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*Retraction*

# Retracted: On Adjacency Metric Dimension of Some Families of Graph

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Manipulated or compromised peer review

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

## References

[1] A. N. A. Koam, A. Ahmad, M. Azeem, A. Khalil, and M. F. Nadeem, "On Adjacency Metric Dimension of Some Families of Graph," *Journal of Function Spaces*, vol. 2022, Article ID 6906316, 8 pages, 2022.

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*Research Article*

# On Adjacency Metric Dimension of Some Families of Graph

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Metric dimension of a graph is a well-studied concept. Recently, adjacency metric dimension of graph has been introduced. A set $Q_a \subset V(G)$ is considered to be an adjacency metric generator for $G$ if $u_1, u_2 \in V \setminus Q_a$ (supposing each pair); there must exist a vertex $q \in Q_a$ along with the condition that $q$ is indeed adjacent to one of $u_1, u_2$. The minimum number of elements in adjacency metric generator is the adjacency metric dimension of $G$, denoted by $\dim_a(G)$. In this work, we compute exact values of the adjacency metric dimension of circulant graph $C_n(1,2)$, Möbius ladder, hexagonal Möbius ladder, and the ladder graph.



## 1. Introduction

Let $G$ be a simple connected graph with vertex set $V$ and edge set $E$. Let $N$ be a set of nonnegative integers; we assume a function $d_G : V \times V \longrightarrow N$ defined as

$$d_G(x, y) = \text{length of the shortest path between } x \text{ and } y. \quad (1)$$

Then, $(V, d_G)$ is a metric space. A subset $A \subset V$ is called a metric generator for $G$ if it is the generator of the metric space $(V, d_G)$; that is, every point of the space is uniquely determined by its distances from the elements of $A$. A minimum metric generator is the metric basis, and its cardinality is the metric dimension of $G$, denoted by dim $(G)$; for further detail of metric and their parameters, see [1–6]. The concept of metric dimension was first introduced by [7] in the problem of uniquely determining the location of an intruder in a network and was named as a locating set instead of metric generators. The same concept of metric generators or locating set was introduced by [8], and this time, they named it as resolving sets. Application of metric dimension to the navigation of robots in networks is studied in [9] and to chemistry in [10, 11]. This concept was further studied by many researchers (for instance, see [12, 13]); different studies of metric and its related parameters are studied in literature; for example, metric dimension of bilinear form graphs is found in [14], distance-regular graph is available in [15], in terms of dominating set [16], study of the corona product is found in [17], and some computer fields are attached to this topic found in [18, 19]. Several metric generators have since been developed and researched, including independent resolving sets [20], resolving dominant sets [21], strong resolving sets [22], and local metric sets [23].

A subset $Q_a \subset V$ of the set of vertices is an adjacency metric generator of $G$ if for every two vertices $u_1, u_2 \in V \setminus Q_a$, there exists a vertex $q \in Q_a$ such that $q$ is indeed adjacent to one of $u_1, u_2$. The minimum number of elements in the adjacency metric generator is the adjacency metric dimension of $G$ and symbolized by $\dim_a(G)$. The idea of adjacency metric dimension was put forward by [24], and it is closed related to 1-locating dominating set [25]. The goal of this definition is to look at the metric dimension of the lexicographic product of graphs in terms of graph adjacency. For two graphs with $n, K$-orders, the researchers in [26, 27] had taken the corona product of both graphs; the resulted graph has (local) metric dimension $n$ times the (local) adjacency metric dimension. They demonstrated that calculating the adjacency metric dimension is NP-hard as a result of this tight relationship.

The adjacency metric generator of any graph $G$ can be a metric generator in a correctly selected metric space, as pointed out in [26, 27]. Assume $q$ is a positive number. Assume the $d_{G,q} : V \times V \longrightarrow N$ is a distance function, which is defined as

$$d_{G,q}(x, y) = \min \{d_G(x, y), q\}. \tag{2}$$

Then, the metric dimension of $(V, d_{G,2})$ is equal to the adjacency metric dimension of $G$. A subset $Q_a \subset V$ is the $k$ adjacency metric generator for a graph $G$, if for each pair of vertices $u_1, u_2 \in V$, there are at least $k$ vertices $v_1, v_2, \cdots, v_k \in Q_a$ as a result $d_{G,2}(u_1, v_i) \neq d_{G,2}(u_2, v_i)$, for each $1 \leq i \leq k$. For the minimum number $(k)$ of members in the adjacency metric generator, the definition will be called as $k$ adjacency metric dimension of a graph $G$, and here, it is symbolized by $\dim_{a,j}(G)$.

The adjacency metric dimension is an NP-hard problem [25, 28], and it is very important to determine its exact values for well-known families of graphs. The primary goal of this work is to determine the exact values of the adjacency metric dimension of particular graph families, notably circulant graphs $C_n(1, 2)$, ladder, Möbius ladder, and hexagonal ladder graphs. To compute the adjacency metric dimension of these classes, we need the following proposition by Moreno et al. [29].

