Summability is a particularly fertile field for functional analysis application. Summability through functional analysis has become one of the most fascinating disciplines since it contains both interesting and challenging issues. In this paper, we aim to introduce four new sectional properties for topological sequence spaces: sectional weakly absolute convergence (WAC), sectional weak boundedness (WB), sectional weakly \( p \)-absolute convergence (\( W_p AC \)), and sectional weakly bounded variation (WBV). We have also investigated some of their relations and identities.

1. Introduction and Preliminaries

Functional analysis studies, particularly in the early twentieth century, underwent significant methodological modifications as a result of the influence of contemporary thinking styles. Banach’s contributions to the theory of linear operators, in particular, are known as fundamental in terms of methodology. With its tight connection to other disciplines of mathematics, topology, which was initially employed exclusively for certain geometry issues, has resulted in technique changes in every subject throughout time and has given solid foundations for the areas in which it is applied. Frechet-Coordinate space (FK space) theory has been important in the creation of certain areas, such as topological sequence spaces, as well as in resolving the overwhelming majority of issues involving summability, particularly matrix fields. Topological vector space is a linear space with a topology that enables continuous vector space operations. If this vector space has a full metric space structure, it is called Frechet space, and furthermore, if it has a topology with continuous coordinate functions, it is referred to as Frechet-Coordinate space. The theory of FK spaces has acquired more prominence in recent years and has found applications in a variety of fields, thanks to the efforts of many researchers. K and FK spaces are now widely used in the research of sequence spaces and matrix transformations. By accumulating from FK spaces, one may build new sequence spaces that include or encompass previously existing sequence spaces. Between these spaces, matrix characterizations may be provided, and certain duals and their characteristics can be disclosed. We will begin with the following definitions, concepts, and properties that will be necessary for our major findings. By \( \omega \), we denote the vector space containing all real- or complex-valued sequences that are topologized through coordinatewise convergence. Any vector subspace of \( \omega \) is said to be a sequence space. A sequence space \( X \) with a locally convex topology \( \tau \) is referred to as a K-space if the inclusion mapping \( (X, \tau) \rightarrow \omega \) is continuous, when \( \omega \) has the topology of coordinatewise convergence. Any vector subspace of \( \omega \) is said to be a sequence space. A sequence space \( X \) with a locally convex topology \( \tau \) is referred to as a K-space if the inclusion mapping \( (X, \tau) \rightarrow \omega \) is continuous, when \( \omega \) has the topology of coordinatewise convergence. Additionally, if \( \tau \) is complete and metrizable, \( (X, \tau) \) is referred to be an FK-space. A BK-space is an FK-space with a normable topology. Among important BK-spaces are as follows:

\[ m: \text{the space of all bounded sequences} \]
\[ c: \text{the space of all convergent sequences} \]
\[ c_0: \text{the space of all null sequences} \]
\[ \ell^p(1 \leq p < \infty): \text{the space of all absolutely } p\text{-summable sequences, that is,} \]
\[
\ell^p = \left\{ x = (x_k)_{k \in \mathbb{N}} \in \omega : \|x\|_p = \left( \sum_{k \in \mathbb{N}} |x_k|^p \right)^{1/p} < \infty \right\}. \tag{1}
\]

Also, \( \|x\|_\infty = \sup\{ |x_k| : k \in \mathbb{N} \} \) is called sup norm. Here and throughout, \( \mathbb{N} \) will denote the set of positive integers. As is conventional, \( \ell^1 \) is replaced by \( \ell \), \( \| \cdot \|_1 \) indicates the norm \( \ell \), and \( \| \cdot \|_\infty \) indicates the norm on \( m, c, \) and \( c_0 \).

