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Research Article

Bi-Univalent Function Classes Defined by Using a Second Einstein Function

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Motivated by q-calculus, subordination principle, and the second Einstein function, we define two families of bi-univalent analytic functions on the open unit disc of the complex plane. We deduce estimates for the first two Maclaurin's coefficients and the Fekete-Sezgö functional inequalities for the functions that belong to these families of functions.

1. Introduction and Basic Concepts

Let $\mathscr{A}$ denotes the collection of all functions f with the following series representation:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic and univalent in the open unit disc $\mathscr{U} = \{z : |z| < 1\}$ and satisfy the usual normalization condition $f(0) = f'(0) - 1 = 0$. Also, an important class of functions will be called $\mathscr{P}$, $\mathscr{P}$ defines the family of functions ϕ with the restrictions that the image domain of ϕ (ϕ is a convex function with Re $(\phi) > 0$ in $\mathscr{U}$) is symmetric along the real axis and starlike about $\phi(0) = 1$ with $\phi'(0) > 0$.

In 1980, Gradshteyn and Ryzhik [1] give an expression of the Bernoulli polynomials which have important applications in number theory and classical analysis. They appear in the integral representation of differentiable periodic functions since they are employed for approximating such functions in terms of polynomials. They are also used for representing the remainder term of the composite Euler-Maclaurin quadrature rule.

The Bernoulli polynomials $B_n(x)$ are usually defined (see, e.g., [2]) by means of the generating function:

$$G(x, t) := \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n, \quad |t| < 2\pi, \tag{2}$$

where $B_n(x)$ are polynomials in x, for each nonnegative integer n.

The Bernoulli polynomials are easily computed by recursion since

$$\sum_{j=0}^{n-1} \binom{n}{j} B_j(x) = nx^{n-1}, \quad n = 2, 3, \cdots. \tag{3}$$

The first few Bernoulli polynomials are

$$\begin{aligned} B_0(x) &= 1, \\ B_1(x) &= x - \frac{1}{2}, \\ B_2(x) &= x^2 - x + \frac{1}{6}, \\ B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \cdots. \end{aligned} \tag{4}$$

Furthermore, Bernoulli numbers $B_n := B_n(0)$ are produced directly by putting $x = 0$ in Bernoulli polynomials. The first few Bernoulli numbers are

$$\begin{aligned} B_0 &= 1, \\ B_1 &= -\frac{1}{2}, \\ B_2 &= \frac{1}{6}, \\ B_4 &= -\frac{1}{30}, \cdots, \\ B_{2n+1} &= 0, \quad \forall n = 1, 2, \cdots. \end{aligned} \tag{5}$$

Moreover, Bernoulli numbers B_n can be generated by means of the so-called Einstein function $E(z)$:

$$E(z) := \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n. \tag{6}$$

In mathematics, the Einstein function is a name occasionally used for one of the functions (see [3–6]):

$$\begin{aligned} E_1(z) &:= \frac{z}{e^z - 1}, \\ E_2(z) &:= \frac{z^2 e^z}{(e^z - 1)^2}, \\ E_3(z) &:= \log(1 - e^{-z}), \\ E_4(z) &:= \frac{z}{e^z - 1} - \log(1 - e^{-z}). \end{aligned} \tag{7}$$

It is easily noticed that both E_1 and E_2 have these nice properties, but E_3 and E_4 are not (see Figure 1); the range of E_1 and E_2 (E_1 and E_2 are convex functions) is symmetric along the real axis and starlike about $E_1(0) = E_2(0) = 1$ and $\Re(E_1(z)) > 0, \Re(E_2(z)) > 0 \forall z \in \mathcal{U}$.

The series representation is given by (one can type "Maclaurin series for $z/(-1 + e^z)$" in [7])

$$\begin{aligned} E_1(z) &= 1 + \sum_{n=1}^{\infty} \frac{B_n}{n!} z^n, \\ E_2(z) &= 1 + \sum_{n=1}^{\infty} \frac{(1-n)B_n}{n!} z^n, \end{aligned} \tag{8}$$

where B_n is the n^{th} Bernoulli number.

El-Qadeem et al. [8] have introduced some results related to the first Einstein function E_1. Here, we will deal with the second Einstein function E_2. Note that $E_2'(0) \not> 0$ (indeed $E_2'(0) = 0$), i.e., $E_2 \notin \mathcal{P}$. Thus, we shall define the function:

$$\mathbb{E}_\mu(z) := E_2(z) + \mu z, \tag{9}$$

where $\mu \in I := [0.28, 0.92]$. It is obvious that $\mathbb{E}_\mu(\mathcal{U})$ is a convex domain, symmetric along the real axis and starlike about $\mathbb{E}_\mu(0) = 1$ and $\Re(\mathbb{E}_\mu(z)) > 0 \forall z \in \mathcal{U}$; moreover, $\mathbb{E}_\mu'(0) = \mu > 0$. This proves that $\mathbb{E}_\mu \in \mathcal{P}$.

Example 1.

