Hindawi
Journal of Function Spaces
Volume 2022, Article ID 6685771, 12 pages
https://doi.org/10.1155/2022/6685771

Research Article

On $p(z)$–Laplacian System Involving Critical Nonlinearities

Ahmed Aberqi,[1] Jaouad Bennouna,[2] Omar Benslimane,[2] and Maria Alessandra Ragusa[3]

[1]*Laboratoy LAMA, Sidi Mohamed Ben Abdellah University, National School of Applied Sciences Fez, Morocco*
[2]*Laboratory LAMA, Sidi Mohamed Ben Abdellah University, Faculty of Sciences Dhar El Mahraz, B.P. 1796 Atlas Fez, Morocco*
[3]*Dipartimento di Matematica e Informatica, Universitá di Catania, Viale A.Doria 6, 95125 Catania, Italy*

Correspondence should be addressed to Maria Alessandra Ragusa; mariaalessandra.ragusa@unict.it

Received 15 February 2022; Accepted 5 April 2022; Published 25 April 2022

Academic Editor: Giovanni Di Fratta

In this paper, we deal with the existence of at least two nonnegative nontrivial solutions to a $p(z)$–Laplacian system involving critical nonlinearity in the context of Sobolev spaces with variable exponents on complete manifolds. We have established our main results by exploring both Nehari's method and doing a refined analysis on the associated fiber map and some variational techniques.

1. Introduction

In the present work, we investigate the existence of nonnegative nontrivial solutions to the following system:

$$(\mathcal{S})\begin{cases} -\Delta_{g,p(z)}w = \alpha|w|^{q(z)-2}w + \dfrac{2\mathfrak{a}(z)}{\mathfrak{a}(z)+\mathfrak{b}(z)}|w|^{\mathfrak{a}(z)-2}w|y|^{\mathfrak{b}(z)} & \text{in } \mathcal{M}, \\ -\Delta_{g,p(z)}y = \beta|y|^{q(z)-2}y + \dfrac{2\,\mathfrak{b}(z)}{\mathfrak{a}(z)+\mathfrak{b}(z)}|w|^{\mathfrak{a}(z)}|y|^{\mathfrak{b}(z)-2}y & \text{in } \mathcal{M}, \\ w|_{\partial\mathcal{M}} = y|_{\partial\mathcal{M}} = 0. \end{cases} \tag{1}$$

Here, $(\mathcal{M}, g)$ is a complete compact Riemannian N-manifold, $\alpha, \beta \in \mathbb{R}_*^+$ to be specified later, and $p, q, \mathfrak{a}, \mathfrak{b} \in C(\bar{M})$ satisfying the assumptions (23) and (24) in Section 3. $-\Delta_{g,p(z)}$ is the Laplacian operator on $(\mathcal{M}, g)$.

In recent years, several researchers have been interested in equations or systems involving the $p(z)$–Laplacian, not only for their application in several scientific fields, such as fluid filtration in porous media, constrained heating, elastoplasticity, and optimal control, but also for their mathematical importance in the theory of function spaces with variable exponents. For example, in [1], Zhang proved the existence of positive solutions under some conditions of the following class of $p(z)$–Laplacian systems:

$$(\mathcal{S}_1)\begin{cases} -\Delta_{p(z)}w = \alpha f(z,y) & \text{in } \mathcal{Q}, \\ -\Delta_{q(z)}w = \alpha\, g(z,w) & \text{in } \mathcal{Q}, \\ w|_{\partial\mathcal{Q}} = y|_{\partial\mathcal{Q}} = 0, \end{cases} \tag{2}$$

in bounded open set $\mathcal{Q} \subset \mathbb{R}^{\mathcal{N}}$, without assuming the symmetric radial conditions. And by using the subsuper solution technique, Boulaaras et al. in [2] have studied the asymptotic behavior of the system $(\mathcal{S}_1)$. In [3], Aberqi et al. established the existence of a renormalized solution for a class of nonlinear parabolic systems using the Gagliardo-Nirenberg theorem, with the source term being less regular. In addition, we refer to the work of Marino and Winkert [4] who studied this kind of system with nongrowth conditions, governed by a double-phase operator. For the systems with singular source data, we refer to Saoudi [5] and Papageorgiou et al. [6]. For more results, we refer to [7–11], as well as to [12], and the references therein.

Before explaining the novelty of this paper, we give an overview of the literature on this kind of system in $W^{1,p}(\mathcal{Q})$. Adriouch and El Hamidi in [13] proved the existence and multiplicity of solutions to the following system:

$$(\mathcal{S}_2)\begin{cases} -\Delta_p w = \alpha|w|^{p_1-2}w + (\mathrm{a}+1)w|w|^{\mathrm{a}-1}|y|^{\mathrm{b}+1} & \text{in } \mathcal{Q}, \\ -\Delta_q y = \beta|y|^{q-2}y + (\mathrm{b}+1)|w|^{\mathrm{a}+1}|y|^{\mathrm{b}-1}y & \text{in } \mathcal{Q}, \end{cases} \quad (3)$$

by using the variational techniques. Chen and Wu in [14] examined the semilinear version of $(\mathcal{S}_2)$ with more general parametric functions f_α, g_β, and convex-concave critical nonlinearity. Mercuri and Willem in [15] proved a representation theorem for Palais-Smale sequences involving the p–Laplacian and critical nonlinearities. For a deeper comprehension, see [16–18].

Next, we will mention some papers that deal with the same problem with the fractional p–Laplacian. We refer to Chen and Squassina [19], Pawan and Sreenadh [20], and Biswas and Tiwari [21] for fractional $p(z)$–Laplacian. Readers may refer to the references given therein for more background.

Our goal in the present contribution is to study this kind of system with nonstandard convex-concave nonlinearity, in the Sobolev spaces on the complete manifold. We prove the existence of nonnegative nontrivial solutions using the Nehari manifold technique. However, we address the challenges due to the fact that $\Delta_{p(z)}$ is not homogeneous, also due to the non-Euclidean framework of the system. Moreover, we do not have enough background on this space, such as embedding results, Hölder inequality, and the relation between the $\|w\|_{p(z)}$ and $\rho_{q(z)}(|Dw|)$, including the pertinent result proven in ([22], Proposition 2.5). This is the first existing result in this field to the best of our knowledge.

The theorem below contains our main result.

Theorem 1. *Let $(\mathcal{M}, g)$ satisfy the $B_{vol}(\alpha, y)$ property. Then, there exists a constant K such that if $0 < \alpha + \beta < K$, then the system $(\mathcal{S})$ admits at least two nonnegative weak solutions.*

The organization of this contribution is given as follows. We start in Section 2 by presenting some definitions and properties of Lebesgue spaces with variable exponents on a bounded set of $\mathcal{Q} \subset \mathbb{R}^N$ and on a complete manifold $\mathcal{M}$. After that, in Section 3, we give some properties of the Nehari manifold and set up the variational framework of the system $(\mathcal{S})$. Then, we establish the existence of two nonnegative nontrivial solutions to the system $(\mathcal{S})$.

2. Notations and Basic Properties

This section is devoted to recalling some definitions and properties which will be used in the next sections (see [22–27]).

Consider an open-bounded set $\mathcal{Q}$ of $\mathbb{R}^{\mathcal{N}}$ with $(\mathcal{N} \geq 2)$. We define the Lebesgue space with variable exponent $L^{\nu(\cdot)}(\mathcal{Q})$ as the set of all measurable function $w : \mathcal{Q} \mapsto \mathbb{R}$ such that

$$\rho_{\nu(\cdot)}(w) = \int_{\mathcal{Q}} |w(z)|^{\nu(z)} dz < \infty, \quad (4)$$

endowed with the Luxembourg norm. And the associated Sobolev space is given by

$$\mathcal{W}^{1,\nu(z)}(\mathcal{Q}) = \left\{ w \in L^{\nu(z)}(\mathcal{Q}) : |Dw| \in L^{\nu(z)}(\mathcal{Q}) \right\}, \quad (5)$$

with the norm

$$\|w\|_{\mathcal{W}^{1,\nu(z)}(\mathcal{Q})} = \|w\|_{L^{\nu(z)}(\mathcal{Q})} + \|Dw\|\Big|_{L^{\nu(z)}(\mathcal{Q})} \quad \forall w \in \mathcal{W}^{1,\nu(z)}(\mathcal{Q}). \quad (6)$$

And $\mathcal{W}_0^{1,\nu(z)}(\mathcal{Q}) = \overline{C_0^\infty(\mathcal{Q})}^{\mathcal{W}^{1,\nu(z)}(\mathcal{Q})}$.

Lemma 2 (see [12]). *Let $\nu_1(z) \in L^\infty(\mathcal{Q})$ such that $\nu_1 \geq 0$, $\nu_1 \neq 0$. Let $\nu_2 : \mathcal{Q} \longrightarrow \mathbb{R}$ be a measurable function such that $\nu_1(z)\nu_2(z) \geq 1$ a.e. in $\mathcal{Q}$. Then, for every $w \in L^{\nu_1(z)\nu_2(z)}(\mathcal{Q})$,*

$$\left\| |w| \right\|_{L^{\nu_2(z)}(\mathcal{Q})}^{\nu_1(\cdot)} \leq \|w\|_{L^{\nu_1(z)\nu_2(z)}(\mathcal{Q})}^{\nu_1^-} + \|w\|_{L^{\nu_1(z)\nu_2(z)}(\mathcal{Q})}^{\nu_1^+}. \quad (7)$$

2.1. Sobolev Spaces on Manifolds

Definition 3 (see [22]). Let $w \in C_c(\mathcal{M})$ and $(\mathcal{Q}_i, \varphi_i)_{i\in I}$ be an atlas of $\mathcal{M}$,

$$\int_{\mathcal{M}} w(z) dv_{\mathrm{g}}(z) = \sum_{k\in J} \int_{\varphi_k(\mathcal{Q}_k)} \left(\left(\det\left(\mathrm{g}_{ij}\right)^{1/2} \eta_k \mathrm{u} \right) \right) o\varphi_k^{-1}(z) dz, \quad (8)$$

where $dv_{\mathrm{g}} = (\det(\mathrm{g}_{ij}))^{1/2} dz$, g_{ij} are the components of the Riemannian metric g in the chart and dz is the Lebesgue measure of $\mathbb{R}^{\mathcal{N}}$.

Definition 4 (see [22]). The Sobolev space $L_k^{q(\cdot)}(\mathcal{M})$ is the completion of $C_k^{q(\cdot)}(\mathcal{M})$ with respect to the norm $\|\mathrm{w}\|_{L_k^{q(\cdot)}}$, where

$$C_k^{q(\cdot)}(\mathcal{M}) = \left\{ w \in C^\infty(\mathcal{M}) \text{ such that } \left|D^j w\right| \in L^{q(\cdot)}(\mathcal{M})\,; j = 0, \cdots, k \right\},$$

$$\|w\|_{L_k^{q(\cdot)}} = \sum_{j=0}^{k} \left\|D^j w\right\|_{L^{q(\cdot)}}, \quad (9)$$

with $|D^k w|^2 = (\prod_{1\leq\alpha\leq k} \mathrm{g}^{i_\alpha j_\alpha})$. $(D^k w)_{i_1\cdots i_k} (D^k w)_{j_1\cdots j_k}$ is the norm of the k-th covariant derivative of w. If $\mathcal{Q}$ is a subset of $\mathcal{M}$, then $L_{k,0}^{q(\cdot)}(\mathcal{Q})$ is the completion of $C_k^{q(\cdot)}(\mathcal{M}) \cap C_0(\mathcal{Q})$ with respect to$\|.\|_{L_k^{q(\cdot)}}$, where $C_0(\mathcal{Q})$ denotes the vector space of continuous functions whose support is a compact subset of $\mathcal{Q}$.

