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Research Article

Characterizations of Lifetime Distributions Using Two Relative Reliability Measures

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In this paper, a general characterization property considering two new dynamic relative reliability measures is obtained. The new dynamic relative reliability measures are expressed as the ratio of hazard rates and as the ratio of reversed hazard rates. The measures are evaluated partially at some sequential random times following a specific distribution. We show that several particular statistics, as random times, fulfill that specific distribution, and thus, the result is applicable in the context of the specified random times. The results are applied to some examples to characterize the Weibull distribution and the inverse Weibull distribution.

1. Introduction

The aging process of lifetime units is very important in various fields of science to quantify it by a mathematical theory. In survival analysis and in some medical problems, the construction of models is based on the aging process of life spans. In the context of reliability, the aging process of individuals or life span subjects is modeled by the residual life (RL) span variable, which includes the current age of the element [1–4]. The hazard rate (h.r.) function which is the density function divided to the survival function (s.f.) is a useful quantity for measuring the instantaneous risk of failure of units operating at different ages [5, 6]. This quantity has attracted the attention of many researchers in the field, who have presented several statistical models specifically for fitting right-censored data [7, 8].

The inspection process, unlike the aging process that considers the future time of failure of objects, is concerned with determining when failure has already occurred. This process has also been useful to researchers in several areas, including reliability and risk. In these contexts, the inactivity time (IT) or past lifetime variable is used to quantify the inspection process and identify earlier times of failure [9–13]. A useful reliability measure that takes into account the risk of previous failures immediately prior to an inspection time is the revered hazard rate (r.h.r.) function [14–17]. The r.h.r. function which equals with density function divided to the distribution function has also been used in reliability analysis to propose new models that are particularly useful for modeling left-censored data [18–21].

In probability theory, equivalent conditions for identical distributions have been always important tools for further purposes (see, e.g., [22–26]). The aim of the current paper is to present some characterizations using two quantities related to h.r. and r.h.r. functions. The results are obtained using the well-known theory of completeness in functional analysis.

In what follows, in Section 2, two new random relative reliability measures in the context of RL at random time (RLRT) and IT at random time (ITRT) are proposed. In Section 3, we give the main result including a general characterization property obtained using the introduced random relative h.r. and the random relative r.h.r. evaluated in some partial sequence of time. In Section 4, some typical random sequences of lifetimes which fulfill the derived characterization property are considered. In Section 5, it is shown that the obtained characterization property can be developed for characterization of particular distributions. In Section 6, we close the paper with further illustrations along with additional considerations to be studied in the future.

2. Relative Measures of Lifetime Distributions

Suppose that X represents a nonnegative random variable (r.v.) with an absolutely continuous distribution function (d.f.) F_X and probability density function (p.d.f.) f_X. Then, the conditional r.v. $X_t = [X - t | X > t]$ for $t : F_X(t) < 1$ is called the residual lifetime of a subject (such as an electrical device) with random lifetime X after the time that subject has age t. Denote by $\bar{F}_X$ the s.f. of X, defined by $\bar{F}_X(t) = 1 - F_X(t)$. Let us assume that X has h.r. function h_X, given by

$$h_X(t) \coloneqq \lim_{\Delta \longrightarrow 0^+} \frac{1}{\Delta} P(X_t \le \Delta) = \frac{f_X(t)}{\bar{F}_X(t)}, \quad (1)$$

which is valid for all $t \ge 0$ for which $F_X(t) < 1$. For instance, if a system has not failed until the time t, then it is considered a used system of age t, and the r.v. X_t denotes the remaining lifetime it has. Thus, a fresh device with the s.f. $\bar{F}(x)$ and p.d.f. $f_X(x)$ when it is in use and has age t will have the updated s.f. and p.d.f.

$$\begin{aligned} \bar{F}_{X_t}(x) &\coloneqq \frac{\bar{F}_X(t+x)}{\bar{F}_X(t)}, x \ge 0, \\ f_{X_t}(x) &\coloneqq \frac{f_X(t+x)}{\bar{F}_X(t)}, x \ge 0, \end{aligned} \quad (2)$$

respectively. Suppose Y is another nonnegative r.v. and denote by $F_Y, f_Y, \bar{F}_Y$, and h_Y its d.f., p.d.f., s.f., and h.r. functions which are defined similarly as X.

For the sake of obtaining a relative measure of aging of X with respect to Y, consider the following r.v.

$$Rl_{X,Y}(t) = h_Y(t) X_t, t : \max \{F_X(t), F_Y(t)\} < 1, \quad (3)$$

which is the RL of X multiplied by the h.r. of Y. Now, the limiting probability for failure of a device with lifetime X right after the time t in comparison with the limiting probability for failure of another device with lifetime Y immediately after the time t (i.e., the h.r. of Y), the following measure can be proposed:

$$\mathfrak{h}_{X,Y}(t) \coloneqq \lim_{\Delta \longrightarrow 0^+} \frac{1}{\Delta} P(Rl_{X,Y}(t) \le \Delta) = \frac{h_X(t)}{h_Y(t)}. \quad (4)$$

Note that (4) corresponds to the p.d.f. of $Rl_{X,Y}(t)$ evaluated at the point 0. The relative measure (4) has been frequently applied in literature, specially for comparison of coherent systems and other lifetime events (see, e.g., [27–33]). The function defined in (4) can be viewed as a relative risk, and it plays a role in the evaluation of the Kullback-Leibler information (see, for instance, [34]). The monotonicity of $\mathfrak{h}_{X,Y}(t)$ with respect to t induces that the device, with lifetime X, ages faster (resp., slower) than another device with lifetime Y when $\mathfrak{h}_{X,Y}(t)$ is increasing (resp., decreasing) in t (see [35]).

