML}\mathbf{D}_{a^+}^{\varrho}\vartheta(\varkappa) = \hbar(\varkappa), \vartheta(a) = c_1, \\ \vartheta'(a) = c_2, \end{cases} \tag{18}$$

is equivalent to the following integral equation

$$\vartheta(\varkappa) = c_1 + c_2(\varkappa - a) + {}^{ML}\mathbf{I}_{a^+}^{\varrho}\hbar(\varkappa), \tag{19}$$

where

$${}^{ML}\mathbf{I}_{a^+}^{\varrho}\hbar(\varkappa) = \frac{2 - \varrho}{\Upsilon(\varrho - 1)}\int_a^{\varkappa}\hbar(s)ds + \frac{\varrho - 1}{\Upsilon(\varrho - 1)\Gamma(\varrho)}\int_a^{\varkappa}(\varkappa - s)^{\varrho - 1}\hbar(s)ds. \tag{20}$$

Lemma 8. *Let $\varrho_l \in (1, 2], l = 1, 2, \cdots, n$ and $\hbar \in C(\mathscr{E}_l, \mathbb{R})$ and let $\Theta = (b_l - b_{l-1}) - \sum_{j=1}^{n}\tau_j(\kappa_j - b_{l-1}) \neq 0, \tau_j \in \mathbb{R}, \kappa_j \in (b_{l-1}, b_l)$, with $b_0 = 0, b_n = b, j = 1, 2, \cdots, n$. Then, the following ML-type linear problem,*

$$\begin{cases} {}^{ML}\mathbf{D}_{b_{l-1}^+}^{\varrho_l}\vartheta(\varkappa) = \hbar(\varkappa), \\ \vartheta(b_{l-1}) = 0, \vartheta(b_l) = \sum_{j=1}^{n}\tau_j\vartheta(\kappa_j), \end{cases} \tag{21}$$

is equivalent to the following integral equation

$$\vartheta(\varkappa) = \frac{(\varkappa - b_{l-1})}{\Theta}\left[\sum_{j=1}^{n}\tau_j^{ML}\mathbf{I}_{b_{l-1}^+}^{\varrho_l}\hbar(\kappa_j) - {}^{ML}\mathbf{I}_{b_{l-1}^+}^{\varrho_l}\hbar(b_l)\right] + {}^{ML}\mathbf{I}_{b_{l-1}^+}^{\varrho_l}\hbar(\varkappa), \tag{22}$$

where

$${}^{ML}\mathbf{I}_{b_{l-1}^+}^{\varrho_l}\hbar(\varkappa) = \frac{2 - \varrho_l}{\Upsilon(\varrho_l - 1)}\int_{b_{l-1}}^{\varkappa}\hbar(s)ds + \frac{\varrho_l - 1}{\Upsilon(\varrho_l - 1)\Gamma(\varrho_l)} \cdot \int_{b_{l-1}}^{\varkappa}(\varkappa - s)^{\varrho_l - 1}\hbar(s)ds. \tag{23}$$

Proof. Suppose that $\vartheta \in C(\mathscr{E}_l, \mathbb{R})$ is a solution of problem (21). Applying the operator ${}^{ML}\mathbf{I}_{b_{l-1}^+}^{\varrho_l}$ to both sides of (21), we find

$$\vartheta(\varkappa) = c_1 + c_2(\varkappa - b_{l-1}) + {}^{ML}\mathbf{I}_{b_{l-1}^+}^{\varrho_l}\hbar(\varkappa). \tag{24}$$

By the condition $\vartheta(b_{l-1}) = 0$, we get $c_1 = 0$.. Hence, Equation (24) reduces to

$$\vartheta(\varkappa) = c_2(\varkappa - b_{l-1}) + {}^{ML}\mathbf{I}_{b_{l-1}^+}^{\varrho_l}\hbar(\varkappa). \tag{25}$$

As per condition $\vartheta(b_l) = \sum_{j=1}^{n}\tau_j\vartheta(\kappa_j)$, we obtain

$$c_2 = \frac{1}{\Theta}\left[\sum_{j=1}^{n}\tau_j^{ML}\mathbf{I}_{b_{l-1}^+}^{\varrho_l}\hbar(\kappa_j) - {}^{ML}\mathbf{I}_{b_{l-1}^+}^{\varrho_l}\hbar(b_l)\right]. \tag{26}$$

Substitute the values of c_1, c_2 into Equation (24), we get Equation (22). Conversely, we assume that ϑ satisfies Equation (22). Then, by applying the operator ${}^{ML}\mathbf{D}_{b_{l-1}^+}^{\varrho_l}$ on both sides of Equation (22) and using the fact ${}^{ML}\mathbf{D}_{b_{l-1}^+}^{\varrho_l}(\varkappa - b_{l-1}) = 0$,, we have

$$
\begin{aligned}
{}^{ML}\mathbf{D}^{\varrho_l}_{b^+_{l-1}}\vartheta(\varkappa) = {}^{ML}\mathbf{D}^{\varrho_l}_{b^+_{l-1}}\frac{(\varkappa-b_{l-1})}{\Theta} \\
\cdot\left[\sum_{j=1}^{n}\tau_j\left({}^{ML}\mathbf{I}^{\varrho_l}_{b^+_{l-1}}\hbar(\kappa_j)\right)-{}^{ML}\mathbf{I}^{\varrho_l}_{b^+_{l-1}}\hbar(b_l)\right] \\
+{}^{ML}\mathbf{D}^{\varrho_l}_{b^+_{l-1}}\left({}^{ML}\mathbf{I}^{\varrho_l}_{b^+_{l-1}}\hbar(\varkappa)\right)=\hbar(\varkappa).
\end{aligned}
\tag{27}
$$

As $\varkappa \longrightarrow \kappa_j$ in (25) and multiply by τ_j, we get

$$
\begin{aligned}
\sum_{j=1}^{n}\tau_j\vartheta(\kappa_j) &= \frac{\sum_{j=1}^{n}\tau_j(\kappa_j-b_{l-1})}{\Theta} \\
&\cdot\left[\sum_{j=1}^{n}\tau_j{}^{ML}\mathbf{I}^{\varrho_l}_{b^+_{l-1}}\hbar(\kappa_j)-{}^{ML}\mathbf{I}^{\varrho_l}_{b^+_{l-1}}\hbar(b_l)\right] \\
&+\sum_{j=1}^{n}\tau_j\left({}^{ML}\mathbf{I}^{\varrho_l}_{b^+_{l-1}}\hbar(\kappa_j)\right)=\frac{(b_l-b_{l-1})-\Theta}{\Theta} \\
&\cdot\left[\sum_{j=1}^{n}\tau_j{}^{ML}\mathbf{I}^{\varrho_l}_{b^+_{l-1}}\hbar(\kappa_j)-{}^{ML}\mathbf{I}^{\varrho_l}_{b^+_{l-1}}\hbar(b_l)\right] \\
&+\sum_{j=1}^{n}\tau_j\left({}^{ML}\mathbf{I}^{\varrho_l}_{b^+_{l-1}}\hbar(\kappa_j)\right)=\vartheta(b_l).
\end{aligned}
\tag{28}
$$

□

Thus, nonlocal conditions are satisfied.

