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Research Article

A New Modified Technique of Adomian Decomposition Method for Fractional Diffusion Equations with Initial-Boundary Conditions

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In general, solving fractional partial differential equations either numerically or analytically is a difficult task. However, mathematicians have tried their best to make the task easy and promoted various techniques for their solutions. In this regard, a very prominent and accurate technique, which is known as the new technique of the Adomian decomposition method, is developed and presented for the solution of the initial-boundary value problem of the diffusion equation with fractional view analysis. The suggested model is an important mathematical model to study the behavior of degrees of memory in diffusing materials. Some important results for the given model at different fractional orders of the derivatives are achieved. Graphs show the obtained results to confirm the accuracy and validity of the suggested technique. These results are in good contact with the physical dynamics of the targeted problems. The obtained results for both fractional and integer orders problems are explained through graphs and tables. Tables and graphs support the physical behavior of each problem and the best of physical analysis. From the results, it is concluded that as the fractional order derivative is changed, the graphs or paths of dynamics are also changed. Therefore, we now choose the best solution or dynamic of the problem at a particular derivative order. It is analyzed that the present technique is one of the best techniques to handle the solutions of fractional partial differential equations having initial and boundary conditions (BCs), which are very rare in literature. Furthermore, a small number of calculations are done to achieve a very high rate of convergence, which is the novelty of the present research work. The proposed method provides the series solution with twice recursive formulae to increase the desired accuracy and is preferred among the best techniques to find the solution of fractional partial differential equations with mixed initials and BCs.

1. Introduction

Fractional Calculus (FC) is the study of derivatives and integration of fractional orders. The idea was first initiated by L'Hospital, who wrote a letter to Leibniz about the noninteger order of derivatives in 1665. After that, the devoted work done by Euler, Lagrange, Abel, and Liouville gives more extension to the field, which is very popular nowadays because of its essential applications in the areas of biology, physics, fluid mechanics, and other sciences [1–3].

Fractional-order differential equations (FPDEs) have a great contribution in the modeling of a variety of complex natural and nonlinear phenomena. FPDEs have made a significant contribution to various scientific research fields, including a diverse range of processes and systems, memories, and various branches of mathematics. The modeling of FPDEs, whether they are with respect to time or space, is more convergent, and many natural phenomena are accurately described by them. The researchers in the fields of anomalous diffusion, dielectric polarization, control theory, and other problems of physical phenomena are interested in FPDEs [4–10]. Since the fractional order of the derivative works more accurately than integer order in describing the properties related to hereditary and where the future state is influenced by the past state, FPDEs give the highest contribution to explaining such types of systems. Psychology, biology, acoustics, chemistry [11], physics, colored noises [12], and continuum mechanics [13] are some of the scientific phenomena and problems that FPDEs are used to model. Special applications of FPDEs can be found in various branches of physics and hydrology, including [14–22]. FPDEs have grown in popularity because, by definition, the fractional derivative is global, whereas the integer order derivative is local [23–25].

In recent years, the development of numerical and analytical methodologies for the solution of FPDEs is a hot topic among the researchers. Obtaining numerical or analytical solutions to FPDEs is never a simple task for mathematicians. Many researchers, on the other hand, have devised a number of novel techniques for dealing with FPDE solutions. Some of the important techniques include the Haar wavelet method (HWM) [26], the Laplace transform method [27], the Elzaki transform decomposition method (ETDM) [28], the Adomian decomposition method (ADM) [29], the finite difference technique [30, 31], the natural transform decomposition approach [32], the Legendre base method [33], the homotopy analysis method [34], the differential transform method [35], the variational iteration approach [36], and the Bernstein polynomial [37].

Many researchers have studied the time-fractional diffusion equations (TFDEs) because of their various applications in science and engineering and other branches of applied sciences. TFDEs are broadly found in physical, biological, and engineering processes [38–41]. The TFDEs are used by Nigmatulin [39] to describe diffusion in media with spectral geometry. The TFDEs are being investigated by many researchers, both analytically and numerically [42, 43]. The solution method of the Laplace transform and a similar method is used to obtain the invariant solution of TFDEs by Gorenlfo et al. [44, 45], and Lin and Zu applied a finite difference scheme in the Legendre spectral method and in space for TFDEs [46]. Dhaigude and Nikam [13] and Schneider and Wyss [43] worked on TFDEs and wave equations to obtain the solution. Moreover, the existence and uniqueness of the targeted problems are shown in [47]. Here, the researchers either used only one from initial boundary conditions (IBCs) to solve the problems. In the current research article, a modified method of ADM is implemented to solve TFDEs with both IBCs suggested by Ali in [48]. The same procedure is applied to the problems of having both initial and BCs in [49] with the homotopy perturbation method, and the results are excellent. Ali applied this new technique in [50] with a variational iteration method to initial-boundary value problems. The procedure becomes accurate because there is a new initial approximate solution with each new iteration. For justification, some examples are discussed in this paper.

ADM was first introduced by Adomian [51] in the 1980s, and it was observed that the technique is beneficial for nonlinear equations. Wazwaz [52] applied the same method to solve different kinds of differential equations. Niu and Wang [53] used the decomposition method to find the solution of fractional heat-like and wave-like equations. Niu and Wang [53] applied this method to boundary value problems to calculate a one-step optimal homotopy analysis method. Pandir and Yildirim used the homotopy perturbation method and ADM in conformable sense [54]. The mathematicians have made several modifications to ADM which have improved the accuracy of the technique, and some of them are [55–57]. Here, a modified technique is implemented to solve the initial and boundary value problems (IBVPs) of TFDEs. In the literature, various authors have used numerical and analytical techniques for the solution of the initial value problems of FPDEs and their systems. However, only a few attempts were made to solve IBVPs of FPDEs and their systems. In this regard, Elaf Jaafar Ali has made a contribution and developed a new technique to solve FPDEs. He modified the existing techniques of ADM to solve IBVPs of FPDEs.

In this paper, we will work on the solution of TFDEs by using a new technique of the Adomian decomposition method (ADM). In literature, some important numerical and analytical techniques have been used for the solutions of time and space fractional diffusion equations [25, 54]. In this article, the work of Elaf Jafaar Ali is further extended to solve IBVPs of TFDEs. The general description of the proposed method is implemented to solve some examples of the suggested problems. The analytical solutions of FPDEs with initial and boundary conditions are very difficult to investigate. In the current work, the analytical solutions of TFDEs are obtained in a very simple and straightforward procedure and provide the closed-form solutions. The less computational work and simplicity are the uniqueness of the present modified technique. The obtained results are displayed through graphs. The graphical representations have shown that there is a close contact between the exact and the approximate solutions of the problems. The solutions are obtained for various fractional-order problems. The fractional order solutions provide useful information about the dynamics of the suggested problems. It is observed that the proposed technique has a very effective procedure for solving FPDEs and their systems with IBCs. However, some limitations are observed while using the present technique, that is, if the FPDEs or their systems have a higher number of IBCs, then the proposed method required a large number of calculations to achieve the results. Mostly, the suggested methods have smaller accuracy at a greater time value and their accuracy increased at a smaller time value. For higher nonlinear problems, the solution components are not easy to compute, so very few terms are calculated to achieve the required solution.