(i) If $\mu < 0.28$, then $E_2(z) + \mu z$ is not a convex function, see Figure 2(a)

(ii) If $\mu > 0.92$, then $\exists z \in \mathcal{U}$ s.t. $\Re(E_2(z) + \mu z) \not> 0$, see Figure 2(b)

(iii) If $0.28 \le \mu \le 0.92$, then $\Re(E_2(z) + \mu z) > 0 \forall z \in \mathcal{U}$; also, $E_2(z) + \mu z$ is a convex function, see Figure 2(c)

Now, let $\mathcal{S}$ be the subfamily of $\mathcal{A}$ consisting of all functions of the form (1) which are univalent in $\mathcal{U}$. It is well known, by using the Koebe one-quarter theorem [9], that every univalent function $f \in \mathcal{S}$ containing a disc of radius 1 /4 has an inverse function f^{-1}, which is defined by

$$\begin{aligned} & f^{-1}(f(z)) = z, \quad z \in \mathcal{U}, \\ & f(f^{-1}(\omega)) = \omega\left(\omega \in \Delta = \left\{\omega \in \mathbb{C} : |\omega| < \frac{1}{4}\right\}\right). \end{aligned} \tag{10}$$

A function $f \in \mathcal{S}$ is said to be bi-univalent in $\mathcal{U}$ if both f and f^{-1} are univalent in $\mathcal{U}$. Let Σ denotes the subfamily of $\mathcal{S}$, consisting of all biunivalent functions defined on the unit disc $\mathcal{U}$. Since $f \in \Sigma$ has the Maclaurin series expansion given by (1), a simple calculation shows that its inverse $g = f^{-1}$ has the series expansion:

$$g(\omega) = f^{-1}(\omega) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - \cdots. \tag{11}$$

Examples of functions in the class Σ are

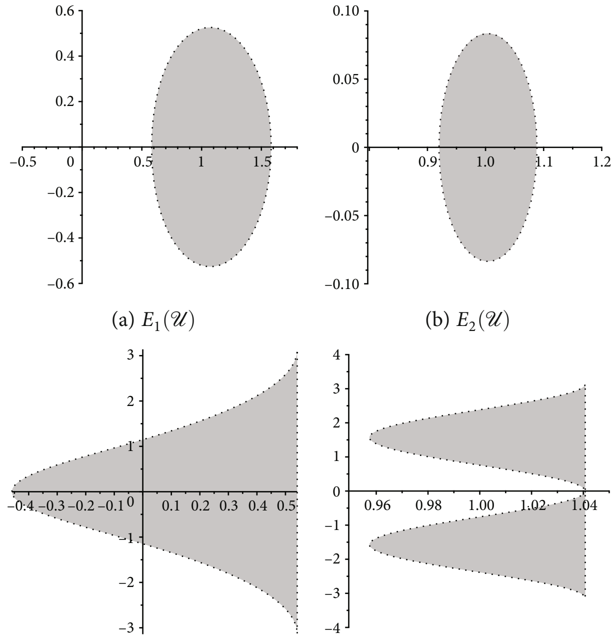
$$\begin{gathered} \frac{z}{1-z}, \\ -\log(1-z), \\ \frac{1}{2}\log\left(\frac{1+z}{1-z}\right), \end{gathered} \tag{12}$$

and so on. However, the familiar Koebe function is not a member of Σ. Other common examples of functions in $\mathcal{S}$ such as

$$\begin{gathered} z - \frac{z^2}{2}, \\ \frac{z}{1-z^2} \end{gathered} \tag{13}$$

are also not members of Σ.

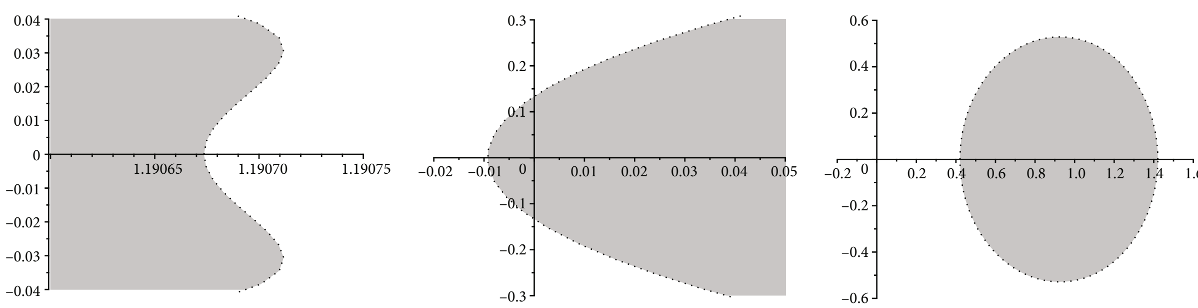
Now, we introduce some notes about the q-difference operator which uses in investigating our main families. In



(a) $E_1(\mathscr{U})$ (b) $E_2(\mathscr{U})$

(c) $E_3(\mathscr{U})$ (d) $E_4(\mathscr{U})$

FIGURE 1: The images of unit disc $\mathscr{U}$ by the Einstein functions E_1, E_2, E_3, and E_4.