Theorem 9. *Let* $\varrho_l \in (1,2], l=1,2,\cdots,n$ *and* $\mathscr{K}:\mathscr{E}_l\times\mathbb{R}^3\longrightarrow\mathbb{R}$ *be continuous function and* $\Theta=(b_l-b_{l-1})-\sum_{j=1}^{n}\tau_j(\kappa_j-b_{l-1})\neq 0, \tau_j\in\mathbb{R}, \kappa_j\in(b_{l-1},b_l)$, *with* $b_0=0, b_n=b, j=1,2,\cdots,n$. *If* $\vartheta\in C(\mathscr{E}_l,\mathbb{R})$ *is a solution of the following ML-type problem*

$$
\begin{cases}
{}^{ML}\mathbf{D}^{\varrho_l}_{b^+_{l-1}}\vartheta(\varkappa)=\mathscr{K}\left(\varkappa,\vartheta(\varkappa),{}^{ML}\mathbf{I}^{\varrho_l}_{b^+_{l-1}}\vartheta(\varkappa),{}^{ML}\mathbf{D}^{\varrho_l}_{b^+_{l-1}}\vartheta(\varkappa)\right), \varkappa\in\mathscr{E}_l \\
\vartheta(b_{l-1})=0, \vartheta(b_l)=\sum_{j=1}^{n}\tau_j\vartheta(\kappa_j),
\end{cases}
\tag{29}
$$

then, ϑ *satisfies the following fractional integral equation*

$$
\begin{aligned}
\vartheta(\varkappa) &= P_1(\varkappa-b_{l-1})\left(\sum_{j=1}^{n}\tau_j\int_{b_{l-1}}^{\kappa_j}\mathscr{K}_\vartheta(s)ds-\int_{b_{l-1}}^{b_l}\mathscr{K}_\vartheta(s)ds\right) \\
&+\frac{P_2(\varkappa-b_{l-1})}{\Gamma(\varrho_l)}\left(\sum_{j=1}^{n}\tau_j\int_{b_{l-1}}^{\kappa_j}(\kappa_j-s)^{\varrho_l-1}\mathscr{K}_\vartheta(s)ds\right. \\
&\left.-\int_{b_{l-1}}^{b_l}(b_l-s)^{\varrho_l-1}\mathscr{K}_\vartheta(s)ds\right)+P_3\int_{b_{l-1}}^{\varkappa}|\mathscr{K}_\vartheta(s)| \\
&\cdot ds+\frac{P_4}{\Gamma(\varrho_l)}\int_{b_{l-1}}^{\varkappa}(\varkappa-s)^{\varrho_l-1}\mathscr{K}_\vartheta(s)ds,
\end{aligned}
\tag{30}
$$

where

$$
\begin{aligned}
\mathscr{K}_\vartheta(\varkappa) &= \mathscr{K}\left(\varkappa,\vartheta(\varkappa),{}^{ML}\mathbf{I}^{\varrho_l}_{b^+_{l-1}}\vartheta(\varkappa),{}^{ML}\mathbf{D}^{\varrho_l}_{b^+_{l-1}}\vartheta(\varkappa)\right), \\
P_1 &= \frac{2-\varrho_l}{\Theta\Upsilon(\varrho_l-1)}, P_2=\frac{\varrho_l-1}{\Theta\Upsilon(\varrho_l-1)}, \\
P_3 &= \frac{2-\varrho_l}{\Upsilon(\varrho_l-1)}, P_4=\frac{\varrho_l-1}{\Upsilon(\varrho_l-1)\Gamma(\varrho_l)}.
\end{aligned}
\tag{31}
$$

3. Existence and Uniqueness of Solutions

This section is devoted to proving the existence and uniqueness theorems for the ML-type problem (14). For simplicity, we set

$$
\begin{aligned}
\mathscr{M} &= \left[\frac{(2-\varrho_l)(b_l-b_{l-1})}{Y(\varrho_l)}+\frac{(\varrho_l-1)(b_l-b_{l-1})}{Y(\varrho_l-1)\Gamma(\varrho_l+1)}\right], \\
\mathscr{R}_P &= P_1(b_l-b_{l-1})\left(\sum_{j=1}^{n}\tau_j(\kappa_j-b_{l-1})+(b_l-b_{l-1})\right) \\
&+\frac{P_2(b_l-b_{l-1})}{\Gamma(\varrho_l+1)}\left(\sum_{j=1}^{n}\tau_j(\kappa_j-b_{l-1})^{\varrho_l}+(b_l-b_{l-1})^{\varrho_l}\right) \\
&+\left(P_3(b_l-b_{l-1})+\frac{P_4}{\Gamma(\varrho_l+1)}(b_l-b_{l-1})^{\varrho_l}\right), \\
\Omega &= \mathscr{R}_P\frac{\mathfrak{N}_f(1+\mathscr{M})}{1-\mathfrak{N}_f}.
\end{aligned}
\tag{32}
$$

Theorem 10. *Suppose that*

$$
\begin{aligned}
(H_1)&: |\mathscr{K}(\varkappa,x,y,z)-\mathscr{K}(\varkappa,\bar{x},\bar{y},\bar{z})| \\
&\leq \mathfrak{N}_f(|x-\bar{x}|+|y-\bar{y}|+|z-\bar{z}|), \mathfrak{N}_f>0,
\end{aligned}
\tag{33}
$$

for all $x,y,z,\bar{x},\bar{y},\bar{z}\in C(\mathscr{E}_l,\mathbb{R})$. *Then, problem (14) has a unique solution provided that* $\Omega<1$.

Proof. As per Theorem 9, we define the operator $\mathbf{\Pi}:C(\mathscr{E}_l,\mathbb{R})\longrightarrow C(\mathscr{E}_l,\mathbb{R})$

$$
\begin{aligned}
(\mathbf{\Pi}\vartheta)(\varkappa) &= P_1(\varkappa-b_{l-1})\left(\sum_{j=1}^{n}\tau_j\int_{b_{l-1}}^{\kappa_j}\mathscr{K}_\vartheta(s)ds-\int_{b_{l-1}}^{b_l}\mathscr{K}_\vartheta(s)ds\right) \\
&+\frac{P_2(\varkappa-b_{l-1})}{\Gamma(\varrho_l)}\left(\sum_{j=1}^{n}\tau_j\int_{b_{l-1}}^{\kappa_j}(\kappa_j-s)^{\varrho_l-1}\mathscr{K}_\vartheta(s)\right. \\
&\left.\cdot ds-\int_{b_{l-1}}^{b_l}(b_l-s)^{\varrho_l-1}\mathscr{K}_\vartheta(s)ds\right) \\
&+P_3\int_{b_{l-1}}^{\varkappa}|\mathscr{K}_\vartheta(s)|ds+\frac{P_4}{\Gamma(\varrho_l)}\int_{b_{l-1}}^{\varkappa}(\varkappa-s)^{\varrho_l-1}|\mathscr{K}_\vartheta(s)|ds.
\end{aligned}
\tag{34}
$$

Let us consider a closed ball Π_φ as

$$\mathbf{K}_{\eta_l} = \{\vartheta \in C(\mathscr{E}_l, \mathbb{R}) : \|\vartheta\| \le \eta_l\}, \tag{35}$$

with the radius $\eta_l \ge \Omega_1/1-\Omega$, where

$$\Omega = \mathscr{R}_P \omega_f, \tag{36}$$

and $\omega_f = \max_{\varkappa \in \mathscr{E}_l} |\mathscr{K}_\vartheta(0)|$. Now, we show that $\mathbf{\Pi K}_{\eta_l} \subset \mathbf{K}_{\eta_l}$. For all $\vartheta \in \mathbf{K}_{\eta_l}$ and $\varkappa \in \mathscr{E}_l$, we have

$$\begin{aligned} |(\mathbf{\Pi}\vartheta)(\varkappa)| &= P_1(\varkappa - b_{l-1}) \left(\sum_{j=1}^n \tau_j \int_{b_{l-1}}^{\kappa_j} |\mathscr{K}_\vartheta(s)| ds + \int_{b_{l-1}}^{b_l} |\mathscr{K}_\vartheta(s)| ds \right) \\ &+ \frac{P_2(\varkappa - b_{l-1})}{\Gamma(\varrho_l)} \left(\sum_{j=1}^n \tau_j \int_{b_{l-1}}^{\kappa_j} (\kappa_j - s)^{\varrho_l - 1} |\mathscr{K}_\vartheta(s)| \right. \\ &\left. \cdot ds + \int_{b_{l-1}}^{b_l} (b_l - s)^{\varrho_l - 1} |\mathscr{K}_\vartheta(s)| ds \right) \\ &+ P_3 \int_{b_{l-1}}^{\varkappa} |\mathscr{K}_\vartheta(s)| ds + \frac{P_4}{\Gamma(\varrho_l)} \int_{b_{l-1}}^{\varkappa} (\varkappa - s)^{\varrho_l - 1} |\mathscr{K}_\vartheta(s)| ds. \end{aligned} \tag{37}$$