2. Preliminaries

Some definitions that are related to our study are considered in this section.

2.1. Definition. The integral operator of Reimann-Liouville having order δ is given by [40]

$$\left(I_\sigma^\delta h\right)(\sigma) = \begin{cases} \frac{1}{\Gamma(\delta)} \int_0^\tau (\sigma - \nu)^{\delta-1} d\nu, & \delta > 0, \\ h(\sigma), & \delta = 0, \end{cases} \tag{1}$$

where Γ is the gamma function and can be written as

$$\Gamma(\omega) = \int_0^\infty e^{-\sigma} \sigma^{\omega-1} d\tau, \quad \omega \in \mathbb{C}. \tag{2}$$

2.2. Definition. The expression for Caputo for fractional order δ is as follows:

$$\left(D^\delta h\right)(\sigma) = \frac{\partial^\delta h(\sigma)}{\partial \tau^\delta} = \begin{cases} I^{m-\delta}\left[\frac{\partial^\delta h(\sigma)}{\partial \tau^\delta}\right], & m-1 < \delta \le m, m \in \mathbb{N}, \\ \frac{\partial^\delta h(\sigma)}{\partial \tau^\delta}, \end{cases} \tag{3}$$

where $m \in \mathbb{N}$, $\sigma > 0$, $g \in \mathbb{C}_\tau$, and $\tau \ge 1$.

2.3. Lemma. For $j-1 < \delta \le j$ with $j \in \mathbb{N}$ and $h \in \mathbb{C}_\tau$ with $\tau \ge -1$, then [58]

$$\begin{cases} I^\delta I^b = I^{\delta+b} h(\sigma), & b, \delta \ge 0, \\ I^\delta \sigma^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\delta+\lambda+1)} \sigma^{\delta+\lambda}, & \delta > 0, \lambda > -1, \sigma > 0, \\ I^\delta D^\delta h(\sigma) = h(\sigma) - \sum_{k=0}^{j-1} h^k(0^+) \frac{\sigma^k}{k!}, \end{cases} \tag{4}$$

where $\sigma > 0$, $j-1 < \delta \le j$.

2.4. Definition. The Mittag-Leffler function $E_\delta(\rho)$ for $\delta > 0$ is

$$E_\delta(\rho) = \sum_{m=0}^\infty \left(\frac{\rho^m}{\Gamma(m\delta+1)}\right), \quad \delta > 0, \rho \in \mathbb{C}. \tag{5}$$

3. Adomian Decomposition Method

This method was discovered by Adomian in 1994 for the solution of linear and nonlinear differential and integrodifferential equations [29]. To understand the method, let us consider an equation of the following form:

$$F(\vartheta(\sigma)) = g(\sigma), \tag{6}$$

where F is a nonlinear differential operator and g is the known function. We will split the linear term in $F(\vartheta(\sigma))$ into the form $\mathscr{L}\vartheta + R\vartheta$, where $\mathscr{L}$ is the invertible operator, chosen as the highest order derivative, R represents the linear operator, and then, Equation (6) has the representation as follows:

$$\mathfrak{R}\vartheta + R\vartheta + N\vartheta = g, \tag{7}$$

where $N\vartheta$ is the nonlinear term of $F(\vartheta(\sigma))$. Apply $\mathfrak{R}^{-1}$ to Equation (7) on both sides.

$$\vartheta = \varphi + \mathfrak{R}^{-1}(g) - \mathfrak{R}^{-1}(R\vartheta) - \mathfrak{R}^{-1}(N\vartheta), \tag{8}$$

where the constant of integration is φ, and $\mathfrak{R}\varphi = 0$.

The following infinite series shows the solution of ADM as

$$\vartheta = \sum_{n=0}^\infty \zeta_n. \tag{9}$$

The $N\vartheta$ is a nonlinear term represented by A_n, defined as follows:

$$N\vartheta = \sum_{n=0}^\infty A_n. \tag{10}$$

Using the following to calculate A_n,

$$A_n = \frac{1}{n!} \frac{d^n}{d\psi^n} N\left(\sum_{k=0}^\infty \left(\psi^k v_k\right)\right), \quad n = 0, 1, \cdots. \tag{11}$$

Equation (6) has a solution in the form of a series as follows:

$$\begin{cases} \vartheta_0 = \varphi + \mathfrak{R}^{-1}(g), & n = 0, \\ \vartheta_{n+1} = \mathfrak{R}^{-1}(R\vartheta_n) - \mathfrak{R}^{-1}(A_n), & n \ge 0. \end{cases} \tag{12}$$

4. Modification of ADM

To understand the main idea of the proposed technique, we will take the following one-dimensional equation [48]:

$$D_\tau^\delta(\vartheta(\sigma,\tau)) = \frac{\partial^2 \vartheta(\sigma,\tau)}{\partial \sigma^2} + w(\sigma,\tau), \quad 0 < \sigma < 1, \tau > 0, 0 < \delta < 1, \tag{13}$$

having the IBCs as follows:

$$\begin{cases} \vartheta(\sigma,\tau) = \ell_0(\sigma), & \frac{\partial \vartheta(\sigma,0)}{\partial \tau} = \ell_1(\sigma), \quad 0 \le \sigma \le 1, \\ \vartheta(0,\tau) = \hbar_0(\tau), & \vartheta(1,\tau) = \hbar_1(\tau), \quad \tau > 0. \end{cases} \tag{14}$$

The source term is represented by $w(\sigma,\tau)$.