**Proposition 1** (see [29]). *If $G$ is a graph with $|G| = n \geq 2$, then $dim_{a,j}(G) = j$ if and only if $j \in \{1, 2\}$ and $G \in \{P_1, P_2,\}$.*

*Remark 2* (see [29]). If $G$ is a graph with $|G| = n \geq 7$, then $\dim_{a,1}(G) \geq 3$.

## 2. Construction of Graphs

Circulant graphs are a type of graph that may be utilised in the construction of local area networks. Let $v_1, v_2, \cdots, v_z z$ and $n$ be the nonnegative with given conditions, $v_\mu \neq v_\nu \forall 1 \leq \mu < \nu \leq z$, where $1 \leq v_\mu \leq \lfloor n/2 \rfloor$. With the collection of vertices in an undirected graph $V = \{v_1, v_2, \cdots, v_n\}$ and the set of edges $E = \{v_\mu v_{\mu + v\nu} : 1 \leq \mu \leq n, 1 \leq \nu \leq z\}$, the indices are considered to be taken in modulo $n$ condition, called a circulant graph, and it is symbolized by $C_n(v_1, v_2, \cdots, v_z)$. The generators are $v_1, v_2, \cdots, v_z$-numbers, and the edge $v_\mu v_{\mu + v\nu}$ is of type $v_\nu$. Actually, we can observe that the $C_n(v_1, v_2, \cdots, v_z)$ circulant graph is an $r$-regular graph, and the $r$ given is

$$c = \begin{cases} 2z, & \text{otherwise}\,; .1\,\text{cm}, \\ 2z - 1, & \text{if } \dfrac{n}{2} \in \{v_1, v_2, \cdots, v_z\}. \end{cases} \tag{3}$$

Möbius ladder $ML_m$ is built by a grid of $m \times 1$, and twist this grid at 180°; now, paste the extreme most left and right path of vertices as seen in Figure 1. It contains $m$-horizontal cycles of order four. For more study on ladder-type networks, see [30, 31]. The metric dimension of $ML_m$ is three [32].

Hexagonal Möbius ladder $HML_m$ is built in [33], it can construct by dividing each horizontal edge of a square grid by inserting a new vertex, and it becomes a grid of $m \times 1$ with each cycle having order six; now, twist this grid at 180° and paste the extreme most left and right path of vertices as shown in Figure 2. This graph contains $m$-horizontal cycles of order six. The metric dimension of the hexagonal Möbius ladder network is three [33].

Let $L_m$ be a ladder graph [34], with $n \geq 3$. We label the ladder graph as shown in Figure 3. The order and size of the ladder graph are $2n$ and $3n - 2$, respectively.

## 3. Results on Adjacency Metric Dimension of Graphs

We compute the adjacency metric dimension of certain graph families in this section. Let $C_n(1, 2)$ be a circulant graph obtained from $C_n$ by joining all the vertices at distance 2. Let $n \geq 6$; then, $|V(C_n(1, 2))| = n$ and $|E(C_n(1, 2))| = 2n$. In the next theorem, we compute the exact value of the adjacency metric dimension of the circulant graph $C_n(1, 2)$.

**Theorem 3.** *Let $G = C_n(1, 2)$ be a circulant graph. Then,*

$$dim_a(G) = \begin{cases} 3, & if\ n = 6, 7, 8\,; .1\,cm, \\ 2\left\lfloor \dfrac{n+3}{6} \right\rfloor, & if\ n \equiv 0, 3, 4, 5 (\text{mod } 6), n \geq 9, \\ 2\left\lfloor \dfrac{n+3}{6} \right\rfloor + 1, & if\ n \equiv 1, 2 (\text{mod } 6), n \geq 13. \end{cases} \tag{4}$$

*Proof.* We divide the proof of the theorem into three parts.

Part A. For the inequality $\dim_a(G) \leq 3$, when $n = 6, 7, 8$, let $Q_a = \{v_1, v_2, v_3\}$ as an adjacency metric generator; then, the representation of $v_1, v_2, v_3$ are as follows $v_1 = (0, 1, 1)$, $v_2 = (1, 0, 1)$, and $v_3 = (1, 1, 0)$. For any vertex $v_k \neq v_1, v_2, v_3$, we have the following representation:

$$r(v_k \mid Q_a) = (a, b, c). \tag{5}$$

where $a = 2$, when $k = 4, \cdots, n - 2$; otherwise, $a = 1$. Also, $b = 1$ when $k = 4, n$, and $c = 1$, when $k = 4, 5$; otherwise, both $b, c$ are 0.

From the above discussion, we can say that $\dim_a(G) \leq 3$.