By (H_1) and definition of ${}^{ML}\mathbf{I}^{\varrho_l}_{b^+_{l-1}}$ in the case of $\varrho_l \in (1, 2]$ defined as Equation (23), we have

$$\begin{aligned} &\left| \mathscr{K}\left(\varkappa, \vartheta(\varkappa), {}^{ML}\mathbf{I}^{\varrho_l}_{b^+_{l-1}} \vartheta(\varkappa), {}^{ML}\mathbf{D}^{\varrho_l}_{b^+_{l-1}} \vartheta(\varkappa)\right) \right| \\ &= \left| \mathscr{K}\left(\varkappa, \vartheta(\varkappa), {}^{ML}\mathbf{I}^{\varrho_l}_{b^+_{l-1}} \vartheta(\varkappa), {}^{ML}\mathbf{D}^{\varrho_l}_{b^+_{l-1}} \vartheta(\varkappa)\right) - f(\varkappa, 0, 0, 0) \right| \\ &+ |f(\varkappa, 0, 0, 0)| \le \frac{\mathfrak{N}_f (1 + \mathscr{M})}{1 - \mathfrak{N}_f} |\vartheta(\varkappa)| + \omega_f. \end{aligned} \tag{38}$$

Hence,

$$\begin{aligned} \|\mathbf{\Pi}\vartheta\| &\le P_1(\varkappa - b_{l-1}) \left(\sum_{j=1}^n \tau_j \int_{b_{l-1}}^{\kappa_j} |\mathscr{K}_\vartheta(s)| ds + \int_{b_{l-1}}^{b_l} |\mathscr{K}_\vartheta(s)| ds \right) \\ &+ \frac{P_2(\varkappa - b_{l-1})}{\Gamma(\varrho_l)} \left(\sum_{j=1}^n \tau_j \int_{b_{l-1}}^{\kappa_j} (\kappa_j - s)^{\varrho_l - 1} |\mathscr{K}_\vartheta(s)| \right. \\ &\left. \cdot ds + \int_{b_{l-1}}^{b_l} (b_l - s)^{\varrho_l - 1} |\mathscr{K}_\vartheta(s)| ds \right) \\ &+ P_3 \int_{b_{l-1}}^{\varkappa} |\mathscr{K}_\vartheta(s)| ds + \frac{P_4}{\Gamma(\varrho_l)} \int_{b_{l-1}}^{\varkappa} (\varkappa - s)^{\varrho_l - 1} |\mathscr{K}_\vartheta(s)| ds \\ &\le \Omega \eta_l + \Omega_1 \le \eta_l. \end{aligned} \tag{39}$$

Thus $\mathbf{\Pi}\vartheta \in D_l$. Now, we need to prove that $\mathbf{\Pi}$ is a contraction map. Let $\vartheta, \widehat{\vartheta} \in D_l$ and $\varkappa \in \mathscr{E}_l$ .Then, we have

$$\begin{aligned} &\left| (\mathbf{\Pi}\vartheta)(\varkappa) - \left(\mathbf{\Pi}\widehat{\vartheta}\right)(\varkappa) \right| \le P_1(\varkappa - b_{l-1}) \\ &\cdot \left(\sum_{j=1}^n \tau_j \int_{b_{l-1}}^{\kappa_j} \left| \mathscr{K}_\vartheta(s) - \mathscr{K}_{\widehat{\vartheta}}(s) \right| ds + \int_{b_{l-1}}^{b_l} \left| \mathscr{K}_\vartheta(s) - \mathscr{K}_{\widehat{\vartheta}}(s) \right| ds \right) \\ &+ \frac{P_2(\varkappa - b_{l-1})}{\Gamma(\varrho_l)} \left(\sum_{j=1}^n \tau_j \int_{b_{l-1}}^{\kappa_j} (\kappa_j - s)^{\varrho_l - 1} \left| \mathscr{K}_\vartheta(s) - \mathscr{K}_{\widehat{\vartheta}}(s) \right| \right. \\ &\left. \cdot ds + \int_{b_{l-1}}^{b_l} (b_l - s)^{\varrho_l - 1} \left| \mathscr{K}_\vartheta(s) - \mathscr{K}_{\widehat{\vartheta}}(s) \right| ds \right) + P_3 \int_{b_{l-1}}^{\varkappa} |\mathscr{K}_\vartheta(s) \\ &- \mathscr{K}_{\widehat{\vartheta}}(s)| ds + \frac{P_4}{\Gamma(\varrho_l)} \int_{b_{l-1}}^{\varkappa} (\varkappa - s)^{\varrho_l - 1} \left| \mathscr{K}_\vartheta(s) - \mathscr{K}_{\widehat{\vartheta}}(s) \right| ds. \end{aligned} \tag{40}$$

From our assumption, we obtain

$$\begin{aligned} &\left| \mathscr{K}_\vartheta(s) - \mathscr{K}_{\widehat{\vartheta}}(s) \right| \le \mathfrak{N}_f \left(\left| \vartheta(s) - \widehat{\vartheta}(s) \right| + \left| {}^{ML}\mathbf{I}^{\varrho_l}_{b^+_{l-1}} \vartheta(\varkappa) - {}^{ML}\mathbf{I}^{\varrho_l}_{b^+_{l-1}} \widehat{\vartheta}(\varkappa) \right| \right. \\ &\left. + \left| \mathscr{K}_\vartheta(s) - \mathscr{K}_{\widehat{\vartheta}}(s) \right| \right) \le \frac{\mathfrak{N}_f (1 + \mathscr{M})}{1 - \mathfrak{N}_f} \left\| \vartheta - \widehat{\vartheta} \right\|. \end{aligned} \tag{41}$$

Hence,

$$\left\| \mathbf{\Pi}\vartheta - \mathbf{\Pi}\widehat{\vartheta} \right\| \le \Omega \left\| \vartheta - \widehat{\vartheta} \right\|. \tag{42}$$

Due to $\Omega < 1$, we conclude that $\mathbf{\Pi}$ is a contraction mapping. Hence, by the Banach fixed point Theorem 4, $\mathbf{\Pi}$ has a unique fixed point. □

Theorem 11. *Under the hypotheses of Theorem 10, the ML-type problem (14) has at least one solution.*

Proof. Let us consider the operator $\mathbf{\Pi}$ defined by Theorem 10. Now, we will divided the operator $\mathbf{\Pi}$ into two operators $\mathbf{\Pi}_1, \mathbf{\Pi}_2$ such that $(\mathbf{\Pi}\vartheta)(\varkappa) = (\mathbf{\Pi}_1\vartheta)(\varkappa) + (\mathbf{\Pi}_2\vartheta)(\varkappa)$, where

$$\begin{aligned} (\mathbf{\Pi}_1\vartheta)(\varkappa) &= P_1(\varkappa - b_{l-1}) \left(\sum_{j=1}^n \tau_j \int_{b_{l-1}}^{\kappa_j} \mathscr{K}_\vartheta(s) ds - \int_{b_{l-1}}^{b_l} \mathscr{K}_\vartheta(s) ds \right) + \frac{P_2(\varkappa - b_{l-1})}{\Gamma(\varrho_l)} \\ &\cdot \left(\sum_{j=1}^n \tau_j \int_{b_{l-1}}^{\kappa_j} (\kappa_j - s)^{\varrho_l - 1} \mathscr{K}_\vartheta(s) ds - \int_{b_{l-1}}^{b_l} (b_l - s)^{\varrho_l - 1} \mathscr{K}_\vartheta(s) ds \right), \\ (\mathbf{\Pi}_2\vartheta)(\varkappa) &= P_3 \int_{b_{l-1}}^{\varkappa} |\mathscr{K}_\vartheta(s)| ds + \frac{P_4}{\Gamma(\varrho_l)} \int_{b_{l-1}}^{\varkappa} (\varkappa - s)^{\varrho_l - 1} \mathscr{K}_\vartheta(s) ds. \end{aligned} \tag{43}$$