Definition 5 (see [22]). Let $\xi : [a, b] \longrightarrow \mathscr{M}$ a curve of class C^1. The length of ξ is

$$\ell(\xi) = \int_a^b g\left(\frac{d\xi}{ds}, \frac{d\xi}{ds}\right)^{1/2} ds. \quad (10)$$

Let $(u, v) \in \mathscr{M}^2$, we define the distance between u and v by

$$d_g(u, v) = \inf\left\{\frac{\ell(\xi)}{\xi} : [a, b] \longrightarrow \mathscr{M} : \xi(a) = u \text{ and } \xi(b) = v\right\}. \quad (11)$$

Definition 6 (see [22]). Log-Hölder continuity: let$t : \mathscr{M} \longrightarrow \mathbb{R}$; we say that t is log-Hölder continuous, if there exists $c > 0$ such that

$$|t(u) - t(v)| \le c.\left(\log\left(e + d_g(u, v)^{-1}\right)\right)^{-1}, \quad \forall \{u, v\} \in \mathscr{M}. \quad (12)$$

The set of Log-Hölder continuous functions on $\mathscr{M}$ will be denoted by $\mathscr{P}^{\log}(\mathscr{M})$, which is linked to $\mathscr{P}^{\log}(\mathbb{R}^{\mathscr{N}})$ by the proposition below.

Proposition 7 (see [24, 25]). *Let $(\mathscr{Q}, \phi)$ be a chart of $\mathscr{M}, q \in \mathscr{P}^{\log}(\mathscr{M})$, such that*

$$\frac{1}{2}\delta_{ij} \le g_{ij} \le 2\delta_{ij}, \quad \left(\delta_{ij} \text{ is the delta Kronecker symbol}\right), \quad (13)$$

like bilinear forms. Then, $q \circ \phi^{-1} \in \mathscr{P}^{\log}(\phi(\mathscr{Q}))$.

Definition 8 (see [22]). We say that $(\mathscr{M}, g)$ has property $B_{\text{vol}}(\alpha, y)$, if the Ricci tensor of g noted by $Rc(g)$ verifies $Rc(g) \ge \alpha(\mathscr{N} - 1)g$ for some α, and for all $u \in \mathscr{M}$, there exists some $y > 0$ such that $|B_1(z)|_g \ge y$ where $B_1(z)$ are the balls of radius 1 centered at some point z in terms of the volume of smaller concentric balls.

To compare the functionals $\|\cdot\|_{q(\cdot)}$ and $\rho_{q(\cdot)}(\cdot)$, one has the relation

$$\begin{aligned} \min\left\{\rho_{q(\cdot)}(w)^{1/q^-}, \rho_{q(\cdot)}(w)^{1/q^+}\right\} &\le \|w\|_{L^{q(\cdot)}} \\ &\le \max\left\{\rho_{q(\cdot)}(w)^{1/q^-}, \rho_{q(\cdot)}(w)^{1/q^+}\right\}. \end{aligned} \quad (14)$$

Proposition 9 (see [23]). *Hölder's inequality: for all $w \in L^{q(\cdot)}(\mathscr{M})$ and $y \in L^{q'(\cdot)}(\mathscr{M})$, we have*

$$\int_{\mathscr{M}} |w(z)y(z)| dv_g(z) \le r_q \|w\|_{L^{q(\cdot)}(\mathscr{M})} \cdot \|y\|_{L^{q'(\cdot)}(\mathscr{M})}, \quad (15)$$

where r_q is a positive constant depending on q^- and q^+.

Definition 10 (see [22, 26]). We define the Sobolev space on $(\mathscr{M}, g)$ by

$$W^{1,q(z)}(\mathscr{M}) = \left\{w \in L^{q(z)}(\mathscr{M}) : D^k w \in L^{q(z)}(\mathscr{M})\ k = 1, 2, \cdots, n\right\}, \quad (16)$$

endowed by the norm

$$\|w\|_{W^{1,q(z)}(\mathscr{M})} = \|w\|_{L^{q(z)}(\mathscr{M})} + \sum_{k=1}^{n} \left\|D^k w\right\|_{L^{q(z)}(\mathscr{M})}, \quad (17)$$

and we define $W_0^{1,q(z)}(\mathscr{M})$ as the closure of $C^\infty(\mathscr{M})$ in $W^{1,q(z)}(\mathscr{M})$.

Theorem 11 (see [22, 23]). *Let $\mathfrak{a}(z), \mathfrak{b}(z), q(z), p(z) \in C(\bar{\mathscr{M}}) \cap L^\infty(\mathscr{M})$, with $\mathscr{M}$ as a compact Riemannian manifold.*

(i) If

$$\begin{aligned} p(z) < q^*(z) = \frac{\mathscr{N} q(z)}{\mathscr{N} - q(z)} \quad &\text{for } z \in \bar{\mathscr{M}}, \\ q(z) < \mathscr{N}. \end{aligned} \quad (18)$$

Then, we have

$$W^{1,q(z)}(\mathscr{M}) \hookrightarrow\hookrightarrow L^{p(z)}(\mathscr{M}). \quad (19)$$

(ii) If

$$\begin{aligned} \mathfrak{a}(z) + \mathfrak{b}(z) < q^*(z) \quad &\text{for } z \in \bar{\mathscr{M}}, \\ q(z) < \mathscr{N}. \end{aligned} \quad (20)$$

Then, we have

$$W^{1,q(z)}(\mathscr{M}) \hookrightarrow\hookrightarrow L^{\mathfrak{a}(z)+\mathfrak{b}(z)}(\mathscr{M}). \quad (21)$$

Proposition 12 (see [24]). *We have $W^{1,q(x)}(\mathscr{M}) = W_0^{1,q(x)}(\mathscr{M})$, if $(\mathscr{M}, g)$ is complete.*

3. Proof of the Main Results

In this section, we prove our main result, and we note that $J = W_0^{1,q(z)}(\mathscr{M}) \times W_0^{1,q(z)}(\mathscr{M})$, endowed with norm $\|(w, y)\|^{p(z)} = \|Dw(z)\|^{p(z)} + \|Dy(z)\|^{p(z)}$. In what follows, $\mathscr{D}(\mathscr{M})$ is the space of C^∞ functions with compact support in $\mathscr{M}$.

3.1. Nehari Manifold Analysis for $(\mathscr{S})$. First, we define the weak solution of system $(\mathscr{S})$ as follows.

Definition 13. We say that $(w, y) \in J$ is a weak solution of the system $(\mathcal{S})$, if $(w, y) \in J$ one has

$$\begin{aligned}
&\int_{\mathcal{M}} |Dw(z)|^{p(z)-2} \mathrm{g}(Dw(z), D\phi(z)) dv_{\mathrm{g}}(z) \\
&\quad + \int_{\mathcal{M}} |Dy(z)|^{p(z)-2} \mathrm{g}(Dy(z), D\psi(z)) dv_{\mathrm{g}}(z) \\
&= \int_{\mathcal{M}} \left(\alpha |w(z)|^{q(z)-2} w(z) \phi(z) + \beta |y(z)|^{q(z)-2} y(z) \psi(z)\right) dv_{\mathrm{g}}(z) \\
&\quad + \int_{\mathcal{M}} \frac{2\,\mathfrak{a}(z)}{\mathfrak{a}(z)+\mathfrak{b}(z)} |w(z)|^{\mathfrak{a}(z)-2} w(z) |y(z)|^{\mathfrak{b}(z)} \phi(z) dv_{\mathrm{g}}(z) \\
&\quad + \int_{\mathcal{M}} \frac{2\,\mathfrak{b}(z)}{\mathfrak{a}(z)+\mathfrak{b}(z)} |w(z)|^{\mathfrak{a}(z)} |y(z)|^{\mathfrak{b}(z)-2} y(z) \psi(z)\, dv_{\mathrm{g}}(z),
\end{aligned} \tag{22}$$

for all $(\phi, \psi) \in \mathcal{D}(\mathcal{M}) \times \mathcal{D}(\mathcal{M})$.

The functions $p, q, \mathfrak{a}, \mathfrak{b} \in C(\bar{\mathcal{M}})$ are assumed to satisfy the following assumption:

$$1 < q^- \le q^+ < p^- \le p^+ < \mathfrak{a}^- + \mathfrak{b}^- \le \mathfrak{a}^+ + \mathfrak{b}^+ < \infty, \tag{23}$$

and the following condition holds

$$\frac{p^-}{\mathfrak{a}^+ + \mathfrak{b}^+} < \left(\frac{p^- - q^+}{(\mathfrak{a}^+ + \mathfrak{b}^+) - q^+}\right)\left(\frac{(\mathfrak{a}^- + \mathfrak{b}^-) - q^+}{p^+ - q^-}\right). \tag{24}$$

To prove our main result, we will use Nehari manifold and fibering maps. The fact that (w, y) is a weak solution is equivalent to being a critical point of the following functional $\mathcal{E}_{\alpha,\beta} : J \longrightarrow \mathbb{R}$ defined as

$$\begin{aligned}
\mathcal{E}_{\alpha,\beta}(w, y) &= \int_{\mathcal{M}} \frac{1}{p(z)} \left(|Dw(z)|^{p(z)} + |Dy(z)|^{p(z)}\right) dv_{\mathrm{g}}(z) \\
&\quad - \int_{\mathcal{M}} \frac{1}{q(z)} \left(\alpha |w(z)|^{q(z)} + \beta |y(z)|^{q(z)}\right) dv_{\mathrm{g}}(z) \\
&\quad - \int_{\mathcal{M}} \frac{2}{\mathfrak{a}(z)+\mathfrak{b}(z)} |w(z)|^{\mathfrak{a}(z)} |y(z)|^{\mathfrak{b}(z)} dv_{\mathrm{g}}(z).
\end{aligned} \tag{25}$$

By a direct calculation, we have $\mathcal{E}_{\alpha,\beta} \in C^1(J, \mathbb{R})$ and

$$\begin{aligned}
\left\langle \mathcal{E}'_{\alpha,\beta}(w, y), (\phi, \psi) \right\rangle &= \int_{\mathcal{M}} |\alpha w|^{p(z)-2} w \phi\, dv_{\mathrm{g}}(z) \\
&\quad + \int_{\mathcal{M}} |\alpha y|^{p(z)-2} y \psi\, dv_{\mathrm{g}}(z) \\
&\quad - \int_{\mathcal{M}} \left(\alpha |w|^{q(z)-2} w \phi + \beta |y|^{q(z)-2} y \psi\right) dv_{\mathrm{g}}(z) \\
&\quad - \int_{\mathcal{M}} \frac{2\,\mathfrak{a}(z)}{\mathfrak{a}(z)+\mathfrak{b}(z)} |w|^{\mathfrak{a}(z)-2} w |y|^{\mathfrak{b}(z)} \phi\, dv_{\mathrm{g}}(z) \\
&\quad - \int_{\mathcal{M}} \frac{2\,\mathfrak{b}(z)}{\mathfrak{a}(z)+\mathfrak{b}(z)} |w|^{\mathfrak{a}(z)} |y|^{\mathscr{b}(z)-2} y \psi\, dv_{\mathrm{g}}(z),
\end{aligned} \tag{26}$$

for any $(\phi, \psi) \in \mathcal{D}(\mathcal{M}) \times \mathcal{D}(\mathcal{M})$.

