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Research Article

Interpolative Prešić Type Contractions and Related Results

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In this article, we will extend the notion of interpolative Kannan contraction by introducing the notions of interpolative Prešić type contractions and interpolative Prešić type proximal contractions for mappings defined on product spaces. Through these notions, we will derive some results to ensure the existence of fixed points and best proximity points for such mappings.

1. Introduction and Preliminaries

The Banach contraction principle is the most significant and basic result of metric fixed point theory. Through this result, we can obtain a unique fixed point of a self-map $N: L \longrightarrow L$, provided that N is a contraction map on a complete metric space (L, d_L). This result motivated Prešić to study about the existence of fixed points of the operators defined on product spaces, that is, $N: L^k \longrightarrow L$, for any fixed $k \in \mathbb{N}$. As an outcome of this motivation, Prešić [1] presented the following noteworthy extension of the Banach contraction principle.

Theorem 1 (see [1]). *Suppose that (L, d_L) be a complete metric space and $N: L^k \longrightarrow L$ be a map, for any fixed $k \in \mathbb{N}$, satisfying the following inequality:*

$$d(N(l_1, l_2, \ldots, l_k), N(l_2, l_3, \ldots, l_{k+1})) \leq \sum_{j=1}^{k} \zeta_j d_L(l_j, l_{j+1}), \quad (1)$$

for every $l_1, l_2, \ldots, l_k, l_{k+1} \in L$, where $\zeta_1, \zeta_2, \ldots, \zeta_k$ are the nonnegative real numbers with $\sum_{j=1}^{k} \zeta_j < 1$. Then, there exists a unique point of L that satisfies the equation $l = N(\underbrace{l, l, \ldots, l}_{k})$.

This result is used to discuss the existence of equilibrium points for the k^{th}-order nonlinear difference equation of the form

$$l_{n+k} = N(l_n, l_{n+1}, \ldots, l_{n+k-1}), \quad (2)$$

where $N: L^k \subset \mathbb{R}^k \longrightarrow L$ is a continuous map. Note that a point $l^* \in L$ is known as an equilibrium point of (2) if $l^* = N(\underbrace{l^*, l^*, \ldots, l^*}_{k})$. Such a point is also known as a fixed point of $N: L^k \longrightarrow L$. Some well-known generalizations of this work have been studied by several authors, for example, [2–5].

Kannan and Chatterjea made a vital contribution in the development of this field through the fixed point results derived in [6, 7], respectively. Recently, Karapınar [8] modified the Kannan contraction by introducing interpolative Kannan contraction, stated as, a map $N: (L, d_L) \longrightarrow (L, d_L)$ is called an interpolative Kannan contraction [8] if

$$d_L(Nk, Nl) \leq \zeta d_L(k, Nk)^{\vartheta} d_L(l, Nl)^{1-\vartheta}, \quad (3)$$

for all $k, l \in L$ with $k \neq Nk$ and $l \neq Nl$, where $\zeta \in [0, 1)$ and $\vartheta \in (0, 1)$.

After that, many existing contraction-type conditions have been generalized in the sense of interpolative Kannan contraction, for example, Karapınar et al. [9] studied interpolative Reich-Rus-Ćirić type contraction in partial metric spaces, Aydi et al. [10] studied interpolative Ćirić-Reich-Rus type contractions in Branciari metric spaces, Mohammadi et al. [11] extended the concept of F-contractions by interpolative Ćirić-Reich-Rus type

F-contractions, Karapınar et al. [12] studied interpolative Hardy–Rogers type contractions, Debnath and Sen [13] studied set-valued interpolative Hardy-Rogers and set-valued Reich-Rus-Ćirić-type contractions, Sarwar et al. [14] presented rational type interpolative contractions, Khan et al. [15] worked on interpolative (ϕ, ψ)-type Z-contractions, Altun and Tasdemir [16] presented interpolative proximal contractions for nonself mappings, Fulga and Yesilkaya [17] studied interpolative Suzuki-type contractions, Karapınar et al. [18] defined $(\alpha, \beta, \psi, \phi)$-interpolative contractions, and Alansari and Ali [19] studied multivalued interpolative Reich-Rus-Ćirić-type contractions.

Gaba and Karapınar [20] extended the notion of interpolative Kannan contraction through exponential powers, stated as, a map $N: (L, d_L) \longrightarrow (L, d_L)$ is called an $(\zeta, \vartheta_1, \vartheta_2)$-interpolative Kannan contraction, if

$$d_L(Nk, Nl) \le \zeta d_L(k, Nk)^{\vartheta_1} d_L(l, Nl)^{\vartheta_2}, \tag{4}$$

for all $k, l \in L$ with $k \ne Nk$ and $l \ne Nl$, where $\vartheta_1, \vartheta_2 \in (0,1)$ with $\vartheta_1 + \vartheta_2 < 1$ and $\zeta \in [0,1)$. Readers can find other similar generalizations in [21].

Consider O_L and M_L be nonvoid subsets of a metric space (L, d_L). It is well-known that a fixed point of a map $N: O_L \longrightarrow M_L$ is a solution of $Nl = l$. If $O_L \cap M_L = \varnothing$, then fixed point of $N: O_L \longrightarrow M_L$ does not exist, that is, $d_L(Nl, l) > 0$ for all $l \in O_L$. In this situation, we try to find $l \in O_L$, such that $d_L(Nl, l)$ attain the minimum value in some sense. It is obvious that the smallest value that can be obtained by $d_L(Nl, l)$ for any $l \in O_L$ will be greater or equal to $D_L(O_L, M_L)$, that is, distance between O_L and M_L. A point $l \in O_L$ is said to be a best proximity point of $N: O_L \longrightarrow M_L$, if $d_L(Nl, l) = D_L(O_L, M_L)$. The existence of such points of nonself maps has been discussed by several researchers in different ways, for example, Caballero et al. [22] studied the existence of best proximity points for nonself maps satisfying Geraghty contraction and P-property in metric spaces, Bilgili et al. [23], Aydi et al. [24], and Pitea [25] extended the work of Caballero et al. [22] by introducing generalized Geraghty contraction, ψ-Geraghty contraction and generalized almost θ-Geraghty contraction for nonself maps, Basha and Shahzad [26] and Basha [27] defined proximal-type contractions to study the existence of best proximity points, Jleli and Samet [28] defined α-ψ-proximal contraction to ensure the existence of best proximity points, Jleli et al. [29] and Aydi et al. [30] defined generalized α-ψ-proximal contractions to extend the work of Jleli and Samet [28], Abkar and Gabeleh [31] and Kumam et al. [32] studied the existence of best proximity points for multivalued nonself maps in metric spaces, Ali et al. [33] defined implicit proximal contractions, Sahin et al. [34] defined proximal nonunique contraction, and Ali et al. [35] studied the existence of best proximity points for Prešić type nonself operators satisfying proximal type contractions.