**Definition 1.**

(i) Let \( \lambda \) and \( \mu \) be any sequence spaces \((\subset \omega)\). Then, the multiplier space \( \lambda^\mu = (\lambda \longrightarrow \mu) \) is defined by
\[
\lambda^\mu = \{ y \in \omega \mid xy \in \mu \text{ for all } x \in \lambda \}, \tag{2}
\]
where \( xy \) is the termwise product, namely, \( xy = (x_k y_k)_{k \in \mathbb{N}} \) for \( x = (x_k)_{k \in \mathbb{N}} \in \omega \) and \( y = (y_k)_{k \in \mathbb{N}} \in \omega \).

(ii) Likewise, for any spaces \( \lambda, \mu, \nu \) in \( \omega \),
\[
(\lambda^\mu) ^\nu = \lambda^{\mu \nu} := \{ y \in \omega \mid xy \in \nu \text{ for all } x \in \lambda^\mu \} \tag{3}
\]

(iii) If \( \lambda = \lambda^{\mu \nu} \) for some spaces \( \lambda, \mu, \) in \( \omega \), the sequence space \( \lambda \) is referred to as \( \mu \)-space

(iv) Recall the following spaces in \( \omega \):
\[
\ell = \left\{ x = (x_k) \in \omega : \|x\|_1 = \sum_{k \in \mathbb{N}} |x_k| < \infty \right\},
\]
\[
cs = \left\{ (x_k) \in \omega : \sum_{k = 1}^{\infty} x_k \text{ is convergent} \right\}, \tag{4}
\]
\[
bs = \left\{ x = (x_k) \in \omega : \|x\|_b = \sup_{n} \left| \sum_{k = 1}^{n} x_k \right| < \infty \right\}
\]
For a space \( \lambda \) in \( \omega \), \( \lambda^f \), \( \lambda^{cs} \), and \( \lambda^{bs} \) are called \( \alpha^f \), \( \beta^f \), and \( \gamma^f \)-duals of \( \lambda \), respectively.

**Remark 2.** For spaces \( \lambda, \mu \) in \( \omega \), the following statements are known:

(i) These are Banach spaces with their natural norms

(ii) \( \phi \subset \lambda^\phi \subset \lambda^{\beta^f} \subset \lambda^f \)

(iii) If \( \mu \subset \lambda \), then \( \mu^f \subset \lambda^f \), for any space \( \xi \subset \omega \), in particular, \( \zeta \) being one of the \( \alpha^f \), \( \beta^f \), and \( \gamma^f \)-duals

(iv) \( \lambda^f = \lambda^\infty \), \( \lambda \subset \lambda^\infty \), where \( \zeta \) is one of the \( \alpha^f \), \( \beta^f \), and \( \gamma^f \)-duals

**Definition 3.**

(i) Suppose that \( e \) represent the sequence of ones \((1, 1, \cdots) \) and \( \mu_1^f = (k \in \mathbb{N}) \) symbolizes the sequences \((0, 0, 0, 1, 0, 0, \cdots) \) with the one in the \( k \)-th position, namely, \( \mu_1^f = (\delta_k)_{k \in \mathbb{N}} \), where \( \delta_k \) denotes the Kro-

(ii) Here and throughout, \( \lambda^f \) will denote the topological dual of the space \( \lambda \). Then, for some \( f \in \lambda^f \),
\[
\lambda^f = \left( f \left( \mu_1^f \right) \right)_{k \in \mathbb{N}}, \tag{5}
\]
is referred to as the \( f \)-dual of a BK space \( \lambda \supset \phi \). Here, \( \lambda^f \) is also a BK space with \( \| f \|_{\lambda^f} = \| f \left( \mu_1^f \right) \|_{k \in \mathbb{N}} \).

(iii) Thereby, if \( \lambda \subset \mu \), then \( \mu^f \subset \lambda^f \). Also, if \( \lambda^f \subset cs \) holds, an \( FK \) space \( \lambda \) containing \( \phi \) is called a semiconservative space (sc).