In many circumstances, the interest is in past events and, specifically, the time elapsed since a failure or death of a subject is important to be measured. The RL can then be considered in a reversed time scale (see, e.g., [9]). The conditional r.v. $X_{(t)} = [t - X | X \le t]$ for $t : F_X(t) > 0$ is considered the IT of a subject with random lifetime X at the time t at which the failure of the subject has been detected for the first time. We suppose that X has r.h.r. function

$$\tilde{h}_X(t) \coloneqq \lim_{\Delta \longrightarrow 0^+} \frac{1}{\Delta} P\left(X_{(t)} \le \Delta\right) = \frac{f_X(t)}{F_X(t)}, \quad (5)$$

which is well-defined for all $t > 0$ for which $F_X(t) > 0$. For example, a system which is under periodically inspection at the time t observed to be failed. The r.v. $X_{(t)}$ indicates the interval time between the failure time of a system and the time at which the inspector finds the system failed. The r.v. $X_{(t)}$ has s.f. and p.d.f.

$$\begin{aligned} \bar{F}_{X_{(t)}}(x) &\coloneqq \frac{F_X(t-x)}{F_X(t)}, x \ge 0, \\ f_{X_{(t)}}(x) &\coloneqq \frac{f_X(t-x)}{F_X(t)}, x \ge 0, \end{aligned} \quad (6)$$

respectively. Let us denote by $\tilde{h}_Y$ the r.h.r. function of Y. Based on the concept of IT, another measure of X relative to Y is

$$\widetilde{Rl}_{X,Y}(t) = \tilde{h}_Y(t) X_{(t)}, t : \min \{F_X(t), F_Y(t)\} > 0, \quad (7)$$

which, indeed, is the IT of X multiplied by the r.h.r. of Y. Now, the following measure can be proposed:

$$\tilde{\mathfrak{h}}_{X,Y}(t) \coloneqq \lim_{\Delta \longrightarrow 0^+} \frac{1}{\Delta} P\left(\widetilde{Rl}_{X,Y}(t) \le \Delta\right) = \frac{\tilde{h}_X(t)}{\tilde{h}_Y(t)}. \quad (8)$$

Note that (8) corresponds to the p.d.f. of $\widetilde{Rl}_{X,Y}(t)$ evaluated at the point 0. The measure (8) has been considered a relative quantity for two lifetime distributions in literature. The monotonicity of $\tilde{\mathfrak{h}}_{X,Y}(t)$ in terms of t concludes that the device, with lifetime X, is faster (slower) in decreasing r.h.r. (DRHR) property than another device with lifetime Y when $\tilde{\mathfrak{h}}_{X,Y}(t)$ decreases (resp., increases) in t (see, e.g., [17, 36]). The monotonicity of the function defined in (8) has also been investigated in Proposition 5.1 of Di Crescenzo and Longobardi [37].

The concept of RL and IT which are related to a given certain time has been developed to random time. The random time T is assumed to be nonnegative. Let T have d.f. H (see, for instance, [38, 39]). Then, $X_T = [X - T | X > T]$ is called the RL of X at T which, when T and X are independent, has s.f.

$$\bar{F}_{X_T}(x) = \frac{\int_0^{+\infty} \bar{F}_X(t+x) dH(t)}{\int_0^{+\infty} \bar{F}_X(t) dH(t)}, x \geq 0, \tag{9}$$

and the associated p.d.f. is

$$f_{X_T}(x) = \frac{\int_0^{+\infty} f(x+t) dH(t)}{\int_0^{+\infty} \bar{F}_X(t) dH(t)}, x \geq 0. \tag{10}$$

In contrast to the RLRT, $X_{(T)} = [T - X | X \leq T]$ is called the ITRT of T. If X and T are independent, then $X_{(T)}$ has s.f.

$$\bar{F}_{X_{(T)}}(x) = \frac{\int_0^{+\infty} F_X(t-x) dH(t)}{\int_0^{+\infty} F_X(t) dH(t)}, \tag{11}$$

with corresponding p.d.f.

$$f_{X_{(T)}}(x) = \frac{\int_0^{+\infty} f(t-x) dH(t)}{\int_0^{+\infty} F_X(t) dH(t)}. \tag{12}$$

To update the quantities (4) and (8) in terms of randomness of, respectively, the current age of a subject and the time of observation of failure of an item, we define their random counterparts. First,

$$\mathfrak{h}_{X,Y}(T) = \lim_{\Delta \longrightarrow 0^+} \frac{1}{\Delta} P(Rl_{X,Y}(T) \leq \Delta) = \int_0^{+\infty} \frac{h_X(t)}{h_Y(t)} dH^*(t), \tag{13}$$

where

$$dH^*(t) = \frac{\bar{F}_X(t) dH(t)}{\int_0^{+\infty} \bar{F}_X(t) dH(t)}. \tag{14}$$

Similarly,

$$\tilde{\mathfrak{h}}_{X,Y}(T) = \lim_{\Delta \longrightarrow 0^+} \frac{1}{\Delta} P\left(\widetilde{Rl}_{X,Y}(T) \leq \Delta\right) = \int_0^{+\infty} \frac{\tilde{h}_X(t)}{h^{\sim}{}_Y(t) dH^{**}(t),} \tag{15}$$

in which

$$dH^{**}(t) = \frac{F_X(t) dH(t)}{\int_0^{+\infty} F_X(t) dH(t)}. \tag{16}$$

In the context of characterizations of distributions, one can see that since the h.r. function is a unique characteristic of the parent distribution, thus in view of (4), $\mathfrak{h}_{X,Y}(t) = 1$, for all $t \geq 0$ if X is equal in distribution with Y. Further, as the r.h.r. function is also a unique characteristic of the underlying distribution, therefore, in spirit of (8), $\tilde{\mathfrak{h}}_{X,Y}(t) = 1$, for all $t > 0$ if X and Y are equally distributed. In the sequel, we seek whether these properties can be developed for quantities (13) and (15). In the residual part of the paper, the terms "increasing" and "decreasing" mean "nondecreasing" and "nonincreasing"; thus, the monotonicity properties of functions are supposed to be nonstrict throughout the paper.

3. Main Characterization Properties

In this section, using a technical lemma to reach completeness property in functional analysis, two characterization properties will be given. We first remind the concept of completeness of a sequence of real function.

Definition 1. The sequence $\delta_1, \delta_2, \cdots$ in a given Hilbert space H is considered complete if the sole member in H having the orthogonality property with respect to every δ_m is the null member, in the way

$$\psi \bullet \delta_m = 0, \forall m \in \mathbb{N} \Rightarrow \psi = 0, \tag{17}$$

where 0 indicates the zero member in H.