This article aims to present the notions of interpolative Prešić type contractions and interpolative Prešić type proximal contractions for mappings defined on product spaces. Through these notions, we will study the existence of fixed points and best proximity points for such mappings.

2. Main Results

We begin this section with the following definition.

Definition 1. A map $N: L \times L \longrightarrow L$ is called an interpolative Prešić type-I contraction, if for each $s, w, t, v \in L \setminus Fix(N)$, we get

$$d_L(N(s,w), N(t,v))^{\min\{\gamma(s,w), \gamma(t,v)\}} \le \zeta d_L(w, N(s,w))^{\vartheta_1} d_L(v, N(t,v))^{\vartheta_2}, \tag{5}$$

where $\gamma: L \times L \longrightarrow \mathbb{R} \setminus \{0\}$ is a map, $\vartheta_1, \vartheta_2 \in (0,1)$ with $\vartheta_1 + \vartheta_2 = 1$, $\zeta \in [0,1)$, and $Fix(N) = \{l \in L: l = N(l,l)\}$.

The following theorem is used to study the existence of fixed points for the above map.

Theorem 2. *Consider an interpolative Prešić type-I contraction map $N: L \times L \longrightarrow L$ on a complete metric space (L, d_L). Also, consider that*

- *(i) If $\min\{\gamma(s,w), \gamma(t,v)\} = 1$, then $\gamma(N(s,w), N(t,v)) = 1$.*
- *(ii) There exist two elements $s, w \in L$ with $\min\{\gamma(s,w), \gamma(w, N(s,w))\} = 1$.*
- *(iii) For every sequence $\{l_m\}$ in L with $\gamma(l_m, l_{m+1}) = 1\, \forall m \ge m_0$ for some natural number m_0 and $l_m \longrightarrow l$, we have $\gamma(l,l) = 1$.*

Then, there exists at least one point of L that satisfies the equation $l = N(l,l)$.

Proof. Hypothesis (ii) assures that there are two points, say, l_0 and l_1 in L with

$$\min\{\gamma(l_0, l_1), \gamma(l_1, N(l_0, l_1))\} = 1. \tag{6}$$

By using these two points, we can define a sequence $\{l_m\}$ with $l_{m+1} = N(l_{m-1}, l_m) \forall m \in \mathbb{N}$. From hypothesis (i), it can be concluded that $\gamma(l_m, l_{m+1}) = 1$, $\forall m > 1$. Hence,

$$\min\{\gamma(l_{m-1}, l_m), \gamma(l_m, l_{m+1})\} = 1 \;\forall m \in \mathbb{N}. \tag{7}$$

By (5), we get

$$\begin{aligned} d_L(N(l_{m-1},l_m),N(l_m,l_{m+1})) &= d_L(N(l_{m-1},l_m),N(l_m,l_{m+1}))^{\min\{\gamma(l_{m-1},l_m),\gamma(l_m,l_{m+1})\}},\\ &\leq \zeta d_L(l_m,N(l_{m-1},l_m))^{\vartheta_1} d_L(l_{m+1},N(l_m,l_{m+1}))^{\vartheta_2} \forall m\in\mathbb{N}, \end{aligned} \tag{8}$$

that is,

$$d_L(l_{m+1},l_{m+2}) \leq \zeta d_L(l_m,l_{m+1})^{\vartheta_1} d_L(l_{m+1},l_{m+2})^{\vartheta_2}, \quad \forall m\in\mathbb{N}. \tag{9}$$

By (9), we obtain

$$d_L(l_{m+1},l_{m+2})^{1-\vartheta_2} \leq \zeta d_L(l_m,l_{m+1})^{\vartheta_1}, \quad \forall m\in\mathbb{N}. \tag{10}$$

Since $1-\vartheta_2=\vartheta_1$, thus, by (10), we get

$$\begin{aligned} d_L(l_{m+1},l_{m+2}) &\leq \zeta^{1/(1-\vartheta_2)} d_L(l_m,l_{m+1}),\\ &\leq \zeta d_L(l_m,l_{m+1}) \forall m\in\mathbb{N}. \end{aligned} \tag{11}$$

Hence, by (11), we conclude that

$$d_L(l_{m+1},l_{m+2}) \leq \zeta^m d_L(l_1,l_2), \quad \forall m\in\mathbb{N}. \tag{12}$$

By triangle inequality and (12), for each $k,n\in\mathbb{N}$ with $k>n$, we obtain

$$d_L(l_n,l_k) \leq \sum_{j=n}^{k-1} d_L(l_j,l_{j+1}) \leq \sum_{j=n}^{k-1} \zeta^{j-1} d_L(l_1,l_2). \tag{13}$$

In view of the above inequality and the convergence of $\sum_{j=1}^{\infty}\zeta^j$, we say that the sequence $\{l_m\}$ is a Cauchy in L. By the completeness of (L,d_L), we get a point $l^*\in L$, such that $l_m \longrightarrow l^*$. Also, by (iii), we get $\gamma(l^*,l^*)=1$, since $\gamma(l_m,l_{m+1})=1, \quad \forall m\in\mathbb{N}$ and $l_m \longrightarrow l^*$.

Here, the claim is $l^*=N(l^*,l^*)$. If the claim is wrong, then by (5), for each $m\in\mathbb{N}$, we get

$$\begin{aligned} d_L(N(l_{m-1},l_m),N(l^*,l^*)) &= d_L(N(l_{m-1},l_m),N(l^*,l^*))^{\min\{\gamma(l_{m-1},l_m),\gamma(l^*,l^*)\}},\\ &\leq \zeta d_L(l_m,N(l_{m-1},l_m))^{\vartheta_1} d_L(l^*,N(l^*,l^*))^{\vartheta_2},\\ &\leq \zeta d_L(l_{m+1},l_{m+2})^{\vartheta_1} d_L(l^*,N(l^*,l^*))^{\vartheta_2}. \end{aligned} \tag{14}$$

By triangle inequality and (14), we obtain

$$\begin{aligned} d_L(l^*,N(l^*,l^*)) &\leq d_L(l^*,N(l_m,l_{m+1})) + d_L(N(l_m,l_{m+1}),N(l^*,l^*)),\\ &\leq d_L(l^*,l_{m+2}) + \zeta d_L(l_{m+1},l_{m+2})^{\vartheta_1} d_L(l^*,N(l^*,l^*))^{\vartheta_2}. \end{aligned} \tag{15}$$

Letting $m\longrightarrow\infty$ in (15), we get $d_L(l^*,N(l^*,l^*))=0$. Hence, the claim is true, that is, $l^*=N(l^*,l^*)$. □

Example 1. Consider $L=\mathbb{R}$ equipped with a metric $d_L(k,l)=|k-l|$ for each $k,l\in L$. Define $N: L\times L \longrightarrow L$ and $\gamma: L\times L \longrightarrow \mathbb{R}\setminus\{0\}$ by

$$N(k,l)=\begin{cases} \dfrac{k+l}{2}, & \text{if } k,l\geq 0,\\ 0, & \text{otherwise,} \end{cases} \tag{16}$$

and

$$\gamma(k,l)=\begin{cases} 1, & \text{if } k,\ l\geq 0.\\ 1/2, & \text{otherwise.} \end{cases} \tag{17}$$

Then, one can easily verify that the axioms of Theorem 2 are satisfied. Hence, there is at least one element $l^*\in L$, such that $l^*=N(l^*,l^*)$.