**Definition 4** (see [1–4]). For an FK space \( \lambda \supset \phi \), we denote the \( n \)-th section of a sequence \( x = (x_k) \in \lambda \) by
\[
x^{[n]} = \sum_{k = 1}^{n} x_k \mu_1^f (n \in \mathbb{N}). \tag{6}
\]
Then, the following is valid for the sequence \( x \):

(i) Section boundedness (abschnitt beschränktheit) denoted by \( ABf \) \( \sup_{n \in \mathbb{N}} \| x^{[n]} \|_\lambda < \infty \)

(ii) Section convergence (abschnitt konvergenz) denoted by \( AK \) if \( \lim_{n \longrightarrow \infty} \| x - x^{[n]} \|_\lambda = 0 \)

(iii) Coordinatewise boundedness (koordinatenweise beschränkt) denoted by \( KB \) if \( \sup_{n \in \mathbb{N}} \| x - x^{[n]} \|_\lambda < \infty \)

(iv) Functional section convergence denoted by \( FAK \) if \( \{ f(x^{[n]}) \} \) converges for every \( f \in \lambda^f \) (equivalently, if \( \{ x^{[n]} \} \) is Cauchy in the weak topology \( \sigma(\lambda, \lambda^f) \))

(v) Weak section convergence denoted by \( SAK \) if \( \{ x^{[n]} \} \) converges to \( x \) in the weak topology \( \sigma(\lambda, \lambda^f) \)

(vi) Section density (abschnitt dichte) denoted by \( AD \) if \( \{ x^{[n]} \} \in \phi \) (closure of \( \phi \) in \( \lambda )

A K space \( \lambda \) is AD space if \( \lambda = \phi \), where \( \phi \) is closure of \( \phi \) in \( \lambda \). Via Hahn-Banach theorem, it is clear that \( \lambda^f = \phi^f \).
Definition 5 (see [2, 3, 5]). Let $\Omega$ be the collection of all finite subsets of $\mathbb{N}$.

(i) A series $\sum x_k$ in a topological vector space $X$ is unconditionally convergent to $x \in X$, if the net $(\sum_{k \in F} x_k)_{F \in \Omega}$ converges to $x$ directed by set inclusion.

(ii) The series $\sum x_k$ is unconditionally Cauchy, if the net $(\sum_{k \in F} x_k)_{F \in \Omega}$ is a Cauchy net.

Definition 6 (see [2, 3]). Assume that $H$ denotes the set of all sequences in $\phi$ consisting of $0$’s and $1$’s, such that,

$$H = \{h = (h_k) \in \phi : h_k = 0 \text{ or } h_k = 1, k \in \mathbb{N}\}.$$  \hspace{1cm} (7)

If $x \in \omega$, we have $H(x) = \{hx : h \in H\}$, where $hx$ denotes the coordinatewise product $hx = (h_kx_k)$. If $X$ is a K-space, then $H(x) \subset X$ for any sequence $x$. An unrestricted section of a sequence $x = (x_k)$ is any sequence of the form $\sum_{k \in F} x_k e^i$, where $F \in \Omega$.

Let $x$ be a K-space. Then, the following is valid for a sequence $x \in \omega$:

(i) Unconditional section boundedness (UAB) in $X$, if $H(x)$ is a bounded subset of $E$

(ii) Unconditional section convergence (UAK), if the net $H(x)$ converges to $x$ in $X$

(iii) Unconditional weak section convergence (USAK), if the net $H(x)$ converges to $x$ in $\sigma(X, X')$

(iv) Unconditional functional section convergence (UFAK), if the net $H(x)$ is Cauchy in $\sigma(X, X')$

Recently, some researchers have attempted to assign properties to AK, AB, KB, SAK, FAK, AD, UAB, UAK, USAK, and UFAK in FK spaces and topological properties of sequence spaces (see, e.g., [6, 7]). Additional information and techniques on summability and related concepts can be found in [4, 8–12].

FK-space theory, which has an important role in the characterization of matrix transformations between sequence spaces and contributes to the proof of the results in the summability theory in an easier and shorter way, is a concept that has been extensively studied by researchers. Sequence spaces are also widely used concepts in many areas of functional analysis and mathematical analysis. In this context, as in the studies on sequence spaces, the main motivation in this study is to reveal new sequence spaces and to determine the relationships between them by looking at the topological structures and properties of these sequence spaces. In addition, investigations have been made on the duals of these new sequence spaces and their relations.