(a) $E_2(z) + 0.27z$ (b) $E_2(z) + 0.93z$ (c) $E_2(z) + 0.5z$

FIGURE 2: $\mathbb{E}_\mu(\mathscr{U})$ for three different values of μ.

view of Annaby and Mansour [10], the q-difference operator is defined by

$$\partial_q f(z) = \begin{cases} \dfrac{f(qz) - f(z)}{z(q-1)}, & z \neq 0\,; \\ f'(0), & z = 0\,; \end{cases}$$
$$\partial_q^0 f(z) = f(z), \tag{14}$$
$$\partial_q^1 f(z) = \partial_q f(z),$$
$$\partial_q^m f(z) = \partial_q\left(\partial_q^{m-1} f(z)\right)(m \in \mathbb{N}).$$

Thus, for the function $f \in \Sigma$ denoted by (1), we have

$$\partial_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1} (z \neq 0), \tag{15}$$

where

$$[n]_q = \frac{q^n - 1}{q - 1} = \sum_{j=0}^{n-1} q^j, \quad n \in \mathbb{N}. \tag{16}$$

Definition 1 (see [11, 12]). An analytic function f is said to be subordinate to another analytic function g, written as $f(z) \prec g(z) (z \in \mathscr{U})$, if there exists a Schwarz function ω, which is analytic in $\mathscr{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1 (z \in \mathscr{U})$, such that $f(z) = g(w(z))$. In particular, if the function g is univalent in $\mathscr{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0), f(\mathscr{U}) \subset g(\mathscr{U}). \tag{17}$$

Our aim in this article is to introduce two families of analytic bi-univalent function related to the modified Einstein function $\mathbb{E}_\mu(z)$. Furthermore, we get estimations to $|a_2|$, $|a_3|$, and also the Fekete Sezgö inequalities for the functions that belong to these two families.

Definition 2. Consider $0 \le \delta \le 1$, $0 \le \lambda \le 1$, $q \in (0,1)$, and $\mu \in [0.28, 0.92]$. The function $f \in \Sigma$ is said to be in $\mathscr{M}_{\Sigma}^{q}(\delta, \lambda; \mathbb{E}_{\mu})$ if it is satisfying

$$(1-\delta)\left(\frac{z}{f(z)}\right)^{1-\lambda}\partial_q f(z) + \delta\frac{\partial_q\left(z\partial_q f(z)\right)}{\partial_q f(z)} \prec \mathbb{E}_{\mu}(z), \quad (18)$$

$$(1-\delta)\left(\frac{\omega}{g(\omega)}\right)^{1-\lambda}\partial_q g(\omega) + \delta\frac{\partial_q\left(\omega\partial_q g(\omega)\right)}{\partial_q g(\omega)} \prec \mathbb{E}_{\mu}(\omega), \quad (19)$$

where $g = f^{-1}$ is given by (11) and $z, \omega \in \mathscr{U}$.

Definition 3. Consider $0 \le \alpha \le 1$, $0 \le \beta \le 1$, $q \in (0,1)$, and $\mu \in [0.28, 0.92]$. The function $f \in \Sigma$ is said to be in $\mathscr{S}_{\Sigma}^{q}(\alpha, \beta; \mathbb{E}_{\mu})$ if it is satisfying

$$\begin{aligned}(1-\alpha)\frac{f(z)}{z} + \alpha\partial_q f(z) + \beta z\partial_q^2 f(z) \prec \mathbb{E}_{\mu}(z),\\ (1-\alpha)\frac{g(\omega)}{\omega} + \alpha\partial_q g(\omega) + \beta\omega\partial_q^2 g(\omega) \prec \mathbb{E}_{\mu}(\omega),\end{aligned} \quad (20)$$

where $g = f^{-1}$ is given by (11) and $z, \omega \in \mathscr{U}$.

Remark 4. Specializing the parameters δ, λ, α, β, and q in Definitions 2 and 3, then we have the following subfamilies:

(i) If $\delta = 0$ and $q \longrightarrow 1^{-}$, then $\mathscr{M}_{\Sigma}(\lambda; \mathbb{E}_{\mu})$ is the subfamily of Σ, consisting of functions f which satisfy the following conditions:

$$\begin{aligned}\left(\frac{z}{f(z)}\right)^{1-\lambda} f'(z) \prec \mathbb{E}_{\mu}(z),\\ \left(\frac{\omega}{g(\omega)}\right)^{1-\lambda} g'(\omega) \prec \mathbb{E}_{\mu}(\omega),\end{aligned} \quad (21)$$

where $g = f^{-1}$ is given by (11), $\mathbb{E}_{\mu}$ is defined by (9), and $z, \omega \in \mathscr{U}$

(ii) If $\delta = 1$, $\lambda = 0$, and $q \longrightarrow 1^{-}$, then let $\mathscr{K}\mathscr{M}_{\Sigma}(\mathbb{E}_{\mu})$ be the family of bi-univalent convex functions with respect to Einstein function, consisting of the functions f which satisfy the following conditions:

$$\begin{aligned}1 + \frac{zf''(z)}{f'(z)} \prec \mathbb{E}_{\mu}(z),\\ 1 + \frac{\omega g''(\omega)}{g'(\omega)} \prec \mathbb{E}_{\mu}(\omega),\end{aligned} \quad (22)$$

where $g = f^{-1}$ is given by (11), $\mathbb{E}_{\mu}$ is defined by (9), and $z, \omega \in \mathscr{U}$