□

Consider a closed ball $\mathbf{K}_{\eta_l}$ defined as in Theorem 10. In order to fulfill the conditions in Theorem 5, we split the proof into the following steps:

Step 1. $\mathbf{\Pi}_1\vartheta+\mathbf{\Pi}_2\widehat{\vartheta}\in\mathbf{K}_{\eta_l}$ for all $\vartheta,\widehat{\vartheta}\in\mathbf{K}_{\eta_l}$. First, in order to operator $\mathbf{\Pi}_1$. For $\vartheta\in\mathbf{K}_{\eta_l}$ and $\varkappa\in\mathscr{E}_l$, we have

$$|(\mathbf{\Pi}_1\vartheta)(\varkappa)|\le P_1(\varkappa-b_{l-1})\left(\sum_{j=1}^{n}\tau_j\int_{b_{l-1}}^{\kappa_j}|\mathscr{K}_\vartheta(s)|ds+\int_{b_{l-1}}^{b_l}|\mathscr{K}_\vartheta(s)|ds\right)+\frac{P_2(\varkappa-b_{l-1})}{\Gamma(\varrho_l)}\left(\sum_{j=1}^{n}\tau_j\int_{b_{l-1}}^{\kappa_j}(\kappa_j-s)^{\varrho_l-1}|\mathscr{K}_\vartheta(s)|ds+\int_{b_{l-1}}^{b_l}(b_l-s)^{\varrho_l-1}|\mathscr{K}_\vartheta(s)|ds\right). \quad (44)$$

By Equation (38), we have

$$\|\mathbf{\Pi}_1\vartheta\|\le\left[P_1(b_l-b_{l-1})\left(\sum_{j=1}^{n}\tau_j(\kappa_j-b_{l-1})+(b_l-b_{l-1})\right)+\frac{P_2(b_l-b_{l-1})}{\Gamma(\varrho_l+1)}\left(\sum_{j=1}^{n}\tau_j(\kappa_j-b_{l-1})^{\varrho_l}+(b_l-b_{l-1})^{\varrho_l}\right)\right]\cdot\left(\frac{\mathfrak{N}_f(1+\mathscr{M})}{1-\mathfrak{N}_f}\eta_l+\omega_f\right). \quad (45)$$

Next, for the operator $\mathbf{\Pi}_2$, we have

$$\|\mathbf{\Pi}_2\vartheta\|\le\left(P_3(b_l-b_{l-1})+\frac{P_4}{\Gamma(\varrho_l+1)}(b_l-b_{l-1})^{\varrho_l}\right)\left(\frac{\mathfrak{N}_f(1+\mathscr{M})}{1-\mathfrak{N}_f}\eta_l+\omega_f\right). \quad (46)$$

Inequalities (45) and (46) give

$$\|\mathbf{\Pi}_1\vartheta+\mathbf{\Pi}_2\vartheta\|\le\|\mathbf{\Pi}_1\vartheta\|+\|\mathbf{\Pi}_2\vartheta\|<\eta_l. \quad (47)$$

Thus, $\mathbf{\Pi}_1\vartheta+\mathbf{\Pi}_2\widehat{\vartheta}\in\mathbf{K}_{\eta_l}$.

Step 2. $\mathbf{\Pi}_2$ is a contraction map. Due to the operator $\mathbf{\Pi}$ is a contraction map, we conclude that $\mathbf{\Pi}_1$ is contraction too.

Step 3. $\mathbf{\Pi}_1$ is continuous and compact. Since $\mathscr{K}_\vartheta$ is continuous, $\mathbf{\Pi}_1$ is continuous too. Also, by Equation (45), $\mathbf{\Pi}_1$ is uniformly bounded on $\mathbf{K}_{\eta_l}$. Now, we show that $\mathbf{\Pi}_1(\mathbf{K}_{\eta_l})$ is an equicontinuous. For this purpose, let $\vartheta\in\mathbf{K}_{\eta_l}, a\le\varkappa_1<\varkappa_2\le b$. Then, we have

$$|(\mathbf{\Pi}_1\vartheta)(\varkappa_2)-(\mathbf{\Pi}_1\vartheta)(\varkappa_1)|\le P_1[(\varkappa_2-b_{l-1})-(\varkappa_1-b_{l-1})]\cdot\left(\sum_{j=1}^{n}\tau_j\int_{b_{l-1}}^{\kappa_j}\mathscr{K}_\vartheta(s)ds-\int_{b_{l-1}}^{b_l}\mathscr{K}_\vartheta(s)ds\right)+\frac{P_2(\varkappa_2-b_{l-1})-(\varkappa_1-b_{l-1})}{\Gamma(\varrho_l)}\cdot\left(\sum_{j=1}^{n}\tau_j\int_{b_{l-1}}^{\kappa_j}(\kappa_j-s)^{\varrho_l-1}\mathscr{K}_\vartheta(s)ds-\int_{b_{l-1}}^{b_l}(b_l-s)^{\varrho_l-1}\mathscr{K}_\vartheta(s)ds\right). \quad (48)$$

Thus,

$$\|(\mathbf{\Pi}_1\vartheta)(\varkappa_2)-(\mathbf{\Pi}_1\vartheta)(\varkappa_1)\|\longrightarrow 0\ as\ \varkappa_2\longrightarrow\varkappa_1. \quad (49)$$

In view of the previous steps along with the theorem of Arzela-Ascoli, we deduce that $(\mathbf{\Pi}_1\mathbf{K}_{\eta_l})$ is relatively compact. Consequently, $\mathbf{\Pi}_1$ is completely continuous. Hence, by Theorem 5, there exists a solution ϑ_l of problem (14). For $l\in\{1,2,\cdots,n\}$, we define the function

$$\vartheta_l=\begin{cases}0, & \varkappa\in[0,b_{l-1}],\\ \tilde{\vartheta}_l, & \varkappa\in\mathscr{E}_l.\end{cases} \quad (50)$$

As a result of this, it is well known that $\vartheta_l\in C(\mathscr{E}_l,\mathbb{R})$ given by Equation (50) satisfies the following problem

$$\frac{\Upsilon(\varrho(\varkappa))}{1-\varrho(\varkappa)}\int_0^{b_1}E_{\varrho(\varkappa)}\left(\frac{\varrho(\varkappa)}{\varrho(\varkappa)-1}(\varkappa_1-\theta)^{\varrho(\varkappa)}\right)\vartheta_l'(\theta)d\theta+\cdots\cdots+\frac{\Upsilon(\varrho(\varkappa))}{1-\varrho(\varkappa)}\cdot\int_{b_{l-1}}^{\varkappa}E_{\varrho(\varkappa)}\left(\frac{\varrho(\varkappa)}{\varrho(\varkappa)-1}(b-\theta)^{\varrho(\varkappa)}\right)\vartheta_l'(\theta)d\theta=\mathscr{K}\left(\varrho,\vartheta_l(\varrho),{}^{ML}\mathbf{I}_{0^+}^{\varrho(\varkappa)}\vartheta_l(\varrho),{}^{ML}\mathbf{D}_{0^+}^{\varrho(\varkappa)}\vartheta_l(\varrho)\right), \quad (51)$$

where ϑ_l is a solution to Equation (12) equipped with $\vartheta_l(0)=\vartheta_l(b_l)=\tilde{\vartheta}_l(b_l)=0$. Then,

$$\vartheta(\varkappa)=\begin{cases}\vartheta_1(\varkappa), & \varkappa\in\mathscr{E}_1,\\ \vartheta_2(\varkappa)=\begin{cases}0, & \varkappa\in\mathscr{E}_1,\\ \tilde{\vartheta}_2, & \varkappa\in\mathscr{E}_2,\end{cases}\\ \cdot\\ \cdot\\ \cdot\\ \vartheta_n(\varkappa)=\begin{cases}0, & \varkappa\in[0,b_{l-1}],\\ \tilde{\vartheta}_n, & \varkappa\in\mathscr{E}_n,\end{cases}\end{cases} \quad (52)$$

is the solution of problem (14).

