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Research Article

On Pata Convex-Type Contractive Mappings

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In this work, we introduce weak Pata convex contractions and weak E-Pata convex contractions via simulation functions in metric spaces to prove some fixed point results for such mappings. Also, we consider an example related to weak Pata convex contractions. Consequently, our results generalize and unify some results in the literature.

1. Introduction and Preliminaries

It is well known that Banach [1] pioneered in fixed point theory by introducing a novel notion, namely, Banach contraction principle in 1922. After this date, several authors generalized and extended this principle. A generalization was given by Pata [2] known as Pata contraction. Recently, Pata contraction has been studied by many authors. Some of the studies were for Pata contraction presented by [3–13].

Firstly, the concept of $\phi-$weak contraction was given by Alber et al. [14]. Zhang et al. and Rhoades's results [15, 16] extend previous results given by Alber et al., and they obtained fixed point results for single-valued mappings in Banach spaces, and Rhoades [15] got a unique common fixed point of such contractions, respectively.

In 2012, Samet et al. [17] suggested a novel notion, the so-called α-admissible. Later, Karapinar et al. [18] presented triangular α-admissible mappings, and then, Arshad et al. [19] introduced α-orbital admissible and triangular α-orbital admissible mappings. Due to the importance, many authors studied such mappings. For more knowledge and different examples related to admissible mappings, one can see [20–25].

Istratescu [26–28] gave the concept of contractions known as the convex contraction of order 2 and two-sided convex contraction mappings. Very recently on, Karapinar et al. [10] introduced the notion of α-almost Istratescu contraction of type E. Some notable generalizations related to Istratescu's results were obtained by [29–35].

In a recent work, Khojasteh et al. [36] introduced the notion of Z-contraction using simulation functions. Later, Karapinar [37] and Argoubi et al. [38] studied such contractions. After that, some new studies were obtained related to simulation functions in [39–44].

The aim of this paper is to establish some fixed point results for weak Pata convex contractive mapping and weak E-Pata convex contractive mapping via α-admissible mappings by using simulation functions in metric spaces. Our results are generalization of recent fixed point results derived by Karapinar et al. ([10, 32, 45]), Alber et al. [14], Zhang et al. [16], Istratescu [26], Pata [2], and Banach [1] and some other related results in the literature.

Firstly, we start this section by recalling some definitions related to our work.

In the course of this manuscript, $\mathbb{R}$, $\mathbb{N}$ denote the set of real numbers and the set of natural numbers, respectively. Let $FixS = \{w \in W : Sw = w.\}$

Alber et al. [14] gave the definition of $\phi-$weak contraction, stated below.

Definition 1. See [14]. Let (W, ρ) be a metric space. A mapping $S : W \longrightarrow W$ is called $\phi-$weak contraction, if there exists a map $\phi : [0,+\infty) \longrightarrow [0,+\infty)$ with $\phi(0) = 0$ and $\phi(w) > 0$ for all $w > 0$ such that

$$\rho(Sw, Sv) \leq \rho(w, v) - \phi(\rho(w, v)), \tag{1}$$

for all $w, v \in W$.

The concept of ϕ-weak contraction was generalized by Zhang et al. [16] as generalized ϕ-weak contraction.

Definition 2. See [16]. Let (W, ρ) be a metric space. A mapping $S : W \longrightarrow W$ is called generalized ϕ -weak contraction, if there exists a map $\phi : [0,+\infty) \longrightarrow [0,+\infty)$ with $\phi(0) = 0$ and $\phi(w) > 0$ for all $w > 0$ such that

$$\rho(Sw, Sv) \leq M(w, v) - \phi(M(w, v)), \tag{2}$$

for all $w, v \in W$, where

$$M(w, v) = \max \left\{ \rho(w, v), \rho(w, Sw), \rho(v, Sv), \frac{\rho(w, Sv) + \rho(v, Sw)}{2} \right\}. \tag{3}$$

Samet et al. [17] and Karapinar et al. [18] introduced the following concepts, respectively.

Definition 3. Let (W, ρ) be a metric space, $S : W \longrightarrow W$ be a map, and $\alpha : W \times W \longrightarrow [0,+\infty)$ be a function.