Consider the Nehari manifold

$$\mathcal{N}_{\alpha,\beta} = \left\{(w, y) \in J \setminus \{(0, 0)\}: \ < \mathcal{E}'_{\alpha,\beta}(w, y), (w, y) \ge 0\right\}. \tag{27}$$

Then, $(w, y) \in \mathcal{N}_{\alpha,\beta}$ if and only if

$$\begin{aligned}
&\int_{\mathcal{M}} \left(|Dw|^{p(z)} + |Dy|^{p(z)}\right) dv_{\mathrm{g}}(z) - \int_{\mathcal{M}} \left(\alpha |w|^{q(z)} + \beta |y|^{q(z)}\right) dv_{\mathrm{g}}(z) \\
&\quad - \int_{\mathcal{M}} \frac{2\,\mathfrak{a}(z)}{\mathfrak{a}(z)+\mathfrak{b}(z)} |w|^{\mathfrak{a}(z)} |y|^{\mathfrak{b}(z)} dv_{\mathrm{g}}(z) \\
&\quad - \int_{\mathcal{M}} \frac{2\,\mathfrak{b}(z)}{\mathfrak{a}(z)+\mathfrak{b}(z)} |w|^{\mathfrak{a}(z)} |y|^{\mathfrak{b}(z)} dv_{\mathrm{g}}(z) = 0,
\end{aligned} \tag{28}$$

which implies that

$$\begin{aligned}
&\int_{\mathcal{M}} \left(|Dw|^{p(z)} + |Dy|^{p(z)}\right) dv_{\mathrm{g}}(z) \\
&\quad - \int_{\mathcal{M}} \left(\alpha |w|^{q(z)} + \beta |y|^{q(z)}\right) dv_{\mathrm{g}}(z) \\
&\quad - 2\int_{\mathcal{M}} |w|^{\mathfrak{a}(z)} |y|^{\mathfrak{b}(z)} dv_{\mathrm{g}}(z) = 0.
\end{aligned} \tag{29}$$

The Nehari manifold $\mathcal{N}_{\alpha,\beta}$ is closely linked to the behavior of the function of the form $\zeta_{(w,y)} : s \mapsto \mathcal{E}_{\alpha,\beta}(sw, sy)$ for $s > 0$ defined by

$$\begin{aligned}
\zeta_{(w,y)}(s) = \mathcal{E}'_{\alpha,\beta}(sw, sy) &= \int_{\mathcal{M}} \frac{s^{p(z)}}{p(z)} \left(|\mathfrak{a} w(z)|^{p(z)} + |\mathfrak{a} y(z)|^{p(z)}\right) dv_{\mathrm{g}}(z) \\
&\quad - \int_{\mathcal{M}} \frac{s^{q(z)}}{q(z)} \left(\alpha |w(z)|^{q(z)} + \beta |y(z)|^{q(z)}\right) dv_{\mathrm{g}}(z) \\
&\quad - \int_{\mathcal{M}} \frac{2 s^{\mathfrak{a}(z)+\mathfrak{b}(z)}}{\mathfrak{a}(z)+\mathfrak{b}(z)} |w(z)|^{\mathfrak{a}(z)} |y(z)|^{\mathfrak{b}(z)} dv_{\mathrm{g}}(z).
\end{aligned} \tag{30}$$

Lemma 14. *Let $(w, y) \in J \setminus \{(0, 0)\}$, then $(sw, sy) \in \mathcal{N}_{\alpha,\beta}$ if and only if $\zeta'_{(w,y)}(s) = 0$.*

Proof. The result is a consequence of the fact that

$$\zeta'_{(w,y)}(s) = \left\langle \mathcal{E}'_{\alpha,\beta}(sw, sy), (w, y) \right\rangle = 0. \tag{31}$$

□

From this lemma, we have that the elements in $\mathcal{N}_{\alpha,\beta}$ correspond to stationary points of the maps $\zeta_{(w,y)}$.

Hence, we note that

$$\zeta'_{(w,y)}(s)=\int_{\mathcal{M}} s^{p(z)-1}\left(|Dw(z)|^{p(z)}+|Dy(z)|^{p(z)}\right)dv_g(z) - \int_{\mathcal{M}} s^{q(z)-1}\left(\alpha|w(z)|^{q(z)}+\beta|y(z)|^{q(z)}\right)dv_g(z) - 2\int_{\mathcal{M}} s^{(\mathfrak{a}(z)+\mathfrak{b}(z))-1}|w(z)|^{\mathfrak{a}(z)}|y(z)|^{\mathfrak{b}(z)}dv_g(z), \quad (32)$$

$$\zeta''_{(w,y)}(s)=\int_{\mathcal{M}} (p(z)-1)s^{p(z)-2}\left(|Dw(z)|^{p(z)}+|Dy(z)|^{p(z)}\right)dv_g(z) - \int_{\mathcal{M}} (q(z)-1)s^{q(z)-2}\left(\alpha|w(z)|^{q(z)}+\beta|y(z)|^{q(z)}\right)dv_g(z) - 2\int_{\mathcal{M}} ((\mathfrak{a}(z)+\mathfrak{b}(z))-1)s^{(\mathfrak{a}(z)+\mathfrak{b}(z))-2}|w(z)|^{\mathfrak{a}(z)}|y(z)|^{\mathfrak{b}(z)}dv_g(z). \quad (33)$$

By Lemma 14, $(w,y)\in\mathcal{N}_{\alpha,\beta}$ if and only if $\zeta'_{(w,y)}(1)=0$. Hence, according to (32), we have for $(w,y)\in\mathcal{N}_{\alpha,\beta}$ that

$$\begin{aligned}\zeta''_{(w,y)}(1)&=\int_{\mathcal{M}} (p(z)-1)\left(|Dw(z)|^{p(z)}+|Dy(z)|^{p(z)}\right)dv_g(z) - \int_{\mathcal{M}} (q(z)-1)\left(\alpha|w(z)|^{q(z)}+\beta|y(z)|^{q(z)}\right)dv_g(z) \\ &\quad - 2\int_{\mathcal{M}} (\mathfrak{a}(z)+\mathfrak{b}(z)-1)|w(z)|^{\mathfrak{a}(z)}|y(z)|^{\mathfrak{b}(z)}dv_g(z) \\ &= 2\int_{\mathcal{M}} (p(z)-(\mathfrak{a}(z)+\mathfrak{b}(z)))|w(z)|^{\mathfrak{a}(z)}|y(z)|^{\mathfrak{b}(z)}dv_g(z) + \int_{\mathcal{M}} (p(z)-q(z))\left(\alpha|w(z)|^{q(z)}+\beta|y(z)|^{q(z)}\right)dv_g(z) \\ &= \int_{\mathcal{M}} (p(z)-q(z))\left(|Dw(z)|^{p(z)}+|Dy(z)|^{p(z)}\right)dv_g(z) - 2\int_{\mathcal{M}} ((\mathfrak{a}(z)+\mathfrak{b}(z))-q(z))|w(z)|^{\mathfrak{a}(z)}|y(z)|^{\mathfrak{b}(z)}dv_g(z) \\ &= \int_{\mathcal{M}} (p(z)-(\mathfrak{a}(z)+\mathfrak{b}(z)))\left(|Dw(z)|^{p(z)}+|Dy(z)|^{p(z)}\right)dv_g(z) + \int_{\mathcal{M}} ((\mathfrak{a}(z)+\mathfrak{b}(z))-q(z))\left(\alpha|w(z)|^{q(z)}+\beta|y(z)|^{q(z)}\right)dv_g(z).\end{aligned} \quad (34)$$

Thus, it is natural to split $\mathcal{N}_{\alpha,\beta}$ into three corresponding to local minima, local maxima, and points of inflexion of $\zeta_{(w,y)}$, i.e.,

$$\begin{aligned}\mathcal{N}^{+}_{\alpha,\beta}&=\left\{(w,y)\in\mathcal{N}_{\alpha,\beta}:\zeta''_{(w,y)}(1)>0\right\},\\ \mathcal{N}^{-}_{\alpha,\beta}&=\left\{(w,y)\in\mathcal{N}_{\alpha,\beta}:\zeta''_{(w,y)}(1)<0\right\},\\ \mathcal{N}^{0}_{\alpha,\beta}&=\left\{(w,y)\in\mathcal{N}_{\alpha,\beta}:\zeta''_{(w,y)}(1)=0\right\}.\end{aligned} \quad (35)$$

Lemma 15. *Let $(w,y)\in J$, then we have*

(i)

$$\int_{\mathcal{M}}\left(\alpha|w(z)|^{q(z)}+\beta|y(z)|^{q(z)}\right)dv_g(z)\le c_1(\alpha+\beta)\times\max\left[\max\left\{\|w\|^{q^-}_{W_0^{1,q(z)}(\mathcal{M})},\|y\|^{q^-}_{W_0^{1,q(z)}(\mathcal{M})}\right\},\max\cdot\left\{\|w\|^{q^+}_{W_0^{1,q(z)}(\mathcal{M})},\|y\|^{q^+}_{W_0^{1,q(z)}(\mathcal{M})}\right\}\right]. \quad (36)$$

(ii)

$$\int_{\mathcal{M}}|w(z)|^{\mathfrak{a}(z)}|y(x)|^{\mathfrak{b}(z)}dv_g(z)\le c_2\max\left[\max\left\{\|w\|^{\mathfrak{a}^-+\mathfrak{b}^-}_{W_0^{1,q(z)}(\mathcal{M})},\|y\|^{\mathfrak{a}^-+\mathfrak{b}^-}_{W_0^{1,q(z)}(\mathcal{M})}\right\},\max\cdot\left\{\|w\|^{\mathfrak{a}^++\mathfrak{b}^+}_{W_0^{1,q(z)}(\mathcal{M})},\|y\|^{\mathfrak{a}^++\mathfrak{b}^+}_{W_0^{1,q(z)}(\mathcal{M})}\right\}\right], \quad (37)$$

for some constants $c_1,c_2>0$.

Proof. (i) Using Theorem 11 (i), and Lemma 2, we get

$$\begin{aligned}\int_{\mathcal{M}}\left(\alpha|w(z)|^{q(z)}+\beta|y(z)|^{q(z)}\right)dv_g(z)&\le 2\alpha\|w\|^{q(\cdot)}_{L^{(\mathfrak{a}(z)+\mathfrak{b}(z))/q(z)}(\mathcal{M})}+2\beta\|y\|^{q(\cdot)}_{L^{(\mathfrak{a}(z)+\mathfrak{b}(z))/q(z)}(\mathcal{M})}\le 2\alpha\left\{\|w\|^{q^-}_{L^{\mathfrak{a}(z)+\mathfrak{b}(z)}(\mathcal{M})}+\|w\|^{q^+}_{L^{\mathfrak{a}(z)+\mathfrak{b}(z)}(\mathcal{M})}\right\}\\ &\quad+2\beta\left\{\|y\|^{q^-}_{L^{\mathfrak{a}(z)+\mathfrak{b}(z)}(\mathcal{M})}+\|y\|^{q^+}_{L^{\mathfrak{a}(z)+\mathfrak{b}(z)}(\mathcal{M})}\right\}\\ &\le 2\max\left\{c(n,p^-,q^-,\mathfrak{a}^-+\mathfrak{a}^-)^{q^-},c(n,p^-,q^-,\mathfrak{a}^-+\mathfrak{b}^-)^{q^+}\right\}\\ &\quad\times\left[\alpha\left\{\|w\|^{q^-}_{W_0^{1,q(z)}(\mathcal{M})}+\|w\|^{q^+}_{W_0^{1,q(z)}(\mathcal{M})}\right\}+\beta\left\{\|y\|^{q^-}_{W_0^{1,q(z)}(\mathcal{M})}+\|y\|^{q^+}_{W_0^{1,q(z)}(\mathcal{M})}\right\}\right].\end{aligned} \quad (38)$$

Hence,

$$\int_{\mathcal{M}}\left(\alpha|w(z)|^{q(z)}+\beta|y(z)|^{q(z)}\right)dv_g(z)\le c_1(\alpha+\beta)\times\max\cdot\left[\max\left\{\|w\|^{q^-}_{W_0^{1,q(z)}(\mathcal{M})},\|y\|^{q^-}_{W_0^{1,q(z)}(\mathcal{M})}\right\},\max\cdot\left\{\|w\|^{q^+}_{W_0^{1,q(z)}(\mathcal{M})},\|y\|^{q^+}_{W_0^{1,q(z)}(\mathcal{M})}\right\}\right]. \quad (39)$$

(ii) By Young's inequality, Lemma 2, and Theorem 11 (ii), we have that

$$\int_{\mathscr{M}} |w(z)|^{\mathfrak{a}(z)}|y(z)|^{\mathfrak{b}(z)}dv_g(z) \le \int_{\mathscr{M}}\left(\frac{\mathfrak{a}(z)}{\mathfrak{a}(z)+\mathfrak{b}(z)}|w(z)|^{\mathfrak{a}(z)+\mathfrak{b}(z)} + \frac{\mathfrak{b}(z)}{\mathfrak{a}(z)+\mathfrak{b}(z)}|y(z)|^{\mathfrak{a}(z)+\mathfrak{b}(z)}\right)dv_g(z)$$
$$\le \left\{\|w\|^{\mathfrak{a}^++\mathfrak{b}^+}_{L^{\mathfrak{a}(z)+\mathfrak{b}(z)}(\mathscr{M})} + \|w\|^{\mathfrak{a}^-+\mathfrak{b}^-}_{L^{\mathfrak{a}(z)+\mathfrak{b}(z)}(\mathscr{M})}\right\} + \left\{\|y\|^{\mathfrak{a}^++\mathfrak{b}^+}_{L^{\mathfrak{a}(z)+\mathfrak{b}(z)}(\mathscr{M})} + \|y\|^{\mathfrak{a}^-+\mathfrak{b}^-}_{L^{\mathfrak{a}(z)+\mathfrak{b}(z)}(\mathscr{M})}\right\}$$
$$\le 2\max\left\{c(n,\mathfrak{a}^-+\mathfrak{b}^-,q^-)^{q^-}, c(n,\mathfrak{a}^-+\mathfrak{b}^-,q^-)^{q^+}\right\} \times \left[\left\{\|w\|^{\mathfrak{a}^++\mathfrak{b}^+}_{W_0^{1,q(z)}(\mathscr{M})} + \|w\|^{\mathfrak{a}^-+\mathfrak{b}^-}_{W_0^{1,q(z)}(\mathscr{M})}\right\} + \left\{\|y\|^{\mathfrak{a}^++\mathfrak{b}^+}_{W_0^{1,q(z)}(\mathscr{M})} + \|y\|^{\mathfrak{a}^-+\mathfrak{b}^-}_{W_0^{1,q(z)}(\mathscr{M})}\right\}\right]. \tag{40}$$