(iii) If $\delta = \lambda = 0$ and $q \longrightarrow 1^{-}$, then let $\mathscr{S}^{*}\mathscr{M}_{\Sigma}(\mathbb{E}_{\mu})$ be the family of bi-univalent starlike functions with respect to Einstein function, consisting of functions f which satisfy the following conditions:

$$\begin{aligned}\frac{zf'(z)}{f(z)} \prec \mathbb{E}_{\mu}(z),\\ \frac{\omega g'(\omega)}{g(\omega)} \prec \mathbb{E}_{\mu}(\omega),\end{aligned} \quad (23)$$

where $g = f^{-1}$ is given by (11), $\mathbb{E}_{\mu}$ is defined by (9), and $z, \omega \in \mathscr{U}$

(iv) If $\alpha = \beta = 1$ and $q \longrightarrow 1^{-}$, then $\mathscr{K}_{\Sigma}(\mathbb{E}_{\mu})$ is the subfamily of Σ, consisting of functions f which satisfy the following conditions:

$$\begin{aligned}f'(z) + zf''(z) \prec \mathbb{E}_{\mu}(z),\\ g'(\omega) + \omega g''(\omega) \prec \mathbb{E}_{\mu}(\omega),\end{aligned} \quad (24)$$

where $g = f^{-1}$ is given by (11), $\mathbb{E}_{\mu}$ is defined by (9), and $z, \omega \in \mathscr{U}$

(v) If $\alpha = 1$, $\beta = 0$, and $q \longrightarrow 1^{-}$, then $\mathscr{S}_{\Sigma}(\mathbb{E}_{\mu})$ is the subfamily of Σ, consisting of functions f which satisfy the following conditions:

$$\begin{aligned}f'(z) \prec \mathbb{E}_{\mu}(z),\\ g'(\omega) \prec \mathbb{E}_{\mu}(\omega),\end{aligned} \quad (25)$$

where $g = f^{-1}$ is given by (11), $\mathbb{E}_{\mu}$ is defined by (9), and $z, \omega \in \mathscr{U}$

Lemma 5 (see [13, 14]). *Let $l_1, l_2 \in \mathbb{R}$ and $p_1, p_2 \in \mathbb{C}$. If $|p_1|, |p_2| < \zeta$, then*

$$|(l_1 + l_2)p_1 + (l_1 - l_2)p_2| \le \begin{cases} 2|l_1|\zeta, & |l_1| \ge |l_2|, \\ 2|l_2|\zeta, & |l_1| \le |l_2|. \end{cases} \quad (26)$$

Lemma 6 (see [9]). *Suppose that $\chi(z)$ is analytic in the unit open disc $\mathscr{U}$ with $\chi(0) = 0$, $|\chi(z)| < 1$, and that*

$$\chi(z) = \rho_1 z + \sum_{n=2}^{\infty} \rho_n z^n \text{ for all } z \in \mathscr{U}, \quad (27)$$

then

$$|\rho_1| \le 1,\quad |\rho_n| \le 1 - |\rho_1|^2 \ (n \in \mathbb{N} \setminus \{1\}). \tag{28}$$

2. Main Results

Unless otherwise mentioned, we assume in the remainder of this article that $0 \le \delta \le 1$, $0 \le \lambda \le 1$, $0 \le \alpha \le 1$, $0 \le \beta \le 1$, $q \in (0,1)$, $\mu \in [0.28, 0.92]$, and $z, \omega \in \mathscr{U}$.

Theorem 7. *Let $f \in \mathscr{M}_\Sigma^q(\delta, \lambda; \mathbb{E}_\mu)$, then*

$$|a_2| \le \sqrt{\frac{2\mu}{\left|K_2 + K_4 + \left(K_1^2/6\mu^2\right)\right| + (2/\mu)K_1^2}},\quad |a_3| \le \frac{\mu(|K_2| + |K_4|)}{K_3|K_2 + K_4|}, \tag{29}$$

where

$$\begin{aligned} K_1 &= (1-\delta)\left([2]_q + \lambda - 1\right) + \delta[2]_q\left([2]_q - 1\right)\\ K_2 &= (1-\delta)(\lambda-1)\left([2]_q + \frac{\lambda}{2} - 1\right) - \delta[2]_q\left([2]_q - 1\right)\\ K_3 &= (1-\delta)\left([3]_q + \lambda - 1\right) + \delta[3]_q\left([3]_q - 1\right)\\ K_4 &= (1-\delta)(\lambda-1)\left([2]_q + \frac{\lambda}{2} + 1\right) - \delta[2]_q^2\left([2]_q - 1\right) + 2(1-\delta)[3]_q + 2\delta[3]_q\left([3]_q - 1\right) \}. \end{aligned} \tag{30}$$