The notation $\bullet$ represents the inner product of H. The Hilbert space $L^2[a, b]$, across this paper, is assumed to have an inner product as

$$f_1 \bullet f_2 = \int_a^b f_1(x) f_2(x) dx, \tag{18}$$

where $f_i, i = 1, 2$ is a real-valued square integrable function in the domain $[a, b]$. It is noticeable that if $\delta_1, \delta_2, \cdots$ is a complete sequence in the Hilbert space H, then $\sum c_m \delta_m$ where $c_m = f \bullet \delta_m$ converges in H whenever $\sum_{m=1}^{+\infty} |c_m|^2 < +\infty$, and the limit becomes identical to f. Higgins [40] provided further detailed discussion regarding this area.

Lemma 2 [41]. *Let φ be a function on $[a, b]$ which is absolutely continuous so that $\varphi(a)\varphi(b) \geq 0$, and suppose that $\varphi'(x) \neq 0$ almost everywhere on (a, b). Then, under the assumption*

$$\frac{1}{\nu_1} + \frac{1}{\nu_2} + \cdots = +\infty, \text{in which } 1 \leq \nu_1 < \nu_2 < \cdots, \tag{19}$$

the sequence $\varphi^{\nu_1}(x), \varphi^{\nu_2}(x), \cdots$ is complete on $a < x < b$ if φ is monotone on (a, b).

The special case where $\nu_m = m$, for $m = 1, 2, \cdots$, fulfills the result of Lemma 2, because it is well-known that $\sum_{m=1}^{+\infty} 1/m = +\infty$. Therefore, when φ is absolutely continuous and monotone, as a result, $\varphi(x), \varphi^2(x), \cdots$ constitutes a complete sequence of functions.

We consider a sequential family of distributions for T as the random time and apply it to update H^* and H^{**} in (13) and (15), respectively. Two characterization properties will then be secured.

Let us suppose that, for $m \in \mathbb{N}$, the random time T'_m has p.d.f.

$$f_{T'_m}(t) = \frac{w(t)\varphi^m(t)}{\int_0^{+\infty} w(t)\varphi^m(t)dt}, t \ge 0, \quad (20)$$

where $w(\cdot)$ and $\varphi(\cdot)$ are two functions which $w(t) \ge 0$ and $\varphi(t) \ge 0$ for all $t \ge 0$ so that $0 < \int_0^{+\infty} w(t)\varphi^m(t)dt < +\infty$. We shall denote the c.d.f. corresponding to T'_m by $F_{T'_m}$. The random h.r. (13) and the random r.h.r. (13) with $\{T'_m, m \in \mathbb{N}\}$ in place of T as a sequence of random times, two characterization relations, are obtained as follows:

Theorem 3. *Let T'_m be a sequence of r.v.s independent of X and Y. Then,*

(i) *X and Y are equally distributed, if there exists a fixed positive integer m_0 such that $\mathfrak{h}_{X,Y}(T'_m) = 1$, for all $m = m_0, m_0 + 1, \cdots$ where T'_m has p.d.f. (20) with a monotone function φ*

(ii) *X and Y are equally distributed, if there exists a fixed positive integer m_0 such that $\tilde{\mathfrak{h}}_{X,Y}(T'_m) = 1$, for all $m = m_0, m_0 + 1, \cdots$ in which T'_m has p.d.f. (20) with a function φ which is monotone*

Proof. We first prove assertion (i). We assume that X and Y are two nonnegative r.v.s with c.d.f.s F and G, and p.d.f.s f and g, respectively. By (13), we get

$$\mathfrak{h}_{X,Y}\left(T'_m\right) = E\left[\frac{h_X(T^*_m)}{h_Y(T^*_m)}\right] = \int_0^{+\infty} \frac{h_X(t)}{h_Y(t)} dF_{T^*_m}(t) = \int_0^{+\infty} \frac{h_X(t)}{h_Y(t)} \frac{w(t)\bar{F}_X(t)\varphi^m(t)}{\int_0^{+\infty} w(t)\bar{F}_X(t)\varphi^m(t)dt} dt, \quad (21)$$

where T^*_m has p.d.f

$$f_{T^*_m}(t) = \frac{w(t)\bar{F}_X(t)\varphi^m(t)}{\int_0^{+\infty} w(t)\bar{F}_X(t)\varphi^m(t)dt}, t \ge 0. \quad (22)$$

It is straightforward that if X and Y have equal distribution, then $\mathfrak{h}_{X,Y}(T'_m) = 1$, for all $m = m_0, m_0 + 1, \cdots$. To prove the converse, note that $\mathfrak{h}_{X,Y}(T'_m) - 1 = 0$, for all $m = m_0, m_0 + 1, \cdots$, if

$$\int_0^{+\infty} \left(\frac{h_X(t)}{h_Y(t)} - 1\right) \frac{w(t)\bar{F}_X(t)\varphi^m(t)}{\int_0^{+\infty} w(t)\bar{F}_X(t)\varphi^m(t)dt} dt = 0, \text{for all } m = m_0, m_0 + 1, \cdots, \quad (23)$$

which holds, equivalently if, $\int_0^{+\infty} l(t)\varphi^m(t)dt = 0$, for all $m = 1, 2, \cdots$, that is $l \bullet \varphi^m = 0$, for all $m = 1, 2, \cdots$ where $l(t) = (((h_X(t))/(h_Y(t))) - 1)w(t)\varphi^{m_0-1}(t)\bar{F}_X(t)$. By Lemma 2, $l(t) = 0$, for all $t \ge 0$, i.e., $h_X(t) = h_Y(t)$, for all $t \ge 0$, i.e., X is equal in distribution with Y. We now prove assertion (ii). By (15), we obtain

$$\tilde{\mathfrak{h}}_{X,Y}\left(T'_m\right) = E\left[\frac{h^{\sim}_X(T^{**}_m)}{h^{\sim}_Y(T^{**}_m)]} = \int_0^{+\infty}\frac{\tilde{h}_X(t)}{h^{\sim}_Y(t)dF_{T^{**}_m}(t) = \int_0^{+\infty}\frac{\tilde{h}_X(t)}{h^{\sim}_Y(t)\frac{w(t)F_X(t)\varphi^m(t)}{\int_0^{+\infty} w(t)F_X(t)\varphi^m(t)dt} dt,}}}\right. \quad (24)$$

where T^{**}_m has p.d.f

$$f_{T^{**}_m}(t) = \frac{w(t)F_X(t)\varphi^m(t)}{\int_0^{+\infty} w(t)F_X(t)\varphi^m(t)dt}, t \ge 0. \quad (25)$$