(i) [17] If $\alpha(w, v) \geq 1$ implies $\alpha(Sw, Sv) \geq 1$ for all $w, v \in W$, then S is called $\alpha -$ admissible

(ii) [18] If S *is* $\alpha -$ admissible and $\alpha(w, z) \geq 1$ and $\alpha(z, v) \geq 1$ imply $\alpha(w, v) \geq 1$, then S is called triangular $\alpha -$ admissible

Example 4. Let $W = \mathbb{R}$, the mappings $S : W \longrightarrow W$ by

$$S(w) = \begin{cases} \dfrac{w^2 + 1}{3}, & w \in [0, 1), \\ \dfrac{1}{2}, & w \notin [0, 1), \end{cases} \tag{4}$$

and $\alpha : W \times W \longrightarrow [0,+\infty)$ by

$$\alpha(w, v) = \begin{cases} 1, & w, v \in [0, 1], \\ 0, & w, v \notin [0, 1]. \end{cases} \tag{5}$$

Thus, S is a triangular α-admissible mapping.

Khojasteh et al. [36] gave the simulation function and Z -contraction as follows.

Definition 5. See [36]. A mapping $\zeta : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ is called a simulation function if it satisfies the following conditions:

(ζ_1) $\zeta(0, 0) = 0$

(ζ_2) $\zeta(w, v) < w - v$

(ζ_3) if $\{w_n\}$ and $\{v_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\longrightarrow+\infty} w_n = \lim_{n\longrightarrow+\infty} v_n > 0$, then $\limsup_{n\longrightarrow+\infty} \zeta(w_n, v_n) < 0$.

Definition 6. See [36]. Let (W, ρ) be a metric space and $S : W \longrightarrow W$ be a mapping. If there exists $\zeta \in Z$ such that

$$\zeta(\rho(Sw, Sv), \rho(w, v)) \geq 0, \quad \text{for all } w, v \in W, \tag{6}$$

then, S is called $Z -$ contraction with respect to ζ.

(ζ_1) condition was removed in the above definition of simulation function by Argoubi et al. [38] in 2015. Also, Z' denotes the set of all simulation functions.

Example 7. See [36, 42, 44]. Let $\zeta : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ and $\varphi_i : [0,\infty) \longrightarrow [0,\infty)$, $i = 1, 2, 3$ be continuous functions with $\varphi_i(w) = 0 \Leftrightarrow t = 0$.

$\zeta(w, v) = \varphi_1(w) - \varphi_2(v)$, for all $w, v \in [0,\infty)$, where $\varphi_1(w) < w \leq \varphi_2(v)$ for all $w > 0$.

$$\zeta(w, v) = \nu - \varphi_3(w) - w. \tag{7}$$

For the above examples and other examples related to simulation functions, one can see [36, 37, 42, 44] and references therein.

The following two concepts were defined by Istratescu [26] as follows.

Definition 8. See [26]. Let (W, ρ) be a metric space and $S : W \longrightarrow W$ be a self-mapping. For all $w, v \in W$, S is called convex contraction of order 2 if there exist $d_1, d_2 \in (0, 1)$ such that $d_1 + d_2 < 1$ and

$$\rho\left(S^2 w, S^2 v\right) \leq d_1 \rho(Sw, Sv) + d_2 \rho(w, v). \tag{8}$$

S is called two-sided convex contraction mappings if there exist $d_1, d_2, d_3, d_4 \in (0, 1)$ such that $d_1 + d_2 + d_3 + d_4 < 1$ and

$$\begin{aligned} \rho\left(S^2 w, S^2 v\right) \leq{}& d_1 \rho(w, Sw) + d_2 \rho\left(Sw, S^2 w\right) + d_3 \rho(v, Sv) \\ &+ d_4 \rho\left(Sv, S^2 v\right). \end{aligned} \tag{9}$$

In the course of this work, Ψ denotes the set of all increasing function $\psi : [0, 1] \longrightarrow [0,\infty)$, which vanishes with continuity at zero. For a random $w_0 \in W$, we denote $\|w\| = \rho(w, w_0), \forall w \in W$.

Introducing a novel generalization of the Banach contraction principle, Pata [2] proved Theorem 9.