The new initial solution (ϑ_n^*) calculated for Equation (13) can be written in operator form as

$$\vartheta_n^* = \vartheta_n(\sigma,\tau) + (1-\sigma)[\hbar_0(\tau) - \vartheta_n(0,\tau)] + \sigma[\hbar_1(\tau) - \vartheta_n(1,\tau)]. \tag{15}$$

In operator form, Equation (13) can be written as

$$\mathfrak{R}\vartheta = \frac{\partial^2 \vartheta(\sigma,\tau)}{\partial\sigma^2} + w(\sigma,\tau). \tag{16}$$

$\mathfrak{R}$ is

$$\mathfrak{R} = \frac{\partial^\delta}{\partial\tau^\delta}, \tag{17}$$

so $\mathfrak{R}^{-1}$ is

$$\mathfrak{R}^{-1}(.) = I^\delta(.)d\tau. \tag{18}$$

Applying $\mathfrak{R}^{-1}$ to Equation (16), we obtained

$$\vartheta(\sigma,\tau) = \vartheta(\sigma,0) + \mathfrak{R}^{-1}\left(\frac{\partial^2\vartheta(\sigma,\tau)}{\partial\sigma^2} + w(\sigma,\tau)\right), \tag{19}$$

where $n = 0, 1, \cdots$.

The initial approximation can be written as

$$\vartheta_0(\sigma,\tau) = \vartheta(\sigma,0) + \mathfrak{R}^{-1}(w(\sigma,\tau)), \tag{20}$$

and hence, the iteration formula is

$$\vartheta_{n+1}(\sigma,\tau) = \mathfrak{R}^{-1}\left(\frac{\partial^2\vartheta_*(\sigma,\tau)}{\partial\sigma^2}\right). \tag{21}$$

The initial solutions u_n^* of Equation (13) satisfied both the IBCs, as given in the following:

$$\begin{aligned} at \quad \tau = 0, \quad & \vartheta_n^*(\sigma,0) = \vartheta_n(\sigma,0), \\ \sigma = 0, \quad & \vartheta_n^*(0,\tau) = \hbar_0(\tau), \\ \sigma = 1, \quad & \vartheta_n^*(1,\tau) = \hbar_1(\tau). \end{aligned} \tag{22}$$

The proposed technique works effectively for two-dimensional problems.

5. Numerical Results

In this section, some illustrative examples are solved by the new technique of ADM.

5.1. Example. Consider TFDE of the following form [59]:

$$\begin{aligned} \frac{\partial^\delta\vartheta(\sigma,\tau)}{\partial\tau^\delta} = {} & \frac{\partial^2\vartheta(\sigma,\tau)}{\partial\sigma^2} + \frac{\Gamma(4+\delta)}{6}\sigma^4(2-\sigma)\tau^3 \\ & - 4\sigma^2(6-5\sigma)\tau^{3+\sigma}, \quad 0 \le \sigma \le 2, 0 < \delta \le 1, \end{aligned} \tag{23}$$

having the IBCs as follows:

$$\begin{aligned} \vartheta(\sigma,0) &= 0, \\ \vartheta(0,\tau) &= \vartheta(2,\tau) = 0. \end{aligned} \tag{24}$$

The problem has the analytical solution at $\delta = 1$ as follows:

$$\vartheta(\sigma,\tau) = \sigma^4(2-\sigma)\tau^{3+\delta}. \tag{25}$$

Applying the suggested method of ADM to Equation (23), we have

$$\vartheta_n^*(\sigma,\tau) = \vartheta_n(\sigma,\tau) + (1-\sigma)[0 - \vartheta_n(0,\tau)] + \sigma[0 - \vartheta_n(2,\tau)], \tag{26}$$

where $n = 0, 1, \cdots$.

Applying $\mathfrak{R}$ to Equation (23), we have

$$\mathfrak{R}\vartheta = \frac{\partial^2\vartheta(\sigma,\tau)}{\partial\sigma^2} + \frac{\Gamma(4+\delta)}{6}\sigma^4(2-\sigma)\tau^3 - 4\sigma^2(6-5\sigma)\tau^{3+\sigma}, \tag{27}$$

where $\mathfrak{R} = \partial^\delta/\partial\tau^\delta$ and $\mathfrak{R}^{-1}$ is

$$\mathfrak{R}^{-1}(.) = I^\delta(.)d\tau. \tag{28}$$

Operating Equation (23) by mathcal$\mathfrak{R}^{-1}$, we have

$$\begin{aligned} \vartheta(\sigma,\tau) = {} & u(\sigma,0) + \mathfrak{R}^{-1} \\ & \cdot\left(\frac{\partial^2\vartheta(\sigma,\tau)}{\partial\sigma^2} + \frac{\Gamma(4+\delta)}{6}\sigma^4(2-\sigma)\tau^3 - 4\sigma^2(6-5\sigma)\tau^{3+\sigma}\right). \end{aligned} \tag{29}$$

Using ADM solution, the initial approximation becomes

$$\begin{aligned} \vartheta_0(\sigma,\tau) &= \vartheta(\sigma,0) + \mathfrak{R}^{-1}\left(\frac{\Gamma(4+\delta)}{6}\sigma^4(2-\sigma)\tau^3 - 4\sigma^2(6-5\sigma)\tau^{3+\sigma}\right) \\ &= 0 + \frac{\Gamma(4+\delta)\Gamma(4)\sigma^4(2-\sigma)\tau^{3+\delta}}{6\Gamma(4+\delta)} - \frac{4\sigma^2(6-5\sigma)\Gamma(4+\delta)\tau^{3+2\delta}}{\Gamma(4+2\delta)} \\ &= \sigma^4(2-\sigma)\tau^{3+\delta} - \frac{(24\sigma^2 + 20\sigma^3)\Gamma(4+\delta)\tau^{3+2\delta}}{\Gamma(4+2\delta)}, \end{aligned}$$

$$\vartheta_0(\sigma,\tau) = \sigma^4(2-\sigma)\tau^{3+\delta} - \frac{24\sigma^2\Gamma(4+\delta)\tau^{3+2\delta}}{\Gamma(4+2\delta)} + \frac{20\sigma^3\Gamma(4+\delta)\tau^{3+2\delta}}{\Gamma(4+2\delta)}. \tag{30}$$

With the help of initial approximation ϑ_n^*, the formula for iterations is

$$\vartheta_{n+1}(\sigma,\tau) = \mathfrak{R}^{-1}\left(\frac{\partial^2 \vartheta_n^*}{\partial\sigma^2}\right). \tag{31}$$