Proof. Let f and g be in $\mathscr{M}_\Sigma^q(\delta, \lambda; \mathbb{E}_\mu)$, then, they satisfy (18) and (19), respectively. But according to Definition 1 and Lemma 6, there exist two functions $u(z)$ and $v(\omega)$ of the form

$$u(z) = \sum_{n=1}^{\infty} c_n z^n,\quad v(\omega) = \sum_{n=1}^{\infty} d_n \omega^n, \tag{31}$$

such that

$$(1-\delta)\left(\frac{z}{f(z)}\right)^{1-\lambda} \partial_q f(z) + \delta \frac{\partial_q\left(z\partial_q f(z)\right)}{\partial_q f(z)} = \mu u(z) + E_2(u(z)), \tag{32}$$

$$(1-\delta)\left(\frac{\omega}{g(\omega)}\right)^{1-\lambda} \partial_q g(\omega) + \delta \frac{\partial_q\left(\omega\partial_q g(\omega)\right)}{\partial_q g(\omega)} = \mu v(\omega) + E_2(v(\omega)). \tag{33}$$

After some simple calculations, we deduce

$$\begin{aligned} \mu u(z) + E_2(u(z)) &= 1 + \mu u(z) - \frac{(u(z))^2}{12} + \frac{(u(z))^4}{240} + \cdots\\ &= 1 + \mu c_1 z + \left(\mu c_2 - \frac{c_1^2}{12}\right)z^2 + \cdots, \end{aligned} \tag{34}$$

$$\begin{aligned} \mu v(\omega) + E_2(v(\omega)) &= 1 + \mu v(\omega) - \frac{(v(\omega))^2}{12} + \frac{(v(\omega))^4}{240} + \cdots\\ &= 1 + \mu d_1 \omega + \left(\mu d_2 - \frac{d_1^2}{12}\right)\omega^2 + \cdots. \end{aligned} \tag{35}$$

Also,

$$\begin{aligned} &(1-\delta)\left(\frac{z}{f(z)}\right)^{1-\lambda} \partial_q f(z) + \delta \frac{\partial_q\left(z\partial_q f(z)\right)}{\partial_q f(z)}\\ &= 1 + K_1 a_2 z + \left(K_3 a_3 + K_2 a_2^2\right)z^2 + \cdots. \end{aligned} \tag{36}$$

$$\begin{aligned} &(1-\delta)\left(\frac{\omega}{g(\omega)}\right)^{1-\lambda} \partial_q g(\omega) + \delta \frac{\partial_q\left(\omega\partial_q g(\omega)\right)}{\partial_q g(\omega)}\\ &= 1 - K_1 a_2 \omega + \left(K_4 a_2^2 - K_3 a_3\right)\omega^2 + \cdots, \end{aligned} \tag{37}$$

where $K_j : j = 1, 2, 3, 4$ are stated in (30).

By substituting from (34), (35), (36), and (37) into (32) and (33) with comparing the coefficient in both sides, we obtain

$$K_1 a_2 = \mu c_1, \tag{38}$$

$$K_3 a_3 + K_2 a_2^2 = \mu c_2 - \frac{c_1^2}{12}, \tag{39}$$

$$-K_1 a_2 = \mu d_1, \tag{40}$$

$$-K_3 a_3 + K_4 a_2^2 = \mu d_2 - \frac{d_1^2}{12}. \tag{41}$$

As a direct result of equations (38) and (40), we get

$$c_1 = -d_1, \tag{42}$$

and also,

$$c_1^2 + d_1^2 = \frac{2}{\mu^2} K_1^2 a_2^2. \tag{43}$$

By adding (39) to (41), then using (43), we obtain

$$\left(K_2 + K_4 + \frac{K_1^2}{6\alpha^2}\right) a_2^2 = \mu(c_2 + d_2). \tag{44}$$

Equations (42) and (44) together with using Lemma 6 implies that

$$\left|K_2 + K_4 + \frac{K_1^2}{6\alpha^2}\right| |a_2|^2 \le 2\mu\left(1 - |c_1|^2\right). \tag{45}$$

But from equation (38), we can deduce

$$|c_1|^2 = \frac{K_1^2}{\mu^2} |a_2|^2. \tag{46}$$

By using (46) into (45), we obtain

$$|a_2| \le \sqrt{\frac{2\mu}{\left|K_2 + K_4 + \left(K_1^2/6\mu^2\right)\right| + \left(2K_1^2/\mu\right)}}. \tag{47}$$

Further, from (39) and (41) with using (42), we get

$$K_3(K_2 + K_4)a_3 = \mu(c_2K_4 - K_2d_2) - \frac{c_1^2}{12}(K_4 - K_2). \tag{48}$$

Thus, by virtue of Lemma 6, we find

$$K_3|K_2 + K_4||a_3| \le (|K_2| + |K_4|)\left(\mu + |c_1|^2\left(\frac{1}{12} - \mu\right)\right). \tag{49}$$