In the following, we present interpolative Prešić type-II contraction map and related fixed point result.

Definition 2. A map $N: L\times L\longrightarrow L$ is called an interpolative Prešić type-II contraction, if for each $s,w,t,v\in L\setminus\mathrm{Fix}(N)$ with $\min\{\gamma(s,w),\gamma(t,v)\}\geq 1$, we get

$$d_L(N(s,w),N(t,v)) \leq \zeta d_L(w,N(s,w))^{\vartheta_1} d_L(v,N(t,v))^{\vartheta_2}, \tag{18}$$

where $\gamma: L\times L\longrightarrow\mathbb{R}$ is a map, $\vartheta_1,\vartheta_2\in(0,1)$ with $\vartheta_1+\vartheta_2=1$, $\zeta\in[0,1)$, and $\mathrm{Fix}(N)=\{l\in L: l=N(l,l)\}$.

Theorem 3. *Consider an interpolative Prešić type-II contraction map $N: L\times L\longrightarrow L$ on a complete metric space (L,d_L). Also, consider that*

- *(i) If $\min\{\gamma(s,w),\gamma(t,v)\}\geq 1$, then $\gamma(N(s,w),N(t,v))\geq 1$.*
- *(ii) There exist two elements $s,w\in L$ with $\min\{\gamma(s,w),\gamma(w,N(s,w))\}\geq 1$.*
- *(iii) For every sequence $\{l_m\}$ in L with $\gamma(l_m,l_{m+1})\geq 1\ \forall m\geq m_0$ for some natural number m_0 and $l_m\longrightarrow l$, we have $\gamma(l,l)\geq 1$.*

Then, there exists at least one point of L that satisfies the equation $l = N(l, l)$.

Proof. In view of the hypothesis (ii), we get

$$\min\{\gamma(l_0, l_1), \gamma(l_1, N(l_0, l_1))\} \geq 1. \tag{19}$$

For some l_0 and l_1 in L. Through these two points, we can construct a sequence $\{l_m\}$ with $l_{m+1} = N(l_{m-1}, l_m) \forall m \in \mathbb{N}$. Also, hypothesis (i) implies that $\gamma(l_m, l_{m+1}) \geq 1, \quad \forall m > 1$. Hence,

$$\min\{\gamma(l_{m-1}, l_m), \gamma(l_m, l_{m+1})\} \geq 1 \ \forall\, m \in \mathbb{N}. \tag{20}$$

By (18), we get

$$d_L(N(l_{m-1}, l_m), N(l_m, l_{m+1})) \leq \zeta d_L(l_m, N(l_{m-1}, l_m))^{\vartheta_1} d_L(l_{m+1}, N(l_m, l_{m+1}))^{\vartheta_2} \forall m \in \mathbb{N}. \tag{21}$$

By performing some calculations, we get

$$d_L(l_{m+1}, l_{m+2}) \leq \zeta d_L(l_m, l_{m+1})^{\vartheta_1} \leq \zeta^m d_L(l_1, l_2), \quad \forall m \in \mathbb{N}. \tag{22}$$

Hence, it can be seen that $\{l_m\}$ is a Cauchy sequence in L with $l_m \longrightarrow l^* \in L$. Also, by (iii), we get $\gamma(l^*, l^*) \geq 1$. Suppose that $l^* \neq N(l^*, l^*)$. Then, by (18), for each $m \in \mathbb{N}$, we get

$$\begin{aligned} d_L(N(l_m, l_{m+1}), N(l^*, l^*)) &\leq \zeta d_L(l_{m+1}, N(l_m, l_{m+1}))^{\vartheta_1} d_L(l^*, N(l^*, l^*))^{\vartheta_2}, \\ &\leq \zeta d_L(l_{m+1}, l_{m+2})^{\vartheta_1} d_L(l^*, N(l^*, l^*))^{\vartheta_2}. \end{aligned} \tag{23}$$

By triangle inequality and (23), we obtain

$$\begin{aligned} d_L(l^*, N(l^*, l^*)) &\leq d_L(l^*, N(l_m, l_{m+1})) + d_L(N(l_m, l_{m+1}), N(l^*, l^*)), \\ &\leq d_L(l^*, l_{m+2}) + \zeta d_L(l_{m+1}, l_{m+2})^{\vartheta_1} d_L(l^*, N(l^*, l^*))^{\vartheta_2}. \end{aligned} \tag{24}$$

Letting $m \longrightarrow \infty$ in (24), we get $d_L(l^*, N(l^*, l^*)) = 0$. Hence, our supposition is wrong and $l^* = N(l^*, l^*)$. □

Example 2. Consider $L = \mathbb{Z}$ equipped with a metric $d_L(k, l) = |k - l|$ for each $k, l \in L$. Define $N\colon L \times L \longrightarrow L$ and $\gamma\colon L \times L \longrightarrow \mathbb{R}$ by

$$N(k, l) = \begin{cases} l, & \text{if } k, l \geq 0, \\ |k| + |l| & \text{otherwise,} \end{cases} \tag{25}$$

and

$$\gamma(k, l) = \begin{cases} 1, & \text{if } k,\ l \geq 0. \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

Then, one can easily verify that the axioms of Theorem 3 are satisfied. Hence, there is at least one element $l^* \in L$, such that $l^* = N(l^*, l^*)$.

Some consequences of the above results can be obtained in the form of the following listed corollaries. The following corollary is obtained from Theorem 2 by considering $t = w$.