Use the IBCs in Equation (26), for $n = 0$:

$$\begin{aligned}\vartheta_0^*(\sigma,\tau) &= \vartheta_0(\sigma,\tau) + (1-\sigma)[0-\vartheta_0(0,\tau)] + \sigma[0-\vartheta_0(2,\tau)]\\ &= \sigma^4(2-\sigma)\tau^{3+\delta} - \frac{24\sigma^2\Gamma(4+\delta)\tau^{3+2\delta}}{\Gamma(4+2\delta)} + \frac{20\sigma^3\Gamma(4+\delta)\tau^{3+2\delta}}{\Gamma(4+2\delta)}\\ &= \sigma^4(2-\sigma)\tau^{3+\delta} - \frac{\left(24\sigma^2+20\sigma^3\right)\Gamma(4+\delta)\tau^{3+2\delta}}{\Gamma(4+2\delta)}\\ &\quad + (1-\sigma)[0-0] + \sigma\left[0-\left(0-\frac{-16\Gamma(4+\delta)\tau^{3+2\delta}}{\Gamma(4+2\delta)}\right)\right],\end{aligned}$$

$$\begin{aligned}\vartheta_0^*(\sigma,\tau) &= \sigma^4(2-\sigma)\tau^{3+\delta} - \frac{24\sigma^2\Gamma(4+\delta)\tau^{3+2\delta}}{\Gamma(4+2\delta)}\\ &\quad + \frac{20\sigma^3\Gamma(4+\delta)\tau^{3+2\delta}}{\Gamma(4+2\delta)} - \frac{16\sigma\Gamma(4+\delta)\tau^{3+2\delta}}{\Gamma(4+2\delta)}.\end{aligned} \tag{32}$$

From Equation (31), we have

$$\begin{aligned}\vartheta_1(\sigma,\tau) &= \mathfrak{R}^{-1}\left(\frac{\partial^2_\vartheta 0^*}{\partial\sigma^2}\right)\\ &= \mathfrak{R}^{-1}\left(\left(24\sigma^2-20\sigma^3\right)\tau^{3+\delta} - \frac{48-120\sigma\Gamma(4+\delta)\tau^{3+2\delta}}{\Gamma(4+2\delta)}\right),\end{aligned}$$

$$\begin{aligned}\vartheta_1(\sigma,\tau) &= \frac{24\sigma^2\Gamma(4+\delta)\tau^{3+2\delta}}{\Gamma(4+2\delta)} - \frac{20\sigma^3\Gamma(4+\delta)\tau^{3+2\delta}}{\Gamma(4+2\delta)}\\ &\quad - \frac{48\Gamma(4+\delta)\tau^{3+3\delta}}{\Gamma(4+3\delta)} + \frac{120\sigma\Gamma(4+\delta)\tau^{3+3\delta}}{\Gamma(4+3\delta)}.\end{aligned} \tag{33}$$

For $n = 1$, Equation (26) becomes

$$\begin{aligned}\vartheta_1^*(\sigma,\tau) &= \vartheta_1(\sigma,\tau) + (1-\sigma)[0-\vartheta_1(0,\tau)] + \sigma[0-\vartheta_1(2,\tau)]\\ &= \frac{24\sigma^2\Gamma(4+\delta)\tau^{3+2\delta}}{\Gamma(4+2\delta)} - \frac{20\sigma^3\Gamma(4+\delta)\tau^{3+2\delta}}{\Gamma(4+2\delta)}\\ &\quad - \frac{48\Gamma(4+\delta)\tau^{3+3\delta}}{\Gamma(4+3\delta)} + \frac{120\sigma\Gamma(4+\delta)\tau^{3+3\delta}}{\Gamma(4+3\delta)}(1-\sigma)\\ &\quad \cdot\left[0+\frac{48\Gamma(4+\delta)\tau^{3+3\delta}}{\Gamma(4+3\delta)}\right]\\ &\quad + \sigma\left[0+\frac{64\Gamma(4+\delta)\tau^{3+2\delta}}{\Gamma(4+3\delta)} - \frac{192\Gamma(4+\delta)\tau^{3+3\delta}}{\Gamma(4+3\delta)}\right],\end{aligned}$$

$$\begin{aligned}\vartheta_1^*(\sigma,\tau) &= \frac{24\sigma^2\Gamma(4+\delta)\tau^{3+2\delta}}{\Gamma(4+2\delta)} - \frac{20\sigma^3\Gamma(4+\delta)\tau^{3+2\delta}}{\Gamma(4+2\delta)}\\ &\quad - \frac{120\sigma\Gamma(4+\delta)\tau^{3+3\delta}}{\Gamma(4+3\delta)} + \frac{64\sigma\Gamma(4+\delta)\tau^{3+2\delta}}{\Gamma(4+2\delta)}.\end{aligned} \tag{34}$$

From Equation (31), we have

$$\begin{aligned}\vartheta_2(\sigma,\tau) &= L^{-1}\left(\frac{\partial^2\vartheta_1^*}{\partial\sigma^2}\right)\\ &= L^{-1}\left(\frac{48\Gamma(4+\delta)tau^{3+2\delta}}{\Gamma(4+2\delta)} - \frac{120\sigma\Gamma(4+\delta)\tau^{3+2\delta}}{\Gamma(4+2\delta)}\right),\end{aligned}$$

$$\vartheta_2(\sigma,\tau) = \frac{48\Gamma(4+\delta)\tau^{3+3\delta}}{\Gamma(4+3\delta)} - \frac{120\sigma\Gamma(4+\delta)\tau^{3+3\delta}}{\Gamma(4+3\delta)}. \tag{35}$$

For $n = 2$, Equation (26) becomes

$$\begin{aligned}\vartheta_2^*(\sigma,\tau) &= \vartheta_2(\sigma,\tau) + (1-\sigma)[0-\vartheta_2(0,\tau)] + \sigma[0-\vartheta_2(2,\tau)]\\ &= \frac{(48-120\sigma)\Gamma(4+\delta)\tau^{3+3\delta}}{\Gamma(4+3\delta)}\\ &\quad + (1-\sigma)[0-\vartheta_2(0,\tau)] + \sigma[0-\vartheta_2(2,\tau)],\\ &= \frac{(48-120\sigma)\Gamma(4+\delta)\tau^{3+3\delta}}{\Gamma(4+3\delta)}\\ &\quad + (1-\sigma)\left[0-\frac{48\Gamma(4+\delta)\tau^{3+3\delta}}{\Gamma(4+3\delta)}\right]\\ &\quad + \sigma\left[0+\frac{192\Gamma(4+\delta)\tau^{3+3\delta}}{\Gamma(4+3\delta)}\right]\\ &= \frac{-120\sigma\Gamma(4+\delta)\tau^{3+3\delta}}{\Gamma(4+3\delta)} + \frac{48\sigma\Gamma(4+\delta)\tau^{3+3\delta}}{\Gamma(4+3\delta)}\\ &\quad + \frac{192\sigma\Gamma(4+\delta)\tau^{3+3\delta}}{\Gamma(4+3\delta)},\\ \vartheta_2^*(\sigma,\tau) &= \frac{120\sigma\Gamma(4+\delta)\tau^{3+3\delta}}{\Gamma(4+3\delta)}.\end{aligned} \tag{36}$$