In this article, we aim to introduce four new sectional properties for FK spaces, WAC (sectional weakly absolute convergence), WB (sectional weak boundedness), WpAC (sectional weakly $p$-absolute convergence), and WBV (sectional weakly bounded variation), and explore some of their relations and identities. Also, we examine certain relationships between those subspaces of FK spaces which are made from various existing sequence properties.

2. Relationships between the Subspaces of FK Spaces

This section explores further relationships between the subspaces of FK spaces. We will start with some well-known properties on the sections and duals of a sequence space, as stated in the following lemmas.

Lemma 7 (see [1, 2]). Assume that $X$ is a BK space which contains the space $\phi$. Then

(1) $X$ has AB-property ($X_{AB} \supset X$) if and only if the inclusion $b\nu_{\phi} X \subset X$ holds \hspace{1cm} (8)

(2) $X$ has KB-property ($X_{KB} \supset X$) if and only if the inclusion $\ell X \subset X$ holds \hspace{1cm} (9)

(3) $X$ has UAB-property ($X_{UAB} \supset X$) if and only if the inclusion $cX \subset X$ holds \hspace{1cm} (10)

Here, $X_{AB}$ denotes the set of all sequences $x \in X$ that have property AB, that is,

$$X_{AB} = \{x \in X : x \text{ has AB}\}.$$ \hspace{1cm} (11)

So are the others.

Lemma 8 (see [4, 9]). Assume that $X$ is an FK space which contains $\phi$. Then

(1) $X^{\beta} \subset X^{\gamma} \subset X^{i}$

(2) If $X$ has property AK, then $X^{\beta} = X^{i}$

(3) If $X$ has property AD, then $X^{\beta} = X^{\gamma}$

Lemma 9 (see [4]). Let $X$ has the property AD. Then, for any FK spaces $Y$, $Y \supset X$ if and only if $Y^{i} \subset X^{i}$.
**Lemma 10** (see [4]). Let $X$ be an FK space, $z$ a sequence, and
\[ Y = z^{-1} \ast X := \{ x \in \omega : xz \in X \}, \] where $xz = \{ x \omega z_n \}$. Then, $Y$ is an FK space and $f \in Y'$ if and only if $f(x) = ax + g(zx)$, $\alpha \in \phi$, $g \in X'$.

**Definition 11** (see [1, 3, 4]). Let $X$ be an FK space including $\phi$. Then,
1. $(i) X_{AB} = B^\ast (X) = \{ x \in w : x \text{ has AB in } X \} = X^f$
2. $(ii) X_{FAK} = F^\ast (X) = \{ x \in w : x \text{ has FAK in } X \} = X^\ell
3. $(iii) X_{UAB} = \{ x \in w : x \text{ has UAB in } X \}$
4. $(iv) X_{KB} = \{ x \in w : x \text{ has KB in } X \}$

**Remark 12.** For the properties in FK space $E$ such as AK, AB, KB, SAK, FAK, AD, UAB, UAK, USAK, UFAK, and $X_\phi$ are equivalent to $E(X)$. By Definition 11, obviously, we can write $F^\ast (X) \subset B^\ast (X)$.

**Theorem 13.** Assume that $X$ and $Y$ are FK spaces containing $\phi$. If $X \subset Y \subset \omega$, then $X^\mu \supset Y^\mu$, where $\mu$ is one of the $\alpha$, $\beta$, $\gamma$, $bv$, and $\ell_{co}$ duals.

**Proof.** Suppose that $X \subset Y$ and $z \in Y^\mu$. Then, $yz \in \mu$ for all $y \in Y$. Since, $X \subset Y$, $xz \in \mu$ for all $x \in X$. This implies that $z \in X^\mu$. Therefore, $Y^\mu \supset X^\mu$. 