Converse: on the contrary, assume that $\dim_a(G) = 2$, for $n = 7, 8$. Let $Q'_a = (v_1, v_2)$ be a resolving set for adjacency metric dimension. Then, for any $v \in V(G) \setminus Q'_a$, the distance

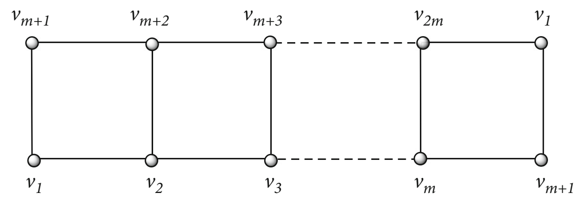


FIGURE 1: Möbius ladder graph $ML_m$.

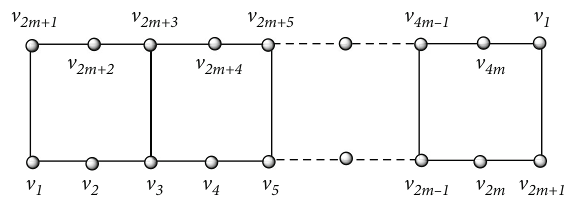


FIGURE 2: Hexagonal Möbius ladder graph $HML_m$.

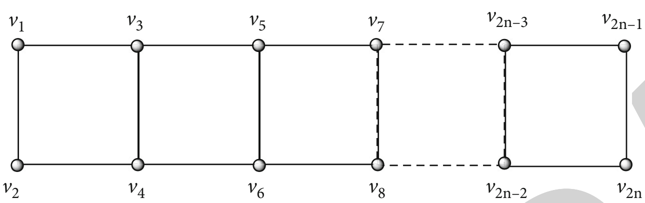


FIGURE 3: Ladder graph $L_n$.

$(d(v_1, v), d(v_2, v)) \in \{(1,1), (1,2), (2,1), (2,2)\}$. Given $|V(G) - Q'_a| \geq 5$, by the principle of Dirichlet's box, two or more components of $V(G) \setminus Q'_a$ resulted in identical vectors of distance, and further, this implied as a contradiction. Now, for $n = 6$, let $Q'_a = \{v_1, v_i\}, 2 \leq i \leq 6$; we have the same representations, that is, either $r(v_2 \mid Q'_a) = r(v_5 \mid Q'_a)$ or $r(v_3 \mid Q'_a) = r(v_6 \mid Q'_a)$. Let the adjacency metric generator $Q'_a = \{v_2, v_i\}, 3 \leq i \leq 6$; then, we have the same representations, that is, either $r(v_1 \mid Q'_a) = r(v_4 \mid Q'_a)$ or $r(v_3 \mid Q'_a) = r(v_6 \mid Q'_a)$. Let $Q'_a = \{v_i, v_j\}, 3 \leq i \leq 5, 4 \leq j \leq 6$; then, we have the same representations, that is, $r(v_1 \mid Q'_a) = r(v_4 \mid Q'_a)$, $r(v_3 \mid Q'_a) = r(v_5 \mid Q'_a)$, or $r(v_2 \mid Q'_a) = r(v_5 \mid Q'_a)$. Therefore, $\dim_a(G) \geq 3$, concluding that $\dim_a(G) = 3$.

Part B. When $n \equiv 0, 3, 4, 5 \pmod 6), n \geq 9$, we intend to use the induction method on the order of cycle; set the base step as $n = 9$, which implies that $\dim_a(C_9) = 4$. Let $Q_a = \{v_1, v_2, v_3, v_4\}$ be a resolving set for adjacency metric dimension. All vertices have the following representations:

$$r(v_k \mid Q_a) = (a, b, c, d), \quad k = 1, 2, \cdots, n. \tag{6}$$

where $a, b, c, d = 0$, when $k = 1, 2, 3, 4$, respectively, and $a = 1$ when $k = 2, 3, 8, 9$, $b = 1$ when $k = 1, 3, 4, 9$, $c = 1$ when $k = 1, 2, 4, 5$, and $d = 1$ when $k = 2, 3, 5, 6$; otherwise, all values of $a, b, c, d$ are 2. All the representations are different so

$$\dim_a(C_9) \leq 4. \tag{7}$$

On the contrary, assume that $\dim_a(C_9) = 3$. On this contradiction, the following are some cases to defend it. □

*Case 1.* Let the adjacency metric generator $Q'_a = \{v_i, v_j, v_k\}$, $1 \leq i, j, k \leq 9$, taking vertices in $Q'_a$ with 0-size gap (consecutive vetices). Then, we have the same representations in $\{v_i, v_j, v_k\}$-$\{v_{i'}, v_{j'}\}$; it means that $Q'_a$ have the same representations with the vertices which are not adjacent to any of the vertex of $Q'_a$.

*Case 2.* Let $Q'_a = \{v_i, v_{i+2}, v_{i+3}\}, 1 \leq i \leq 6$, taking vertices in $Q'_a$ with 1- and 0-size gap, respectively. Then, we have the same representations in $v_i \sim \{v_{i'}, v_{j'}\}$; it means that $Q'_a$ have the same representations with the vertices which are adjacent to $v_i$ only.