Hence,

$$\int_{\mathscr{M}} \left|w(z)^{\mathfrak{a}(z)}\right|y(z)\Big|^{\mathfrak{b}(z)}dv_g(z) \le c_2 \max \cdot \left[\max\left\{\|w\|^{\mathfrak{a}^-+\mathfrak{b}^-}_{W_0^{1,q(z)}(\mathscr{M})}, \|y\|^{\mathfrak{a}^-+\mathfrak{b}^-}_{W_0^{1,q(z)}(\mathscr{M})}\right\}\max \cdot \left\{\|w\|^{\mathfrak{a}^++\mathfrak{b}^+}_{W_0^{1,q(z)}(\mathscr{M})}, \|y\|^{\mathfrak{a}^++\mathfrak{b}^+}_{W_0^{1,q(z)}(\mathscr{M})}\right\}\right]. \tag{41}$$

□

Lemma 16. *For each $(\alpha,\beta)\in\mathbb{R}^2\setminus\{(0,0)\}$, there exists a constant $K_1>0$ such that for any $0<\alpha+\beta<K_1$, we have $\mathscr{N}^0_{\alpha,\beta}=\varnothing$.*

Proof. Suppose otherwise, that $\mathscr{N}^0_{\alpha,\beta}\neq\varnothing$ for all $(\alpha,\beta)\in\mathbb{R}^2\setminus\{(0,0)\}$. Let $(w,y)\in\mathscr{N}^0_{\alpha,\beta}$, such that $\|(w,y)\|>1$. Then, by Lemma 15, (34), and the definition of $\mathscr{N}^0_{\alpha,\beta}$, we have

$$0=\zeta''_{(w,y)}(1)\le(p^+-(\mathfrak{a}^-+\mathfrak{b}^-))\|(w,y)\|^{p^-} + c_1(\alpha,\beta)((\mathfrak{a}^++\mathfrak{b}^+)-q^-)\|(w,y)\|^{q^-}, \tag{42}$$

that is,

$$((\mathfrak{a}^-+\mathfrak{b}^-)-p^+)\|(w,y)\|^{p^-}\le((\mathfrak{a}^++\mathfrak{b}^+)-q^-)c_1(\alpha,\beta)\|(w,y)\|^{q^-}. \tag{43}$$

Then,

$$\|(w,y)\|\le\left(\frac{(\mathfrak{a}^++\mathfrak{b}^+)-q^-}{(\mathfrak{a}^-+\mathfrak{b}^-)-p^+}c_1(\alpha+\beta)\right)^{1/(p^--q^-)}. \tag{44}$$

Analogously,

$$0=\zeta''_{(w,y)}(1)\ge(p^--q^+)\|(w,y)\|^{p^-}-2((\mathfrak{a}^++\mathfrak{b}^+)-q^-)c_2\|(w,y)\|^{\mathfrak{a}^++\mathfrak{b}^+},$$
$$2((\mathfrak{a}^++\mathfrak{b}^+)-q^-)c_2\|(w,y)\|^{\mathfrak{a}^++\mathfrak{b}^+}\ge(p^--q^+)\|(w,y)\|^{p^-}. \tag{45}$$

Then,

$$\|(w,y)\|^{\mathfrak{a}^++\mathfrak{b}^+-p^-}\ge\frac{p^--q^+}{2((\mathfrak{a}^++\mathfrak{b}^+)-q^-)c_2}, \tag{46}$$

thus,

$$\|(w,y)\|\ge\left(\frac{p^--q^+}{2c_2((\mathfrak{a}^++\mathfrak{b}^+)-q^-)}\right)^{1/((\mathfrak{a}^++\mathfrak{b}^+)-p^-)}. \tag{47}$$

According to (44) and (47), we deduce that

$$\alpha+\beta>\frac{1}{c_1}\left(\frac{(\mathfrak{a}^-+\mathfrak{b}^-)-p^+}{(\mathfrak{a}^++\mathfrak{b}^+)-q^-}\right)\left(\frac{p^--q^+}{2c_2((\mathfrak{a}^++\mathfrak{b}^+)-q^-)}\right)^{(p^--q^-)/((\mathfrak{a}^++\mathfrak{b}^+)-p^-)}=K_1, \tag{48}$$

which is a contradiction. Hence, we can conclude that for any $0<\alpha+\beta<K_1$, we have $\mathscr{N}_{\alpha,\beta}=\varnothing\forall(\alpha,\beta)\in\mathbb{R}^2\setminus\{(0,0)\}$. □

Lemma 17. *If (w,y) is a minimizing of $\mathscr{E}_{\alpha,\beta}$ on $\mathscr{N}_{\alpha,\beta}$ such that $(w,y)\notin\mathscr{N}^0_{\alpha,\beta}$. Then, (w,y) is a critical point of $\mathscr{E}_{\alpha,\beta}$.*

Proof. Let (w,y) be a local minimizing of $\mathscr{E}_{\alpha,\beta}$ in any subset of $\mathscr{N}_{\alpha,\beta}$. Then, in any case, (w,y) is a minimizer of $\mathscr{E}_{\alpha,\beta}$, under the constraint

$$L_{\alpha,\beta}=\left\langle\mathscr{E}'_{\alpha,\beta}(w,y),(w,y)\right\rangle=0. \tag{49}$$

Since $(w,y)\notin\mathscr{N}^0_{\alpha,\beta}$, the constraint is nondegenerate in (w,y), then by the theory of Lagrange multipliers, there exists $\sigma\in\mathbb{R}$ such that

$$\mathscr{E}_{\alpha,\beta}(w,y)=\sigma L_{\alpha,\beta}(w,y). \tag{50}$$

Thus,

$$\left\langle\mathscr{E}'_{\alpha,\beta}(w,y),(w,y)\right\rangle=\sigma\left\langle L'_{\alpha,\beta}(w,y),(w,y)\right\rangle=\sigma\zeta''_{(w,y)}(1). \tag{51}$$

Since $\zeta''_{(w,y)}(1)\neq0$ and $(w,y)\notin\mathscr{N}^0_{\alpha,\beta}$, we obtain that $\sigma=0$, which completes the proof. □

Lemma 18. *For every $(\alpha,\beta)\in\mathbb{R}^2\setminus\{(0,0)\}$ such that $0<\alpha+\beta<K_1$. The functional $\mathscr{E}_{\alpha,\beta}$ is bounded and coercive on $\mathscr{N}_{\alpha,\beta}$.*

Proof. For any $(w, y) \in \mathcal{N}_{\alpha,\beta}$, according to (23), (24), (29), and Lemma 15, we get

$$\begin{aligned}\mathcal{E}_{\alpha,\beta}(w,y) &= \int_{\mathcal{M}} \frac{1}{p(z)}\left(|Dw(z)|^{p(z)} + |Dy(z)|^{p(z)}\right) dv_g(z) \\ &\quad - \int_{\mathcal{M}} \frac{1}{q(z)}\left(\alpha|w(z)|^{q(z)} + \beta|y(z)|^{q(z)}\right) dv_g(z) \\ &\quad - \int_{\mathcal{M}} \frac{1}{\mathfrak{a}(z)+\mathfrak{b}(z)}\left(Dw(z)\right|^{p(z)} + \left\|Dy(z)\right|^{p(z)}\right) dv_g(z) \\ &\quad + \int_{\mathcal{M}} \frac{1}{\mathfrak{a}(z)+\mathfrak{b}(z)}\left(\alpha|w(z)|^{q(z)} + \beta|y(z)|^{q(z)}\right) dv_g(z) \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\mathfrak{a}^- + \mathfrak{b}^-}\right)\int_{\mathcal{M}}\left(Dw(z)|^{p(z)} + |Dy(z)|^{p(z)}\right) dv_g(z) \\ &\quad + \left(\frac{1}{\mathfrak{a}^+ + \mathfrak{b}^+} - \frac{1}{q^-}\right)\int_{\mathcal{M}}\left(\alpha|w(z)|^{q(z)} + \beta|y(z)|^{q(z)}\right) dv_g(z) \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\mathfrak{a}^- + \mathfrak{b}^-}\right)\|(w,y)\|^{p^-} \\ &\quad + \left(\frac{1}{\mathfrak{a}^+ + \mathfrak{b}^+} - \frac{1}{q^-}\right) c_1(\alpha+\beta)\|(w,y)\|^{q^+}.\end{aligned} \tag{52}$$

As $p^- > q^+$, then $\mathcal{E}_{\alpha,\beta}(w, y) \longrightarrow \infty$ as $\|(w, y)\| \longrightarrow \infty$. It follows that $\mathcal{E}_{\alpha,\beta}$ is coercive and bounded below on $\mathcal{N}_{\alpha,\beta}$ for $0 < \alpha + \beta < K_1$. □

Lemma 19.

(i) If $(w, y) \in \mathcal{N}^+_{\alpha,\beta}$, then $\int_{\mathcal{M}}(\alpha|w(z)|^{q(z)} + \beta|y(z)|^{q(z)})d v_g(z) > 0$.

(ii) If $(w, y) \in \mathcal{N}^-_{\alpha,\beta}$, then $\int_{\mathcal{M}}|w(z)|^{\mathfrak{a}(z)}|y(z)|^{\mathfrak{b}(z)}\, dv_g(z) > 0$.

Proof. (i) Since $(w, y) \in \mathcal{N}^+_{\alpha,\beta}$, we have $\zeta''_{(w,y)}(1) > 0$. Then, using (23) and (34), we get

$$\begin{aligned}0 < \zeta''_{(w,y)}(1) &< (p^+ - (\mathfrak{a}^- + \mathfrak{b}^-))\int_{\mathcal{M}}\left(|Dw(z)|^{p(z)}\right. \\ &\quad \left.+ |Dy(z)|^{p(z)}\right) dv_g(z) + ((\mathfrak{a}^+ + \mathfrak{b}^+) - q^-)\int_{\mathcal{M}}\left(\alpha|w(z)|^{q(z)}\right. \\ &\quad \left.+ \beta|y(z)|^{q(z)}\right) dv_g(z),\end{aligned} \tag{53}$$

then,

$$\begin{aligned}((\mathfrak{a}^- + \mathfrak{b}^-) - p^+)\|(w,y)\|^{p^+} &< ((\mathfrak{a}^+ + \mathfrak{b}^+) - q^-) \\ &\quad \cdot \int_{\mathcal{M}}\left(\alpha|w|^{q(z)} + \beta|y|^{q(z)}\right) dv_g(z).\end{aligned} \tag{54}$$

Hence,

$$\int_{\mathcal{M}}\left(\alpha|w(z)|^{q(z)} + \beta|y(z)|^{q(z)}\right) dv_g(z) > \frac{(\mathfrak{a}^- + \mathfrak{b}^-) - p^+}{(\mathfrak{a}^+ + \mathfrak{b}^+) - q^-}\|(w,y)\|^{p^+} > 0. \tag{55}$$

(ii) Since $(w, y) \in \mathcal{N}^-_{\alpha,\beta}$, we have $\zeta''_{(w,y)}(1) < 0$. Thus, according to (23) and (34), we obtain that

$$\begin{aligned}0 > \zeta''_{(w,y)}(1) &\geq (p^- - q^+)\int_{\mathcal{M}}\left(|Dw(z)|^{p(z)} + |Dy(z)|^{p(z)}\right) dv_g(z) \\ &\quad - 2((\mathfrak{a}^+ + \mathfrak{b}^+) - q^-)\int_{\mathcal{M}}|w(z)|^{\mathfrak{a}(z)}|y(z)|^{\mathfrak{b}(z)} dv_g(z),\end{aligned} \tag{56}$$

then,

$$\begin{aligned}2((\mathfrak{a}^+ + \mathfrak{b}^+) - q^-)\int_{\mathcal{M}}|w|^{\mathfrak{a}(z)}|y|^{\mathfrak{b}(z)} dv_g(z) &> (p^- - q^+) \\ &\quad \cdot \int_{\mathcal{M}}\left(|Dw|^{p(z)} + |Dy|^{p(z)}\right) dv_g(z).\end{aligned} \tag{57}$$