**Theorem 14.** Assume that $X$ and $Y$ are FK spaces containing $\phi$. If $X \subset Y$, then $F^\ast (X) \subset F^\ast (Y)$ and $B^\ast (X) \subset B^\ast (Y)$.

**Proof.** Suppose that $X \subset Y$. Taking $f$-dual of the resultant inclusion $X \subset Y$ gives $Y^f \subset X^f$. Again, taking $\beta$-dual of the resultant inclusion $Y^f \subset X^f$ yields $X^\ell \subset Y^\ell$. It follows from Definition 11 that $F^\ast (X) \subset F^\ast (Y)$. Likewise, the other inclusion may be established.

**Theorem 15.** Let $X$ and $Y$ be FK spaces. Also, let $X$ contain $\phi$ and has the property AD. Then, the following two statements hold:

1. $(i)$ Assume that $Y^f$ and $X^f$ are $\beta$-spaces. Then
   \[ X \subset Y \Leftrightarrow F^\ast (X) \subset F^\ast (Y) \] (13)
2. $(ii)$ Assume that $Y^f$ and $X^f$ are $\gamma$-spaces. Then
   \[ X \subset Y \Leftrightarrow B^\ast (X) \subset B^\ast (Y) \] (14)

**Proof.** For (i), let $X \subset Y$. By Theorem 14, the inclusion $F^\ast (X) \subset F^\ast (Y)$ follows. Conversely, let $F^\ast (X) \subset F^\ast (Y)$. It can be concluded from Definition 11 that $X^\ell \supset Y^\ell$.

By the assumption, since $Y^f$ and $X^f$ are $\beta$-spaces, we have $X^f \supset Y^f$. Since, $X$ has the property AD, by Lemma 9, we find $X \subset Y$. This completes the proof of the statement (i).

Likewise, the assertion (ii) may be established. We omit the details.

**Theorem 16.** Let $X$ be an FK space containing $\phi$. If $X$ is a $(sc)$ space, then $B^+ \supset bv$.

**Proof.** Since $X$ is $(sc)$ space, $X^f \subset cs$ holds. Then, taking $\gamma$-dual of the inclusion $X^f \subset cs$ gives $X^\ell \supset cs^\ell$. From Definition 11 and $cs^f = bv$, we have $B^+ \supset bv$.

The following corollary summarizes some inclusion relations between spaces.

**Corollary 17.** Let $X$ be an FK space containing $\phi$. Then

1. $(i) X \supset \ell \Leftrightarrow F^+ \supset \ell$ or $B^+ \supset \ell$
2. $(ii) X \supset bv \Rightarrow F^+ \supset bv_0$ or $B^+ \supset bv$
3. $(iii) X \supset bs \Rightarrow F^+ \supset cs$ or $B^+ \supset bs$
4. $(iv) X \supset bv_0 \Rightarrow F^+ \supset bv_0$ or $B^+ \supset bv$
5. $(v) X \supset c_0 \Rightarrow F^+ \supset \ell_{co}$ or $B^+ \supset \ell_{co}$
6. $(vi) X \supset cs \Rightarrow F^+ \supset cs$ or $B^+ \supset bs$

**Proof.** We will only prove the statement (i). Let be $X \supset \ell$. Obviously, $F^+ \supset \ell$. Conversely, assume that $F^+ \supset \ell$. Taking $\beta$-dual on the assumed inclusion, we have $\ell^\beta \supset (F^\ast)^\beta$. Using $\ell^\beta = \ell_{co}$, we can find $X^\ell \subset X^{\ell^\beta} \subset \ell^\beta = \ell_{co}$. Since, $\ell$ has the property AD, we obtain $X^f \subset \ell_{co}$, which implies $X \supset \ell$. Similarly, the statement $X \supset \ell \Rightarrow B^+ \supset \ell$ can be proved.

Likewise, the other statements can be shown. The details are omitted.

### 3. Sectional Weakly Properties

We define and investigate four new sectional properties: sectional weakly absolute convergence (WAC), sectional weak boundedness (WB), sectional weakly $p$-absolute convergence (W$p$AC), and sectional weakly bounded variation (WBV) of a sequence in FK space, as given in the following definition.