It is evident that if X and Y have equal distribution, then $\tilde{\mathfrak{h}}_{X,Y}(T'_m) = 1$, for all $m = m_0, m_0 + 1, \cdots$. To prove the reversed implication, we have $\tilde{\mathfrak{h}}_{X,Y}(T'_m) - 1 = 0$, for all $m = m_0, m_0 + 1, \cdots$, if

$$\int_0^{+\infty} \left(\frac{\tilde{h}_X(t)}{\tilde{h}_Y(t)} - 1\right) \frac{w(t)F_X(t)\varphi^m(t)}{\int_0^{+\infty} w(t)F_X(t)\varphi^m(t)dt} dt = 0, \text{for all } m = m_0, m_0 + 1, \cdots, \quad (26)$$

which is satisfied, equivalently, if $\int_0^{+\infty} l^*(t)\varphi^m(t)dt = 0$, for all $m = 1, 2, \cdots$, that is, $l^* \bullet \varphi^m = 0$, for all $m = 1, 2, \cdots$ where $l^*(t) = (((\tilde{h}_X(t))/(\tilde{h}_Y(t))) - 1)w(t)\varphi^{m_0-1}(t)\bar{F}_X(t)$. By applying Lemma 2, $l^*(t) = 0$, for all $t \ge 0$, which concludes that $\tilde{h}_X(t) = \tilde{h}_Y(t)$, for all $t \ge 0$, that is, X is equal in distribution with Y. □

Remark 4. The result of Theorem 3 remains valid if T'_m has p.d.f.

$$f_{T'_m}(t) = \frac{w(t)\varphi^{s_m}(t)}{\int_0^{+\infty} w(t)\varphi^{s_m}(t)dt}, t \ge 0, \quad (27)$$

where $s_1, s_2, \cdots$ are selected such that $\varphi^{s_1}, \varphi^{s_2}, \cdots$ is a complete sequence of functions when φ is monotone due to Lemma 2. In this case, we see that in the context of Theorem 3 (i), one has

$$\mathfrak{h}_{X,Y}\left(T'_m\right) = E\left(\frac{h_X(T^*_m)}{h_Y(T^*_m)}\right), \tag{28}$$

where T^*_m has p.d.f.

$$f_{T^*_m}(t) = \frac{w(t)\bar{F}_X(t)\varphi^{s_m}(t)}{\int_0^{+\infty} w(t)\bar{F}_X(t)\varphi^{s_m}(t)dt}, t \geq 0. \tag{29}$$

In parallel, in Theorem 3 (ii), we have

$$\tilde{\mathfrak{h}}_{X,Y}\left(T'_m\right) = E\left(\frac{h^{\sim}{}_X(T^{**}_m)}{h^{\sim}{}_Y(T^{**}_m)}\right), \tag{30}$$

in which T^{**}_m is considered an r.v. with p.d.f.

$$f_{T^{**}_m}(t) = \frac{w(t)F_X(t)\varphi^{s_m}(t)}{\int_0^{+\infty} w(t)F_X(t)\varphi^{s_m}(t)dt}, t \geq 0. \tag{31}$$

Arnold and Villasenor [42] pointed out that characterizations are particularly of interest when they can be used to assess the conceivable of certain assumptions on distributions via suitable tests of hypotheses. Characterization properties of distributions can be, particularly, applied to build goodness-of-fit tests of distributions. By consideration of a proper random variable T'_m in Theorem 3, the relation $\mathfrak{h}_{X,Y}(T'_m) = 1$, for all $m = m_0, m_0 + 1, \cdots$, and also the relation $\tilde{\mathfrak{h}}_{X,Y}(T'_m) = 1$, for all $m = m_0, m_0 + 1, \cdots$, can be potential indices to construct tests for the hypothesis $H_0 : X$ and Y which are equal in distribution, versus $H_1 : X$ and Y which are note equal in distribution which is an appropriate alternative.

4. Fulfilling Random Sequences as Random Times

In this section, several situations where particular statistics may be adopted as random times are provided. In the context of random sampling from a population, many statistics can be considered.

4.1. Order Statistics from Homogenous Populations. In reliability engineering, the lifetime of a coherent system is represented in terms of consecutive order statistics (see [43]) arisen from the components lifetimes in the system, while the lifetime of a standby system is stated based on the partial sum of the component lifetimes in the system (see, for example, [44]). Suppose that $T_1, T_2, \cdots$ is a sequence of independent and identically distributed (i.i.d.) nonnegative r.v.s with p.d.f. γ, c.d.f. H, and s.f. $\bar{H}$. In the sequel, assume that the lifetime r.v. X is independent of T_i's. Let $T_{1:m} \leq T_{2:m} \leq \cdots \leq T_{m:m}$ be the order statistics from the first m elements of the sequence of $T_1, T_2, \cdots$. In the context of Theorem 3, as i is fixed, the ith order statistic $T_{i:m}$ can be considered as T'_m where $m = i, i+1, \cdots$, so that in the setting of Theorem 3, one can choose $m_0 = i$. It is known that T'_m has p.d.f.

$$f_{T'_m}(t) = \frac{m!}{(i-1)!(m-i)!} H^{i-1}(t)\bar{H}^{m-i}(t)\gamma(t), \text{for all } t \geq 0, \tag{32}$$

which coincides with (20) if we take $w(t) = ((H^{i-1}(t))/(\bar{H}^i(t)))\gamma(t)$ and $\varphi(t) = \bar{H}(t)$. Note that φ is a monotone decreasing function. Hence, Theorem 3 (i) is applicable and concludes that X and Y are equally distributed, if there exists an $i \in \mathbb{N}$ such that $\mathfrak{h}_{X,Y}(T_{i:m}) = 1$, for all $m = i, i+1, \cdots$. In view of (32), T^*_m in the proof of Theorem 3 (i), for a fixed $i = 1, 2, \cdots$ and $m = i, i+1, \cdots$, has p.d.f.