Corollary 1. *Consider a map $N\colon L \times L \longrightarrow L$ on a complete metric space (L, d_L), such that for each $s, w, v \in L \setminus Fix(N)$, we get*

$$d_L(N(s, w), N(w, v))^{\min\{\gamma(s,w), \gamma(w,v)\}} \leq \zeta d_L(w, N(s, w))^{\vartheta_1} d_L(v, N(w, v))^{\vartheta_2}, \tag{27}$$

where $\gamma\colon L \times L \longrightarrow \mathbb{R}\setminus\{0\}$ *is a map,* $\vartheta_1, \vartheta_2 \in (0,1)$ *with* $\vartheta_1 + \vartheta_2 = 1$*, and* $\zeta \in [0,1)$*. Also, consider that*

(i) *If* $\min\{\gamma(s,w), \gamma(w,v)\} = 1$*, then* $\gamma(N(s,w), N(w,v)) = 1$

(ii) *There exist two elements* $s, w \in L$ *with* $\min\{\gamma(s,w), \gamma(w, N(s,w))\} = 1$

(iii) *For every sequence* $\{l_m\}$ *in* L *with* $\gamma(l_m, l_{m+1}) = 1\ \forall m \geq m_0$ *for some natural number* m_0 *and* $l_m \longrightarrow l$*, we have* $\gamma(l,l) = 1$

Then, there exists at least one point of L that satisfies the equation $l = N(l,l)$.

The following corollary is a special case of Theorem 3 which can be obtained by considering $t = w$.

Corollary 2. *Consider a map* $N\colon L \times L \longrightarrow L$ *on a complete metric space* (L, d_L)*, such that for each* $s, w, v \in L\setminus Fix(N)$ *with* $\min\{\gamma(s,w), \gamma(w,v)\} \geq 1$*, we get*

$$d_L(N(s,w), N(w,v)) \leq \zeta d_L(w, N(s,w))^{\vartheta_1} d_L(v, N(w,v))^{\vartheta_2}, \tag{28}$$

where $\gamma\colon L \times L \longrightarrow \mathbb{R}$ *is a map,* $\vartheta_1, \vartheta_2 \in (0,1)$ *with* $\vartheta_1 + \vartheta_2 = 1$ *and* $\zeta \in [0,1)$*. Also, consider that*

(i) *If* $\min\{\gamma(s,w), \gamma(w,v)\} \geq 1$*, then* $\gamma(N(s,w), N(w,v)) \geq 1$

(ii) *There exist two elements* $s, w \in L$ *with* $\min\{\gamma(s,w), \gamma(w, N(s,w))\} \geq 1$

(iii) *For every sequence* $\{l_m\}$ *in* L *with* $\gamma(l_m, l_{m+1}) \geq 1\ \forall m \geq m_0$ *for some natural number* m_0 *and* $l_m \longrightarrow l$*, we have* $\gamma(l,l) \geq 1$

Then, there exists at least one point of L that satisfies the equation $l = N(l,l)$.

By defining $\gamma(s,w) = 1$ for each $s, w \in L$ in Theorem 2 or Theorem 3, we get the following.

Corollary 3. *Consider a map* $N\colon L \times L \longrightarrow L$ *on a complete metric space* (L, d_L) *that satisfies*

$$d_L(N(s,w), N(t,v)) \leq \zeta d_L(w, N(s,w))^{\vartheta_1} d_L(v, N(t,v))^{\vartheta_2}, \tag{29}$$

for each $s, w, v \in L\setminus Fix(N)$*, where* $\vartheta_1, \vartheta_2 \in (0,1)$ *with* $\vartheta_1 + \vartheta_2 = 1$ *and* $\zeta \in [0,1)$*. Then, there exists at least one point of L that satisfies the equation* $l = N(l,l)$.

From the above corollary, we can also obtain the following result.

Corollary 4. *Consider a map* $N\colon L \times L \longrightarrow L$ *on a complete metric space* (L, d_L) *that satisfies*

$$d_L(N(s,w), N(w,v)) \leq \zeta d_L(w, N(s,w))^{\vartheta_1} d_L(v, N(w,v))^{\vartheta_2}, \tag{30}$$

for each $s, w, v \in L\setminus Fix(N)$*, where* $\vartheta_1, \vartheta_2 \in (0,1)$ *with* $\vartheta_1 + \vartheta_2 = 1$ *and* $\zeta \in [0,1)$*. Then, there exists at least one point of L that satisfies the equation* $l = N(l,l)$.

In the following, we will study about the interpolative Prešić type proximal contractions and related results.

Let (L, d_L) be a metric space and O_L, M_L be nonvoid subsets of L. We will use the following notations.

$$\begin{aligned} D_L(O_L, M_L) &= \inf\{d_L(o,m)\colon\ o \in O_L,\ m \in M_L\},\\ d_L(o, M_L) &= \inf\{d_L(o,m)\colon\ \ m \in M_L\},\\ O_{L0} &= \{o \in O_L\colon\ d_L(o,m) = D_L(O_L, M_L) \text{ for some } m \in M_L\},\\ M_{L0} &= \{m \in M_L\colon\ d_L(o,m) = D_L(O_L, M_L) \text{ for some } o \in O_L\}. \end{aligned} \tag{31}$$

Note that a point $o^* \in O_L$ is known as a best proximity point of $N\colon O_L \times O_L \longrightarrow M_L$ if $d_L(o^*, N(o^*, o^*)) = D_L(O_L, M_L)$. The collection of all such points for $N\colon O_L \times O_L \longrightarrow M_L$ is denoted by $Bes(N)$.

Definition 3. A map $N\colon O_L \times O_L \longrightarrow M_L$ is called an interpolative Prešić type-I proximal contraction, if for each $s, w, t, v, p, q \in O_L\setminus Bes(N)$ with $d_L(p, N(s,w)) = D_L(O_L, M_L) = d_L(q, N(t,v))$, we get

$$d_L(p,q)^{\min\{\gamma(s,w),\gamma(t,v)\}} \leq \zeta d_L(w,p)^{\vartheta_1} d_L(v,q)^{\vartheta_2}, \tag{32}$$

where $\gamma\colon O_L \times O_L \longrightarrow \mathbb{R}\setminus\{0\}$ is a map, $\vartheta_1, \vartheta_2 \in (0,1)$ with $\vartheta_1 + \vartheta_2 = 1$ and $\zeta \in [0,1)$.

The following theorem is used to ensure the existence of best proximity points for the above defined maps.

Theorem 4. *Consider an interpolative Prešić type-I proximal contraction map* $N\colon O_L \times O_L \longrightarrow M_L$ *on a metric space* (L, d_L)*. Also, consider that*

(i) *If* $\min\{\gamma(s,w), \gamma(t,v)\} = 1$ *and* $d_L(p, N(s,w)) = D_L(O_L, M_L) = d_L(q, N(t,v))$*, then* $\gamma(p,q) = 1$.

(ii) *There exist elements* $s, w, p \in O_L$ *with* $d_L(p, N(s,w)) = D_L(O_L, M_L)$ *and* $\min\{\gamma(s,w), \gamma(w,p)\} \geq 1$.