**Definition 18.** Let $X$ be an FK space containing $\phi$. Then, a sequence $x = (x_n)$ in $X$ has the following properties:

1. Sectional weakly absolute convergence (denoted by WAC), if
   \[ \sum_n |f(x_n e^n)| < \infty \text{ for all } f \in X' \] (15)
2. Sectional weakly boundedness (denoted by WB), if
Theorem 19. Let $X$ be an FK space containing $\phi$. Then,

(i) $X$ has $WAC^+$ if and only if $\ell^f \subset X^f$, i.e., $X^f = X^f_{\ell}\subset X^f_{\ell^f}$.

(ii) $X$ has $WBV$ if and only if $\ell^f \subset X^f_{\ell^f}$, i.e., $X^f = X^f_{\ell^f}$.

Proof. For (i), let $X$ be an FK space containing $\phi$. By part (i) of Theorem 19, $X \subset WAC^+(X) = X^f_{\ell^f}$. Therefore, $X^f \supset X^f_{\ell^f} \supset X^f_{\ell^f}$. Also, by Lemma 8, we have $X^f \subset X^f_{\ell^f}$. Hence, $X^f = X^f_{\ell^f}$. The opposite implication can be proved by a similar way.

The proof of the statement in (ii) would run in parallel with the proof of the statement (i). We omit the details.

Theorem 20. Let $X$ be an FK space containing $\phi$. Then,

(i) $X$ has $WAC^+$ if and only if $\ell^f \subset X^f$, i.e., $X^f = X^f_{\ell}\subset X^f_{\ell^f}$.

(ii) $X$ has $WBV$ if and only if $\ell^f \subset X^f_{\ell^f}$, i.e., $X^f = X^f_{\ell^f}$.

Proof. For (i), let $X$ be an FK space containing $\phi$. By part (i) of Theorem 19, $X \subset WAC^+(X) = X^f_{\ell^f}$. Therefore, $X^f \supset X^f_{\ell^f} \supset X^f_{\ell^f}$. Also, by Lemma 8, we have $X^f \subset X^f_{\ell^f}$. Hence, $X^f = X^f_{\ell^f}$. The opposite implication can be proved by a similar way.

The proof of the statement in (ii) would run in parallel with the proof of the statement (i). We omit the details.

Theorem 21. Let $X$ and $Y$ be FK spaces containing $\phi$ which have one (say $E$) of the properties $WAC$, $WB$, $W\rho AC$, and $WBV$. If $X \subset Y$, then $E^+(X) \subset E^+(Y)$.

Proof. By Definition 18 and the subsequent notations, the result is obvious.

Theorem 22. Let $Y$ be an FK space $\varphi$ and $X$ have the property $AD$.

(i) If $X$ and $Y$ have property $WAC$, then

$$X \subset Y \Leftrightarrow WAC^+(X) \subset WAC^+(Y)$$  \hspace{1cm} (21)

(ii) If $X$ and $Y$ have property $WBV$, then

$$X \subset Y \Leftrightarrow WBV^+(X) \subset WBV^+(Y)$$  \hspace{1cm} (22)

Proof. We will only prove the statement (i). Suppose that $X \subset Y$. By Theorem 21, $WAC^+(X) \subset WAC^+(Y)$. Conversely, let $WAC^+(X) \subset WAC^+(Y)$. By Theorem 19, $X^f_{\ell^f} \subset Y^f_{\ell^f}$. Then, by part (iii) in Remark 2, we have $X^f_{\ell^f} \supset Y^f_{\ell^f}$. By part (i) in Theorem 20, $X^f_{\ell^f} \supset Y^f_{\ell^f}$. By part (iv) in Remark 2, we find $X^f \supset Y^f$. By part (i) in Theorem 20 again, we obtain $X \subset Y$. Since $X$ has the property $AD$, we obtain $X \subset Y$ from Lemma 9.