Theorem 9. See [2]. *Let (W, ρ) be a metric space and $\Lambda \geq 0$, $\xi \geq 1$ and $\vartheta \in [0, \xi]$ be fixed constants. $\psi \in \Psi$ and $S : W \longrightarrow W$ be functions. If for all $w, v \in W$, the inequality*

$$\rho(Sw, Sv) \leq (1 - \varepsilon)\rho(w, v) + \Lambda \varepsilon^{\xi} \psi(\varepsilon)[1 + \|w\| + \|v\|]^{\vartheta} \tag{10}$$

is satisfied for all $\varepsilon \in [0, 1]$; then S has a unique fixed point, $\omega = S\omega$, $\omega \in W$.

Pata-type contractions were studied by some authors. Karapinar et al. [11] introduced Pata-Ciric type contraction at a point. Alqahtani et al. [5] gave the α-Pata–Suzuki contraction and fixed point results for such contractions. After that, Karapinar and Himabindu [11] proved some common fixed point results for Pata–Suzuki Z-contraction.

We recall here the following important Lemma 10 that we will use to proof of our main results.

Lemma 10. See [46]. *Let (W,ρ) be a metric space and $\{w_n\}$ be a sequence in W such that $\rho(w_{n+1},w_n)\longrightarrow 0$ as $n\longrightarrow\infty$. If $\{w_n\}$ is not a Cauchy sequence, then there exist a $\varsigma>0$ and subsequences $\{w_{m_j}\}$ and $\{w_{n_j}\}$ of $\{w_n\}$ such that* $\lim_{j\longrightarrow\infty}\rho(x_{m_j+1},x_{n_j+1})=\varsigma$, $\lim_{j\longrightarrow\infty}\rho(x_{m_j},x_{n_j})=\varsigma$, $\lim_{j\longrightarrow\infty}\rho(x_{m_j+1},x_{n_j})=\varsigma$ *and* $\lim_{j\longrightarrow\infty}\rho(x_{m_j},x_{n_j+1})=\varsigma$.

2. Main Results

The main objective of this work is to give some new fixed point theorems via a combination of convex contraction, weak contraction and Pata type contractive mappings by introducing the concept of weak E-Pata convex contractions and weak Pata convex contractions in metric spaces. We will use simulation functions and admissible mappings when combining these concepts. Also, we will give an example that supports our conclusion.

In definitions and results in this paper, $\Lambda\geq 0$, $\xi\geq 1$, and $\vartheta\in[0,\xi]$ will be considered as fixed constants, and also, we will consider the following equations:

$$
\begin{aligned}
E_I(w,v)&=\rho(Sw,Sv)+\left|\rho\left(Sw,S^2w\right)-\rho\left(Sv,S^2v\right)\right|,\\
M_I(w,v)&=\max\left\{\rho(w,v),\rho(Sw,Sv),\rho(w,Sw),\rho(v,Sv),\right.\\
&\qquad\left.\rho\left(Sw,S^2w\right),\rho\left(Sv,S^2v\right)\right\},\\
P_I(w,v)&=\left[1+\|w\|+\|v\|+\|Sw\|+\|Sv\|+\left\|S^2w\right\|+\left\|S^2v\right\|\right]^{\vartheta}.
\end{aligned}
\tag{11}
$$

At first, we begin our work by giving the following definitions.

Definition 11. Let (W,ρ) be a metric space. We say that $S:W\longrightarrow W$ is weak Pata convex contractive mapping via simulation function if for all $w,v\in W$, and $\varepsilon\in[0,1]$, there exist three functions $\zeta\in Z'$, $\psi\in\Psi$, and $\alpha:W\times W\longrightarrow[0,+\infty)$ such that S satisfies the inequality

$$
\begin{aligned}
&\zeta\left(\alpha(w,v)\rho\left(S^2w,S^2v\right),(1-\varepsilon)(M_I(w,v)-\phi(M_I(w,v)))\right.\\
&\qquad\left.+\Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w,v)\right)\geq 0,
\end{aligned}
\tag{12}
$$

where $\phi:[0,+\infty)\longrightarrow[0,+\infty)$ is a continuous and nondecreasing function with $\phi(0)=0$ and $\phi(w)>0$, for all $w>0$.