From Equation (31), we have

$$\begin{aligned}\vartheta_3(\sigma,\tau) &= L^{-1}\left(\frac{\partial^2\vartheta_2^*}{\partial\sigma^2}\right),\\ &= \mathfrak{R}^{-1}(0) = 0.\\ &\vdots\end{aligned} \tag{37}$$

Thus, the series form of ADM solution is

$$\begin{aligned}\vartheta(\sigma,\tau) &= \vartheta_0(\sigma,\tau) + \vartheta_1(\sigma,\tau) + \vartheta_2(\sigma,\tau) + \vartheta_3(\sigma,\tau) + \cdots\\ &= \sigma^4(2-\sigma)\tau^{3+\delta} - \frac{24\sigma^2\Gamma(4+\delta)\tau^{3+2\delta}}{\Gamma(4+2\delta)}\\ &\quad + \frac{20\sigma^3\Gamma(4+\delta)\tau^{3+2\delta}}{\Gamma(4+2\delta)} + \frac{24\sigma^2\Gamma(4+\delta)\tau^{3+2\delta}}{\Gamma(4+2\delta)}\\ &\quad - \frac{20\sigma^3\Gamma(4+\delta)\tau^{3+2\delta}}{\Gamma(4+2\delta)} - \frac{48\Gamma(4+\delta)\tau^{3+3\delta}}{\Gamma(4+3\delta)}\\ &\quad + \frac{120\sigma\Gamma(4+\delta)\tau^{3+3\delta}}{\Gamma(4+3\delta)} + \frac{48\Gamma(4+\delta)\tau^{3+3\delta}}{\Gamma(4+3\delta)}\\ &\quad - \frac{120\sigma\Gamma(4+\delta)\tau^{3+3\delta}}{\Gamma(4+3\delta)} + 0 + \cdots,\end{aligned}$$

$$\vartheta(\sigma,\tau) = \sigma^4(2-\sigma)\tau^{3+\delta}. \tag{38}$$

5.2. Example. Consider the TFDE of the following form [59]:

$$\frac{\partial^\delta \vartheta(\sigma,\tau)}{\partial \tau^\delta} = \frac{\partial^2 \vartheta(\sigma,\tau)}{\partial \sigma^2} + 3\frac{\Gamma(1/2)}{4}\tau\sigma^4(\sigma-1) - 4\sigma^2(5\sigma-3)\tau^{3/2}, \quad 0 \le \sigma \le 1, 0 < \delta \le 1, \tag{39}$$

having the IBCs as follows:

$$\begin{aligned} \vartheta(\sigma,0) &= 0, \\ \vartheta(0,\tau) &= \vartheta(1,\tau) = 0. \end{aligned} \tag{40}$$

With analytical solution at $\delta = 1/2$ as follows:

$$\vartheta(\sigma,\tau) = \sigma^4(\sigma-1)\tau^{3/2}. \tag{41}$$

Apply the suggested method of ADM to Equation (39), we have

$$\vartheta_n^*(\sigma,\tau) = \vartheta_n(\sigma,\tau) + (1-\sigma)[0 - \vartheta_n(0,\tau)] + \sigma[0 - \vartheta_n(1,\tau)], \tag{42}$$

where $n = 0, 1, \cdots$.

Applying $\mathfrak{R}$ to Equation (39), we have

$$\mathfrak{R}\vartheta = \frac{\partial^2 \vartheta(\sigma,\tau)}{\partial \sigma^2} + \Gamma\left(\frac{1}{2}\right)4\tau\sigma^4(\sigma-1) - 4\sigma^2(5\sigma-3)\tau^{3/2}, \tag{43}$$

where $\mathfrak{R} = \partial^\delta/\partial\tau^\delta$ and $\mathfrak{R}^{-1}$ is defined as

$$\mathfrak{R}^{-1}(.) = I^\delta(.)d\tau. \tag{44}$$

Operating Equation (39) by $\mathfrak{R}^{-1}$, we have

$$\vartheta(\sigma,\tau) = \vartheta(\sigma,0) + \mathfrak{R}^{-1}\left(3\frac{\Gamma(1/2)}{4}\tau\sigma^4(\sigma-1) - 4\sigma^2(5\sigma-3)\tau^{3/2}\right). \tag{45}$$

Using ADM solution, the initial approximation becomes

$$\begin{aligned} \vartheta_0(\sigma,\tau) &= \vartheta(\sigma,0) + \mathfrak{R}^{-1}\left(3\frac{\Gamma(1/2)}{4}\tau\sigma^4(\sigma-1) - 4\sigma^2(5\sigma-3)\tau^{3/2}\right) \\ &= 0 + \frac{3\Gamma(1/2)\Gamma(2)\tau^{\delta+1}\sigma^4(\sigma-1)}{4\Gamma(\delta+2)} - \frac{4\sigma^2(5\sigma-3)\Gamma(5/2)\tau^{3/2+\delta}}{\Gamma(5/2+\delta)}, \end{aligned}$$

$$\vartheta_0(\sigma,\tau) = \frac{3\Gamma(1/2)\Gamma(2)\tau^{\delta+1}\sigma^4(\sigma-1)}{4\Gamma(\delta+2)} - \frac{20\sigma^3\Gamma(5/2)\tau^{3/2+\delta}}{\Gamma(5/2+\delta)} + \frac{12\sigma^2\Gamma(5/2)\tau^{3/2+\delta}}{\Gamma(5/2+\delta)}. \tag{46}$$

With the help of initial approximation u_n^*, the formula for iterations is

$$\vartheta_{n+1}(\sigma,\tau) = \mathfrak{R}^{-1}\left(\frac{\partial^2 \vartheta_n^*}{\partial \sigma^2}\right). \tag{47}$$

For $n = 0$, put the IBCs into Equation (42).