Since $(1/12) - \mu < 0$ for all $\mu \in [0.28, 0.92]$. Then, we conclude

$$|a_3| \le \frac{\mu(|K_2| + |K_4|)}{K_3|K_2 + K_4|}. \tag{50}$$

Thus, the proof is completed. □

Theorem 8. *Let $f \in \mathcal{S}_\Sigma^q(\alpha, \beta; \mathbb{E}_\mu)$, then*

$$|a_2| \le \sqrt{\frac{2\mu}{Y(\alpha,\beta,\mu;q) + (2/\mu)\left(1 + \alpha\left([2]_q - 1\right) + \beta[2]_q\right)^2}},$$

$$|a_3| \le \begin{cases} \dfrac{\mu}{1 + \alpha\left([3]_q - 1\right) + \beta[2]_q[3]_q}, & \dfrac{\left(1 + \alpha\left([2]_q - 1\right) + \beta[2]_q\right)^2}{\mu\left(1 + \alpha\left([3]_q - 1\right) + \beta[2]_q[3]_q\right)} \ge 1, \\ \Theta(\alpha,\beta,\mu,q) + \dfrac{\mu}{1 + \alpha\left([3]_q - 1\right) + \beta[2]_q[3]_q}, & \dfrac{\left(1 + \alpha\left([2]_q - 1\right) + \beta[2]_q\right)^2}{\mu\left(1 + \alpha\left([3]_q - 1\right) + \beta[2]_q[3]_q\right)} < 1, \end{cases} \tag{51}$$

where

$$Y(\alpha,\beta,\mu;q) = 2\left(1 + \alpha\left([3]_q - 1\right) + \beta[2]_q[3]_q\right) + \frac{1}{12\mu^2}\left(1 + \alpha\left([2]_q - 1\right) + \beta[2]_q\right)^2, \tag{52}$$

$$\Theta(\alpha,\beta,\mu,q) = \frac{2\mu\left(1 + \alpha\left([3]_q - 1\right) + \beta[2]_q[3]_q\right) - 2\left(1 + \alpha\left([2]_q - 1\right) + \beta[2]_q\right)^2}{\left(1 + \alpha\left([3]_q - 1\right) + \beta[2]_q[3]_q\right)\left(Y(\alpha,\beta,\mu;q) + (2/\mu)\left(1 + \alpha\left([2]_q - 1\right) + \beta[2]_q\right)^2\right)}. \tag{53}$$

Proof. Suppose f and g be in $\mathcal{S}_{\Sigma}^{q}(\alpha,\beta;\mathbb{E}_{\mu})$, then it satisfies (18) and (19). According to Definition 1 and Lemma 6, there exist two functions $u(z)$ and $v(\omega)$ of the form

$$\begin{aligned} u(z) &= \sum_{n=1}^{\infty} c_n z^n, \\ v(\omega) &= \sum_{n=1}^{\infty} d_n \omega^n, \end{aligned} \tag{54}$$

such that

$$(1-\alpha)\frac{f(z)}{z} + \alpha\partial_q f(z) + \beta z\partial_q^2 f(z) = \mu u(z) + E_2(u(z)), \tag{55}$$

$$(1-\alpha)\frac{g(\omega)}{\omega} + \alpha\partial_q g(\omega) + \beta\omega\partial_q^2 g(\omega) = \mu v(\omega) + E_2(v(\omega)). \tag{56}$$

With some simple calculations, we get

$$\begin{aligned} &(1-\alpha)\frac{f(z)}{z} + \alpha\partial_q f(z) + \beta z\partial_q^2 f(z) \\ &\quad = 1 + \sum_{n=2}^{\infty}\left(1+\alpha\left([n]_q-1\right)+\beta[n-1]_q[n]_q\right)a_n z^n, \end{aligned} \tag{57}$$

$$\begin{aligned} &(1-\alpha)\frac{g(\omega)}{\omega} + \alpha\partial_q g(\omega) + \beta\omega\partial_q^2 g(\omega) \\ &\quad = 1 - \left(1+\alpha\left([2]_q-1\right)+\beta[2]_q\right)a_2\omega \\ &\qquad + \left(1+\alpha\left([3]_q-1\right)+\beta[2]_q[3]_q\right)a_2\omega^2 + \cdots. \end{aligned} \tag{58}$$

By substituting from (34), (35), (57), and (58) into (55) and (56) with comparing the coefficient in both sides, we conclude

$$\left(1+\alpha\left([2]_q-1\right)+\beta[2]_q\right)a_2 = \mu c_1, \tag{59}$$

$$\left(1+\alpha\left([3]_q-1\right)+\beta[2]_q[3]_q\right)a_3 = \mu c_2 - \frac{c_1^2}{12}, \tag{60}$$

$$-\left(1+\alpha\left([2]_q-1\right)+\beta[2]_q\right)a_2 = \mu d_1, \tag{61}$$

$$\left(1+\alpha\left([3]_q-1\right)+\beta[2]_q[3]_q\right)\left(2a_2^2-a_3\right) = \mu d_2 + \frac{d_1^2}{12}. \tag{62}$$

From (59) and (61), we obtain

$$c_1 = -d_1, \tag{63}$$

and also,

$$c_1^2 + d_1^2 = \frac{\left(1+\alpha\left([2]_q-1\right)+\beta[2]_q\right)^2}{\mu^2}a_2^2. \tag{64}$$

By adding (60) to (62) with using (64), we get

$$Y(\alpha,\beta,\mu;q)a_2^2 = \mu(c_2+d_2). \tag{65}$$

In view of Lemma 6, equation (65) together with (63) implies that

$$Y(\alpha,\beta,\mu;q)|a_2|^2 \le 2\mu\left(1-|c_1|^2\right). \tag{66}$$

On the other hand, from equation (59), we can write

$$|c_1|^2 = \frac{\left(1+\alpha\left([2]_q-1\right)+\beta[2]_q\right)^2}{\mu^2}|a_2|^2. \tag{67}$$

By using (67) into (66), we get

$$|a_2| \le \sqrt{\frac{2\mu}{Y(\alpha,\beta,\mu;q) + (2/\mu)\left(1+\alpha\left([2]_q-1\right)+\beta[2]_q\right)^2}}, \tag{68}$$

where $Y(\alpha,\beta,\mu;q)$ is defined in (52).