Definition 12. Let (W,ρ) be a metric space. We say that $S:W\longrightarrow W$ is weak E -Pata convex contractive mapping via simulation function if for all $w,v\in W$, and $\varepsilon\in[0,1]$, there exist three functions $\psi\in\Psi$, $\zeta\in Z'$, and $\alpha:W\times W\longrightarrow[0,+\infty)$ such that S satisfies the inequality

$$
\begin{aligned}
&\zeta\left(\alpha(w,v)\rho\left(S^2w,S^2v\right),(1-\varepsilon)(E_I(w,v)-\phi(E_I(w,v)))\right.\\
&\qquad\left.+\Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w,v)\right)\geq 0,
\end{aligned}
\tag{13}
$$

where $\phi:[0,+\infty)\longrightarrow[0,+\infty)$ is a continuous and nondecreasing function with $\phi(0)=0$ and $\phi(w)>0$, for all $w>0$.

Now, we are in a position to present our main theorems.

Theorem 13. *Let (W,ρ) be a complete metric space, $\alpha:W\times W\longrightarrow[0,+\infty)$ and $S:W\longrightarrow W$ be a weak E -Pata convex mapping via simulation function. Suppose that*

(i) S is triangular α -admissible

(ii) there exists $w_0\in W$ such that $\alpha(w_0,Sw_0)\geq 1$

(iii) S is continuous

(iv) for all $w,v\in FixS$, $\alpha(w,v)\geq 1$.

Then S has a unique fixed point in W.

Proof. From hypothesis (ii) of the Theorem 13, there exists $w_0\in W$ such that $\alpha(w_0,Sw_0)\geq 1$. Firstly, we will show that $\alpha(S^nw_0,S^{n+1}w_0)\geq 1$ for all $n\in\mathbb{N}$. Since S is an α-admissible mapping, we have

$$
\begin{aligned}
&\alpha(w_0,w_1)\geq 1=\alpha(w_0,Sw_0)\geq 1\Rightarrow\alpha\left(Sw_0,S^2w_0\right)\geq 1,\\
&\alpha\left(Sw_0,S^2w_0\right)\geq 1\Rightarrow\alpha\left(S^2w_0,S^3w_0\right)\geq 1.
\end{aligned}
\tag{14}
$$

By induction, we obtain that

$$
\alpha\left(S^nw_0,S^{n+1}w_0\right)\geq 1,\quad\text{for all } n\in\mathbb{N}. \tag{15}
$$

Taking into account hypothesis (i) of the Theorem 13, we have

$$
\begin{aligned}
&\alpha\left(S^nw_0,S^{n+1}w_0\right)\geq 1\text{ and }\alpha\left(S^{n+1}w_0,S^{n+2}w_0\right)\\
&\qquad\geq 1\Rightarrow\alpha\left(S^nw_0,S^{n+2}w_0\right)\geq 1.
\end{aligned}
\tag{16}
$$

Again by induction, we obtain that

$$
\alpha(S^nw_0,S^mw_0)\geq 1,\quad\text{for all } m>n\geq 0. \tag{17}
$$

Now, we will show that $\{\rho(S^nw_0,S^{n+1}w_0)\}$ is a nonincreasing sequence. Since S is a weak E-Pata convex contractive mapping via simulation function, we have

$$\zeta\left(\alpha(w_0, Sw_0)\rho(S^2w_0, S^3w_0), (1-\varepsilon)\begin{pmatrix} E_I(w_0, Sw_0) \\ -\phi(E_I(w_0, Sw_0)) \end{pmatrix} + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w_0, Sw_0)\right) \geq 0,$$

$$(1-\varepsilon)\begin{pmatrix} E_I(w_0, Sw_0) \\ -\phi(E_I(w_0, Sw_0)) \end{pmatrix} + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w_0, Sw_0) - \alpha(w_0, Sw_0)\rho(S^2w_0, S^3w_0) \geq 0. \tag{18}$$