4. Stability Results for ML-Type Problem (14)

In this section, we discuss Ulam-Hyers (UH) and generalized Ulam-Hyers (GUH) stability results for problem (14). Let $\varepsilon>0$ and ϑ be a function such that $\vartheta(\varkappa)\in C(\mathscr{E}_l,\mathbb{R})$ satisfies the following inequations:

$$\left|{}^{ML}\mathbf{D}_0^{\varrho(\varkappa)}\vartheta(\varkappa)-\mathscr{K}_\vartheta(\varkappa)\right|\le\varepsilon,\varkappa\in\mathscr{E}. \quad (53)$$

Define the functions $\vartheta_l(\varkappa)$, $\widehat{\vartheta}_l(\varkappa)$, $\Bbbk_l(\varkappa)$,$\varkappa\in\mathscr{E}_l$ as follows.

$$\vartheta_l(\varkappa)=\begin{cases}0, & \text{if}\varkappa\in[0,b_{l-1}],\\ \vartheta(\varkappa) & \text{if}\varkappa\in\mathscr{E}_l,\end{cases} \quad (54)$$

$$\widehat{\vartheta}_l(\varkappa) = \begin{cases} 0, & \text{if } \varkappa \in [0, b_{l-1}], \\ \widehat{\vartheta}(\varkappa) & \text{if } \varkappa \in \mathscr{E}_l, \end{cases} \tag{55}$$

$$\Bbbk_l(\varkappa) = \begin{cases} 0, & \text{if } \varkappa \in [0, b_{l-1}], \\ \Bbbk(\varkappa) & \text{if } \varkappa \in \mathscr{E}_l. \end{cases} \tag{56}$$

Definition 12. Problem (14) is UH stable if there exists a real number $C_{\mathscr{K}} > 0$ such that, for each $\varepsilon > 0$ and for each solution $\widehat{\vartheta} \in C(\mathscr{E}_l, \mathbb{R})$ of inequality (53), there exists a unique solution $\vartheta \in C(\mathscr{E}_l, \mathbb{R})$ of problem (14) with

$$\left|\widehat{\vartheta}(\varkappa) - \vartheta(\varkappa)\right| \le C_{\mathscr{K}}\varepsilon, \tag{57}$$

where ϑ and $\widehat{\vartheta}$ are defined by Equation (54) and Equation (55), respectively.

Remark 13. Let $\widehat{\vartheta} \in C(\mathscr{E}_l, \mathbb{R})$ be the solution of inequality (53) if and only if we have a function $\Bbbk \in C(\mathscr{E}_l, \mathbb{R})$ which depends on ϑ such that

(i) $|\Bbbk(\varkappa)| \le \varepsilon$ for all $\varkappa \in \mathscr{E}_l$

(ii) ${}^{ML}\mathbf{D}^{\varrho_l}_{b^+_{l-1}}\widehat{\vartheta}(\varkappa) = \mathscr{K}_{\widehat{\vartheta}}(\varkappa) + \Bbbk(\varkappa)$, for all $\varkappa \in \mathscr{E}$.

Lemma 14. *If $\vartheta \in C(\mathscr{E}_l, \mathbb{R})$ is a solution of inequality (53), then ϑ satisfies the following inequality*

$$\left|\widehat{\vartheta}(\varkappa) - \Psi_{\widehat{\vartheta}} - P_3\int_{b_{l-1}}^{\varkappa} \mathscr{K}_{\widehat{\vartheta}}(s)ds - \frac{P_4}{\Gamma(\varrho_l)}\int_{b_{l-1}}^{\varkappa} (\varkappa - s)^{\varrho_l - 1}\mathscr{K}_{\widehat{\vartheta}}(s)ds\right| \le \varepsilon\mathscr{R}_P, \tag{58}$$

where

$$\Psi_{\widehat{\vartheta}} = P_1(\varkappa - b_{l-1})\left(\sum_{j=1}^{n}\tau_j\int_{b_{l-1}}^{\kappa_j}\mathscr{K}_{\widehat{\vartheta}}(s)ds - \int_{b_{l-1}}^{b_l}\mathscr{K}_{\widehat{\vartheta}}(s)ds\right) + \frac{P_2(\varkappa - b_{l-1})}{\Gamma(\varrho_l)}\left(\sum_{j=1}^{n}\tau_j\int_{b_{l-1}}^{\kappa_j}(\kappa_j - s)^{\varrho_l-1}\mathscr{K}_{\widehat{\vartheta}}(s)ds - \int_{b_{l-1}}^{b_l}(b_l - s)^{\varrho_l-1}\mathscr{K}_{\widehat{\vartheta}}(s)ds\right). \tag{59}$$

Proof. As per Remark 13, we have

$$\begin{cases} {}^{ML}\mathbf{D}^{\varrho_l}_{b^+_{l-1}}\widehat{\vartheta}(\varkappa) = \mathscr{K}_{\widehat{\vartheta}}(\varkappa) + \Bbbk(\varkappa), \varkappa \in \mathscr{E}_l \\ \widehat{\vartheta}(b_{l-1}) = \widehat{\vartheta}(b_l) = 0. \end{cases} \tag{60}$$

Then, by Lemma 8, we get

$$\begin{aligned} \widehat{\vartheta}(\varkappa) = {} & \Psi_{\widehat{\vartheta}} + P_3\int_{b_{l-1}}^{\varkappa}\mathscr{K}_{\widehat{\vartheta}}(s)ds + \frac{P_4}{\Gamma(\varrho_l)}\int_{b_{l-1}}^{\varkappa}(\varkappa - s)^{\varrho_l-1}\mathscr{K}_{\widehat{\vartheta}}(s) \\ & \cdot ds + P_1(\varkappa - b_{l-1})\left(\sum_{j=1}^{n}\tau_j\int_{b_{l-1}}^{\kappa_j}\Bbbk(s)ds - \int_{b_{l-1}}^{b_l}\Bbbk(s)ds\right) + \frac{P_2(\varkappa - b_{l-1})}{\Gamma(\varrho_l)} \\ & \cdot\left(\sum_{j=1}^{n}\tau_j\int_{b_{l-1}}^{\kappa_j}(\kappa_j - s)^{\varrho_l-1}\Bbbk(s)ds - \int_{b_{l-1}}^{b_l}(b_l - s)^{\varrho_l-1}\Bbbk(s)ds\right) \\ & + P_3\int_{b_{l-1}}^{\varkappa}\Bbbk(s)ds + \frac{P_4}{\Gamma(\varrho_l)}\int_{b_{l-1}}^{\varkappa}(\varkappa - s)^{\varrho_l-1}\Bbbk(s)ds, \end{aligned} \tag{61}$$

which implies

$$\left|\widehat{\vartheta}(\varkappa) - \Psi_{\widehat{\vartheta}} - P_3\int_{b_{l-1}}^{\varkappa}\mathscr{K}_{\widehat{\vartheta}}(s)ds - \frac{P_4}{\Gamma(\varrho_l)}\int_{b_{l-1}}^{\varkappa}(\varkappa - s)^{\varrho_l-1}\mathscr{K}_{\widehat{\vartheta}}(s)ds\right| \le \varepsilon\mathscr{R}_P. \tag{62}$$

□

Theorem 15. *Suppose that (H_1) holds. If*

$$\left(P_3(b_l - b_{l-1}) + \frac{P_4(b_l - b_{l-1})^{\varrho_l}}{\Gamma(\varrho_l + 1)}\right)\frac{\mathfrak{N}_f(1 + \mathscr{M})}{1 - \mathfrak{N}_f} < 1, \tag{63}$$

then the ML-type problem (14) is UH and GUH stable.