$$f_{T^*_m}(t) = \frac{\bar{F}_X(t)H^{i-1}(t)\bar{H}^{m-i}(t)\gamma(t)}{\int_0^{+\infty} \bar{F}_X(t)H^{i-1}(t)\bar{H}^{m-i}(t)\gamma(t)dt}, \text{for all } t \geq 0. \tag{33}$$

In terms of (32), the p.d.f. of T^{**}_m in the proof of Theorem 3 (ii), for a fixed $i = 1, 2, \cdots$ and $m = i, i+1, \cdots$, is

$$f_{T^{**}_m}(t) = \frac{F_X(t)H^{i-1}(t)\bar{H}^{m-i}(t)\gamma(t)}{\int_0^{+\infty} F_X(t)H^{i-1}(t)\bar{H}^{m-i}(t)\gamma(t)dt}, \text{for all } t \geq 0. \tag{34}$$

Theorem 3 (ii) implies that X and Y are equally distributed, if there exists an $i \in \mathbb{N}$ for which $\tilde{\mathfrak{h}}_{X,Y}(T_{i:m}) = 1$, for all $m = i, i+1, \cdots$.

In spirit of (32), $T'_m \coloneqq T_{m:m}$ for $m = 1, 2, \cdots$, has p.d.f.

$$f_{T'_m(t)} = \frac{w(t)H^m(t)}{\int_0^{+\infty} w(t)H^m(t)dt}, \tag{35}$$

where $w(t) = (\gamma(t))/(H(t))$ and, thus, (20) is reached. The p.d.f. of T^*_m in the proof of Theorem 3 (i) for $m = 1, 2, \cdots$, is given here by

$$f_{T^*_m}(t) = \frac{\bar{F}_X(t)H^{m-1}(t)\gamma(t)}{\int_0^{+\infty} \bar{F}_X(t)H^{m-1}(t)\gamma(t)dt}, \text{for all } t \geq 0. \tag{36}$$

By considering $\varphi(t) = H(t)$ which increases in t, Theorem 3 (i) applies and concludes that X and Y are equally distributed, if $\mathfrak{h}_{X,Y}(T_{m:m}) = 1$, for all $m = 1, 2, \cdots$. From (32), T^{**}_m in the proof of Theorem 3 (ii), for $m = 1, 2, \cdots$, has p.d.f.

$$f_{T^{**}_m}(t) = \frac{F_X(t)H^{m-1}(t)\gamma(t)}{\int_0^{+\infty} F_X(t)H^{m-1}(t)\gamma(t)dt}, \text{for all } t \geq 0. \tag{37}$$

In this case, from Theorem 3 (ii) it concludes that X and Y are equally distributed, if $\tilde{\mathfrak{h}}_{X,Y}(T_{m:m}) = 1$, for all $m = 1, 2, \cdots$.

4.2. Order Statistics from Heterogenous Populations. In previous subsection, order statistics from i.i.d. r.v.s have been considered. In the context of the proportional hazard rate model (PHRM) and the proportional revered hazard rate model (PRHRM), the condition that $T_1, T_2, \cdots$ are identically distributed can be relaxed, respectively, when the smallest order statistic and the greatest order statistic are considered in these models as two coming random times. Let us assume that $T_1, T_2, \cdots$ are nonnegative independent r.v.s and, furthermore, assume that they are independent of X such that T_i has s.f. $\bar{H}^{\lambda_i}, i = 1, 2, \cdots$ where $\lambda_1 \geq 1$ and $\lambda_i > 0, i = 2, 3, \cdots$. The r.v. T_i follows the PHRM. Let us take $s_m = \sum_{i=1}^{m} \lambda_i$ such that $\sum_{m=1}^{+\infty} (s_m)^{-1} = +\infty$ in which $s_1 = \lambda_1 \geq 1$. In light of (32) and also from (27), the p.d.f. of $T'_m := T_{1:m}$ for $m = 1, 2, \cdots$, is

$$f_{T'_m}(t) = \frac{w(t)\bar{H}^{s_m}(t)}{\int_0^{+\infty} w(t)\bar{H}^{s_m}(t)dt}, \tag{38}$$

where $w(t) = (\gamma(t))/(\bar{H}(t))$. By (32), the p.d.f. of T_m^* in the proof of Theorem 3 (i) for $m = 1, 2, \cdots$ is replaced by the p.d.f. (29) which yields

$$f_{T_m^*}(t) = \frac{\bar{F}_X(t)\bar{H}^{s_m-1}(t)\gamma(t)}{\int_0^{+\infty} \bar{F}_X(t)\bar{H}^{s_m-1}(t)\gamma(t)dt}, \text{ for all } t \geq 0. \tag{39}$$

Therefore, if $\varphi(t) = \bar{H}(t)$ which is a decreasing function, then Theorem 3 (i) concludes that X and Y are equally distributed, if $\mathfrak{h}_{X,Y}(T_{1:m}) = 1$, for all $m = 1, 2, \cdots$. The p.d.f. of T_m^{**} in the proof of Theorem 3 (ii), for $m = 1, 2, \cdots$, is replaced by the p.d.f. (31) which gives

$$f_{T_m^{**}}(t) = \frac{F_X(t)\bar{H}^{s_m-1}(t)\gamma(t)}{\int_0^{+\infty} F_X(t)\bar{H}^{s_m-1}(t)\gamma(t)dt}, \text{ for all } t \geq 0. \tag{40}$$

Thus, by assigning $\varphi(t) = H(t)$ which is an increasing function, Theorem 3 (ii) presents that X and Y are equal in distribution, if $\tilde{\mathfrak{h}}_{X,Y}(T_{1:m}) = 1$, for all $m = 1, 2, \cdots$.