*Case 3.* Let $Q'_a = \{v_i, v_{i+1}, v_{i+3}\}$, $1 \leq i \leq 6$, taking vertices again in $Q'_a$ with 0- and 1-size gap, respectively. Then, we have the same representations in $v_{i+3} \sim \{v_{i'}, v_{j'}\}$; it means that $Q'_a$ have the same representations with the vertices which are adjacent to $v_{i+3}$ only.

*Case 4.* Let $Q'_a = \{v_i, v_{i+2}, v_{i+4}\}$, $1 \leq i \leq 5$, taking vertices in $Q'_a$ with 1-size gap. Then, we have the same representations in $v_{i+4} \sim \{v_{i'}, v_{j'}\}$; it means that $Q'_a$ have the same representations with the vertices which are adjacent to $v_{i+4}$ only.

*Case 5.* Let $Q'_a = \{v_i, v_{i+1}, v_{i+4}\}$, $1 \leq i \leq 5$, with 0- and 2-size gap, respectively. Then, we have the same representations in $v_{i+4} \sim \{v_{i'}, v_{j'}\}$, which simplifies that $Q'_a$ have the same representations with the vertices which are adjacent to $v_{i+4}$ only.

*Case 6.* Let $Q'_a = \{v_i, v_{i+2}, v_{i+5}\}$, $1 \leq i \leq 4$, taking vertices in $Q'_a$ with 1- and 2-size gap, respectively. Then, we have the same representations in $v_{i+5} \sim \{v_{i'}, v_{j'}\}$, which means that $Q'_a$ have the same representations with the vertices which are adjacent to $v_{i+5}$ only.

Due to the nature of the adjacency metric generator with three cardinalities, $Q'_a$ have the following gap size possibilities:

(i) The first gap is even, and second is odd

(ii) The first gap is odd and the next is even

(iii) Both gaps are even

(iv) Both gaps are odd

*Case 7.* Let $Q'_a = \{v_i, v_j, v_k\}$, $1 \leq i, j, k \leq 9$, taking vertices in above any sizes of gap possibilities. Then, we have the same representations in either $\{v_i, v_j, v_k\}$-$\{v_{i'}, v_{j'}\}$ or only single vertex of $Q'_a$ adjacent to both vertices, which have the same representations. From all above cases, $\dim_a(C_9) \neq 3$. Hence, our base step is true:

$$\dim_a(C_9) = 4. \quad (8)$$

Now, assume that it is also true for $n = m$, and we have to show that it is also true for $n = m + 1$ which implies that

$$\dim_a(C_{m+1}) = 2\left\lfloor \frac{m+4}{6} \right\rfloor. \quad (9)$$

Assume the assertion that

$$\dim_a(C_{m+1}) = \dim_a(C_m) + \dim_a(C_9) - 4. \quad (10)$$

Putting equations (8) and (9) in equation (10), we have

$$\dim_a(C_{m+1}) = 2\left\lfloor \frac{n+3}{6} \right\rfloor + 4 - 4 = 2\left\lfloor \frac{n+4}{6} \right\rfloor, \quad (11)$$

where $\lfloor (n+3)/6 \rfloor = \lfloor (n+4)/6 \rfloor$.

Part C. Remaining cases when $n \equiv 1, 2 (\text{mod } 6)$, $n \geq 13$.

Again, the method of induction can be used, and this implies that the base step becomes true for $n = 13$ and as well $\dim_a(C_{13}) = 5$; for this purpose, let $Q_a = \{v_1, v_2, v_5, v_7, v_{11}\}$ be a resolving set for adjacency metric dimension; all the vertices have the following representations:

$$r(v_k \mid Q) = (a, b, c, d, e), \quad k = 1, 2, \cdots, n. \quad (12)$$

where $a, b, c, d = 0$ when $k = 1, 2, 5, 7, 11$, respectively, and $a = 1$ when $k = 2, 3, 12, 13$, $b = 1$ when $k = 1, 3, 4, 13$, $c = 1$ when $k = 3, 4, 6, 7$, $d = 1$ when $k = 5, 6, 8, 9$, and $e = 1$ when $k = 9,10,12,13$; otherwise, all values of $a, b, c, d$ are 2.

$$\dim_a(C_{13}) \leq 5. \quad (13)$$

On the contrary, assume that $\dim_a(C_{13}) = 4$. Let $Q'_a$ be the adjacency metric generator with cardinality 4 having the following gap size possibilities.