Hence,

$$\int_{\mathcal{M}}|w(z)|^{\mathfrak{a}(z)}|y(z)|^{\mathfrak{b}(z)}\, dv_g(z) > \frac{p^- - q^+}{2(\mathfrak{a}^+ + \mathfrak{b}^+ - q^-)}\|(w,y)\|^{p^-} > 0. \tag{58}$$

□

Remark 20. As a consequence of Lemmas 16–18, we have for every $(\alpha, \beta) \in (\mathbb{R}^+)^2$ with $\alpha + \beta < K_1$, $\mathcal{N}_{\alpha,\beta} = \mathcal{N}^-_{\alpha,\beta} \cup \mathcal{N}^+_{\alpha,\beta}$, and $\mathcal{E}_{\alpha,\beta}$ is coercive and bounded below on $\mathcal{N}^-_{\alpha,\beta}$ and $\mathcal{N}^+_{\alpha,\beta}$. We define

$$\begin{aligned}\sigma_{\alpha,\beta} &= \inf_{(w,y)\in\mathcal{N}_{\alpha,\beta}} \mathcal{E}_{\alpha,\beta}(w,y), \sigma^+_{\alpha,\beta} = \inf_{(w,y)\in\mathcal{N}^+_{\alpha,\beta}} \mathcal{E}_{\alpha,\beta}(w,y), \text{and}\, \sigma^-_{\alpha,\beta} \\ &= \inf_{(w,y)\in\mathcal{N}^-_{\alpha,\beta}} \mathcal{E}_{\alpha,\beta}(w,y).\end{aligned} \tag{59}$$

Lemma 21. *The following facts hold:*

(i) If $\alpha + \beta < K_1$, then $\sigma_{\alpha,\beta} \leq \sigma^+_{\alpha,\beta} < 0$

(ii) If $\alpha + \beta < K_2$, then we have $\sigma^-_{\alpha,\beta} > c_0$ for some $c_0(\mathfrak{a}, \mathfrak{b}, p, q, \alpha, \beta) > 0$

Proof. (i) Let $(w, y) \in \mathcal{N}^+_{\alpha,\beta}$; by (34), we have

$$\begin{aligned}0 &< (p^+ - q^-)\int_{\mathcal{M}}\left(|Dw|^{p(z)} + |Dy|^{p(z)}\right) dv_g(z) \\ &\quad - 2((\mathfrak{a}^- + \mathfrak{b}^-) - q^+)\int_{\mathcal{M}}|w|^{\mathfrak{a}(z)}|y|^{\mathfrak{b}(z)} dv_g(z),\end{aligned} \tag{60}$$

then,

$$\int_{\mathscr{M}}|w|^{\mathfrak{a}(z)}|y|^{\mathfrak{b}(z)}dv_{\mathrm{g}}(z)<\frac{p^{+}-q^{-}}{2((\mathfrak{a}^{-}+\mathfrak{b}^{-})-q^{+})}\cdot\int_{\mathscr{M}}\left(|Dw|^{p(z)}+|Dy|^{p(z)}\right)dv_{\mathrm{g}}(z). \tag{61}$$

Hence, by (61) and (29), we have

$$\begin{aligned}
\mathscr{E}_{\alpha,\beta}(w,y) &\leq \frac{1}{p^{-}}\int_{\mathscr{M}}\left(|Dw(z)|^{p(z)}+|Dy(z)|^{p(z)}\right)dv_{\mathrm{g}}(z)\\
&\quad-\frac{1}{q^{+}}\int_{\mathscr{M}}\left(\alpha|w(z)|^{q(z)}+\beta|y(z)|^{q(z)}\right)dv_{\mathrm{g}}(z)\\
&\quad-\frac{2}{\mathfrak{a}^{+}+\mathfrak{b}^{+}}\int_{\mathscr{M}}|w(z)|^{\mathfrak{a}(z)}|y(z)|^{\mathfrak{b}(z)}dv_{\mathrm{g}}(z)\\
&=\left(\frac{1}{p^{-}}-\frac{1}{q^{+}}\right)\int_{\mathscr{M}}\left(|Dw(z)|^{p(z)}+|Dy(z)|^{p(z)}\right)dv_{\mathrm{g}}(z)\\
&\quad+2\left(\frac{1}{q^{+}}-\frac{1}{\mathfrak{a}^{+}+\mathfrak{b}^{+}}\right)\int_{\mathscr{M}}|w(z)|^{\mathfrak{a}(z)}|y(z)|^{\mathfrak{a}(z)}dv_{\mathrm{g}}(z)\\
&\leq\left(\frac{1}{p^{-}}-\frac{1}{q^{+}}\right)\int_{\mathscr{M}}\left(|Dw(z)|^{p(z)}+|Dy(z)|^{p(z)}\right)dv_{\mathrm{g}}(z)\\
&\quad+2\left(\frac{1}{q^{+}}-\frac{1}{\mathfrak{a}^{+}+\mathfrak{b}^{+}}\right)\cdot\left(\frac{p^{+}-q^{-}}{2((\mathfrak{a}^{-}+\mathfrak{b}^{-})-q^{+})}\right)\\
&\quad\times\int_{\mathscr{M}}\left(|Dw(z)|^{p(z)}+|Dy(z)|^{p(z)}\right)dv_{\mathrm{g}}(z)\\
&=\left[\left(\frac{1}{p^{-}}-\frac{1}{q^{+}}\right)+\left(\frac{1}{q^{+}}-\frac{1}{\mathfrak{a}^{+}+\mathfrak{b}^{+}}\right)\cdot\left(\frac{p^{+}-q^{-}}{(\mathfrak{a}^{-}+\mathfrak{b}^{-})-q^{+}}\right)\right]\\
&\quad\times\int_{\mathscr{M}}\left(|Dw(z)|^{p(z)}+|Dy(z)|^{p(z)}\right)dv_{\mathrm{g}}(z)\\
&=(q^{+}-p^{-})(\mathfrak{a}^{+}+\mathfrak{b}^{+})+p^{-}((\mathfrak{a}^{+}+\mathfrak{b}^{+})-q^{+})\\
&\quad\times\left(\frac{p^{+}-q^{-}}{(\mathfrak{a}^{-}+\mathfrak{b}^{-})-q^{+}}\right)\times\left(p^{-}q^{+}(\mathfrak{a}^{+}+\mathfrak{b}^{+})\right)^{-1}\|(w,y)\|^{p^{+}}.
\end{aligned} \tag{62}$$

According to (24) and (23), we get $\mathscr{E}_{\alpha,\beta}(w,y)<0$. Therefore, from the definition of $\sigma_{\alpha,\beta}$ and $\sigma^{+}_{\alpha,\beta}$, it follows that$\sigma_{\alpha,\beta}\leq\sigma^{+}_{\alpha,\beta}<0$.(73)(ii) Let $(w,y)\in\mathscr{N}^{-}_{\alpha,\beta}$; by (47), we have

$$\frac{p^{-}-q^{+}}{2((\mathfrak{a}^{+}+\mathfrak{b}^{+})-q^{-})}\|(w,y)\|^{p^{-}}<\int_{\mathscr{M}}|w(z)|^{\mathfrak{a}(z)}|y(z)|^{\mathfrak{b}(z)}dv_{\mathrm{g}}(z), \tag{63}$$

and by Lemma 15 (ii), we get

$$\frac{p^{-}-q^{+}}{2((\mathfrak{a}^{+}+\mathfrak{b}^{+})-q^{-})}\|(w,y)\|^{p^{-}}<c_{2}\|(w,y)\|^{\mathfrak{a}^{+}+\mathfrak{b}^{+}}, \tag{64}$$

then,

$$\|(w,y)\|^{(\mathfrak{a}^{+}+\mathfrak{b}^{+})-p^{-}}>\frac{p^{-}-q^{+}}{2c_{2}((\mathfrak{a}^{+}+\mathfrak{b}^{+})-q^{-})}. \tag{65}$$

Hence,

$$\|(w,y)\|>\left(\frac{p^{-}-q^{+}}{2c_{2}((\mathfrak{a}^{+}+\mathfrak{b}^{+})-q^{-})}\right)^{1/((\mathfrak{a}^{+}+\mathfrak{b}^{+})-p^{-})}. \tag{66}$$

According to (29), (66), and Lemma 15 (i), we deduce that

$$\begin{aligned}
\mathscr{E}_{\alpha,\beta}(w,y) &\geq \frac{1}{p^{+}}\int_{\mathscr{M}}\left(|Dw(z)|^{p(z)}+|Dy(z)|^{p(z)}\right)dv_{\mathrm{g}}(z)\\
&\quad-\frac{1}{q^{-}}\int_{\mathscr{M}}\left(\alpha|w(z)|^{q(z)}+\beta|y(z)|^{q(z)}\right)dv_{\mathrm{g}}(z)\\
&\quad-\frac{2}{\mathfrak{a}^{-}+\mathfrak{b}^{-}}\int_{\mathscr{M}}|w(z)|^{\mathfrak{a}(z)}|y(z)|^{\mathfrak{b}(z)}dv_{\mathrm{g}}(z)\\
&=\left(\frac{1}{p^{+}}-\frac{1}{\mathfrak{a}^{-}+\mathfrak{b}^{-}}\right)\int_{\mathscr{M}}\left(|Dw(z)|^{p(z)}+|Dy(z)|^{p(z)}\right)dv_{\mathrm{g}}(z)\\
&\quad+\left(\frac{1}{\mathfrak{a}^{-}+\mathfrak{b}^{-}}-\frac{1}{q^{-}}\right)\int_{\mathscr{M}}\left(\alpha|w(z)|^{q(z)}+\beta|y(z)|^{q(z)}\right)dv_{\mathrm{g}}(z)\\
&\geq\left(\frac{1}{p^{+}}-\frac{1}{\mathfrak{a}^{-}+\mathfrak{b}^{-}}\right)\|(w,y)\|^{p^{+}}+c_{1}(\alpha+\beta)\\
&\quad\cdot\left(\frac{1}{\mathfrak{a}^{-}+\mathfrak{b}^{+}}-\frac{1}{q^{-}}\right)\|(w,y)\|^{q^{-}}\geq\left(\frac{1}{p^{+}}-\frac{1}{\mathfrak{a}^{-}+\mathfrak{b}^{-}}\right)\\
&\quad\cdot\left(\frac{p^{-}-q^{+}}{2c_{2}((\mathfrak{a}^{+}+\mathfrak{b}^{+})-q^{-})}\right)^{\frac{p^{+}}{(\mathfrak{a}^{+}+\mathfrak{b}^{+})-p^{-}}}\\
&\quad+\left(\frac{1}{\mathfrak{a}^{-}+\mathfrak{b}^{+}}-\frac{1}{q^{-}}\right)c_{1}(\alpha+\beta)\\
&\quad\cdot\left(\frac{p^{-}-q^{+}}{2c_{2}((\mathfrak{a}^{+}+\mathfrak{b}^{+})-q^{-})}\right)^{\frac{q^{-}}{(\mathfrak{a}^{+}+\mathfrak{b}^{+})-p^{-}}}=c_{0}.
\end{aligned} \tag{67}$$

Thus, if we choose

$$\alpha+\beta<\frac{1}{c_{1}}\cdot\frac{q^{-}((\mathfrak{a}^{-}+\mathfrak{b}^{-})-p^{+})}{p^{+}((\mathfrak{a}^{-}+\mathfrak{b}^{-})-q^{-})}\cdot\left(\frac{p^{-}-q^{+}}{2c_{2}((\mathfrak{a}^{+}+\mathfrak{b}^{+})-q^{-})}\right)^{(p^{+}-q^{-})/((\mathfrak{a}^{+}+\mathfrak{b}^{+})-p^{-})}=k_{2}, \tag{68}$$

we deduce that $\mathscr{E}_{\alpha,\beta}(w,y)>c_{0}$ for some positive constant c_{0} depending on $\mathfrak{a},\mathfrak{b},p,q,\alpha$, and β. □