We now discuss the case when the PRHRM is used. In this setting, assume that $T_1, T_2, \cdots$ are nonnegative independent r.v.s which are independent of X so that T_i has c.d.f. $H^{\lambda_i}, i = 1, 2, \cdots$ where $\lambda_1 \geq 1$ and $\lambda_i > 0, i = 2, 3, \cdots$. We again take $s_m = \sum_{i=1}^{m} \lambda_i$ with the requirement $\sum_{m=1}^{+\infty} (s_m)^{-1} = +\infty$ in which $s_1 = \lambda_1 \geq 1$. In terms of (32), the p.d.f of $T'_m := T_{m:m}$ for $m = 1, 2, \cdots$ is

$$f_{T'_m}(t) = \frac{w(t)H^{s_m}(t)}{\int_0^{+\infty} w(t)H^{s_m}(t)dt}, \tag{41}$$

where $w(t) = (\gamma(t))/(H(t))$, and this is a particular case for the p.d.f. (27) in Remark 4. The p.d.f. of T_m^* in the proof of Theorem 3 (i) for $m = 1, 2, \cdots$ can be replaced by the p.d.f. (29) implying that

$$f_{T_m^*}(t) = \frac{\bar{F}_X(t)H^{s_m-1}(t)\gamma(t)}{\int_0^{+\infty} \bar{F}_X(t)H^{s_m-1}(t)\gamma(t)dt}, \text{ for all } t \geq 0. \tag{42}$$

Hence, if $\varphi(t) = H(t)$ which is an increasing function, then Theorem 3 (i) concludes that X and Y are equally distributed, if $\mathfrak{h}_{X,Y}(T_{m:m}) = 1$, for all $m = 1, 2, \cdots$. The p.d.f. of T_m^{**} in the proof of Theorem 3 (ii), for $m = 1, 2, \cdots$, can be replaced by the p.d.f. (31) which gives

$$f_{T_m^{**}}(t) = \frac{F_X(t)H^{s_m-1}(t)\gamma(t)}{\int_0^{+\infty} F_X(t)H^{s_m-1}(t)\gamma(t)dt}, \quad \text{for all } t \geq 0. \tag{43}$$

By making the choice of $\varphi(t) = H(t)$ which is increasing, Theorem 3 (ii) concludes that X and Y are equally distributed, if $\tilde{\mathfrak{h}}_{X,Y}(T_{m:m}) = 1$, for all $m = 1, 2, \cdots$.

4.3. Record Values. Now, we consider random times in the framework of record statistics. The epoch times associated with a nonhomogeneous Poisson process can be thought as the consecutive record values of a sequence of i.i.d. nonnegative r.v.s (see, for instance, [45]).

4.3.1. Upper Records. The r.v. T_i upon its observation is an upper record, if the value it takes is greater than the corresponding value for previous observations. Hence, T_j is considered to be an upper record if $T_j > T_i$ for every $i < j$. By realizing the amounts of consecutive records, a random sequence of times is produced at which the records appear. Let us denote the ith element of this sequence by U_i, considered to be the time at which the ith upper record is reached. The origin of time is considered U_0 which is assumed to be zero with probability one and, for $j \geq 1$, $U_j = \min\{i : T_i > T_{U_{i-1}}\}$. The upper records are then $\{T_{U_m} : m = 1, 2, \cdots\}$. Since T_i's are lifetime r.v.s, thus $T_0 = 0$. The r.v. $T'_m := T_{U_m}$ as the mth upper record follows the p.d.f.

$$f_{T'_m}(t) = \frac{w(t)\left(-\log\left(\bar{H}(t)\right)\right)^m}{\int_0^{+\infty} w(t)\left(-\log\left(\bar{H}(t)\right)\right)^m dt}, \tag{44}$$

where $w(t) = \gamma(t)$ which coincides with (20). The p.d.f. of T_m^* in the proof of Theorem 3 (i) for $m = 1, 2, \cdots$ is here

$$f_{T_m^*}(t) = \frac{\bar{F}_X(t)\left(-\log\left(\bar{H}(t)\right)\right)^m \gamma(t)}{\int_0^{+\infty} \bar{F}_X(t)\left(-\log\left(\bar{H}(t)\right)\right)^m \gamma(t)dt}. \tag{45}$$

Thus, if $\varphi(t) = -\log(\bar{H}(t))$ which is increasing in t, then Theorem 3 (i) is applicable and concludes that X and Y are equally distributed, if $\mathfrak{h}_{X,Y}(T_{U_m}) = 1$, for all $m = 1, 2, \cdots$. The p.d.f. of T_m^{**} in the proof of Theorem 3 (ii), for $m = 1, 2, \cdots$ is given here by

$$f_{T_m^{**}}(t) = \frac{F_X(t)\left(-\log\left(\bar{H}(t)\right)\right)^m \gamma(t)}{\int_0^{+\infty} F_X(t)\left(-\log\left(\bar{H}(t)\right)\right)^m \gamma(t)dt}, \quad \text{for all } t \geq 0. \tag{46}$$

Theorem 3 (ii) yields X and Y are equally distributed, if $\tilde{\mathfrak{h}}_{X,Y}(T_{U_m}) = 1$, for all $m = 1, 2, \cdots$.

4.3.2. Lower Records. In contrast to the upper records, the r.v. T_i upon its observation is a lower record, if its value is smaller than the corresponding value for previous observations. In such situation, T_j is considered to be a lower record if $T_j < T_i$ for every $i < j$. Holding the amount of these records, a random sequence of times is produced at which the lower records occur. We denote the ith element of this sequence by R_i, considered to be the time at which the ith lower record is achieved. The origin of time is considered here to be R_0 assumed to be zero and, for $j \geq 1$, $R_j = \min\{i : T_i < T_{U_{i-1}}\}$. The lower records are recognized as $\{T_{L_m} : m = 1, 2, \cdots\}$. It is known that $T_0 = 0$. The r.v. $T'_m \coloneqq T_{L_m}$ as the mth lower record follows the p.d.f.

$$f_{T'_m}(t) = \frac{w(t) - \log(H(t)))^m}{\int_0^{+\infty} w(t)(-\log(H(t))^m dt}, \qquad (47)$$

in which $w(t) = \gamma(t)$, and this fulfills (20). The p.d.f. of T^*_m in the proof of Theorem 3 (i) for $m = 1, 2, \cdots$ is

$$f_{T^*_m}(t) = \frac{\bar{F}_X(t)(-\log(H(t)))^m \gamma(t)}{\int_0^{+\infty} \bar{F}_X(t)(-\log(H(t)))^m \gamma(t) dt}. \qquad (48)$$

Hence, taking $\varphi(t) = -\log(H(t))$ which is decreasing in t, by Theorem 3 (i), we deduce that X and Y are equally distributed, if $\mathfrak{h}_{X,Y}(T_{L_m}) = 1$, for all $m = 1, 2, \cdots$. The p.d.f. of T^{**}_m in the proof of Theorem 3 (ii), for $m = 1, 2, \cdots$ is

$$f_{T^{**}_m}(t) = \frac{F_X(t)(-\log(H(t)))^m \gamma(t)}{\int_0^{+\infty} F_X(t)(-\log(H(t)))^m \gamma(t) dt}, \quad \text{for all } t \geq 0. \qquad (49)$$

Theorem 3 (ii) provides that X and Y are equally distributed, if $\tilde{\mathfrak{h}}_{X,Y}(T_{L_m}) = 1$, for all $m = 1, 2, \cdots$.