(iii) $N(O_L \times O_{L0}) \subseteq M_{L0}$.

(iv) O_{L0} *is nonempty and complete with respect to* d_L.

(v) *For every sequence* $\{l_m\}$ *in* O_{L0} *with* $\gamma(l_m, l_{m+1}) = 1\ \forall m \geq m_0$ *for some natural number* m_0 *and* $l_m \longrightarrow l$*, we have* $\gamma(l,l) = 1$.

Then, there exists at least one point of L that satisfies the equation $d_L(l, N(l,l)) = D_L(O_L, M_L)$.

Proof. From hypothesis (ii), we have l_0, l_1, and l_2 in O_L with $\min\{\gamma(l_0,l_1)\gamma(l_1,l_2)\} = 1$ and $d_L(l_2, N(l_0,l_1)) = D_L(O_L, M_L)$. Hypothesis (iii) implies that $N(l_1,l_2) \in M_{L_0}$, and there is $l_3 \in O_{L0}$ satisfying $d_L(l_3, N(l_1,l_2)) = D_L(O_L, M_L)$. Thus, from hypothesis (i), we get $\gamma(l_2,l_3) = 1$. Hence, by using hypotheses (i) and (ii) repeatedly, we conclude that $\gamma(l_{m-1},l_m) = 1$ and $d_L(l_{m+1}, N(l_{m-1},l_m)) = D_L(O_L, M_L)$ for all $m \in \mathbb{N}$.

Since $d_L(l_{m+1}, N(l_{m-1},l_m)) = D_L(O_L, M_L) = d_L(l_{m+2}, N(l_m,l_{m+1}))$ for each $m \in \mathbb{N}$ and $\min\{\gamma(l_{m-1},l_m), \gamma(l_m,l_{m+1})\} = 1$ for each $m \in \mathbb{N}$, then, by (32), we get

$$\begin{aligned} d_L(l_{m+1},l_{m+2}) &\le \zeta d_L(l_{m+1},l_{m+2})^{\min\{\gamma(l_{m-1},l_m),\gamma(l_m,l_{m+1})\}}, \\ &\le \zeta d_L(l_m,l_{m+1})^{\vartheta_1} d_L(l_{m+1},l_{m+2})^{\vartheta_2}, \quad \forall m \in \mathbb{N}. \end{aligned} \tag{33}$$

Now, by following the proof of Theorem 2, we say that $\{l_m\}_{m\ge 2}$ is a Cauchy sequence in O_{L0}. Since O_{L0} is complete, we have a point $l^* \in O_{L0}$, such that $l_m \longrightarrow l^*$. Also, by (v), we get $\gamma(l^*,l^*) = 1$, since $\gamma(l_m,l_{m+1}) = 1$ and $l_m \longrightarrow l^*$. Clearly, $N(l^*,l^*) \in M_{L0}$, and there is $w^* \in O_{L0}$ with $d_L(w^*, N(l^*,l^*)) = D_L(O_L, M_L)$. Here, the claim is $w^* = l^*$. Suppose it is wrong, then by (33), for each $m \in \mathbb{N}$, we get

$$\begin{aligned} d_L(l_{m+1},w^*) &= d_L(l_{m+1},w^*)^{\min\{\gamma(l_{m-1},l_m),\gamma(l^*,l^*)\}}, \\ &\le \zeta d_L(l_m,l_{m+1})^{\vartheta_1} d_L(l^*,w^*)^{\vartheta_2}. \end{aligned} \tag{34}$$

Letting $m \longrightarrow \infty$ in (34), we obtain $d_L(l^*,w^*) = 0$, and it contradicts our assumption. Hence, our claim is true, that is, $l^* = w^*$. Therefore, $d_L(l^*, N(l^*,l^*)) = D_L(O_L, M_L)$.

In the following, we present the notion of interpolative Prešić type-II proximal contraction. ☐

Definition 4. A map $N: O_L \times O_L \longrightarrow M_L$ is called an interpolative Prešić type-II proximal contraction, if for each $s, w, t, v, p, q \in O_L \setminus Bes(N)$ with $d_L(p, N(s,w)) = D_L(O_L, M_L) = d_L(q, N(t,v))$ and $\min\{\gamma(s,w), \gamma(t,v)\} \ge 1$, we get

$$d_L(p,q) \le \zeta d_L(w,p)^{\vartheta_1} d_L(v,q)^{\vartheta_2}, \tag{35}$$

where $\gamma: O_L \times O_L \longrightarrow \mathbb{R}$ is a map, $\vartheta_1, \vartheta_2 \in (0,1)$ with $\vartheta_1 + \vartheta_2 = 1$ and $\zeta \in [0,1)$.

The existence of best proximity points for above map can be ensured by the result given.

Theorem 5. *Consider an interpolative Prešić type-II proximal contraction map* $N: O_L \times O_L \longrightarrow M_L$ *on a metric space* (L, d_L). *Also, consider that*

- *(i) If* $\min\{\gamma(s,w), \gamma(t,v)\} \ge 1$ *and* $d_L(p, N(s,w)) = D_L(O_L, M_L) = d_L(q, N(t,v))$, *then* $\gamma(p,q) \ge 1$.
- *(ii) There exist elements* $s, w, p \in O_L$ *with* $d_L(p, N(s,w)) = D_L(O_L, M_L)$ *and* $\min\{\gamma(s,w), \gamma(w,p)\} \ge 1$.
- *(iii)* $N(O_L \times O_{L0}) \subseteq M_{L0}$.
- *(iv)* O_{L0} *is nonempty and complete with respect to* d_L.
- *(v) For every sequence* $\{l_m\}$ *in* O_{L0} *with* $\gamma(l_m,l_{m+1}) \ge 1\ \forall m \ge m_0$ *for some natural number* m_0 *and* $l_m \longrightarrow l$, *we have* $\gamma(l,l) \ge 1$.

Then, there exists at least one point of L that satisfies the equation $d_L(l, N(l,l)) = D_L(O_L, M_L)$.

Proof. The proof can be derived on the same steps as the proof of Theorem 4 is done. ☐

2.1. Results for Extended Interpolative Prešić Type Maps. This subsection presents the extensions of the above listed results. Theorems 6 and 7 can be considered as an extended version of Theorem 2 and Theorem 3, respectively.