Lemma 22. *For each $(w,y)\in J\setminus\{(0,0)\}$, there exists a constant $K_{3}>0$ such that for all $\alpha+\beta<K_{3}$, we have the following:*

(i) *If $\int_{\mathscr{M}}(\alpha|w(z)|^{q(z)}+\beta|y(z)|^{q(z)})dv_{g}(z)=0$, then there exists a unique $(s^{-}w,s^{-}y)>0$ such that $(s^{-}w,s^{-}y)\in\mathscr{N}^{-}_{\alpha,\beta}$ and $\mathscr{E}_{\alpha,\beta}(s^{-}w,s^{-}y)=\sup_{s>0}\mathscr{E}_{\alpha,\beta}(sw,sy)$.*

(ii) *If $\int_{\mathscr{M}}(\alpha|w(z)|^{q(z)}+\beta|y(z)|^{q(z)})dv_{g}(z)>0$, then there exist $s_{\max}>0$ and unique numbers $0<s^{+}<s_{\max}<s^{-}$, such as $(s^{+}w,s^{+}y)\in\mathscr{N}^{+}_{\alpha,\beta}$, $(s^{-}w,s^{-}y)\in\mathscr{N}^{-}_{\alpha,\beta}$ and*

$$\mathscr{E}_{\alpha,\beta}(s^{+}w,s^{+}y)=\inf_{0<s<s_{\max}}\mathscr{E}_{\alpha,\beta}(sw,sy)\,;\,\mathscr{E}_{\alpha,\beta}(s^{-}w,s^{-}y)=\sup_{s\geq 0}\mathscr{E}_{\alpha,\beta}(sw,sy). \tag{69}$$

Proof. Before tackling our proof, we define $s_{\max}$ as follows:

$$s_{\max} = \frac{\int_{\mathscr{M}}(p(z)-q(z))\left(|Dw(z)|^{p(z)}+|Dy(z)|^{p(z)}\right)dv_g(z)}{2\int_{\mathscr{M}}((\mathfrak{a}(z)+\mathfrak{b}(z))-q(z))|w(z)|^{\mathfrak{a}(z)}|y(z)|^{\mathfrak{b}(z)}\,dv_g(z)}, \tag{70}$$

for every $(w,y)\in\mathscr{N}^-_{\alpha,\beta}$.

Hence, we have that $\zeta_{(w,y)}(s)$ is increasing for $s\in[0,s_{\max}]$ and decreasing for $s\in(s_{\max},+\infty)$ and achieves its maximum. We set $\mathscr{R}(s)=\int_{\mathscr{M}}s^{p(z)-q(z)}\left(|Dw(z)|^{p(z)}+|Dy(z)|^{p(z)}\right)dv_g(z) - 2\int_{\mathscr{M}}s^{(\mathfrak{a}(z)+\mathfrak{b}(z))-q(z)}|w(z)|^{\mathfrak{a}(z)}|y(z)|^{\mathfrak{b}(z)}\,dv_g(z)$; by Lemma 19, we have that

$$\mathscr{R}(0)=0 \text{ and } \mathscr{R}(s)\longrightarrow -\infty \text{ when } s\longrightarrow\infty, \tag{71}$$

with

$$\begin{aligned}\mathscr{R}'(s)=&\int_{\mathscr{M}}(p(z)-q(z))s^{p(z)-q(z)-1}\\ &\cdot\left(|Dw(z)|^{p(z)}+|Dy(z)|^{p(z)}\right)dv_g(z)\\ &-2\int_{\mathscr{M}}((\mathfrak{a}(z)+\mathfrak{b}(z))\\ &-q(z))s^{\mathfrak{a}(z)+\mathfrak{b}(z)-q(z)-1}|w(z)|^{\mathfrak{a}(z)}|y(z)|^{\mathfrak{b}(z)}\,dv_g(z).\end{aligned} \tag{72}$$

(i) For $0<s<1$ which is sufficiently small, we have

$$\begin{aligned}\zeta_{(w,y)}(s) >& \frac{s^{p^+}}{p^+}\int_{\mathscr{M}}\left(|Dw(z)|^{p(z)}+|Dy(z)|^{p(z)}\right)dv_g(z)\\ &-\frac{2t^{\mathfrak{a}^++\mathfrak{b}^+}}{\mathfrak{a}^++\mathfrak{b}^+}\int_{\mathscr{M}}|w(z)|^{\mathfrak{a}(z)}|y(z)|^{\mathfrak{b}(z)}\,dv_g(z)>0,\end{aligned} \tag{73}$$

and for $s>1$ which is sufficiently large, we get

$$\begin{aligned}\zeta_{(w,y)}(s) <& \frac{t^{p^+}}{p^+}\int_{\mathscr{M}}\left(|Dw(z)|^{p(z)}+|Dy(z)|^{p(z)}\right)dv_g(z)\\ &-\frac{2s^{\mathfrak{a}^++\mathfrak{b}^+}}{\mathfrak{a}^++\mathfrak{b}^+}\int_{\mathscr{M}}|w(z)|^{\mathfrak{a}(z)}|y(z)|^{\mathfrak{b}(z)}\,dv_g(z)<0.\end{aligned} \tag{74}$$

Since $\zeta_{(w,y)}(s)$ achieves its maximum, then by Lemma 14, $\zeta'_{(w,y)}(s)=\langle \mathscr{E}'_{\alpha,\beta}(s^-w,s^-y),(w,y)\rangle=0$. On the other hand, if $\int_{\mathscr{M}}(\alpha|w(z)|^{q(z)}+\beta|y(z)|^{q(z)})dv_g(z)\le 0$, then there is a unique $s^->s_{\max}$ such that $\mathscr{R}(s^-)=\int_{\mathscr{M}}(\alpha|w(z)|^{q(z)}+\beta|y(z)|^{q(z)})dv_g(z)$ and since

$$\begin{aligned}\left\langle \zeta'_{(s^-w,s^-y)},(s^-w,s^-y)\right\rangle =& \int_{\mathscr{M}}(p(z)-q(z))(s^-)^{p(z)}\\ &\cdot\left(|Dw(z)|^{p(z)}+|Dy(z)|^{p(z)}\right)dv_g(z)\\ &-2\int_{\mathscr{M}}((\mathfrak{a}(z)+\mathfrak{b}(z))\\ &-q(z))(s^-)^{\mathfrak{a}(z)+\mathfrak{b}(z)}|w(z)|^{\mathfrak{a}(z)}|y(z)|^{\mathfrak{b}(z)}\,dv_g(z)\\ \le& (s^-)^{1+q^-}\mathscr{R}'(s_{\max}),\end{aligned}$$

$$\begin{aligned}\left\langle \mathscr{E}'_{\alpha,\beta}(s^-w,s^-y),(s^-w,s^-y)\right\rangle \le& (s^-)^{p^--1}\|(w,y)\|^{p^-}-(s^-)^{q^+-1}\int_{\mathscr{M}}\\ &\cdot\left(\alpha|w(z)|^{q(z)}+\beta|y(z)|^{q(z)}\right)dv_g(z)\\ &-2(s^-)^{(\mathfrak{a}^-+\mathfrak{b}^-)-q^+}\int_{\mathscr{M}}|w(z)|^{\mathfrak{a}(z)}|y(z)|^{\mathfrak{b}(z)}\,dv_g(z)\\ \le& (s^-)^{q^+-1}\left[\mathscr{R}(s^-)-\int_{\mathscr{M}}\left(\alpha|w(z)|^{q(z)}\right.\right.\\ &\left.\left.+\beta|y(z)|^{q(z)}\right)dv_g(z)\right]=0.\end{aligned} \tag{75}$$

We obtain that $(s^-w,s^-y)\in\mathscr{N}^-_{\alpha,\beta}$.

For $s>1$, we get by (74) and (34) that

$$\begin{aligned}\left\langle \mathscr{E}'_{\alpha,\beta}(ss^-w,ss^-y),(s^-w,s^-y)\right\rangle \le& s^{p^--1}(s^-)^{p^--1}\|(w,y)\|^{p^-}\\ &-4s^{(\mathfrak{a}^++\mathfrak{b}^+)-1}(s^-)^{\mathfrak{a}^++\mathfrak{b}^+}\\ &\cdot\int_{\mathscr{M}}|w(z)|^{\mathfrak{a}(z)}|y(z)|^{\mathfrak{b}(z)}dv_g(z)<0,\end{aligned} \tag{76}$$

and for $0<s<1$, we deduce again by (73) and (34) that

$$\begin{aligned}\left\langle \mathscr{E}'_{\alpha,\beta}(ss^-w,ss^-y),(s^-w,s^-y)\right\rangle \ge& s^{p^+-1}(s^-)^{p^+-1}\|(w,y)\|^{p^+}\\ &-4s^{(\mathfrak{a}^-+\mathfrak{b}^-)-1}(s^-)^{\mathfrak{a}^-+\mathfrak{b}^-}\\ &\cdot\int_{\mathscr{M}}|w(z)|^{\mathfrak{a}(z)}|y(z)|^{\mathfrak{b}(z)}\,dv_g(z)>0.\end{aligned} \tag{77}$$

Thus, s^- is unique, which achieves the proof.

(ii) If $\int_{\mathscr{M}}(\alpha|w(z)|^{q(z)}+\beta|y(z)|^{q(z)})\,dv_g(z)>0$, we have

$$\begin{aligned}0<&\int_{\mathscr{M}}\left(\alpha|w(z)|^{q(z)}+\beta|y(z)|^{q(z)}\right)dv_g(z)\\ &\le c_1(\alpha,\beta)\|(w,y)\|^{q^-}\le\mathscr{R}(s_{\max})\quad\text{for }\alpha+\beta<K_3.\end{aligned} \tag{78}$$

Therefore, there are unique s^+ and s^- such that $0<s^+<s_{\max}<s^-$,

$$\begin{aligned}\mathscr{R}(s^+)=\int_{\mathscr{M}}\left(\alpha|w(z)|^{q(z)}+\beta|y(z)|^{q(z)}\right)dv_g(z)=\mathscr{R}(s^-),\\ \mathscr{R}'(s^+)>0>\mathscr{R}'(s^-),\end{aligned} \tag{79}$$

thus, by (i), we have $(s^+w, s^+y) \in \mathcal{N}^+_{\alpha,\beta}$, $(s^-w, s^-y) \in \mathcal{N}^-_{\alpha,\beta}$, $\mathcal{E}_{\alpha,\beta}(s^-w, s^-y) \geq \mathcal{E}_{\alpha,\beta}(sw, sy) \geq \mathcal{E}_{\alpha,\beta}(s^+w, s^+y)$for each $s \in [s^+, s^-]$ and $\mathcal{E}_{\alpha,\beta}(s^+w, s^+y) \leq \mathcal{E}_{\alpha,\beta}(sw, sy)$for each $s \in [0, s^+]$. Hence,

$$\begin{aligned}\mathcal{E}_{\alpha,\beta}(s^+w, s^+y) &= \inf_{0<s<s_{\max}} \mathcal{E}_{\alpha,\beta}(sw, sy)\,;\, \mathcal{E}_{\alpha,\beta}(s^-w, s^-y)\\ &= \sup_{s\geq 0} \mathcal{E}_{\alpha,\beta}(sw, sy).\end{aligned} \tag{80}$$

□

3.2. Existence of Nonnegative Solutions. This section is devoted to proving the existence of minimizers in $\mathcal{N}^+_{\alpha,\beta}$, $\mathcal{N}^-_{\alpha,\beta}$, also to show the existence of two nonnegative solutions of system $(\mathcal{S})$.