The proof of the statement (ii) would flow in parallel with the proof of statement (i). The details are omitted.

Theorem 23. Let $X$ be an FK space $\varphi$. Then,

$$WAC^+(X) \subset WBV^+(X) \subset WB^+(X).$$  \hspace{1cm} (23)

Proof. By applying the inclusion $\ell \subset bv \subset \ell_{\infty}$ to Theorem 19, this yields the result.
Proof. We will prove the statement (i). Assume that $f \in (z^{-1} \ast X)^{\prime}$. Then, we find from Lemma 10 that

$$f(e^n) = a_n + g(z e^n) = a_n + g(z_n e^n) = a_n + z_n g(e^n), \quad (24)$$

for all $a \in \phi$ and $g \in X'$. Hence, $\{f(e^n)\} \in \ell$ if and only if $z \in WAC^+(X)$.

Similarly, the other statements (ii), (iii), and (iv) can be proved. We omit the details. $\square$

**Theorem 25.** Let $X$ be an FK space including $\phi$. For $x \in X$, $x \in UAB(X)$ if and only if $x \in WAC^+(X)$.

**Proof.** The assertion follows from the definitions of $WAC^+(X)$ and $UAB(X)$. $\square$

**Corollary 26.** Let $X$ be an FK space including $\phi$ and $X$ has the property $WAC^+$. Then, the following statements are equivalent:

(i) $X$ has the property UAB

(ii) $cX = X$

(iii) $cX \subset WAC^+(X)$

**Proof.** The assertion is a consequence of Theorem 25. $\square$

**Theorem 27.** Let $X$ be an FK space including $\phi$. Then, for $x \in X$, $x \in KB(X)$ if and only if $x \in WB^+(X)$.

**Proof.** The assertion follows from the definitions of $WB^+(X)$ and KB(X). $\square$

From Theorem 27, we conclude the following result.

**Corollary 28.** Let $X$ be an FK space including $\phi$ and $X$ has the property $WB^+$. Then, the following statements hold true:

(i) $X$ has the property KB

(ii) $X^f \subset X^m$

(iii) $\ell X \subset X$

(iv) $X \subset (X^f)^m$

**Theorem 29.** The property $WB^+$ is weaker than the property $AB$.

Proof. Let $X$ has the property $AB$ and $x = (x_k) \in X$. Since $X$ has $AB$ property, there exists $M > 0$ such that $\|x^{[k]}\| \leq M$ for all $k \in \mathbb{N}$. It follows that

$$\|x_k e^k\| = \|x^{[k]} - x^{[k-1]}\| \leq \|x^{[k]}\| + \|x^{[k-1]}\| \leq 2M, \quad (25)$$

for all $k \in \mathbb{N} \setminus \{1\}$. This implies that the sequence $(x_k e^k)_{k \in \mathbb{N}}$ is bounded. The Banach-Mackey theorem states that $X$ is bounded if and only if it is weakly bounded (see [13]). Therefore, $(x_k e^k)$ is weakly bounded, namely,

$$\sup_{k \in \mathbb{N}} \|x_k f(e^k)\| < \infty \text{ for all } f \in X' \cdot \quad (26)$$

This means that $X$ has the property $WB^+$. $\square$

**Corollary 30.** Property $FAK$ is weaker than the property $WAC^+$.

**Proof.** Let $X$ has the property $WAC^+$. Then, for all $x \in X$ and $f \in X'$, we have $(x, f(e^n)) \in \ell$. Since $\ell \subset c\ell$, $(x, f(e^n)) \in c\ell$ for those $x \in X$ and $f \in X'$. This implies that $X$ has the property $FAK$. $\square$

**Definition 31.** Let $X$ and $\lambda$ be FK spaces including $\phi$. Also, assume that $f \in X'$. A sequence $z = (z_n) \in X$ has the property $W_{\lambda^+}$, if $(z_n g(e^n)) \in \lambda^f$ for all $g \in X'$. Further, consider the following set

$$W_{\lambda^+}^+(X) := \left\{ z \in \omega : (z_n g(e^n)) \in \lambda^f \text{ for all } g \in X' \right\}. \quad (27)$$

Clearly, $W_{\lambda^+}^+(X) = W_{\lambda^+}^+(X) \cap X$. Equivalently, an FK space $X$ including $\phi$ has the property $W_{\lambda^+}$, if $W_{\lambda^+}^+(X) \supset X$; that is, $W_{\lambda^+}^+(X) = X$.