Further, by subtracting (62) from (60) and using (63), we have

$$a_3 = a_2^2 + \frac{\mu(c_2-d_2)}{2\left(1+\alpha\left([3]_q-1\right)+\beta[2]_q[3]_q\right)}. \tag{69}$$

In view of Lemma 6, equation (69) together with (67) implies that

$$|a_3| \le \left(1 - \frac{\left(1+\alpha\left([2]_q-1\right)+\beta[2]_q\right)^2}{\mu\left(1+\alpha\left([3]_q-1\right)+\beta[2]_q[3]_q\right)}\right)|a_2|^2 + \frac{\mu}{1+\alpha\left([3]_q-1\right)+\beta[2]_q[3]_q}. \tag{70}$$

By the virtue of (68), we can get the desired result. Thus, we complete the proof. □

Theorem 9. *Suppose $f \in \mathscr{M}_{\Sigma}^{q}(\delta, \lambda; \mathbb{E}_\mu)$ and $\eta \in \mathbb{C}$, then*

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \dfrac{\mu|\Phi(\delta,\lambda,\mu,\eta;q)|}{K_3}, & |\Phi(\delta,\lambda,\mu,\eta;q)| \geq 1, \\ \dfrac{\mu}{K_3}, & |\Phi(\delta,\lambda,\mu,\eta;q)| \leq 1, \end{cases} \tag{71}$$

where

$$\Phi(\delta,\lambda,\mu,\eta;q) = \frac{K_4 - K_2 - 2\eta K_3}{K_2 + K_4 + \left(K_1^2/6\mu^2\right)}, \tag{72}$$

and K_1, K_2, K_3, and K_4 are given by (30).

Proof. To investigate the desired result, we first subtract (41) from (39) with using (42), we get

$$a_3 = \frac{K_4 - K_2}{2K_3} a_2^2 + \frac{\mu(c_2 - d_2)}{2K_3}. \tag{73}$$

Thus,

$$a_3 - \eta a_2^2 = \left(\frac{K_4 - K_2 - 2\eta K_3}{2K_3}\right) a_2^2 + \frac{\mu(c_2 - d_2)}{2K_3}. \tag{74}$$

As a result of subsequent computations performed by using (44), we obtain

$$\left|a_3 - \eta a_2^2\right| \leq \frac{\mu}{2K_3} \left|(\Phi(\delta,\lambda,\mu,\eta;q) + 1)c_2 + (\Phi(\delta,\lambda,\mu,\eta;q) - 1)d_2\right|, \tag{75}$$

where $\Phi(\delta,\lambda,\mu,\eta;q)$ given by (72).

But in view of Kanas et al. [15] and (28), we can obtain

$$\begin{aligned} |c_2| &\leq 1 - |c_1|^2 \leq 1, \\ |d_2| &\leq 1 - |d_1|^2 \leq 1. \end{aligned} \tag{76}$$

Now, applying Lemma 5 to (75), we can obtain the desired result directly. Thus, we complete the proof. □

Theorem 10. *Let us consider $f \in \mathscr{S}_{\Sigma}^{q}(\alpha, \beta; \mathbb{E}_\mu)$ and $\eta \in \mathbb{C}$, then*

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \dfrac{2\mu|1-\eta|}{Y(\alpha,\beta,\mu;q)}, & \dfrac{|1-\eta|}{Y(\alpha,\beta,\mu;q)} \geq \dfrac{1}{2\left(1 + \alpha\left([3]_q - 1\right) + \beta[2]_q[3]_q\right)}, \\ \dfrac{\mu}{1 + \alpha\left([3]_q - 1\right) + \beta[2]_q[3]_q}, & \dfrac{|1-\eta|}{Y(\alpha,\beta,\mu;q)} \leq \dfrac{1}{2\left(1 + \alpha\left([3]_q - 1\right) + \beta[2]_q[3]_q\right)}, \end{cases} \tag{77}$$

where $Y(\alpha, \beta, \mu; q)$ is defined by (52).

Proof. In order to investigate the desired result (77), we first subtract (62) from (60) with taking in consideration (63), we conclude

$$a_3 - \eta a_2^2 = (1-\eta) a_2^2 + \frac{\mu(c_2 - d_2)}{2\left(1 + \alpha\left([3]_q - 1\right) + \beta[2]_q[3]_q\right)}. \tag{78}$$

By virtue of (65), we can get that

$$a_3 - \eta a_2^2 = \mu c_2 \left(\frac{1-\eta}{Y(\alpha,\beta,\mu;q)} + \frac{1}{2\left(1 + \alpha\left([3]_q - 1\right) + \beta[2]_q[3]_q\right)}\right) + \mu d_2 \left(\frac{1-\eta}{Y(\alpha,\beta,\mu;q)} - \frac{1}{2\left(1 + \alpha\left([3]_q - 1\right) + \beta[2]_q[3]_q\right)}\right). \tag{79}$$

By applying Lemma 5 to (79) with using (76), we obtain the required result which completes the proof. □

3. Set of Corollaries

In this part, we introduce some corollaries by specializing the values of the given parameters λ, δ, α, and β, and taking the limit when $q \longrightarrow 1^-$, in our main results.