Theorem 6. *Consider an extended interpolative Prešić type-I contraction map* $N: L^k \longrightarrow L$, *for any fixed* $k \in \mathbb{N}$, *on a complete metric space* (L, d_L), *that is, for each* $s_1, s_2, \ldots, s_k, w_1, w_2, \ldots, w_k \in L \setminus Fix(N)$, *we get*

$$\begin{aligned} &d_L(N(s_1,s_2,\ldots,s_k), N(w_1,w_2,\ldots,w_k))^{\min\{\gamma(s_1,w_1),\gamma(s_2,w_2),\ldots,\gamma(s_k,w_k)\}} \\ &\quad \le \zeta d_L(s_k, N(s_1,s_2,\ldots,s_k))^{\vartheta_1} d_L(w_k, N(w_1,w_2,\ldots,w_k))^{\vartheta_2}, \end{aligned} \tag{36}$$

where $\gamma: L \times L \longrightarrow \mathbb{R} \setminus \{0\}$ *is a map,* $\vartheta_1, \vartheta_2 \in (0,1)$ *with* $\vartheta_1 + \vartheta_2 = 1$, $\zeta \in [0,1)$, *and* $Fix(N) = \{l \in L: l = N(l,l,\ldots,l)\}$. *Also, consider that*

- *(i) If* $\min\{\gamma(s_1,w_1), \gamma(s_2,w_2), \ldots, \gamma(s_k,w_k)\} = 1$, *then* $\gamma(N(s_1,s_2,\ldots,s_k), N(w_1,w_2,\ldots,w_k)) = 1$.
- *(ii) There exist elements* $s_1, s_2, \ldots, s_k \in L$, *such that*

$$\min\{\gamma(s_1,s_2), \gamma(s_2,s_3), \ldots, \gamma(s_k, N(s_1,s_2,\ldots,s_k))\} = 1. \tag{37}$$

- *(iii) For every sequence* $\{l_m\}$ *in* L *with* $\gamma(l_m,l_{m+1}) = 1\ \forall m \ge m_0$ *for some natural number* m_0 *and* $l_m \longrightarrow l$, *we have* $\gamma(l_m,l) = 1\ \forall m \ge m_0$.

Then, there exists at least one point of L that satisfies the equation $l = N(\underbrace{l,l,\ldots,l}_{k\text{-times}})$.

Proof. Hypothesis (ii) implies the existence of elements $l_1, l_2, \ldots, l_k$ in L with

$$\min\{\gamma(l_1,l_2),\gamma(l_2,l_3),\ldots,\gamma(l_k,N(l_1,l_2,\ldots,l_k))\} = 1. \tag{38}$$

Through these points, we can define a sequence $\{l_m\}$ with $l_{m+k} = N(l_m, l_{m+1}, \ldots, l_{m+k-1})$ for all $m \in \mathbb{N}$. Hence, by considering hypothesis (i), it can be concluded that $\gamma(l_m, l_{m+1}) = 1\ \forall m \geq m_0$. Then, we say that

$$\min\{\gamma(l_m,l_{m+1}),\gamma(l_m,l_{m+1}),\ldots,\gamma(l_{m+k-1},l_{m+k})\} = 1\ \forall m \in \mathbb{N}. \tag{39}$$

By (36), we get

$$\begin{aligned} &d_L(N(l_m,l_{m+1},\ldots,l_{m+k-1}),N(l_{m+1},l_{m+2},\ldots,l_{m+k})) \\ &\quad= d_L(N(l_m,l_{m+1},\ldots,l_{m+k-1}),N(l_{m+1},l_{m+2},\ldots,l_{m+k}))^{\min\{\gamma(l_m,l_{m+1}),\gamma(l_{m+1},l_{m+2}),\ldots,\gamma(l_{m+k-1},l_{m+k})\}} \\ &\quad\leq \zeta d_L(l_{m+k-1},N(l_m,l_{m+1},\ldots,l_{m+k-1}))^{\vartheta_1} d_L(l_{m+k},N(l_{m+1},l_{m+2},\ldots,l_{m+k}))^{\vartheta_2}\forall m \in \mathbb{N}, \end{aligned} \tag{40}$$

that is,

$$d_L(l_{m+k},l_{m+k+1}) \leq \zeta d_L(l_{m+k-1},l_{m+k})^{\vartheta_1} d_L(l_{m+k},l_{m+k+1})^{\vartheta_2}, \quad \forall m \in \mathbb{N}. \tag{41}$$

By (41), we obtain

$$d_L(l_{m+k},l_{m+k+1})^{1-\vartheta_2} \leq \zeta d_L(l_{m+k-1},l_{m+k})^{\vartheta_1}, \quad \forall m \in \mathbb{N}. \tag{42}$$

Since $1-\vartheta_2 = \vartheta_1$, thus, by (42), we get

$$\begin{aligned} d_L(l_{m+k},l_{m+k+1}) &\leq \zeta^{1/(1-\vartheta_2)} d_L(l_{m+k-1},l_{m+k})^{\vartheta_1}, \\ &\leq \zeta d_L(l_{m+k-1},l_{m+k}) \quad \forall m \in \mathbb{N}. \end{aligned} \tag{43}$$

Hence, by (43), we get

$$d_L(l_{m+k},l_{m+k+1}) \leq \zeta^m d_L(l_k,l_{k+1}), \quad \forall m \in \mathbb{N}, \tag{44}$$

that is,

$$d_L(l_m,l_{m+1}) \leq \zeta^{m-k} d_L(l_k,l_{k+1}), \quad \forall m \geq k+1. \tag{45}$$

From triangle inequality and (45), for each $q, n \in \mathbb{N}$ with $q > n \geq k+1$, we obtain

$$d_L(l_n,l_q) \leq \sum_{j=n}^{q-1} d_L(l_j,l_{j+1}) \leq \sum_{j=n}^{q-1} \zeta^{j-k} d_L(l_k,l_{k+1}). \tag{46}$$

Above inequality yields that $\{l_m\}$ is a Cauchy sequence in a complete space L. Hence, we get a point $l^* \in L$ with $l_m \longrightarrow l^*$. Also, by (iii), we get $\gamma(l_m, l^*) = 1, \quad \forall m \in \mathbb{N}$, since $\gamma(l_m, l_{m+1}) = 1, \quad \forall m \in \mathbb{N}$ and $l_m \longrightarrow l^*$.