(i) All the gap sizes are even

(ii) All the gap sizes are odd

(iii) One gap size is odd, and others are even

(iv) One gap size is even, and others are odd

*Case 1.* Let $Q'_a = \{v_i, v_j, v_k, v_l\}$, $1 \leq i, j, k, l \leq 13$, with any sizes of gap possibilities above defined. Then, we have the same representations in either $\{v_i, v_j, v_k, v_l\}$-$\{v_{i'}, v_{j'}\}$ or only single vertex of $Q'_a$ adjacent to both vertices, which have the same representations. From all above cases, $\dim_a(C_{13}) \neq 4$. Hence,

$$\dim_a(C_{13}) = 5. \quad (14)$$

Now, assume that it is also true for $n = m$, and we have to show that it is also true for $n = m + 1$, which leads to the assertion that

$$\dim_a(C_{m+1}) = 2\left\lfloor \frac{m+4}{6} \right\rfloor + 1. \quad (15)$$

Assuming that

$$\dim_a(C_{m+1}) = \dim_a(C_m) + \dim_a(C_{13}) - 4. \quad (16)$$

Putting equations (14) and (15) in equation (16), we have

$$\dim_a(C_{m+1}) = 2\left\lfloor \frac{n+3}{6} \right\rfloor + 5 - 4 = 2\left\lfloor \frac{n+4}{6} \right\rfloor + 1, \quad (17)$$

where $\lfloor (n+3)/6 \rfloor = \lfloor (n+4)/6 \rfloor$. This completes the proof of the theorem with all possible cases on order of the $C_n$ graph.

**Theorem 4.** *Let $ML_m$ be a Möbius ladder graph. Then,*

$$dim_a(ML_m) = \begin{cases} 3, & \text{if } m = 4\,;\,.1\,cm, \\ 4, & \text{if } m = 3, 5, \\ m-2, & \text{if } m \geq 6. \end{cases} \quad (18)$$

*Proof.* We divide the proof of the theorem into three parts.

Part A. Let $Q_a = \{v_1, v_2, v_5\}$ be an adjacency metric generator, and those shown in Table 1 are the representations of all vertices for $\dim_a(\text{ML}_4) \leq 3$.

All the vertices have different representations, and it proves that $\dim_a(\text{ML}_4) \leq 3$. On the contrary, suppose that $\dim_a(\text{ML}_4) = 2$. By using Proposition 1, it is not possible, concluding that

$$\dim_a(\text{ML}_4) = 3. \quad (19)$$

Part B. When $m = 3, 5$, this adjacency metric generator is $Q_a = \{v_1, v_2, v_4, v_{4+\lfloor m/2 \rfloor}\}$ which claims that $\dim_a(\text{ML}_m) \leq 4$. All vertices have the following representations:

$$r(v_k \mid Q) = (a, b, c, d), \quad k = 1, 2, \cdots, 2m, \quad (20)$$

where $a, b, c, d = 0$ when $k = 1, 2, 4, 4 + \lfloor m/2 \rfloor$, respectively, and $a = 1$ when $k = 2, m + 1, 2m$, $b = 1$ when $k = 1, 3, m + 2$, $c = 1$ when $k = m - 2, n, 2m - 1$, and $d = 1$ when $m - 1, m + 1, 2m$; otherwise, all $a, b, c, d$ are 2. It proves the claim that $\dim_a(\text{ML}_m) \leq 4$; on the other hand, suppose that $\dim_a(\text{ML}_m) = 3$, when $m = 3, 5$. □

*Case 1.* Let $Q'_a = \{v_i, v_j, v_k\}$, $1 \leq i, j, k \leq 2m$, taking vertices in $Q'_a$ with 0-size gap. Then, we have the same representations in $r(v_l \mid Q'_a) = r(v_{l'} \mid Q'_a)$, $1 \leq l \leq m + 1$ and $m + 2 \leq l' \leq 2m$.

*Case 2.* Let $Q'_a = \{v_i, v_j, v_k\}$, $1 \leq i, j, k \leq 2m$, with the vertices in $Q'_a$ with any size of gap discussed in Theorem 3, part 6 Then, we have the same representations in $r(v_l \mid Q'_a) = r(v_{l'}$

Table 1: Representations of vertices with respect to $Q_a = \{v_1, v_2, v_5\}$.

| $v_k$ | $r(v_k \mid v_1)$ | $r(v_k \mid v_2)$ | $r(v_k \mid v_5)$ | $v_k$ | $r(v_k \mid v_1)$ | $r(v_k \mid v_2)$ | $r(v_k \mid v_5)$ |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 2 | 5 | 1 | 2 | 1 |
| 2 | 1 | 0 | 1 | 6 | 2 | 1 | 0 |
| 3 | 2 | 1 | 2 | 7 | 2 | 2 | 1 |
| 4 | 2 | 2 | 2 | 8 | 1 | 2 | 2 |

$\mid Q_a')$, $1 \le l \le m+1$ and $m+2 \le l' \le 2m$. It means that when choosing the resolving set for adjacency metric dimension $Q_a'$ with cardinality three and vertices with any gap size, there must exist two vertices with the same representations which have one belonging to $l$-domain and the second from $l'$-domain. From all above cases, $\dim_a(\mathrm{ML}_m) \ne 3$ for the values of $m = 3, 5$. Hence,

$$\dim_a(\mathrm{ML}_m) = 4. \tag{21}$$

Part C. Here, we will use the induction method. For the base step, we choose $m = 6$; the $\dim_a(\mathrm{ML}_6) = 4$ having the adjacency metric generator $Q_a = \{v_1, v_3, v_5, v_7\}$, and the following are the representations of all vertices:

$$r(v_k \mid Q_a) = (a, b, c, d), \quad k = 1, 2, \cdots, 2m, \tag{22}$$

where $a, b, c, d = 0$ when $k = 1, 3, 5, 7$, respectively, and $a = 1$ when $k = 2, 7, 12$, $b = 1$ when $k = 2, 4, 9$, $c = 1$ when $k = 4, 6, 11$ , and $d = 1$ when $k = 1, 6, 8$; otherwise, all values of $a, b, c, d$ are 2. It concludes that

$$\dim_a(\mathrm{ML}_6) \le 4. \tag{23}$$

The contradiction side gives us $\dim_a(\mathrm{ML}_6) = 3$. For this contradiction, possible cases are discussed in the converse of part b of the theorem. One can evaluate from the discussion that $Q_a$ with cardinality three is not possible which implies that

$$\dim_a(\mathrm{ML}_6) = 4. \tag{24}$$

Now, assume that it is also true for $m = l$, and we have to show that $m = l + 1$ which leads to the inductive step:

$$\dim_a(\mathrm{ML}_{l+1}) = l - 1. \tag{25}$$

Assume that

$$\dim_a(\mathrm{ML}_{l+1}) = \dim_a(\mathrm{ML}_l) + \dim_a(\mathrm{ML}_6) - 3. \tag{26}$$

Using equations (24) and (25) in equation (26), we have

$$\dim_a(\mathrm{ML}_{l+1}) = m - 2 + 4 - 3 = m - 1. \tag{27}$$

It completes the proof of the theorem with all possible cases on $m$-cycles of the Möbius ladder graph.

**Theorem 5.** *Let $HML_m$ be a hexagonal Möbius ladder graph. Then,*

$$dim_a(HML_m) = \begin{cases} 3, & if\ m = 2, \\ 4, & if\ m = 3, \\ 6, & if\ m = 4, \\ m+3, & if\ m \ge 5. \end{cases} \tag{28}$$

*Proof.* We break the proof of the theorem into four parts according to the adjacency metric dimension.

Part A. In this part, we claim that $\dim_a(\mathrm{HML}_2) = 3$. For the case $\dim_a(\mathrm{HML}_2) \le 3$, let the adjacency metric generator be $Q_a = \{v_1, v_2, v_6\}$, and those shown in Table 2 are the representations of all vertices of $\mathrm{HML}_2$ with respect to $Q_a$.

In Table 2, the given representations are different, and it proves that $\dim_a(\mathrm{HML}_2) \le 3$; on the reverse, for inequality which is $\dim_a(\mathrm{HML}_2) = 2$, using Proposition 1 is not true, concluding that

$$\dim_a(\mathrm{HML}_2) = 3. \tag{29}$$

Part B. This part contains the adjacency metric dimension of the hexagonal Möbius ladder graph with $m = 3$; for this, let the adjacency metric generator be $Q_a = \{v_1, v_3, v_5, v_7\}$, and those shown in Table 3 are the representations of all vertices according to $Q_a$, which are again different, and it proves that $\dim_a(\mathrm{HML}_3) \le 4$; on the reverse, inequality which is $\dim_a(\mathrm{HML}_3) = 3$ is not true, and the following are some discussions to this side. □

*Case 1.* Let $Q_a' = \{v_i, v_j, v_k\}$, $1 \le i, j, k \le 4m$, taking vertices in $Q_a'$ with any size of gap discussed in Theorem 3, part 6 Then, we have the same representations in either $r(v_1 \mid Q_a') = r(v_{l'} \mid Q_a')$ or $r(v_l \mid Q_a') = r(v_{l'} \mid Q_a')$, $1 \le l \le 2m+1$ and $2m+2 \le l' \le 4m$. It means that when choosing the resolving set of adjacency metric dimension $Q_a'$ with cardinality three and vertices with any gap size, there must exist two vertices with the same representations which have one belonging to $l$-domain and the second from $l'$-domain. It is also possible that the same representations are in two vertices, and $v_1$ is one of them, and the second vertex belongs to $l'$-domain. From all above cases, $\dim_a(\mathrm{ML}_3) \ne 3$. Hence,

$$\dim_a(G) = 4. \tag{30}$$

Part C. For $\dim_a(\mathrm{HML}_4) \le 6$, let the adjacency metric

Table 2: Representations of vertices with respect to $Q_a = \{v_1, v_2, v_6\}$.

| $v_k$ | $r(v_k \mid v_1)$ | $r(v_k \mid v_2)$ | $r(v_k \mid v_6)$ | $v_k$ | $r(v_k \mid v_1)$ | $r(v_k \mid v_2)$ | $r(v_k \mid v_6)$ |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 2 | 5 | 1 | 2 | 1 |
| 2 | 1 | 0 | 2 | 6 | 2 | 1 | 0 |
| 3 | 2 | 1 | 2 | 7 | 2 | 2 | 1 |
| 4 | 2 | 2 | 2 | 8 | 1 | 2 | 2 |