$$\begin{aligned} \vartheta_0^*(\sigma,\tau) &= \vartheta_0(\sigma,\tau) + (1-\sigma)[0 - \vartheta_0(0,\tau)] + \sigma[0 - \vartheta_0(1,\tau)] \\ &= \frac{3\Gamma(1/2)\Gamma(2)\tau^{\delta+1}\sigma^4(\sigma-1)}{4\Gamma(\delta+2)} - \frac{20\sigma^3\Gamma(5/2)\tau^{3/2+\delta}}{\Gamma(5/2+\delta)} \\ &\quad + \frac{12\sigma^2\Gamma(5/2)\tau^{3/2+\delta}}{\Gamma(5/2+\delta)} + (1-\sigma)[0-0] + \sigma \\ &\quad \cdot \left[\frac{0 + 8\Gamma(5/2)\tau^{3/2+\delta}}{\Gamma(5/2+\delta)}\right], \end{aligned}$$

$$\vartheta_0^*(\sigma,\tau) = \frac{3\Gamma(1/2)\Gamma(2)\tau^{\delta+1}\sigma^4(\sigma-1)}{4\Gamma(\delta+2)} - \frac{20\sigma^3\Gamma(5/2)\tau^{3/2+\delta}}{\Gamma(5/2+\delta)} + \frac{12\sigma^2\Gamma(5/2)\tau^{3/2+\delta}}{\Gamma(5/2+\delta)} + \frac{8\sigma\Gamma(5/2)\tau^{3/2+\delta}}{\Gamma(5/2+\delta)}. \tag{48}$$

From Equation (47), we have

$$\begin{aligned} \vartheta_1(\sigma,\tau) &= \mathfrak{R}^{-1}\left(\frac{\partial^2 \vartheta_0^*}{\partial \sigma^2}\right) \\ &= \mathfrak{R}^{-1}\left(\frac{3\Gamma(1/2)\Gamma(2)\tau^{\delta+1}\left(20\sigma^3 - 12\sigma^2\right)}{4\Gamma(\delta+2)} - \frac{4(30\sigma-6)\Gamma(5/2)\tau^{3/2+\delta}}{\Gamma(5/2+\delta)}\right), \end{aligned}$$

$$\vartheta_1(\sigma,\tau) = \frac{3\Gamma(1/2)\Gamma(2)\tau^{2\delta+1}\left(20\sigma^3 - 12\sigma^2\right)}{4\Gamma(2\delta+2)} - \frac{120\sigma\Gamma(5/2)\tau^{3/2+2\delta}}{\Gamma(5/2+2\delta)} + \frac{24\Gamma(5/2)\tau^{3/2+2\delta}}{\Gamma(5/2+2\delta)}. \tag{49}$$

For $n = 1$ Equation (42), we get

$$\begin{aligned} \vartheta_1^*(\sigma,\tau) &= \vartheta_1(\sigma,\tau) + (1-\sigma)[0 - \vartheta_1(0,\tau)] + \sigma[0 - \vartheta_1(1,\tau)] \\ &= \frac{3\Gamma(1/2)\Gamma(2)\tau^{2\delta+1}\left(20\sigma^3 - 12\sigma^2\right)}{4\Gamma(2\delta+2)} - \frac{120\sigma\Gamma(5/2)\tau^{3/2+2\delta}}{\Gamma(5/2+2\delta)} \\ &\quad + \frac{24\Gamma(5/2)\tau^{3/2+2\delta}}{\Gamma(5/2+2\delta)} + (1-\sigma)\left(0 - \frac{24\Gamma(5/2)\tau^{3/2+2\delta}}{\Gamma(5/2+2\delta)}\right) \\ &\quad + \sigma\left(0 - \frac{24\Gamma(1/2)\Gamma(2)\tau^{2\delta+1}}{4\Gamma(2\delta+2)} + \frac{96\Gamma(5/2)\tau^{3/2+2\delta}}{\Gamma(5/2+2\delta)}\right), \end{aligned}$$

$$\vartheta_1^*(\sigma,\tau) = \frac{3\Gamma(1/2)\Gamma(2)\tau^{2\delta+1}\left(20\sigma^3 - 12\sigma^2\right)}{4\Gamma(2\delta+2)} - \frac{24\Gamma(1/2)\Gamma(2)\tau^{2\delta+1}}{4\Gamma(2\delta+2)}. \tag{50}$$

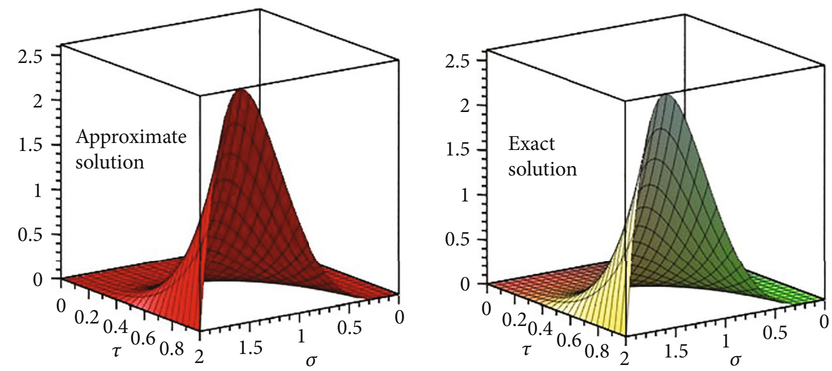


Figure 1: 3D plots of the exact and approximate solution for $\delta = 1$ of Example 5.1.

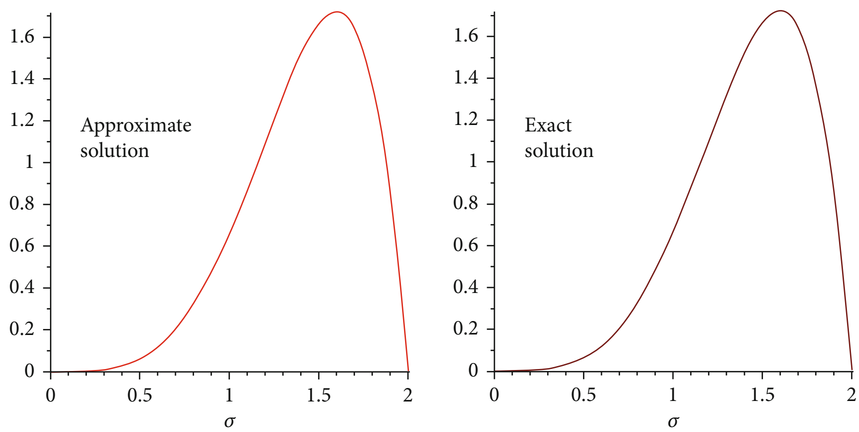


Figure 2: 2D plots of the exact and approximate solution for $\delta = 1$ of Example 5.1.