Put $\delta = 0$ and $q \longrightarrow 1^-$in Theorems 7, 9, then we have the following corollary:

Corollary 11. *Let $f(z)$ belong to $\mathscr{M}_{\Sigma}(\lambda;\mathbb{E}_{\mu})$, then*

$$|a_2| \le \sqrt{\frac{12\mu^3}{(\lambda+1)[\lambda(6\mu^2+12\mu+1)+12\mu^2+12\mu+1]}},$$

$$|a_3| \le \frac{2\mu}{\lambda^2+3\lambda+2}, |a_3-\eta a_2^2| \le \begin{cases} \dfrac{2\mu|1-\eta|}{(\lambda+1)(\lambda+2+((\lambda+1)/6\mu^2))}, & |1-\eta| \ge \dfrac{(\lambda+1)\left(\lambda+2+\left((\lambda+1)/6\mu^2\right)\right)}{2\lambda+4}, \\ \dfrac{\mu}{\lambda+2}, & |1-\eta| \le \dfrac{(\lambda+1)\left(\lambda+2+\left((\lambda+1)/6\mu^2\right)\right)}{2\lambda+4}. \end{cases} \quad (80)$$

Put $\delta=\lambda=0$ and $q \longrightarrow 1^-$in Theorems 7, 9, then we get the following corollary:

Corollary 12. *Let $f(z)$ belong to $\mathcal{S}^*\mathscr{M}_{\Sigma}(\mathbb{E}_{\mu})$, then*

$$\begin{aligned} |a_2| &\le \sqrt{\frac{12\mu^3}{12\mu^2+12\mu+1}}, \\ |a_3| &\le \mu, \\ |a_3-\eta a_2^2| &\le \begin{cases} \dfrac{12\mu^3|1-\eta|}{12\mu^2+1}, & |1-\eta| \ge \dfrac{12\mu^2+1}{24\mu^2}, \\ \dfrac{\mu}{2}, & |1-\eta| \le \dfrac{12\mu^2+1}{24\mu^2}. \end{cases} \end{aligned} \quad (81)$$

Put $\delta=1, \lambda=0$, and $q \longrightarrow 1^-$in Theorems 7, 9, then we obtain the following:

Corollary 13. *Let $f(z)$ belong to $\mathscr{K}\mathscr{M}_{\Sigma}(\mathbb{E}_{\mu})$, then*

$$\begin{aligned} |a_2| &\le \sqrt{\frac{3\mu^3}{9\mu^2+12\mu+1}}, \\ |a_3| &\le \frac{5}{18}\mu, \\ |a_3-\eta a_2^2| &\le \begin{cases} \dfrac{\mu^3|5-6\eta|}{2(9\mu^2+1)}, & |5-6\eta| \ge \dfrac{9\mu^2+1}{3\mu^2}, \\ \dfrac{\mu}{6}, & |5-6\eta| \le \dfrac{9\mu^2+1}{3\mu^2}. \end{cases} \end{aligned} \quad (82)$$

Put $\alpha=\beta=1$ and $q \longrightarrow 1^-$in Theorems 8, 10, then we have the following corollary:

Corollary 14. *Let $f(z)$ belong to $\mathscr{K}_{\Sigma}(\mathbb{E}_{\mu})$, then*

$$\begin{aligned} |a_2| &\le \sqrt{\frac{3\mu^3}{27\mu^2+48\mu+2}}, \\ |a_3| &\le \frac{\mu}{9}, \\ |a_3-\eta a_2^2| &\le \mu \begin{cases} \dfrac{6\mu^2|1-\eta|}{54\mu^2+4}, & |1-\eta| \ge \dfrac{54\mu^2+4}{27\mu^2}, \\ \dfrac{1}{9}, & |1-\eta| \le \dfrac{54\mu^2+4}{27\mu^2}. \end{cases} \end{aligned} \quad (83)$$

Put $\alpha=1, \beta=0$, and $q \longrightarrow 1^-$ in Theorems 8, 10, then we have the following corollary:

Corollary 15. *Let $f(z)$ belong to $\mathcal{S}_{\Sigma}(\mathbb{E}_{\mu})$, then*

$$\begin{aligned} |a_2| &\le \sqrt{\frac{6\mu^3}{18\mu^2+24\mu+1}}, \\ |a_3| &\le \frac{\mu}{3}, \\ |a_3-\eta a_2^2| &\le \mu \begin{cases} \dfrac{6\mu^2|1-\eta|}{18\mu^2+1}, & |1-\eta| \ge \dfrac{18\mu^2+1}{18\mu^2}, \\ \dfrac{1}{3}, & |1-\eta| \le \dfrac{18\mu^2+1}{18\mu^2}. \end{cases} \end{aligned} \quad (84)$$

Data Availability

No data have been used.

Conflicts of Interest

The authors confirm no competing interests.

Authors' Contributions

The authors read and approved the final manuscript.

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