Table 3: Representations of vertices with respect to $Q_a = \{v_1, v_3, v_5, v_7\}$.

| $v_k$ | $r(v_k \mid v_1)$ | $r(v_k \mid v_3)$ | $r(v_k \mid v_5)$ | $r(v_k \mid v_7)$ | $v_k$ | $r(v_k \mid v_1)$ | $r(v_k \mid v_3)$ | $r(v_k \mid v_5)$ | $r(v_k \mid v_7)$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 2 | 2 | 1 | 7 | 1 | 2 | 2 | 0 |
| 2 | 1 | 1 | 2 | 2 | 8 | 2 | 2 | 2 | 1 |
| 3 | 2 | 0 | 2 | 2 | 9 | 2 | 1 | 2 | 2 |
| 4 | 2 | 1 | 1 | 2 | 10 | 2 | 2 | 2 | 2 |
| 5 | 2 | 2 | 0 | 2 | 11 | 2 | 2 | 1 | 2 |
| 6 | 2 | 2 | 1 | 1 | 12 | 1 | 2 | 2 | 2 |

generator be $Q_a = \{v_1, v_2, v_5, v_7, v_9, v_{12}\}$, and representations are shown in Table 4.

The representations given in Table 4 prove that $\dim_a(\mathrm{HML}_4) \le 6$. On contrary, suppose that $\dim_a(\mathrm{HML}_4) = 5$; the following are some discussions to the side of this contradiction.

*Case 1.* Let $Q'_a = \{v_i, v_j, v_k, v_l, v_m\}, 1 \le i, j, k, l, m \le 4m$, taking vertices in $Q'_a$ with any size of gap already discussed above in the theorem. Then, we have the same representations in either $r(v_1 \mid Q'_a) = r(v_{l'} \mid Q'_a)$ or $r(v_l \mid Q'_a) = r(v_{l'} \mid Q'_a), 1 \le l \le 2m+1$ and $2m+2 \le l' \le 4m$. It means that when choosing the resolving set for adjacency metric dimension $Q'_a$ with cardinality five and vertices with any gap size, there must exist two vertices with the same representations which have one belonging to $l$-domain and the second from $l'$-domain. It is also possible that the same representations are in two vertices and $v_1$ is one of them, and the second vertex belongs to $l'$-domain. From all above cases, $\dim_a(\mathrm{ML}_4) \ne 5$. Hence,

$$\dim_a(G) = 6. \tag{31}$$

Part D. In this part for the proof of $\dim_a(\mathrm{HML}_m) = m + 3$ where $m \ge 5$, we will apply the induction method and the base case for $m = 5$, which implies that $\dim_a(\mathrm{HML}_5) = 8$, to prove that $\dim_a(\mathrm{HML}_5) \le 8$; the following are the representations with respect to the adjacency metric generator $Q_a = \{v_1, v_2, v_4, v_7, v_9, v_{11}, v_{13}, v_{16}\}$:

$$r(v_k \mid Q_a) = (a, b, c, d, e, f, g, h), \quad k = 1, 2, \cdots, 4m. \tag{32}$$

where $a, b, c, d, e, f, g, h = 0$ when $k = 1, 2, 4, 7, 9, 11, 13, 16$, and $a = 1$ when $k = 2, 11, 20$, $b = 1$ when $k = 1, 3$, $c = 1$ when $k = 3, 5$, $d = 1$ when $k = 6, 8, 17$, $e = 1$ when $k = 8, 10, 19$, $f = 1$ when $k = 1, 10, 12$, $g = 1$ when $k = 3, 12, 14$, and $h = 1$ when $k = 15, 17$; otherwise, all values of $a, b, c, d, e, f, g, h$ are 2.

Table 4: Representations $r(v_k \mid Q_a)$ of vertices with respect to $Q_a = \{v_1, v_2, v_5, v_7, v_9, v_{12}\}$.

| $v_k$ | $v_1$ | $v_2$ | $v_5$ | $v_7$ | $v_9$ | $v_{12}$ | $v_k$ | $v_1$ | $v_2$ | $v_5$ | $v_7$ | $v_9$ | $v_{12}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 2 | 2 | 1 | 2 | 9 | 1 | 2 | 2 | 2 | 0 | 2 |
| 2 | 1 | 0 | 2 | 2 | 2 | 2 | 10 | 2 | 2 | 2 | 2 | 1 | 2 |
| 3 | 2 | 1 | 2 | 2 | 2 | 2 | 11 | 2 | 2 | 2 | 2 | 2 | 1 |
| 4 | 2 | 2 | 1 | 2 | 2 | 2 | 12 | 2 | 2 | 2 | 2 | 2 | 0 |
| 5 | 2 | 2 | 0 | 2 | 2 | 2 | 13 | 2 | 2 | 1 | 2 | 2 | 1 |
| 6 | 2 | 2 | 1 | 1 | 2 | 2 | 14 | 2 | 2 | 2 | 2 | 2 | 2 |
| 7 | 2 | 2 | 2 | 0 | 2 | 2 | 15 | 2 | 2 | 2 | 1 | 2 | 2 |
| 8 | 2 | 2 | 2 | 1 | 1 | 2 | 16 | 1 | 2 | 2 | 2 | 2 | 2 |

All the representations show that there are no two vertices with the same representation; hence,

$$\dim_a(\mathrm{HML}_5) \le 8. \tag{33}$$

On the contrary, suppose that $\dim_a(\mathrm{HML}_5) = 7$; the following are some discussions to the side of contradiction.