**Theorem 32.** Let $\lambda$ be an FK space including $\phi$ and $z \in \omega$. If $\lambda$ is an AD space, then $z^{-1} \ast X \cap \lambda$ if and only if $z \in W_{\lambda^+}^+(X)$.

**Proof.** It is obvious from Definition 31 that $W_{\lambda^+}^+(X) = (X' \setminus \lambda)$. Using above definition, it is obvious that $W_{\lambda^+}^+(X) = (X' \setminus \lambda)^f$.

Sufficiency: let $z \in W_{\lambda^+}^+(X)$. For each $u \in X'$, $zu \in \lambda^f$;

$$zu = (z_k u_k)_{k \in \mathbb{N}} \in \lambda^f. \quad (28)$$

and $\{g(z_k e^k)\} \in \lambda^f$. By the equality $f(x) = ax + g(xz)$, $(a \in \phi)$, for $f(e^k) \in (z^{-1} \ast X)^{\prime}$, we get $f(e^k) = a_k + g(z_k e^k)$ and $g(z_k e^k) \in \lambda^f$ and so $f(e^k) \in \lambda^f$. Thus, we obtain $(z^{-1} \ast X)^{\prime} \subset \lambda^f$. Since $\lambda$ is an AD space, then $z^{-1} \ast X \cap \lambda$. 

Necessity: let \( z^{-1} \ast X \ni \lambda \). For \( u = (u_k) \in X^f \), there exists a \( g \in X' \) such that \( u_k = g(e^k) \) for all \( k \in \mathbb{N} \). Hence, we get

\[
zu = (z_k u_k) = \left( z_k g(e^k) \right) = \left( g\left( z_k e^k \right) \right).
\]

(29)

Since \( z^{-1} \ast X \ni \lambda \), we have \( (z^{-1} \ast X)' \subset \lambda' \). Let \( v \in (z^{-1} \ast X)' \). Then, there exists an \( f \in (z^{-1} \ast X)' \) such that \( v_k = f(e^k) \) for all \( k \in \mathbb{N} \). Then, we get \( f(e^k) = u_k + g(z_k e^k) \) for \( \alpha \in \phi \) and \( g \in X' \). So, \( \{ g(z_k e^k) \} \in \lambda' \). Thus, \( \{ g(z_k e^k) \} \in W_{\lambda'}^f (X) \) hold. □

4. Concluding Remarks

It is known that the foundations of the theory of FK spaces were set in the first half of the twentieth century by mathematicians such as Mazur and Orlicz, and then, Zeller began his work in 1951. One of the primary benefits of FK space theory is that it is the most powerful and widely used instrument for proving various classical results in summability theory and for describing matrix transformations between sequence spaces in an easy and concise manner. Along with the theory of summability, sequence spaces are important in other fields where functional analysis and summability are employed. Numerous duals of these spaces play a significant role in summability theory and topological sequence spaces. Due to the discoveries acquired from K and FK spaces in the work on sequence spaces and matrix transformations, new sequence spaces may be constructed that are contained or encompassed by an array of previously existing spaces.

Motivated by this idea, in this article, we introduced four new sectional properties for topological sequence spaces, sectional weakly absolute convergence (WAC), sectional weak boundedness (WB), sectional weakly \( p \)-absolute convergence (W\( p \)AC), and sectional weakly bounded variation (WBV), and investigated some of their properties and identities. Further studies will focus on advanced features and inclusions of these spaces.

Researchers interested in the subject can define new spaces that can be functional in terms of summability theory and sequence spaces.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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