Here, the claim is $l^* = N(l^*, l^*, \ldots, l^*)$. If the claim is wrong, then by (36), for each $m \in \mathbb{N}$, we get

$$\begin{aligned} d_L(N(l_m,l_{m+1},\ldots,l_{m+k-1}),N(l^*,l^*,\ldots,l^*)) &= d_L(N(l_m,l_{m+1},\ldots,l_{m+k-1}),N(l^*,l^*,\ldots,l^*))^{\min\{\gamma(l_m,l^*),\gamma(l_{m+1},l^*),\ldots,\gamma(l_{m+k-1},l^*)\}} \\ &\leq \zeta d_L(l_{m+k-1},N(l_m,l_{m+1},\ldots,l_{m+k-1}))^{\vartheta_1} d_L(l^*,N(l^*,l^*,\ldots,l^*))^{\vartheta_2}, \\ &\leq d_L(l_{m+k-1},l_{m+k})^{\vartheta_1} d_L(l^*,N(l^*,l^*,\ldots,l^*))^{\vartheta_2}. \end{aligned} \tag{47}$$

By triangle inequality and (47), for each m, we obtain

$$\begin{aligned} d_L(l^*,N(l^*,l^*,\ldots,l^*)) &\leq d_L(l^*,N(l_m,l_{m+1},\ldots,l_{m+k-1})) + d_L(N(l_m,l_{m+1},\ldots,l_{m+k-1}),N(l^*,l^*,\ldots,l^*)) \\ &\leq d_L(l^*,l_{m+k}) + \zeta d_L(l_{m+k-1},l_{m+k})^{\vartheta_1} d_L(l^*,N(l^*,l^*,\ldots,l^*))^{\vartheta_2}. \end{aligned} \tag{48}$$

Letting $m \longrightarrow \infty$ in(48), we get $d_L(l^*, N(l^*, l^*, \ldots, l^*)) = 0$. Hence, the claim is true, that is, $l^* = N(l^*, l^*, \ldots, l^*)$. □

Theorem 7. *Consider an extended interpolative Prešić type-II contraction map $N: L^k \longrightarrow L$, for any fixed $k \in \mathbb{N}$, on a complete metric space (L, d_L), that is, for each $s_1, s_2, \ldots, s_k$,*

$w_1, w_2, \ldots, w_k, \in L \setminus Fix(N)$ with $\min\{\gamma(s_1, w_1), \gamma(s_2, w_2), \ldots, \gamma(s_k, w_k)\} \geq 1$, we get

$$d_L(N(s_1, s_2, \ldots, s_k), N(w_1, w_2, \ldots, w_k)) \leq \zeta d_L(s_k, N(s_1, s_2, \ldots, s_k))^{\vartheta_1} d_L(w_k, N(w_1, w_2, \ldots, w_k))^{\vartheta_2}, \tag{49}$$

where $\gamma: L \times L \longrightarrow \mathbb{R}$ is a map, $\vartheta_1, \vartheta_2 \in (0,1)$ with $\vartheta_1 + \vartheta_2 = 1$, $\zeta \in [0,1)$, and $Fix(N) = \{l \in L: l = N(l, l, \ldots, l)\}$. Also, consider that

- *(i) If $\min\{\gamma(s_1, w_1), \gamma(s_2, w_2), \ldots, \gamma(s_k, w_k)\} \geq 1$, then $\gamma(N(s_1, s_2, \ldots, s_k), N(w_1, w_2, \ldots, w_k)) \geq 1$.*
- *(ii) There exist elements $s_1, s_2, \ldots, s_k \in L$, such that*

$$\min\{\gamma(s_1, s_2), \gamma(s_2, s_3), \ldots, \gamma(s_k, N(s_1, s_2, \ldots, s_k))\} \geq 1. \tag{50}$$

- *(iii) For every sequence $\{l_m\}$ in L with $\gamma(l_m, l_{m+1}) \geq 1\ \forall m \geq m_0$ for some natural number m_0 and $l_m \longrightarrow l$, we have $\gamma(l_m, l) \geq 1\ \forall m \geq m_0$.*

Then, there exists at least one point of L that satisfies the equation $l = N(\underbrace{l, l, \ldots, l}_{k\text{–times}})$.

Proof. The proof can be obtained on the same steps as the proofs of Theorems 6 and 2 are done.

The following theorems can be considered as an extended form of Theorems 3 and 5, respectively. □

Theorem 8. *Consider an extended interpolative Prešić type-I proximal contraction map $N: O_L^k \longrightarrow M_L$, for any fixed $k \in \mathbb{N}$, on a metric space (L, d_L), that is, for each $s_1, s_2, \ldots, s_k$, $w_1, w_2, \ldots, w_k, p, q \in O_L \setminus Bes(N)$ with*

$$d_L(p, N(s_1, s_2, \ldots, s_k)) = D_L(O_L, M_L) = d_L(q, N(w_1, w_2, \ldots, w_k)), \tag{51}$$

we get

$$d_L(p, q)^{\min\{\gamma(s_1, w_1), \gamma(s_2, w_2), \ldots, \gamma(s_k, w_k)\}} \leq \zeta d_L(s_k, p)^{\vartheta_1} d_L(w_k, q)^{\vartheta_2}, \tag{52}$$

where $\gamma: O_L \times O_L \longrightarrow \mathbb{R} \setminus \{0\}$ is a map, $\vartheta_1, \vartheta_2 \in (0,1)$ with $\vartheta_1 + \vartheta_2 = 1$, $\zeta \in [0,1)$, $Bes(N) = \{o \in O_L: d_L(o, N(o, o, \ldots, o)) = D_L(O_L, M_L)\}$, and O_L, M_L are the nonvoid subsets of L. Also, consider that

- (i) If $\min\{\gamma(s_1, w_1), \gamma(s_2, w_2), \ldots, \gamma(s_k, w_k)\} = 1$ and $d_L(p, N(s_1, s_2, \ldots, s_k)) = D_L(O_L, M_L) = d_L(q, N(w_1, w_2, \ldots, w_k))$, then $\gamma(p, q) = 1$.
- (ii) There exist elements $s_1, s_2, \ldots, s_k, p \in O_L$ with $d_L(p, N(s_1, s_2, \ldots, s_k)) = D_L(O_L, M_L)$ and

$$\min\{\gamma(s_1, s_2), \gamma(s_2, s_3), \ldots, \gamma(s_k, p)\} = 1. \tag{53}$$

- (iii) $N\underbrace{(O_L \times \cdots \times O_L \times O_{L0})} \subseteq M_{L0}$.
- (iv) O_{L_0} is nonempty and complete with respect to d_L.
- (v) For every sequence $\{l_m\}$ in O_{L_0} with $\gamma(l_m, l_{m+1}) = 1\ \forall m \geq m_0$ for some natural number m_0 and $l_m \longrightarrow l$, we have $\gamma(l_m, l) = 1\ \forall m \geq m_0$.

Then, there exists at least one point of L that satisfies the equation $d_L(l, N(\underbrace{l, l, \ldots, l}_{k\text{–times}})) = D_L(O_L, M_L)$.