Lemma 23. *For* $\alpha + \beta < K = \min\{K_1, k_2\}$, *the functional* $\mathcal{E}_{\alpha,\beta}$ *has a minimizer* (w_0, y_0) *in* $\mathcal{N}^+_{\alpha,\beta}$, *which satisfies the following assumptions:*

(i) $\mathcal{E}_{\alpha,\beta}(w_0^+, y_0^+) = \sigma^+_{\alpha,\beta} < 0$

(ii) (w_0^+, y_0^+) *is a solution of* $(\mathcal{S})$

Proof. (i) Thanks to Lemma 18, $\mathcal{E}_{\alpha,\beta}$ is bounded below on $\mathcal{N}_{\alpha,\beta}$, which in particular is bounded below in $\mathcal{N}^+_{\alpha,\beta}$. Then, there exists a minimizing sequence $\{w_n^+, y_n^+\} \subset \mathcal{N}^+_{\alpha,\beta}$ such that

$$\lim_{n\longrightarrow+\infty} \mathcal{E}_{\alpha,\beta}(w_n^+, y_n^+) = \inf_{(w,y)\in\mathcal{N}^+_{\alpha,\beta}} \mathcal{E}_{\alpha,\beta}(w, y) = \sigma^+_{\alpha,\beta} < 0. \tag{81}$$

Since, $\mathcal{E}_{\alpha,\beta}$ is coercive, $\{(w_n^+, y_n^+)\}$ is bounded on J. Hence, we suppose that, without loss generality, $(w_n^+, y_n^+) \longrightarrow (w_0^+, y_0^+)$ on J, and by the compact embedding (Theorem 11), we have

$$\begin{aligned}&w_n^+ \longrightarrow w_0^+ \text{ strongly in } L^{p(z)}(\mathcal{M}) \text{ and } L^{\mathfrak{a}(z)+\mathfrak{b}(z)}(\mathcal{M}) \text{ as } n \longrightarrow +\infty.\\ &y_n^+ \longrightarrow y_0^+ \text{ strongly in } L^{p(z)}(\mathcal{M}) \text{ and } L^{\mathfrak{a}(z)+\mathfrak{b}(z)}(\mathcal{M}) \text{ as } n \longrightarrow +\infty.\\ &w_n^+(z) \longrightarrow w_0^+ \text{ and } y_n^+(z) \longrightarrow y_0^+(z) \text{ a.e.in } \mathcal{M} \text{ as } n \longrightarrow +\infty.\end{aligned} \tag{82}$$

Now, we shall demonstrate that $w_n^+ \longrightarrow w_0^+$ and $y_n^+ \longrightarrow y_0^+$ in $W_0^{1,q(z)}(\mathcal{M})$ as $n \longrightarrow +\infty$. Otherwise, let $w_n^+ \longrightarrow w_0^+$ or $y_n^+ \longrightarrow y_0^+$ in $W_0^{1,q(z)}(\mathcal{M})$ as $n \longrightarrow +\infty$. Then, we have

$$\begin{aligned}\rho_{q(\cdot)}(w_0^+) &< \lim_{n\longrightarrow+\infty} \inf\ \rho_{q(\cdot)}(w_n^+),\\ \rho_{q(\cdot)}(y_0^+) &< \lim_{n\longrightarrow+\infty} \inf\ \rho_{q(\cdot)}(y_n^+),\end{aligned} \tag{83}$$

using (82), we obtain that

$$\begin{aligned}\int_{\mathcal{M}} |w_0^+|^{p(z)}\, dv_g(z) &= \lim_{n\longrightarrow+\infty} \inf \int_{\mathcal{M}} |w_n|^{p(z)}\, dv_g(z),\\ \int_{\mathcal{M}} |y_0^+|^{p(z)} dv_g(z) &= \lim_{n\longrightarrow+\infty} \inf \int_{\mathcal{M}} |y_n|^{p(z)}\, dv_g(z),\end{aligned} \tag{84}$$

since, $\langle \mathcal{E}'_{\alpha,\beta}(w_n^+, y_n^+), (w_n^+, y_n^+)\rangle = 0$, we get

$$\begin{aligned}\mathcal{E}_{\alpha,\beta}(w_n^+, y_n^+) \geq &\left(\frac{1}{p^+} - \frac{1}{\mathfrak{a}^- + \mathfrak{b}^-}\right)\|(w_n^+, y_n^+)\|^{p^+}\\ &+ \left(\frac{1}{\mathfrak{a}^- + \mathfrak{b}^-} - \frac{1}{q^-}\right)\|(w_n^+, y_n^+)\|^{q^-}.\end{aligned} \tag{85}$$

That is,

$$\begin{aligned}\lim_{n\longrightarrow+\infty} \mathcal{E}_{\alpha,\beta}(w_n^+, y_n^+) \geq &\left(\frac{1}{p^+} - \frac{1}{\mathfrak{a}^- + \mathfrak{b}^-}\right)\lim_{n\longrightarrow+\infty}\|(w_n^+, y_n^+)\|^{p^+}\\ &+ \left(\frac{1}{\mathfrak{a}^- + \mathfrak{b}^-} - \frac{1}{q^-}\right)\lim_{n\longrightarrow+\infty}\|(w_n^+, y_n^+)\|^{q^-}.\end{aligned} \tag{86}$$

By (82) and (83), we have

$$\sigma^+_{\alpha,\beta} > \left(\frac{1}{p^+} - \frac{1}{\mathfrak{a}^- + \mathfrak{b}^-}\right)\|(w_0^+, y_0^+)\|^{p^+} + \left(\frac{1}{\mathfrak{a}^- + \mathfrak{b}^-} - \frac{1}{q^-}\right)\|(w_0^+, y_0^+)\|^{q^-}. \tag{87}$$

Since, $p^+ > q^-$ for $\|(w, y)\| > 1$, we deduce that

$$\sigma^+_{\alpha,\beta} = \inf_{(w,y)\in\mathcal{N}^+_{\alpha,\beta}} \mathcal{E}_{\alpha,\beta}(w, y) > 0, \tag{88}$$

which is a contradiction with Lemma 21. Hence,

$$\begin{aligned}&w_n^+ \longrightarrow w_0^+ \text{ in } W_0^{1,q(z)}(\mathcal{M}),\\ &y_n^+ \longrightarrow y_0^+ \text{ in } W_0^{1,q(z)}(\mathcal{M}),\\ &\lim_{n\longrightarrow+\infty} \mathcal{E}_{\alpha,\beta}(w_n^+, y_n^+) = \mathcal{E}_{\alpha,\beta}(w_0^+, y_0^+) = \sigma^+_{\alpha,\beta}.\end{aligned} \tag{89}$$

Consequently, (w_0^+, y_0^+) is a minimizer of $\mathcal{E}_{\alpha,\beta}$ on $\mathcal{N}^+_{\alpha,\beta}$.

(ii) According to Lemma 17, we deduce that (w_0^+, y_0^+) is a solution of $(\mathcal{S})$. □

Lemma 24. *Let* $\{w_n^-\}$ *and* $\{y_n^-\}$ *be any two bounded sequences in* $W_0^{1,q(z)}(\mathcal{M})$. *Then,*

$$\begin{aligned}&\lim_{n\longrightarrow+\infty} \int_{\mathcal{M}} |w_n^-(z)|^{\mathfrak{a}(z)}|y_n^-(x)|^{\mathfrak{b}(z)} dv_g(z)\\ &= \int_{\mathcal{M}} |w_0^-(z)|^{\mathfrak{a}(z)}|y_0^-(z)|^{\mathfrak{b}(z)}\, dv_g(z).\end{aligned} \tag{90}$$

Proof. Similar to the proof ([21], Theorem 5.2), we will omit it. □

Lemma 25. *If $\alpha+\beta<K=\min\{K_1,K_2\}$, then $\mathscr{E}_{\alpha,\beta}$ has a minimizer (w_0^-,y_0^-) in $\mathscr{N}_{\alpha,\beta}^-$ such that*

(i) $\mathscr{E}_{\alpha,\beta}(w_0^-,y_0^-)=\sigma_{\alpha,\beta}^->0$

(ii) (w_0^-,y_0^-) is a solution of $(\mathscr{S})$

Proof. (i) As $\mathscr{E}_{\alpha,\beta}$ is bounded below on $\mathscr{N}_{\alpha,\beta}$ and so on $\mathscr{N}_{\alpha,\beta}^-$. Then, there exists a minimizing sequence $\{w_n^-,y_n^-\}\subset\mathscr{N}_{\alpha,\beta}^-$ such that

$$\lim_{n\longrightarrow+\infty}\mathscr{E}_{\alpha,\beta}(w_n^-,y_n^-)=\inf_{(w,y)\in\mathscr{N}_{\alpha,\beta}^-}\mathscr{E}_{\alpha,\beta}(w,y)=\sigma_{\alpha,\beta}^->0. \quad (91)$$

As $\mathscr{E}_{\alpha,\beta}$ is coercive, $\{w_n^-,y_n^-\}$ is bounded in J, and thus, there exists $(w_0^-,y_0^-)\in J$ such that up to a subsequence $(w_n^-,y_n^-)\subset(w_0^-,y_0^-)$ and according to Theorem 11, we obtain

$$\begin{aligned}&w_n^-\longrightarrow w_0^- \text{ strongly in } L^{p(z)}(\mathscr{M})\text{and } L^{\mathfrak{a}(z)+\mathfrak{b}(z)}(\mathscr{M})\text{as } n\longrightarrow+\infty,\\&y_n^-\longrightarrow y_0^- \text{ strongly in } L^{p(z)}(\mathscr{M})\text{and } L^{\mathfrak{a}(z)+\mathfrak{b}(z)}(\mathscr{M})\text{as } n\longrightarrow+\infty,\\&w_n^-(x)\longrightarrow w_0^- \text{ and } y_n^-(x)\longrightarrow y_0^-(x)\text{ a.e.in } \mathscr{M} \text{ as } n\longrightarrow+\infty.\end{aligned} \quad (92)$$

According to (92) and Lemma 24, we deduce that

$$\begin{aligned}&\int_{\mathscr{M}}\left(\alpha\left|w_n^-\right|^{q(z)}+\beta\left|y_n^-\right|^{q(z)}\right)dv_g(z)\longrightarrow\int_{\mathscr{M}}\left(\alpha\left|w_0^-\right|^{q(z)}+\beta\left|y_0^-\right|^{q(z)}\right)dv_g(z),\\&\int_{\mathscr{M}}|w_n^-(z)|^{\mathfrak{a}(z)}|y_n^-(z)|^{\mathfrak{b}(z)}dv_g(z)\longrightarrow\int_{\mathscr{M}}|w_0^-(z)|^{\mathfrak{a}(z)}|y_0^-(z)|^{\mathfrak{b}(z)}dv_g(z).\end{aligned} \quad (93)$$

On the other hand, if $(w_0^-,y_0^-)\in\mathscr{N}_{\alpha,\beta}^-$, then there exists a constant $s>0$ such that $(sw_0^-,sy_0^-)\in\mathscr{N}_{\alpha,\beta}^-$, and according to (92) and (93), we have

$$\begin{aligned}\lim_{n\longrightarrow+\infty}\mathscr{E}_{\alpha,\beta}(sw_n^-,sy_n^-)&=\lim_{n\longrightarrow+\infty}\inf\Bigg[\int_{\mathscr{M}}\frac{1}{p(z)}\Big(|Dsw_n^-(z)|^{p(z)}\\&\quad+|Dsy_n^-(z)|^{p(z)}\Big)dv_g(z)-\int_{\mathscr{M}}\frac{1}{q(z)}\\&\quad\cdot\left(\alpha|sw_n^-(z)|^{q(z)}+\beta|sy_n^-(z)|^{q(z)}\right)dv_g(z)-\\&\quad\cdot\int_{\mathscr{M}}\frac{2}{\mathfrak{a}(z)+\mathfrak{b}(z)}|sw_n^-(z)|^{\mathfrak{a}(z)}|sy_n^-(z)|^{\mathfrak{b}(z)}dv_g(z)\Bigg]\\&\geq\lim_{n\longrightarrow+\infty}\inf\int_{\mathscr{M}}\frac{1}{p(z)}\left(|Dsw_n^-(z)|^{p(z)}+|Dsy_n^-(z)|^{p(z)}\right)dv_g(z))\\&\quad-\lim_{n\longrightarrow+\infty}\inf\int_{\mathscr{M}}\frac{1}{q(z)}\Big(\alpha|sw_n^-(z)|^{q(z)}\\&\quad+\beta|sy_n^-(z)|^{q(z)})dv_g(z)-\lim_{n\longrightarrow+\infty}\inf\\&\quad\cdot\int_{\mathscr{M}}\frac{2}{\mathfrak{a}(z)+\mathfrak{b}(z)}|sw_n^-(z)|^{\mathfrak{a}(z)}|sy_n^-(z)|^{\mathfrak{b}(z)}dv_g(z)>\\&\quad\cdot\int_{\mathscr{M}}\frac{1}{p(z)}\left(|Dsw_0^-(z)|^{p(z)}+|Dsy_0^-(z)\right)dv_g(z)\\&\quad-\int_{\mathscr{M}}\frac{1}{q(z)}\left(\alpha|sw_0^-(z)|^{q(z)}+\beta|sy_0^-(z)|^{q(z)}\right)dv_g(z)\\&\quad-\int_{\mathscr{M}}\frac{2}{\mathfrak{a}(z)+\mathfrak{b}(z)}|sw_0^-(z)|^{\mathfrak{a}(z)}|sy_0^-(z)|^{\mathfrak{b}(z)}dv_g(z)\\&=\mathscr{E}_{\alpha,\beta}(sw_0^-,sy_0^-).\end{aligned} \quad (94)$$