Proof. Let $\varepsilon > 0$ and $\widehat{\vartheta} \in C(\mathscr{E}_l, \mathbb{R})$ be a function that satisfies the inequality (53), and let $\vartheta \in C(\mathscr{E}_l, \mathbb{R})$ be the unique solution of the following problem.

$$\begin{cases} {}^{ML}\mathbf{D}^{\varrho_l}_{b^+_{l-1}}\vartheta(\varkappa) = \mathscr{K}_{\vartheta}(\varkappa), \varkappa \in \mathscr{E}_l \\ \vartheta(b_{l-1}) = \widehat{\vartheta}(b_{l-1}) = 0 \\ \vartheta(b_l) = \widehat{\vartheta}(b_l) = 0. \end{cases} \tag{64}$$

Then, by Lemma 8, the solution of Equation (64) is given by

$$\vartheta(\varkappa) = \Psi_{\widehat{\vartheta}} + P_3\int_{b_{l-1}}^{\varkappa}\mathscr{K}_{\vartheta}(s)ds + \frac{P_4}{\Gamma(\varrho_l)}\int_{b_{l-1}}^{\varkappa}(\varkappa - s)^{\varrho_l-1}\mathscr{K}_{\vartheta}(s)ds. \tag{65}$$

Hence, by Lemma 14, we have

$$\begin{aligned} \left|\widehat{\vartheta}(\varkappa) - \vartheta(\varkappa)\right| \le {} & \left|\widehat{\vartheta}(\varkappa) - \Psi_{\widehat{\vartheta}} - P_3\int_{b_{l-1}}^{\varkappa}\mathscr{K}_{\widehat{\vartheta}}(s)ds - \frac{P_4}{\Gamma(\varrho_l)}\right. \\ & \left.\cdot\int_{b_{l-1}}^{\varkappa}(\varkappa - s)^{\varrho_l-1}\mathscr{K}_{\widehat{\vartheta}}(s)ds\right| + P_3\int_{b_{l-1}}^{\varkappa}\left|\mathscr{K}_{\widehat{\vartheta}}(s) - \mathscr{K}_{\vartheta}(s)\right| \\ & \cdot ds + \frac{P_4}{\Gamma(\varrho_l)}\int_{b_{l-1}}^{\varkappa}(\varkappa - s)^{\varrho_l-1}\left|\mathscr{K}_{\widehat{\vartheta}}(s) - \mathscr{K}_{\vartheta}(s)\right|ds \le \varepsilon\mathscr{R}_P \\ & + \left(P_3(b_l - b_{l-1}) + \frac{P_4}{\Gamma(\varrho_l + 1)}(b_l - b_{l-1})^{\varrho_l}\right)\frac{\mathfrak{N}_f(1 + \mathscr{M})}{1 - \mathfrak{N}_f}\left\|\widehat{\vartheta} - \vartheta\right\|. \end{aligned} \tag{66}$$

Thus,

$$\left\|\widehat{\vartheta}-\vartheta\right\| \leq C_{\mathscr{K}}\varepsilon, \tag{67}$$

where

$$C_{\mathscr{K}}=\frac{\mathscr{R}_P}{1-\left(P_3(b_l-b_{l-1})+\left(P_4/\Gamma(\varrho_l+1)\right)(b_l-b_{l-1})^{\varrho_l}\right)\left(\mathfrak{N}_f(1+\mathscr{M})/1-\mathfrak{N}_f\right)}>0. \tag{68}$$

Therefore, the ML-type problem (14) is UH stable. Finally, by choosing $C_{\mathscr{K}}(\varepsilon)=C_{\mathscr{K}}\varepsilon$ such that $C_{\mathscr{K}}(0)=0$, then the ML-type problem (14) has GUH stability. □

5. Existence of Positive Solution for ML-Type Problem (14)

In this section, we extend and develop the sufficient conditions of the existence and uniqueness of positive solution for problem (14). For the forthcoming analysis, the following assumptions must be satisfied:

(V_1): $\mathscr{K}:\mathscr{E}_l\times\mathbb{R}^3\longrightarrow\mathbb{R}$ is continuous function.

(V_2): There exists constants numbers $n_1, n_1>0, n_1\neq n_2$ such that

$$n_1\leq\mathscr{K}_\vartheta(\varkappa)\leq n_2.$$

$$(V_3):\ \Omega=\mathscr{R}_P\frac{\mathfrak{N}_f(1+\mathscr{M})}{1-\mathfrak{N}_f}. \tag{69}$$

Define the cone $\mathscr{P}\subset C(\mathscr{E}_l,\mathbb{R})$ as

$$\mathscr{P}=\{\vartheta\in C(\mathscr{E}_l,\mathbb{R}):\ \vartheta(\varkappa)\geq 0, \varkappa\in[0,b]\}. \tag{70}$$

Lemma 16. *Assume that (V_1)-(V_2) hold. Then, $\mathbf{\Pi}:\mathscr{P}\longrightarrow\mathscr{P}$ is completely continuous.*

Proof. By Theorem 11, we conclude $\mathbf{\Pi}:\mathscr{P}\longrightarrow\mathscr{P}$ is completely continuous due to $\mathbf{\Pi}:C(\mathscr{E}_l,\mathbb{R})\longrightarrow C(\mathscr{E}_l,\mathbb{R})$ is completely continuous, since $\mathscr{P}\subset C(\mathscr{E}_l,\mathbb{R})$. □

Theorem 17. *Assume that (V_1)-(V_3) hold. Then, (14) has at least one positive solution.*

Proof. First, we have $\mathbf{\Pi}$ is compact due to Lemma 16. Next, we define two sets $\mathscr{A}_1, \mathscr{A}_2$ such that $\mathscr{A}_1=\{\vartheta\in C(\mathscr{E}_l,\mathbb{R}): \|\vartheta\|\leq n_1\Omega\}$ and $\mathscr{A}_2=\{\vartheta\in C(\mathscr{E}_l,\mathbb{R}): \|\vartheta\|\leq n_2\Omega\}$. Now, for $\vartheta\in\mathscr{P}\cap\partial\mathscr{A}_2$, we have $0\leq\vartheta(\varkappa)\leq n_2\Omega, \varkappa\in\mathscr{E}_l$. Since $\mathscr{K}_\vartheta(\varkappa)\leq n_2$, we have

$$\begin{aligned}|(\mathbf{\Pi}\vartheta)(\varkappa)|&\leq P_1(\varkappa-b_{l-1})\left(\sum_{j=1}^n\tau_j\int_{b_{l-1}}^{\kappa_j}|\mathscr{K}_\vartheta(s)|ds+\int_{b_{l-1}}^{b_l}|\mathscr{K}_\vartheta(s)|ds\right)\\&+\frac{P_2(\varkappa-b_{l-1})}{\Gamma(\varrho_l)}\left(\sum_{j=1}^n\tau_j\int_{b_{l-1}}^{\kappa_j}(\kappa_j-s)^{\varrho_l-1}|\mathscr{K}_\vartheta(s)|ds+\int_{b_{l-1}}^{b_l}(b_l-s)^{\varrho_l-1}|\mathscr{K}_\vartheta(s)|ds\right)\\&+P_3\int_{b_{l-1}}^{\varkappa}|\mathscr{K}_\vartheta(s)|ds+\frac{P_4}{\Gamma(\varrho_l)}\int_{b_{l-1}}^{\varkappa}(\varkappa-s)^{\varrho_l-1}|\mathscr{K}_\vartheta(s)|ds\leq\mathscr{R}_P\frac{\mathfrak{N}_f(1+\mathscr{M})}{1-\mathfrak{N}_f}n_2\leq\Omega n_2.\end{aligned} \tag{71}$$