Proof. By hypothesis (ii), we get $l_1, l_2, \ldots, l_k, l_{k+1}$ in O_L with $d_L(l_{k+1}, N(l_1, l_2, \ldots, l_k)) = D_L(O_L, M_L)$ and

$$\min\{\gamma(l_1, l_2), \gamma(l_2, l_3), \ldots, \gamma(l_k, l_{k+1})\} = 1. \tag{54}$$

Hypothesis (iii) implies that $N(l_2, l_3, \ldots, l_{k+1}) \in M_{L0}$, and there is $l_{k+2} \in O_{L_0}$ satisfying

$$d_L(l_{k+2}, N(l_2, l_3, \ldots, l_{k+1})) = D_L(O_L, M_L). \tag{55}$$

Then, from hypothesis (i), we get $\gamma(l_{k+1}, l_{k+2}) = 1$. Repeated use of hypotheses (i), (ii), and (iii) yields $\gamma(l_m, l_{m+1}) = 1$ and $d_L(l_{m+k}, N(l_m, l_{m+1}, \ldots, l_{m+k-1})) = D_L(O_L, M_L)$ for all $m \in \mathbb{N}$. As

$$d_L(l_{m+k}, N(l_m, l_{m+1}, \ldots, l_{m+k-1})) = D_L(O_L, M_L), \tag{56}$$

and

$$d_L(l_{m+k+1}, N(l_{m+2}, l_{m+2}, \ldots, l_{m+k})) = D_L(O_L, M_L), \quad \forall m \in \mathbb{N}, \tag{57}$$

and

$$\min\{\gamma(l_m, l_{m+1}), \gamma(l_{m+1}, l_{m+2}), \ldots, \gamma(l_{m+k-1}, l_{m+k})\} = 1, \quad \forall m \in \mathbb{N}. \tag{58}$$

Then, by (52), we get

$$\begin{aligned} d_L(l_{m+k}, l_{m+k+1}) &= d_L(l_{m+k}, l_{m+k+1})^{\min\{\gamma(l_m,l_{m+1}),\ldots,\gamma(l_{m+k-1},l_{m+k})\}}, \\ &\leq \zeta d_L(l_{m+k-1}, l_{m+k})^{\vartheta_1} d_L(l_{m+k}, l_{m+k+1})^{\vartheta_2}, \quad \forall m \in \mathbb{N}. \end{aligned} \tag{59}$$

Following the proof of Theorem 6, we say that $\{l_m\}_{m\geq k+1}$ is Cauchy in O_{L0}, and by the completeness of O_{L0}, we get a point $l^* \in O_{L0}$ with $l_m \longrightarrow l^*$. Also, by (v), we get $\gamma(l_m, l^*) = 1\ \forall m \in \mathbb{N}$, since $\gamma(l_m, l_{m+1}) = 1$ and $l_m \longrightarrow l^*$. Clearly, $N(l^*, l^*, \ldots, l^*) \in M_{L0}$, and there is $w^* \in O_{L0}$ with $d_L(w^*, N(l^*, l^*, \ldots, l^*)) = D_L(O_L, M_L)$. Here, the claim is $w^* = l^*$. Suppose it is wrong, then by (52), for each $m \in \mathbb{N}$, we get

$$\begin{aligned} &d_L(l_{m+k}, w^*) = d_L(l_{m+1}, w^*)^{\min\{\gamma(l_m,l^*),\gamma(l_{m+1},l^*),\ldots,\gamma(l_{m+k-1},l^*)\}} \\ &\leq \zeta d_L(l_{m+k-1}, l_{m+k})^{\vartheta_1} d_L(l^*, w^*)^{\vartheta_2}. \end{aligned} \tag{60}$$

Letting $m \longrightarrow \infty$ in (60), we obtain $d_L(l^*, w^*) = 0$, and it contradicts our assumption. Hence, our claim is true, that is, $l^* = w^*$. Therefore, $d_L(l^*, N(l^*, l^*, \ldots, l^*)) = D_L(O_L, M_L)$. ☐

Theorem 9. *Consider an extended interpolative Prešić type-II proximal contraction map* $N: O_L^k \longrightarrow M_L$, *for any fixed* $k \in \mathbb{N}$, *on a metric space* (L, d_L), *that is, for each* $s_1, s_2, \ldots, s_k$, $w_1, w_2, \ldots, w_k, p, q \in O_L \setminus Bes(N)$ *with*

$$d_L(p, N(s_1, s_2, \ldots, s_k)) = D_L(O_L, M_L) = d_L(q, N(w_1, w_2, \ldots, w_k)), \tag{61}$$

and

$$\min\{\gamma(s_1, w_1), \gamma(s_2, w_2), \ldots, \gamma(s_k, w_k)\} \geq 1, \tag{62}$$

we get

$$d_L(p, q) \leq \zeta d_L(s_k, p)^{\vartheta_1} d_L(w_k, q)^{\vartheta_2}, \tag{63}$$

where $\gamma: O_L \times O_L \longrightarrow \mathbb{R}$ *is a map,* $\vartheta_1, \vartheta_2 \in (0, 1)$ *with* $\vartheta_1 + \vartheta_2 = 1$, *and* $\zeta \in [0, 1)$. *Also, consider that*

(i) *If* $\min\{\gamma(s_1, w_1), \gamma(s_2, w_2), \ldots, \gamma(s_k, w_k)\} \geq 1$ *and* $d_L(p, N(s_1, s_2, \ldots, s_k)) = D_L(O_L, M_L) = d_L(q, N(w_1, w_2, \ldots, w_k))$, *then* $\gamma(p, q) \geq 1$.

(ii) *There exist elements* $s_1, s_2, \ldots, s_k, p \in O_L$ *with* $d_L(p, N(s_1, s_2, \ldots, s_k)) = D_L(O_L, M_L)$ *and*

$$\min\{\gamma(s_1, s_2), \gamma(s_2, s_3), \ldots, \gamma(s_k, p)\} \geq 1. \tag{64}$$

(iii) $N\underbrace{(O_L \times \cdots \times O_L \times O_{L0})}_{k-1\ times} \subseteq M_{L0}$.

(iv) O_{L_0} *is nonempty and complete with respect to* d_L.

(v) *For every sequence* $\{l_m\}$ *in* O_{L_0} *with* $\gamma(l_m, l_{m+1}) \geq 1\ \forall m \geq m_0$ *for some natural number* m_0 *and* $l_m \longrightarrow l$, *we have* $\gamma(l_m, l) \geq 1\ \forall m \geq m_0$.

Then, there exists at least one point of L that satisfies the equation $d_L(l, N(\underbrace{l, l, \ldots, l}_{k-times})) = D_L(O_L, M_L)$.

The proof of the above theorem can be derived by viewing the proof of Theorem 8.

3. Conclusion

This article provides a few results dealing with fixed points and best proximity points of the mappings defined on product spaces. The notions of interpolative Prešić type contractions and interpolative Prešić type proximal contractions are introduced in the context of metric spaces to discuss the existence of fixed points and best proximity points of such maps, respectively. These notions are derived by considering the concept of interpolative Kannan contraction.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally in this article and approved the final manuscript.

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