From hypothesis (ii) of the Theorem 13, we get

$$\begin{aligned}
\rho(S^2w_0, S^3w_0) &\leq \alpha(w_0, Sw_0)\rho(S^2w_0, S^3w_0) \\
&\leq (1-\varepsilon)(E_I(w_0, Sw_0) - \phi(E_I(w_0, Sw_0))) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w_0, Sw_0) \\
&= (1-\varepsilon)\begin{pmatrix} \rho(Sw_0, S^2w_0) + |\rho(Sw_0, S^2w_0) - \rho(S^2w_0, S^3w_0)| \\ -\phi(\rho(Sw_0, S^2w_0) + |\rho(Sw_0, S^2w_0) - \rho(S^2w_0, S^3w_0)|) \end{pmatrix} \\
&\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)\begin{bmatrix} 1 + \|w_0\| + \|Sw_0\| + \|Sw_0\| \\ + \|S^2w_0\| + \|S^2w_0\| + \|S^3w_0\| \end{bmatrix}^{\vartheta} \\
&\leq (1-\varepsilon)\begin{pmatrix} \rho(Sw_0, S^2w_0) + |\rho(Sw_0, S^2w_0) - \rho(S^2w_0, S^3w_0)| \\ -\phi(\rho(Sw_0, S^2w_0) + |\rho(Sw_0, S^2w_0) - \rho(S^2w_0, S^3w_0)|) \end{pmatrix} \\
&\quad \cdot \left[1 + \|w_0\| + 2\|Sw_0\| + 2\|S^2w_0\| + \|S^3w_0\|\right]^{\vartheta} \\
&\leq (1-\varepsilon)\begin{pmatrix} \rho(Sw_0, S^2w_0) + |\rho(Sw_0, S^2w_0) - \rho(S^2w_0, S^3w_0)| \\ -\phi(\rho(Sw_0, S^2w_0) + |\rho(Sw_0, S^2w_0) - \rho(S^2w_0, S^3w_0)|) \end{pmatrix} \\
&\quad + K\varepsilon^{\xi}\psi(\varepsilon),
\end{aligned} \tag{19}$$

for some $K > 0$. If we assume that $\rho(Sw_0, S^2w_0) < \rho(S^2w_0, S^3w_0)$, then we have $\rho(Sw_0, S^2w_0) + |\rho(Sw_0, S^2w_0) - \rho(S^2w_0, S^3w_0)| = \rho(S^2w_0, S^3w_0)$. Hence, we have

$$\rho(S^2w_0, S^3w_0) \leq (1-\varepsilon)(\rho(S^2w_0, S^3w_0) - \phi(\rho(S^2w_0, S^3w_0))) + K\varepsilon^{\xi}\psi(\varepsilon). \tag{20}$$

The inequality (20) is true for all $\varepsilon \in [0, 1]$. For $\varepsilon = 0$, we obtain $\rho(S^2w_0, S^3w_0) < \rho(S^2w_0, S^3w_0)$ which is a contradiction. Therefore, we obtain

$$\rho(S^2w_0, S^3w_0) \leq \rho(Sw_0, S^2w_0). \tag{21}$$

Analogously, as S is a weak E-Pata convex *c*ontractive mapping via simulation function, we have

$$\zeta\left(\alpha(Sw_0, S^2w_0)\rho(S^3w_0, S^4w_0), (1-\varepsilon)\begin{pmatrix} E_I(Sw_0, S^2w_0) \\ -\phi(E_I(Sw_0, S^2w_0)) \end{pmatrix} + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(Sw_0, S^2w_0)\right) \geq 0,$$

$$\left((1-\varepsilon)\begin{pmatrix} E_I(Sw_0, S^2w_0) \\ -\phi(E_I(Sw_0, S^2w_0)) \end{pmatrix} + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(Sw_0, S^2w_0) - \alpha(Sw_0, S^2w_0)\rho(S^3w_0, S^4w_0)\right) \geq 0. \tag{22}$$