Hence, $\|\mathbf{\Pi}\vartheta\|\leq\Omega n_2$. Next, for $\vartheta\in\mathscr{P}\cap\partial\mathscr{A}_1$, we have $0\leq\vartheta(\varkappa)\leq n_1\Omega, \varkappa\in[0,b]$. Since $\mathscr{K}_\vartheta(\varkappa)\geq n_1$, we have $\|\mathbf{\Pi}\vartheta\|\geq\Omega n_1$. Thus, the operator $\mathbf{\Pi}$ has a fixed point in $\mathscr{P}\cap(\bar{\mathscr{A}}_2\setminus\mathscr{A}_1)$. ,which implies that the ML-problem (14) has a positive solution. □

Theorem 18. *Let $\varrho_l\in(1,2], l=1,2,3,\cdots,n$ and $\mathscr{K}:\mathscr{E}_l\times\mathbb{R}^3\longrightarrow\mathbb{R}$ is nondecreasing continuous function for each $\varkappa\in\mathscr{E}_l$ and let $\vartheta^*, \vartheta_*\in\mathscr{P}$ such that $0<\vartheta_*<\vartheta^*<b$, $\varkappa\in\mathscr{E}_l$, satisfying ${}^{ML}\mathbf{D}_{b_{l-1}^+}^{\varrho_l}\vartheta^*(\varkappa)\leq\vartheta^*$ and ${}^{ML}\mathbf{D}_{b_{l-1}^+}^{\varrho_l}\vartheta_*(\varkappa)\geq\vartheta_*$. Then, problem (14) has a positive solution.*

Proof. Let $\vartheta^*, \vartheta_*\in\mathscr{P}$ such that $0<\vartheta_*<\vartheta^*<b$. Then, we have

$$\begin{aligned}(\mathbf{\Pi}\vartheta_*)(\varkappa)&=P_1(\varkappa-b_{l-1})\left(\sum_{j=1}^n\tau_j\int_{b_{l-1}}^{\kappa_j}\mathscr{K}_{\vartheta_*}(s)ds+\int_{b_{l-1}}^{b_l}\mathscr{K}_{\vartheta_*}(s)ds\right)+\frac{P_2(\varkappa-b_{l-1})}{\Gamma(\varrho_l)}\\&\cdot\left(\sum_{j=1}^n\tau_j\int_{b_{l-1}}^{\kappa_j}(\kappa_j-s)^{\varrho_l-1}|\mathscr{K}_{\vartheta_*}(s)|ds+\int_{b_{l-1}}^{b_l}(b_l-s)^{\varrho_l-1}\mathscr{K}_{\vartheta_*}(s)ds\right)\\&+P_3\int_{b_{l-1}}^{\varkappa}\mathscr{K}_{\vartheta_*}(s)ds+\frac{P_4}{\Gamma(\varrho_l)}\int_{b_{l-1}}^{\varkappa}(\varkappa-s)^{\varrho_l-1}\mathscr{K}_{\vartheta_*}(s)ds\leq P_1(\varkappa-b_{l-1})\\&\cdot\left(\sum_{j=1}^n\tau_j\int_{b_{l-1}}^{\kappa_j}\mathscr{K}_{\vartheta^*}(s)ds+\int_{b_{l-1}}^{b_l}\mathscr{K}_{\vartheta^*}(s)ds\right)+\frac{P_2(\varkappa-b_{l-1})}{\Gamma(\varrho_l)}\\&\cdot\left(\sum_{j=1}^n\tau_j\int_{b_{l-1}}^{\kappa_j}(\kappa_j-s)^{\varrho_l-1}|\mathscr{K}_{\vartheta^*}(s)|ds+\int_{b_{l-1}}^{b_l}(b_l-s)^{\varrho_l-1}\mathscr{K}_{\vartheta^*}(s)ds\right)\\&+P_3\int_{b_{l-1}}^{\varkappa}\mathscr{K}_{\vartheta^*}(s)ds+\frac{P_4}{\Gamma(\varrho_l)}\int_{b_{l-1}}^{\varkappa}(\varkappa-s)^{\varrho_l-1}\mathscr{K}_{\vartheta^*}(s)ds=(\mathbf{\Pi}\vartheta^*)(\varkappa).\end{aligned} \tag{72}$$

Thus, $(\mathbf{\Pi}\vartheta_*)(\varkappa)\leq(\mathbf{\Pi}\vartheta^*)(\varkappa)$. According to Theorem 1.3 in [39], the operator $\mathbf{\Pi}$ is compact and hence $\mathbf{\Pi}$ has a fixed point in the ordered Banach space $\langle\vartheta_*,\vartheta^*\rangle$. Thus, $\mathbf{\Pi}:\langle\vartheta_*,\vartheta^*\rangle\longrightarrow\langle\vartheta_*,\vartheta^*\rangle$ is compact. Accordingly, $\mathbf{\Pi}$ has a fixed point $\vartheta\in\langle\vartheta_*,\vartheta^*\rangle$. Thus, problem (14) has at least one positive solution. □

Corollary 19. *Let $\mathscr{K}:\mathscr{E}_l\times\mathbb{R}_+^3\longrightarrow\mathbb{R}_+$ be nondecreasing continuous function in $\mathscr{E}_l$. If*

$$0<\lim_{\vartheta\longrightarrow\infty}\mathscr{K}_\vartheta(\varkappa)<\infty, \varkappa\in\mathscr{E}_l, \tag{73}$$

then problem (14) has at least one positive solution.

Corollary 20. *Let $\mathscr{K}:\mathscr{E}_l\times\mathbb{R}_+^3\longrightarrow\mathbb{R}_+$ be nondecreasing continuous function in $\mathscr{E}_l$. If*

$$0<\lim_{\|\vartheta\|\longrightarrow\infty}\frac{\mathscr{K}_\vartheta(\varkappa)}{\|\vartheta\|}<\infty, \varkappa\in\mathscr{E}_l, \tag{74}$$

then problem (14) has at least one positive solution.

Corollary 21. *If there exist constants $m_1, m_2>0, \rho\in(0,1]$ such that $\mathscr{K}_\vartheta(\varkappa)=m_1\vartheta(\varkappa)+m_1^\rho$, then problem (14) has at least one positive solution.*

Corollary 22. *If the function $\mathscr{K} : \mathscr{E}_l \times \mathbb{R}_+^3 \longrightarrow \mathbb{R}_+$ is continuous and there is a constant $\mathfrak{N}_f > 0$ and*

$$|\mathscr{K}(\varkappa, x, y, z) - \mathscr{K}(\varkappa, \bar{x}, \bar{y}, \bar{z})| \leq \mathfrak{N}_f(|x-\bar{x}| + |y-\bar{y}| + |z-\bar{z}|), \mathfrak{N}_f > 0, \tag{75}$$

for all $x, y, z, \bar{x}, \bar{y}, \bar{z} \in C(\mathscr{E}_l, \mathbb{R}^+)$ such that $\mathscr{R}_P(\mathfrak{N}_f(1+\mathscr{M})/1 - \mathfrak{N}_f) < 1$, then problem (14) has a positive solution (Theorem 10).