*Case 1.* Let $Q'_a = \{v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}, v_{i_5}, v_{i_6}, v_{i_7}\}, 1 \le i_j \le 4m$ and $1 \le j \le 7$, taking vertices in $Q'_a$ with any size of gap discussed in Theorem 3, part 6 Then, we have the same representations in either $r(v_1 \mid Q'_a) = r(v_2 \mid Q'_a)$ or $r(v_1 \mid Q'_a) = r(v_{l'} \mid Q'_a)$ or $r(v_l \mid Q'_a) = r(v_{l'} \mid Q'_a), 1 \le l \le 2m+1$ and $2m+2 \le l' \le 4m$. It means that when choosing the resolving set of adjacency metric dimension $Q'_a$ with cardinality seven and vertices with any gap size, there must exist two vertices with the same representations which have one belonging to $l$-domain and the second from $l'$-domain. It is also possible that the same representations are in two vertices and $v_1$ is one of them, and the second vertex belongs to $l'$-domain. From all above cases, $\dim_a(\mathrm{HML}_5) \neq 7$. Hence,

$$\dim_a(\mathrm{HML}_5) = 8. \tag{34}$$

Now, assume that the assertion is true for $m = l$:

$$\dim_a(\mathrm{HML}_m) = l + 3. \tag{35}$$

We have to show that it is also true for $m = l + 1$. Suppose

$$\dim_a(\mathrm{HML}_m) = \dim_a(\mathrm{HML}_{m+1}) + \dim_a(\mathrm{HML}_5) - 7. \tag{36}$$

Plugging the values of equations (34) and (35) in equation (36), we have

$$\dim_a(\mathrm{HML}_{l+1}) = l + 3 + 8 - 7 = l + 4. \tag{37}$$

As a result, the result holds for all positive integers $m \geq 5$. It also completes the all possible cases that we assume on the very start of the proof.

**Theorem 6.** *Let $L_n$ be a ladder graph with $n \geq 3$. Then,*

$$dim_a(L_n) = n - \left\lfloor \frac{n+1}{4} \right\rfloor. \tag{38}$$

*Proof.* We prove that $\dim_a(L_n) = n - \lfloor (n+1)/4 \rfloor$ with the induction method and the base step is $n = 3$ which implies that $\dim_a(L_3) = 2$. If we assume on the contrary it is not possible that $\dim_a(L_3) = 1$, for $\dim_a(L_3) \leq 2$, all vertices have the following representations with respect to $Q_a = \{v_1, v_5\}$:

$$r(v_k \mid Q_a) = (a, b), \quad k = 1, 2, \cdots, 4n. \tag{39}$$

where $a, b = 0$ when $k = 1, 5$, respectively, and $a = 1$ when $k = 2, 3$, $b = 1$ when $k = 3, 6$; otherwise, all values of $a, b$ are 2. All the representations show that there are no two vertices with the same representation; hence,

$$\dim_a(L_3) = 2. \tag{40}$$

Now, assume that the assertion is true for $n = k$:

$$\dim_a(L_k) = k - \left\lfloor \frac{k+1}{4} \right\rfloor. \tag{41}$$

We have to show that it is also true for $n = k + 1$. Suppose

$$\dim_a(L_m) = \dim_a(L_{m+1}) + \dim_a(L_3) - 1. \tag{42}$$

Using equations (40) and (41) in equation (42), we have

$$\begin{aligned} \dim_a(L_{k+1}) &= k - \left\lfloor \frac{k+1}{4} \right\rfloor + 2 - 1, = (k+1) - \left\lfloor \frac{k+1}{4} \right\rfloor, \\ &= (k+1) - \left\lfloor \frac{k+2}{4} \right\rfloor, \end{aligned} \tag{43}$$

where $\lfloor (k+1)/4 \rfloor = \lfloor (k+2)/4 \rfloor$. As a consequence, the result holds for all positive integers for $n \geq 3$, which completes the proof. □

## 4. Conclusion

In this article, we studied the adjacency metric dimension of circulant, ladder, Möbius ladder, and hexagonal Möbius ladder graphs. It is known that adjacency metric dimension is a useful parameter in localization, networking, and some robot navigation ideas. Therefore, it is interesting to find adjacency metric generators for generalized classes of graphs. We demonstrated that in adjacency metric generators, all graphs have inconstant numbers of members and that each graph follows the modification of parameters or order.

## Data Availability

There is no data available.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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