4.4. Convolution of Heterogenous Gamma Populations. Before closing this section, we apply the characterization property given in Theorem 3 to partial sums of gamma r.v.s, with restricted shape parameters, as another fulfilling random sequence. In the next example, we show that a complete sequence is generated by the gamma distribution.

Example 1. Suppose T_i has p.d.f. $f_{T_i}(t) = (t^{\alpha_i - 1} \beta^{\alpha_i} e^{-\beta t})/(\Gamma(\alpha_i))$, for $i = 1, 2, \cdots$, where $\alpha_i > 0$ and $\beta > 0$. Denote $T'_m \coloneqq T_1 + T_2 + \cdots + T_m$. It is a well-known result in probability that convolution of independent r.v.s. following gamma distribution again follows the gamma distribution. Hence, $T_m{}'$ has a gamma distribution with p.d.f.

$$f_{T'_m}(t) = \frac{w(t) t^{\eta_m}}{\int_0^{+\infty} w(t) t^{\eta_m}}, \qquad (50)$$

where $w(t) = e^{-\beta t}/t$ and $\eta_m = \sum_{i=1}^m \alpha_i$. If $\sum_{m=1}^{+\infty} \eta_m^{-1} = +\infty$, whenever $\eta_1 = \alpha_1 \geq 1$, then $t^{\eta_m}, m = 1, 2, \cdots$ is a complete sequence due to Lemma 2.

The p.d.f. of T^*_m in the proof of Theorem 3 (i) for $m = 1, 2, \cdots$ is

$$f_{T^*_m}(t) = \frac{\bar{F}_X(t) t^{\eta_m - 1} e^{-\beta t}}{\int_0^{+\infty} \bar{F}_X(t) t^{\eta_m - 1} e^{-\beta t} dt}. \qquad (51)$$

The choice $\varphi(t) = t$ as an increasing function together with the discussion in Example 1 will make Theorem 3 (i) applicable according which X and Y are equally distributed, if $\mathfrak{h}_{X,Y}(T'_m) = 1$, for all $m = 1, 2, \cdots$. The p.d.f. of T^{**}_m in the proof of Theorem 3 (ii), for $m = 1, 2, \cdots$, is

$$f_{T^{**}_m}(t) = \frac{F_X(t) t^{\eta_m - 1} e^{-\beta t}}{\int_0^{+\infty} F_X(t) t^{\eta_m - 1} e^{-\beta t} dt}. \qquad (52)$$

Thus, it is proved that X and Y are equally distributed, if $\tilde{\mathfrak{h}}_{X,Y}(T'_m) = 1$, for all $m = 1, 2, \cdots$.

5. Characterizations of Specific Distributions

In Section 3 and Section 4, general characterizations of distributions have been derived using random relative (reversed) hazard rate measure when applied to some random sequences of time which are considered to be independent of the original random variables. In the context of Theorem 3 if either X or Y is fixed in a particular lifetime distribution, then a characterization of that specific distribution is produced. In this section, our aim is to characterize the Weibull distribution and the inverse Weibull distribution as two typical lifetime distributions.

From Theorem 3 (i), if there exist r.v.s $T'_1, T'_2, \cdots$ so that T'_m follows the p.d.f. $f_{T'_m}(t) = (w(t)\varphi^m(t))/(\int_0^{+\infty} w(t) \varphi^m(t) dt)$, such that $\mathfrak{h}_{X,Y}(T'_m) = 1$, for $m = m_0, m_0 + 1, \cdots$ (for an $m_0 \in \mathbb{N}$), then X and Y are identical in distribution and vice versa. In view of Theorem 3 (i) and its proof if $E((h_X(T^*_m))/(h_Y(T^*_m))) = 1$, for $m = m_0, m_0 + 1, \cdots$ where T^*_m has p.d.f. $f_{T^*_m}(t) = (w(t)\bar{F}_X(t)\varphi^m(t))/(\int_0^{+\infty} w(t)\bar{F}_X(t)\varphi^m(t) dt)$, then X and Y are equally distributed and vice versa. Example 2 and Example 3 are derived using Theorem 3 (i).

Example 2. Suppose Y follows Weibull distribution with s.f. $\bar{F}_Y(t) = \exp(-(\lambda t)^\alpha), t \geq 0$. Then, $h_Y(t) = \alpha \lambda^\alpha t^{\alpha - 1}$, and if there exists an $m_0 \in \mathbb{N}$ so that $E((h_X(T^*_m))/(\alpha \lambda^\alpha (T^*_m)^{\alpha - 1})) = 1$, for $m = m_0, m_0 + 1, \cdots$, then X has Weibull distribution with parameters α and λ (we write $X \sim W(\alpha, \lambda)$) and vice versa. Equivalently, if there exist $c_0 < 1$ and $c_1 > 0$ such that $E[(T^*_m)^{c_0} h_X(T^*_m)] = c_1$, for all $m = m_0, m_0 + 1, \cdots$ then X has Weibull distribution with $\alpha = 1 - c_0$ and $\lambda = \sqrt[1-c_0]{c_1/(1 - c_0)}$ and vice versa.