From Equation (47), we have

$$\vartheta_2(\sigma,\tau) = \mathfrak{R}^{-1}\left(\frac{\partial^2 \vartheta_1^*}{\partial \sigma^2}\right) = \mathfrak{R}^{-1}\left(\frac{3(120\sigma - 24)\Gamma(1/2)\tau^{2\delta+1}}{4\Gamma(2\delta+2)}\right),$$
$$\vartheta_2(\sigma,\tau) = \frac{3(120\sigma - 24)\Gamma(1/2)\tau^{3\delta+1}}{4\Gamma(3\delta+2)}. \tag{51}$$

Equation (42), for $n = 2$, is

$$\begin{aligned}\vartheta_2^*(\sigma,\tau) &= \vartheta_2(\sigma,\tau) + (1-\sigma)[0 - \vartheta_2(0,\tau)] + \sigma[0 - \vartheta_2(1,\tau)] \\ &= \frac{3(120\sigma - 24)\Gamma(1/2)\tau^{3\delta+1}}{4\Gamma(3\delta+2)} + (1-\sigma) \\ &\quad \cdot \left[0 - \frac{72\Gamma(1/2)\tau^{3\delta+1}}{\Gamma(3\delta+2)}\right] + \sigma\left[0 - \frac{288\Gamma(1/2)\tau^{3\delta+1}}{\Gamma(3\delta+2)}\right],\end{aligned}$$

$$\vartheta_2^*(\sigma,\tau) = \frac{144\sigma\Gamma(1/2)\Gamma(2)\tau^{3\delta+1}}{4\Gamma(3\delta+2)} - \frac{144\Gamma(1/2)\Gamma(2)\tau^{3\delta+1}}{4\Gamma(3\delta+2)}. \tag{52}$$

From Equation (47), we get

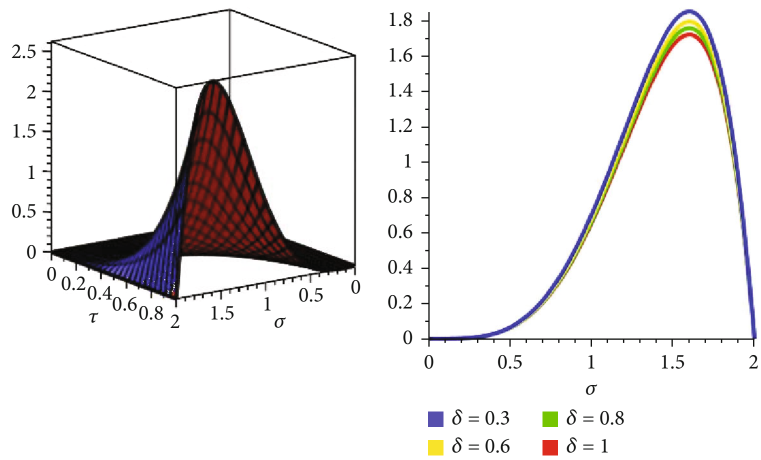
$$\begin{aligned}\vartheta_3(\sigma,\tau) &= \mathfrak{R}^{-1}\left(\frac{\partial^2 \vartheta_2^*}{\partial \sigma^2}\right), \\ &= \mathfrak{R}^{-1}(0) = 0. \\ &\vdots\end{aligned} \tag{53}$$

The series form of ADM solution is

$$\begin{aligned}\vartheta(\sigma,\tau) &= \vartheta_0(\sigma,\tau) + \vartheta_1(\sigma,\tau) + \vartheta_2(\sigma,\tau) \\ &\quad + \vartheta_3(\sigma,\tau) + \cdots \frac{3\Gamma(1/2)\Gamma(2)\tau^{2\delta+1}(20\sigma^3 - 12\sigma^2)}{4\Gamma(2\delta+2)} \\ &\quad - \frac{4(30\sigma - 6)\Gamma(5/2)\tau^{3/2+2\delta}}{\Gamma(5/2+2\delta)} \frac{3(120\sigma - 24)\Gamma(1/2)\tau^{3\delta+1}}{4\Gamma(3\delta+2)} + 0 + \cdots.\end{aligned} \tag{54}$$

6. Results and Discussion

In Figure 1, the 3D graph of exact and approximate solutions to Example 5.1 is presented. The comparison showed that the graphs of exact and obtained solutions are in good agreement and confirms the validity of the proposed method. In Figure 2, the 2D plot of the exact and approximate solution is constructed and again confirms the validity of the



(a) (b)

Figure 3: (a) 3D and (b) 2D plots for different fractional value of δ for Example 5.1.

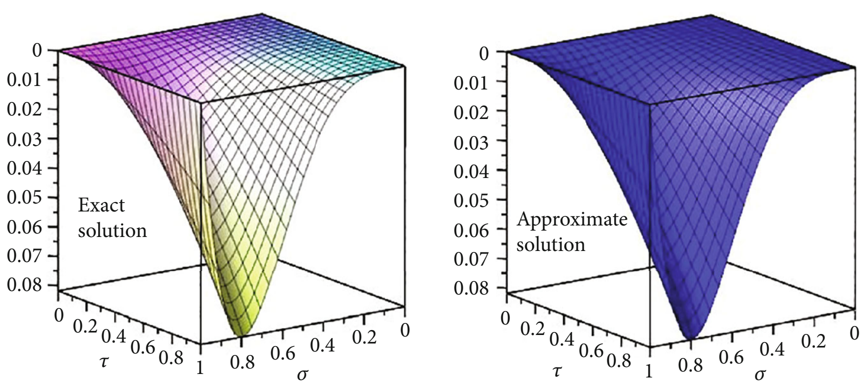


Figure 4: 3D plots of the exact and approximate solution for $\delta = 1$ of Example 5.2.

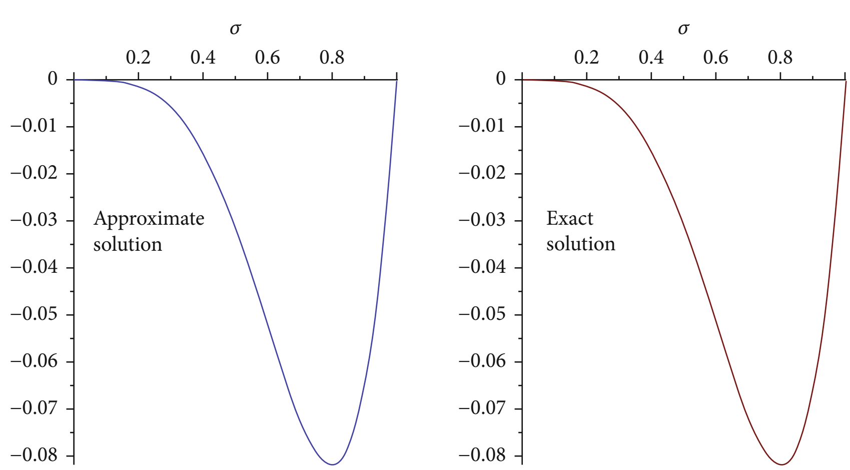


Figure 5: 2D plots of the exact and approximate solution for $\delta = 1$ of Example 5.2.