Considering (32) and (94), we get

$$\begin{aligned}\lim_{n\longrightarrow+\infty}\zeta'_{(w_n^-,y_n^-)}(s)&=\lim_{n\longrightarrow+\infty}\inf\Bigg[\int_{\mathscr{M}}s^{p(z)-1}\left(Dw_n^-(z)|^{p(z)}+|Dy_n^-(z)|^{p(z)}\right)dv_g(z)\\&\quad-\int_{\mathscr{M}}s^{q(z)-1}\left(\alpha|w_n^-(z)|^{q(z)}+\beta|y_n^-(z)|^{q(z)}\right)dv_g(z)\\&\quad-2\int_{\mathscr{M}}s^{(\mathfrak{a}(z)+\mathfrak{b}(z))-1}|w_n^-(z)|^{\mathfrak{a}(z)}|y_n^-(z)|^{\mathfrak{b}(z)}dv_g(z)\Bigg]\\&\geq s^{p^--1}\lim_{n\longrightarrow+\infty}\inf\int_{\mathscr{M}}\left(|Dw_n^-(z)|^{p(z)}+|Dy_n^-(z)|^{p(z)}\right)dv_g(z)\\&\quad-s^{q^+-1}\lim_{n\longrightarrow+\infty}\inf\int_{\mathscr{M}}\left(\alpha|w_n^-(z)|^{q(z)}+\beta|y_n^-(z)|^{q(z)}\right)dv_g(z)\\&\quad-2s^{(\mathfrak{a}^++\mathfrak{b}^+)-1}\lim_{n\longrightarrow+\infty}\inf\int_{\mathscr{M}}|w_n^-(z)|^{\mathfrak{a}(z)}|y_n^-(z)|^{\mathfrak{b}(z)}dv_g(z)>s^{p^--1}\\&\quad\cdot\int_{\mathscr{M}}\left(|Dw_0^-(z)|^{p(z)}+|Dy_0^-(z)|^{p(z)}\right)dv_g(z)-s^{q^+-1}\\&\quad\cdot\int_{\mathscr{M}}\left(\alpha|w_0^-(z)|^{q(z)}+\beta|y_0^-(z)|^{q(z)}\right)dv_g(z)\\&\quad-2s^{(\mathfrak{a}^++\mathfrak{b}^+)-1}\int_{\mathscr{M}}|w_0^-(z)|^{\mathfrak{a}(z)}|y_0^-(z)|^{\mathfrak{b}(z)}dv_g(z)\\&=\zeta'_{(w_0^-,y_0^-)}(s)=0.\end{aligned} \quad (95)$$

For n large enough, $\zeta'_{(w_n^-,y_n^-)}(s)>0$. Since $(w_n^-,y_n^-)\in\mathscr{N}_{\alpha,\beta}^-$ for all $n\in\mathbb{N}$, we have $\zeta'_{(w_n^-,y_n^-)}(1)=0$ and $\zeta'_{(w_n^-,y_n^-)}(s)<0$ for every $n\in\mathbb{N}$. By Lemma 22, we get $\zeta'_{(w_n^-,y_n^-)}(s)<0$ for $s>0$, then from (95), we must have $s<1$. Since $(w_0^-,y_0^-)\in\mathscr{N}_{\alpha,\beta}^-$ and by Lemma 22, we conclude that 1 is the global maximum for $\zeta_{(w_n^-,y_n^-)}(s)$. Therefore, from Lemma 23, it follows that

$$\mathscr{E}_{\alpha,\beta}(w_0^-,y_0^-)<\lim_{n\longrightarrow+\infty}\mathscr{E}_{\alpha,\beta}(w_n^-,y_n^-)\leq\inf_{(w,y)\in\mathscr{N}_{\alpha,\beta}^-}\mathscr{E}_{\alpha,\beta}(w,y). \quad (96)$$

It contradicts that $(w_0^-,y_0^-)\in\mathscr{N}_{\alpha,\beta}^-$. Hence, $(w_n^-,y_n^-)\longrightarrow(w_0^-,y_0^-)$ strongly in J as $n\longrightarrow+\infty$ and $(w_0,y_0)\in\mathscr{N}_{\alpha,\beta}$. Using the fact that

$$\mathscr{E}_{\alpha,\beta}(w_0^-,y_0^-)=\inf_{(w,y)\in\mathscr{N}_{\alpha,\beta}}\mathscr{E}_{\alpha,\beta}(w,y)>0, \quad (97)$$

and Lemma 15, we conclude that $(w_0^-,y_0^-)\in\mathscr{N}_{\alpha,\beta}^-$.

(ii) From Lemma 17, (w_0^-,y_0^-) is a solution of $(\mathscr{S})$. □

Proof of Theorem 1. From Lemma 23 and Lemma 25, there are $(w^+,y^+)\in\mathscr{N}_{\alpha,\beta}^+$ and $(w^-,y^-)\in\mathscr{N}_{\alpha,\beta}^-$ such that

$$\begin{aligned}\mathscr{E}_{\alpha,\beta}(w^+,y^+)&=\inf_{(w,y)\in\mathscr{N}_{\alpha,\beta}^+}\mathscr{E}_{\alpha,\beta}(w,y)\text{ and }\mathscr{E}_{\alpha,\beta}(w^-,y^-)\\&=\inf_{(w,y)\in\mathscr{N}_{\alpha,\beta}^-}\mathscr{E}_{\alpha,\beta}(w,y).\end{aligned} \quad (98)$$

Moreover, $\mathscr{E}_{\alpha,\beta}(w^\pm,y^\pm)=\mathscr{E}_{\alpha,\beta}(|w^\pm|,|y^\pm|)$; hence, we can assume $w^\pm\geq0$, $y^\pm\geq0$. From Lemma 17, $(w^\pm,y^\pm)$ are two critical points of $\mathscr{E}_{\alpha,\beta}$and, thus, are nonnegative nontrivial solutions of system $(\mathscr{S})$. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors declare that their contributions are equal.

Acknowledgments

This paper has been supported by PRIN 2017 n.2017AYM8XW 004.

References

[1] Q. Zhang, "Existence of positive solutions for a class of p (x)-Laplacian systems," *Journal of Mathematical Analysis and Applications*, vol. 333, pp. 1591–1603, 2007.

[2] S. Boulaaras, R. Guefaifia, and S. Kabli, "An asymptotic behavior of positive solutions for a new class of elliptic systems involving of $(p(x), q(x))$-Laplacian systems," *Boletn de la Sociedad Matemática Mexicana*, vol. 25, no. 1, pp. 145–162, 2019.

[3] A. Aberqi, J. Bennouna, and M. Hammoumi, "Existence result for nonlinear degenerated parabolic systems," *Nonlinear Dynamics and Systems Theory*, vol. 17, pp. 217–229, 2017.

[4] G. Marino and P. Winkert, "Existence and uniqueness of elliptic systems with double phase operators and convection terms," *Journal of Mathematical Analysis and Applications*, vol. 492, no. 1, article 124423, 2020.

[5] K. Saoudi, "A singular elliptic system involving the p(x)-Laplacian and generalized Lebesgue–Sobolev spaces," *International Journal of Mathematics*, vol. 30, no. 12, article 1950064, 2019.

[6] N. S. Papageorgiou, D. D. Repovš, and C. Vetro, "Positive solutions for singular double phase problems," *Journal of Mathematical Analysis and Applications*, vol. 501, article 123896, 2020.

[7] O. Benslimane, A. Aberqi, and J. Bennouna, "Existence and uniqueness of weak solution of p(x)-Laplacian in Sobolev spaces with variable exponents in complete manifolds," *Filomat*, vol. 35, pp. 1453–1463, 2021.

[8] O. Benslimane, A. Aberqi, and J. Bennouna, "Existence results for double phase obstacle problems with variable exponents," *Journal of Elliptic and Parabolic Equations*, vol. 7, no. 2, pp. 875–890, 2021.

[9] D. V. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue Spaces: Foundations and Harmonic Analysis*, Springer Science & Business Media, 2013.

[10] B. El Hamdaoui, J. Bennouna, and A. Aberqi, "Renormalized solutions for nonlinear parabolic systems in the Lebesgue Sobolev spaces with variable exponents," *Journal of Mathematical Physics, Analysis, Geometry.*, vol. 14, pp. 27–53, 2018.

[11] X. Fan and D. Zhao, "On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$," *Journal of Mathematical Analysis and Applications*, vol. 263, pp. 424–446, 2001.

[12] J. Giacomoni, S. Tiwari, and G. Warnault, "Quasilinear parabolic problem with p(x)-Laplacian: existence, uniqueness of weak solutions and stabilization," *Nonlinear Differential Equations and Applications NoDEA*, vol. 23, no. 3, p. 24, 2016.

[13] K. Adriouch and A. El Hamidi, "The Nehari manifold for systems of nonlinear elliptic equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 10, pp. 2149–2167, 2006.

[14] C. Y. Chen and T. F. Wu, "The Nehari manifold for indefinite semilinear elliptic systems involving critical exponent," *Applied Mathematics and Computation*, vol. 217, pp. 10817–10828, 2012.

[15] C. Mercuri and M. Willem, "A global compactness result for the p-Laplacian involving critical nonlinearities," *Discrete & Continuous Dynamical Systems*, vol. 28, no. 2, pp. 469–493, 2010.

[16] K. J. Brown, T. F. Wu, and T. F. Wu, "A semilinear elliptic system involving nonlinear boundary condition and sign-changing weight function," *Journal of Mathematical Analysis and Applications*, vol. 337, no. 2, pp. 1326–1336, 2008.

[17] P. Drábek and S. I. Pohozaev, "Positive solutions for the p-Laplacian: application of the fibrering method," *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, vol. 127, no. 4, pp. 703–726, 1997.

[18] A. Farina, C. Mercuri, and M. Willem, "A Liouville theorem for the p-Laplacian and related questions," *Calculus of Variations and Partial Differential Equations*, vol. 58, pp. 1–13, 2019.

[19] W. Chen and M. Squassina, "Critical nonlocal systems with concave-convex powers," *Advanced Nonlinear Studies*, vol. 16, no. 4, pp. 821–842, 2016.

[20] K. M. Pawan and K. Sreenadh, "Fractional p-Kirchhoff system with sign changing nonlinearities," *Revista de la Real Academia de Ciencias Exactas, Fsicas y Naturales. Serie A. Matemáticas*, vol. 111, pp. 281–296, 2017.

[21] R. Biswas and S. Tiwari, "Nehari manifold approach for fractional p (.)-Laplacian system involving concave-convex nonlinearities," *Electronic Journal of Differential Equations*, vol. 2020, 2020.

[22] A. Aberqi, O. Benslimane, A. Ouaziz, and D. D. Repovš, "On a new fractional Sobolev space with variable exponent on complete manifolds," *Boundary Value Problems*, vol. 2022, no. 1, 2022.

[23] A. Aberqi, J. Bennouna, O. Benslimane, and M. A. Ragusa, "Existence results for double phase problem in Sobolev-Orlicz spaces with variable exponents in complete manifold," *Mediteranean Journal of Mathematics*, 2022.

[24] T. Aubin, "Nonlinear analysis on manifolds. Monge-Ampere equations," *Springer Science & Business Media*, vol. 252, 1982.

[25] M. Gaczkowski, P. Górka, and D. J. Pons, "Sobolev spaces with variable exponents on complete manifolds," *Journal of Functional Analysis.*, vol. 270, no. 4, pp. 1379–1415, 2016.

[26] L. Guo, "The Dirichlrt problems for nonlinear elliptic equations with variable exponents on Riemannian manifolds," *Journal of Applied Analysis And Computation*, vol. 5, pp. 562–569, 2015.

[27] E. Hebey, "Nonlinear analysis on manifolds: Sobolev spaces and inequalities," *American Mathematical Society*, vol. 5, 2000.