Example 3. Suppose X follows $W(\alpha, \lambda)$. If there exists an $m_0 \in \mathbb{N}$ so that $E((\alpha \lambda^\alpha (T^*_m)^{\alpha - 1})/(h_Y(T^*_m))) = 1$, for $m = m_0$, $m_0 + 1, \cdots$ then $Y \sim W(\alpha, \lambda)$ and vice versa. Equivalently, if there exist $c_0 > -1$ and $c_1 > 0$ such that $E[(T^*_m)^{c_0}/(h_Y(T^*_m))] = c_1$, for all $m = m_0, m_0 + 1, \cdots$, then Y has Weibull distribution with $\alpha = 1 + c_0$ and $\lambda = \sqrt[1+c_0]{1/(c_1(1 + c_0))}$ and vice versa.

By Theorem 3 (ii), it turns out that if there exist r.v.s $T'_1, T'_2, \cdots$ so that T'_m follows the p.d.f. $f_{T'_m}(t) = (w(t)\varphi^m(t))/(\int_0^{+\infty} w(t)\varphi^m(t)dt)$, such that $\tilde{\mathfrak{h}}_{X,Y}(T'_m) = 1$, for $m = m_0, m_0 + 1, \cdots$ (for an $m_0 \in \mathbb{N}$), then X and Y are identical in distribution and vice versa. This means from the proof of Theorem 3 (ii) that if $E((\tilde{h}_X(T^{**}_m))/(\tilde{h}_Y(T^*_m))) = 1$, for $m = m_0, m_0 + 1, \cdots$ where T^{**}_m has p.d.f. $f_{T^{**}_m}(t) = (w(t)F_X(t)\varphi^m(t))/(\int_0^{+\infty} w(t)F_X(t)\varphi^m(t)dt)$, then X and Y are equally distributed and vice versa. Example 4 and Example 5 are considered as applications of Theorem 3 (ii).

Example 4. Assume that Y follows inverse Weibull distribution with c.d.f. $F_Y(t) = \exp(-1/(\lambda t)^\alpha), t \geq 0$. Then, $\tilde{h}_Y(t) = \alpha/(\lambda^\alpha t^{\alpha+1})$, and if there exists an $m_0 \in \mathbb{N}$ such that $E((\lambda^\alpha (T^{**}_m)^{\alpha+1} \tilde{h}_X(T^{**}_m))/\alpha) = 1$, for $m = m_0, m_0 + 1, \cdots$, then X has inverse Weibull distribution with the shape parameter α and scale parameter λ (we write $X \sim IW(\alpha, \lambda)$) and vice versa. Equivalently, if there exist $c_0 > 1$ and $c_1 > 0$ such that $E[(T^{**}_m)^{c_0} \tilde{h}_X(T^{**})] = c_1$, for all $m = m_0, m_0 + 1, \cdots$ then X has inverse Weibull distribution with $\alpha = c_0 - 1$ and $\lambda = \sqrt[c_0-1]{(c_0 - 1)/c_1}$ and vice versa.

Example 5. Assume that X follows $IW(\alpha, \lambda)$. If there exists an $m_0 \in \mathbb{N}$ so that $E(\alpha/(\lambda^\alpha (T^{**}_m)^{\alpha+1} \tilde{h}_Y(T^{**}_m))) = 1$, for $m = m_0, m_0 + 1, \cdots$, then $Y \sim IW(\alpha, \lambda)$ and vice versa. Equivalently, if there exist $c_0 > 1$ and $c_1 > 0$ such that $E[1/((T^{**}_m)^{c_0} \tilde{h}_Y(T^{**}_m))] = c_1$, for all $m = m_0, m_0 + 1, \cdots$, then Y has inverse Weibull distribution with $\alpha = c_0 - 1$ and $\lambda = \sqrt[c_0-1]{c_1(c_0 - 1)}$ and vice versa.

The characterizations of two standard families of lifetime distributions in the examples in this section can be developed to other (standard or nonstandard) families of lifetime distributions. Furthermore, in this context, considering specific sequences of random times T'_m as identified in Section 4 gives further characterizations of specific distributions. Let us suppose that $X_1, X_2, \cdots, X_m, X_{m+1}, \cdots$ are i.i.d. and that $T'_m = X_{1:m}$ is the minimum order statistic among $X_1, X_2, \cdots, X_m$. Then, X has a distribution equal with distribution of Y if there exists an $m_0 \in \mathbb{N}$ such that

$$E\left(\frac{h_{X_{m+1}}(X_{1:m+1})}{h_Y(X_{1:m+1})}\right) = 1, \text{for all } m = m_0, m_0 + 1, \cdots. \quad (53)$$

In parallel, if we consider $T'_m = X_{m:m}$ which is the maximum order statistic among $X_1, X_2, \cdots, X_m$. Then, X has a distribution equal with distribution of Y, if there exists an $m_0 \in \mathbb{N}$ such that

$$E\left(\frac{\tilde{h}_{X_{m+1}}(X_{m+1:m+1})}{\tilde{h}_Y(X_{m+1:m+1})}\right) = 1, \text{for all } m = m_0, m_0 + 1, \cdots. \quad (54)$$

6. Conclusion

The hazard ratio and the reversed hazard ratio have been developed with random ages. Characterization relations using theses quantities, when evaluated convectively at some random times, have been presented. The general form of the distribution of sequence of random times has been satisfied by a number of particular sequences of random times including minimum and maximum order statistics, as well as moderate order statistics of i.i.d. random times from a general distribution, upper and lower record values of i.i.d. random times taken from a general distribution, minimum and maximum order statistics of independent but not identical random times from the PHRM and PRHM models, and partial sums of gamma distributed random times. In future studies, the generalized order statistics and the sequential order statistics are considered as possible random times to see whether such statistics when constitute a random sequence fulfill the main characterization property derived in this paper.

The random times have been considered to be independent of the original random variable whose RLRT plays the main role. The new measures proposed in this paper can be developed to the case where the original random variable and the sequence of random time are dependent. This may arise in some situations. For example, assume that one considers a coherent system and wants to compare the random h.r. of one (specific) component working in system relative to another (specific) component at the random time at which the system fails. In this situation, it is apparent that the lifetime of system cannot be independent of the lifetime its components have. In the future, the possibility to study the dependency in the random relative h.r. measure as well as the random relative r.h.r. quantity of two lifetime units will be considered.

Data Availability

There is no data used in this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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