Now, we can write

$$\begin{aligned}
\rho(S^3w_0, S^4w_0) &\leq \alpha(Sw_0, S^2w_0)\rho(S^3w_0, S^4w_0) \\
&\leq (1-\varepsilon)\begin{pmatrix} \rho(S^2w_0, S^3w_0) + |\rho(S^2w_0, S^3w_0) - \rho(S^3w_0, S^4w_0)| \\ -\phi(\rho(S^2w_0, S^3w_0) + |\rho(S^2w_0, S^3w_0) - \rho(S^3w_0, S^4w_0)|) \end{pmatrix} \\
&\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)\left[1 + \|Sw_0\| + \|S^2w_0\| + \|S^2w_0\| + \|S^3w_0\| + \|S^3w_0\| + \|S^4w_0\|\right]^{\vartheta} \\
&\leq (1-\varepsilon)\begin{pmatrix} \rho(S^2w_0, S^3w_0) + |\rho(S^2w_0, S^3w_0) - \rho(S^3w_0, S^4w_0)| \\ -\phi(\rho(S^2w_0, S^3w_0) + |\rho(S^2w_0, S^3w_0) - \rho(S^3w_0, S^4w_0)|) \end{pmatrix} + K\varepsilon^{\xi}\psi(\varepsilon),
\end{aligned} \tag{23}$$

for some $K > 0$. In case that $\rho(S^2w_0, S^3w_0) < \rho(S^3w_0, S^4w_0)$; then we have $\rho(S^2w_0, S^3w_0) + |\rho(S^2w_0, S^3w_0) - \rho(S^3w_0, S^4w_0)| = \rho(S^3w_0, S^4w_0)$. So, we have

$$\rho(S^3w_0, S^4w_0) \leq (1-\varepsilon)(\rho(S^3w_0, S^4w_0) - \phi(\rho(S^3w_0, S^4w_0))) + K\varepsilon^{\xi}\psi(\varepsilon). \tag{24}$$

The inequality (24) is true for all $\varepsilon \in [0, 1]$. For $\varepsilon = 0$, we obtain $\rho(S^3w_0, S^4w_0) < \rho(S^3w_0, S^4w_0)$ which is again a contradiction. Therefore, we obtain

$$\rho(S^3w_0, S^4w_0) \leq \rho(S^2w_0, S^3w_0). \tag{25}$$

By induction, since S is a weak E-Pata convex *c*ontractive mapping via simulation function, we have

$$\begin{aligned}
&\zeta\big(\alpha(S^{n-2}w_0, S^{n-1}w_0)\rho(S^nw_0, S^{n+1}w_0), (1-\varepsilon) \\
&\quad \cdot (E_I(S^{n-2}w_0, S^{n-1}w_0) - \phi(E_I(S^{n-2}w_0, S^{n-1}w_0))) \\
&\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(S^{n-2}w_0, S^{n-1}w_0)\big) \geq 0,
\end{aligned}$$

$$\begin{aligned}
&\Bigg((1-\varepsilon)\begin{pmatrix} E_I(S^{n-2}w_0, S^{n-1}w_0) \\ -\phi(E_I(S^{n-2}w_0, S^{n-1}w_0)) \end{pmatrix} \\
&\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(S^{n-2}w_0, S^{n-1}w_0) - \alpha(S^{n-2}w_0, S^{n-1}w_0) \\
&\quad \cdot \rho(S^nw_0, S^{n+1}w_0)\Bigg) \geq 0.
\end{aligned} \tag{26}$$

We have that

$$\begin{aligned}
\rho\left(S^n w_0, S^{n+1} w_0\right) &\leq \alpha\left(S^{n-2} w_0, S^{n-1} w_0\right)\rho\left(S^n w_0, S^{n+1} w_0\right) \\
&\leq (1-\varepsilon)\begin{pmatrix} \rho\left(S^{n-1} w_0, S^n w_0\right) + \left|\rho\left(S^{n-1} w_0, S^n w_0\right) - \rho\left(S^n w_0, S^{n+1} w_0\right)\right| \\ -\phi\left(\rho\left(S^{n-1} w_0, S^n w_0\right) + \left|\rho\left(S^{n-1} w_0, S^n w_0\right) - \rho\left(S^n w_0, S^{n+1} w_0\right)\right|\right) \end{pmatrix} \\
&\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)\left[1 + \left\|S^{n-2} w_0\right\| + \left\|S^{n-1} w_0\right\| + \left\|S^{n-1} w_0\right\| + \left\|S^n w_0\right\| + \left\|S^n w_0\right\| + \left\|S^{n+1} w_0\right\|\right]^{\vartheta} \\
&\leq (1-\varepsilon)\begin{pmatrix} \rho\left(S^{n-1} w_0, S^n w_0\right) + \left|\rho\left(S^{n-1} w_0, S^n w_0\right) - \rho\left(S^n w_0, S^{n+1} w_0\right)\right| \\ -\phi\left(\rho\left(S^{n-1} w_0, S^n w_0\right) + \left|\rho\left(S^{n-1} w_0, S^n w_0\right) - \rho\left(S^n w_0, S^{n+1} w_0\right)\right|\right) \end{pmatrix} + K\varepsilon^{\xi}\psi(\varepsilon),
\end{aligned} \tag{27}$$