suggested technique. Figure 3 represents the fractional order solutions of Example 5.1 at $\delta = 0.3, 0.6, 0.8, 1$. The solutions at different fractional orders of the derivative provide the useful information about the dynamics of Example5.1. In Figures 4 and 5, 3D and 2D graphs of exact and obtained solutions are highlighted. From both the presentations, greater accuracy has been observed and the graphs of the derived results are found to be identical to the exact solution of Example 5.2. In Figure 6, the solution of Example 5.2 at different time levels is calculated and obtained useful dynamics for

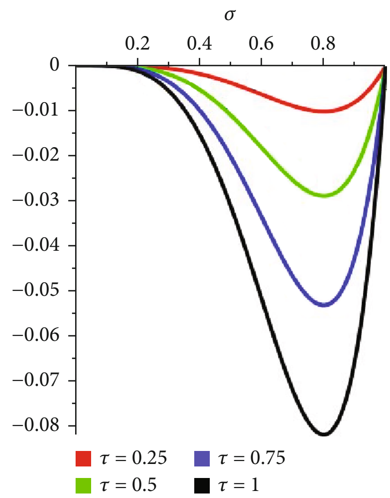


FIGURE 6: 2D plots for approximate solution of Example 5.2 at $\tau = 0.25, 0.5, 0.75, 1$.

TABLE 1: Absolute error of Example 5.1 at $\tau = 0.8$.

σ	$\lvert \text{Exact} - \text{ADM} \rvert$ $\delta = 0.3$	$\lvert \text{Exact} - \text{ADM} \rvert$ $\delta = 0.5$	$\lvert \text{Exact} - \text{ADM} \rvert$ $\delta = 0.7$
0.2	6.26×10^{-11}	4.41×10^{-10}	1.58×10^{-10}
0.4	5.3×10^{-10}	1.2×10^{-10}	6.0×10^{-11}
0.6	7.0×10^{-11}	8.0×10^{-10}	2.8×10^{-10}
0.8	4.3×10^{-9}	3.3×10^{-9}	1.0×10^{-10}
1	4.7×10^{-9}	2.0×10^{-10}	2.0×10^{-10}
1.2	1.9×10^{-9}	5.1×10^{-9}	2.5×10^{-9}
1.4	1.6×10^{-8}	5.0×10^{-9}	2.0×10^{-9}
1.6	2.0×10^{-9}	6.0×10^{-9}	1.0×10^{-9}
1.8	2.6×10^{-8}	8.6×10^{-9}	5.0×10^{-10}
2	2.0×10^{-8}	1.0×10^{-8}	2.0×10^{-9}

TABLE 2: Absolute error of Example 5.2 at $\delta = 1/2$.

σ	$\lvert \text{Exact} - \text{ADM} \rvert$ $\tau = 0.3$	$\lvert \text{Exact} - \text{ADM} \rvert$ $\tau = 0.5$	$\lvert \text{Exact} - \text{ADM} \rvert$ $\tau = 0.7$
0.1	8.748×10^{-21}	3.6348×10^{-20}	5.8352×10^{-20}
0.2	1.5×10^{-22}	2.1×10^{-22}	3.9×10^{-22}
0.3	1.67×10^{-22}	3.20×10^{-20}	4.59×10^{-20}
0.4	1.60×10^{-20}	6.1×10^{-21}	1.443×10^{-19}
0.5	1.30×10^{-22}	1.3×10^{-20}	3.00×10^{-19}
0.6	1.27×10^{-20}	2.1×10^{-20}	1.11×10^{-19}
0.7	4.2×10^{-20}	7.0×10^{-21}	4.40×10^{-19}
0.8	1.5×10^{-20}	1.98×10^{-19}	1.17×10^{-19}
0.9	3.9×10^{-20}	0.000	3.55×10^{-19}
1	0.000	1.722×10^{-19}	0.000

Example 5.2. Table 1 shows the solutions at fractional orders $\delta = 0.3$, 0.5, and 0.7 of Example 5.1. For this purpose, the modified approach of ADM is applied to obtain the solutions. The results are listed in the table, which has confirmed that the suggested method gives the solutions that are in close contact with the analytical solution of the problem. The absolute errors are given for the given analytical and ADM solutions in the table. According to the table, the proposed techniques have the desired degree of accuracy in terms of exact problem solution. In Table 2, the solutions of Example 5.2 are given at different time levels, that is, $\tau = 0.3, 0.5, 0.7$. It is verified from Table 2 that the method provides excellent results at different time levels.

7. Conclusion

In this article, the Adomian decomposition method is implemented along with some new modifications to solve fractional partial differential equation boundary value problems. The proposed technique was found to be very efficient in handling the solution of fractional-order boundary value problems. In particular, the suggested procedure is used to solve some illustrative examples of time-fractional diffusion equations. The solutions are calculated for both fractional and integer order problems, and the present method is observed to be very simple and useful for the solutions to such problems. A comparison between exact and analytical solutions is made with the help of plots and tables. The graphical representation is presented to confirm the validity of the present technique. The solution graphs have confirmed that the derived results are in close contact with the problem's actual solution. Figures 1 and 4 represent 3D solution plots of Example 5.1 and 5.2, respectively, at $\delta = 1$. Both the graphs displayed a very convincing contact between the exact and approximate solutions. In Figures 2 and 5, 2D solution plots are also constructed to confirm the validity of the proposed method. The fractional-order solutions of Example 5.1 and 5.2 are represented in Figures 3 and 6. From the graphical representation of fractional-order problems, it is confirmed that very accurate and useful information is obtained as compared to the integer order of the problems. It is concluded that the solutions at fractional-order derivatives are very useful to analyze the dynamics of the targeted problems of IBVPs. The dual use of initial conditions has made the procedure suitable for using both IBCs simultaneously, which was not the case in the earlier related literature. The solution obtained at each fractional order is found to be converging to the integer order of the targeted problems. Moreover, the proposed method is more accurate and competent to find the solution of nonlinear fractional partial differential equations and, in the future, can be modified for other important fractional nonlinear partial differential equations with higher dimensions.

Data Availability

No data were used to support this study.

Conflicts of Interest

No competing interests are declared.

Authors' Contributions

Saadia Masood performed the investigation, Hajira and Qasim Khan performed the methodology, Hassan Khan performed the supervision, Professor Fairouz Tchier is the project administrator, Rasool Shah has written the original manuscript draft, Saima Mustafa and Gurpreet Singh performed draft writing, and Muhammad Arif contributed to the funding.

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