Corollary 23. *Assume that there exists two continuous functions f_1, f_2 such that $0 < f_1(\varkappa) \leq \mathscr{K}_\vartheta(\varkappa) \leq f_2(\varkappa), \varkappa \in \mathscr{E}_l$. Then, problem (14) has at least one positive solution $\vartheta(\varkappa) \in C(\mathscr{E}_l, \mathbb{R})$.*

6. An Example

Example 24. Let $\varrho \in (1, 2]$ and let us consider the following ML-type problem.

$$\begin{cases} {}^{ML}D_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa) = \dfrac{\varkappa^2}{20e^\varkappa}\left(e^{-\varkappa} + \dfrac{|\vartheta(\varkappa)|}{1+|\vartheta(\varkappa)|} + \dfrac{\left|{}^{ML}\mathbf{I}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa)\right|}{1+{}^{ML}\mathbf{I}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa)} + \dfrac{{}^{ML}\mathbf{D}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa)}{1+{}^{ML}D_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa)}\right) \\ \vartheta(0) = 0, \vartheta(1) = 0. \end{cases} \tag{76}$$

Here, $a = 0, b = 2$ and

$$\mathscr{K}_\vartheta(\varkappa) = \frac{\varkappa^2}{10e^\varkappa}\left(e^{-\varkappa} + \frac{|\vartheta(\varkappa)|}{1+|\vartheta(\varkappa)|} + \frac{{}^{ML}\mathbf{I}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa)}{1+{}^{ML}\mathbf{I}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa)} + \frac{\%^{ML}D_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa)}{1+{}^{ML}D_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa)}\right). \tag{77}$$

Let $\varkappa \in [0, 2]$, and $\vartheta, \bar{\vartheta} \in C(\mathscr{E}_l, \mathbb{R})$. Then,

$$\begin{aligned} |\mathscr{K}_\vartheta(\varkappa) - \mathscr{K}_{\bar{\vartheta}}(\varkappa)| \leq \frac{1}{20}\big(|\vartheta(\varkappa) - \bar{\vartheta}(\varkappa)| \\ + \left|{}^{ML}\mathbf{I}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa) - {}^{ML}\mathbf{I}_{0^+}^{\varrho(\varkappa)}\bar{\vartheta}(\varkappa)\right| + \left|{}^{ML}\mathbf{D}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa) - {}^{ML}\mathbf{D}_{0^+}^{\varrho(\varkappa)}\bar{\vartheta}(\varkappa)\right|\big). \end{aligned} \tag{78}$$

Therefore, (H_1) holds with $\mathfrak{N}_f = 1/20$. Here,

$$\varrho(\varkappa) = \begin{cases} \dfrac{3}{2}, & \text{if } \varkappa \in (0, 1], \\ \dfrac{5}{2} & \text{if } \varkappa \in (1, 2]. \end{cases} \tag{79}$$

For $l = 1$, we have

$$\begin{cases} {}^{ML}D_{0^+}^{\frac{3}{2}}\vartheta(\varkappa) = \dfrac{\varkappa^2}{20e^\varkappa}\left(e^{-\varkappa} + \dfrac{|\vartheta(\varkappa)|}{1+|\vartheta(\varkappa)|} + \dfrac{\left|{}^{ML}\mathbf{I}_{0^+}^{3/2}\vartheta(\varkappa)\right|}{} 1+{}^{ML}\mathbf{I}_{0^+}^{\frac{3}{2}}\vartheta(\varkappa) + \dfrac{{}^{ML}\mathbf{D}_{0^+}^{3/2}\vartheta(\varkappa)}{1+{}^{ML}\mathbf{D}_{0^+}^{3/2}\vartheta(\varkappa)}\right), \varkappa \in (0, 1] \\ \vartheta(0) = 0, \vartheta(1) = 0 \end{cases} \tag{80}$$

Also, $\Omega = 0.68 < 1$. Thus, all conditions of Theorem 10 are satisfied, and hence, the ML-type problem (14) has a unique solution. For every $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\} > 0$ and each $\widehat{\vartheta} \in C(\mathscr{E}_l, \mathbb{R})$ satisfies

$$\left|{}^{ML}\mathbf{D}_0^{\varrho(\varkappa)}\vartheta(\varkappa) - \mathscr{K}_\vartheta(\varkappa)\right| \leq \varepsilon. \tag{81}$$

There exists a solution $\vartheta \in C(\mathscr{E}_l, \mathbb{R})$ of the ML-type problem (14) with

$$\left\|\widehat{\vartheta} - \vartheta\right\| \leq C_{\mathscr{K}}\varepsilon, \tag{82}$$

where

$$C_{\mathscr{K}} = \frac{\mathscr{R}_P}{1 - (P_3(b_l - b_{l-1}) + (P_4(b_l - b_{l-1})^{\varrho_l}/\Gamma(\varrho_l + 1)))(\mathfrak{N}_f(1+\mathscr{M})/1 - \mathfrak{N}_f)} > 0. \tag{83}$$

Therefore, all conditions in Theorem 15 are satisfied, and hence, the ML-problem (14) is UH stable.

Next, for $l = 2$,, we have

$$\begin{cases} {}^{ML}\mathbf{D}_{0^+}^{\frac{5}{2}}\vartheta(\varkappa) = \dfrac{\varkappa^2}{20e^\varkappa}\left(e^{-\varkappa} + \dfrac{|\vartheta(\varkappa)|}{1+|\vartheta(\varkappa)|} + \dfrac{\left|{}^{ML}\mathbf{I}_{0^+}^{5/2}\vartheta(\varkappa)\right|}{1+{}^{ML}\mathbf{I}_{0^+}^{5/2}\vartheta(\varkappa)} + \dfrac{{}^{ML}\mathbf{D}_{0^+}^{5/2}\vartheta(\varkappa)}{1+{}^{ML}\mathbf{D}_{0^+}^{5/2}\vartheta(\varkappa)}\right), \varkappa \in (1, 2] \\ \vartheta(0) = 0, \vartheta(1) = 0, \end{cases} \tag{84}$$

and $\Omega = 0.55 < 1$. Thus, all conditions of Theorem 10 are satisfied, and hence, the ML-type problem (14) has a unique solution. For every $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\} > 0$ and each $\widehat{\vartheta} \in C(\mathscr{E}_l, \mathbb{R})$ satisfies

$$\left|{}^{ML}\mathbf{D}_0^{\varrho(\varkappa)}\vartheta(\varkappa) - \mathscr{K}_\vartheta(\varkappa)\right| \leq \varepsilon. \tag{85}$$

There exists a solution $\vartheta \in C(\mathscr{E}_l, \mathbb{R})$ of the ML-type problem (14) with

$$\left\|\widehat{\vartheta} - \vartheta\right\| \leq C_{\mathscr{K}}\varepsilon, \tag{86}$$

where

$$C_{\mathscr{K}} = \frac{\mathscr{R}_P}{1 - (P_3(b_l - b_{l-1}) + (P_4(b_l - b_{l-1})^{\varrho_l}/\Gamma(\varrho_l + 1)))(\mathfrak{N}_f(1+\mathscr{M})/1 - \mathfrak{N}_f)} > 0. \tag{87}$$

7. Conclusion Remarks

AB fractional operators are very fertile and interesting topic of research recently; thus, there are some researchers who studied and developed some qualitative properties of solutions of FDEs involving such operators. Already significant amount of work on fractional constant order for various operators has been done in literature. But to the best of our information, fractional variable order problems have not been well studied so for fractional calculus. There is a waste gap between constant and variable fractional order problems in literature, the first one has got tremendous attention as compared to the second one. Very recently, the area of variable order has started attention to be investigated. In line with these developments, we developed and investigated sufficient conditions of the existence and uniqueness of solutions for fractional variable order integro-differential equations in the frame of a ML power law. Our approach was based on the reduction of the proposed problem into the fractional integral equation and using some standard fixed point theorems as per the Banach-type and Krasnoselskii-type. Furthermore, through mathematical analysis techniques, we have analyzed the stability results in UH and GUH sense. An example has been provided to justify the main results. Due to the wide recent investigations and applications of the ML power law, we believe that acquired results here will be interesting for future investigations on the theory of fractional calculus.

In future studies, it would be interesting to study the current problem using a Mittag-Leffler power law with respect to another function introduced by Fernandez and Baleanu [40].

Data Availability

Data are available upon request.

Conflicts of Interest

The authors declare no conflict of interest.

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