for some $K > 0$. In case that $\rho(S^{n-1} w_0, S^n w_0) < \rho(S^n w_0, S^{n+1} w_0)$; then we have

$$\begin{aligned}
\rho\left(S^n w_0, S^{n+1} w_0\right) < (1-\varepsilon)\big(\rho\left(S^n w_0, S^{n+1} w_0\right) \\
- \phi\left(\rho\left(S^n w_0, S^{n+1} w_0\right)\right)\big) + K\varepsilon^{\xi}\psi(\varepsilon).
\end{aligned} \tag{28}$$

Again, the inequality (28) is true for all $\varepsilon \in [0, 1]$ for $\varepsilon = 0$; we obtain $\rho(S^n w_0, S^{n+1} w_0) < \rho(S^n w_0, S^{n+1} w_0)$ is again a contradiction. Therefore, we obtain

$$\rho\left(S^n w_0, S^{n+1} w_0\right) \leq \rho\left(S^{n-1} w_0, S^n w_0\right). \tag{29}$$

Consequently, we find that

$$\begin{aligned}
\rho\left(S^n w_0, S^{n+1} w_0\right) &\leq \rho\left(S^{n-1} w_0, S^n w_0\right) \leq \cdots \leq \rho\left(S^3 w_0, S^4 w_0\right) \\
&\leq \rho\left(S^2 w_0, S^3 w_0\right) \leq \rho\left(S w_0, S^2 w_0\right).
\end{aligned} \tag{30}$$

If the point $w_0 \in W$ is taken as the starting point, the sequence $\{w_n\}$ is constructed by $w_n = S w_{n-1} = S^n w_0$, $n \geq 1$. If $w_{n_0+1} = w_{n_0}$ for any $n_0 \in \mathbb{N}$, then w_{n_0} is a fixed point of S. As a result, supposing that $w_{n_0+1} \neq w_{n_0}$ for all $n_0 \in \mathbb{N}$ and let $\rho_n = \rho(w_{n-1}, w_n)$. So, we get that $\{\rho_n\}$ is a nonincreasing sequence. For this reason, there exists a $\delta \geq 0$ such that

$$\lim_{n \longrightarrow \infty} \rho(w_{n-1}, w_n) = \lim_{n \longrightarrow \infty} \rho_n = \delta. \tag{31}$$

We will demonstrate that $\delta = 0$. For this, we should demostrate that the sequence $\{\|w_n\|\}$ is bounded. Since $\{\rho_n\}$ is a nonincreasing sequence, we have

$$\begin{aligned}
\rho_{n+1} = \rho(w_n, w_{n+1}) &\leq \rho(w_{n-1}, w_n) \leq \cdots \leq \rho(w_3, w_4) \\
&\leq \rho(w_2, w_3) \leq \rho(w_1, w_2) = \rho_2 \leq \|w_1\| + \|w_2\|.
\end{aligned} \tag{32}$$

By the triangle inequality, we have

$$\begin{aligned}
\|w_n\| = \rho(w_n, w_0) &\leq \rho(w_n, w_{n+1}) + \rho(w_{n+1}, w_2) + \rho(w_2, w_0) \\
&= \rho_{n+1} + \rho(w_{n+1}, w_2) + \|w_2\| \leq \rho_2 + \rho(w_{n+1}, w_2) \\
&\quad + \|w_2\| \leq \|w_1\| + 2\|w_2\| + \rho(w_{n+1}, w_2).
\end{aligned} \tag{33}$$

Since S is a weak E-Pata convex contractive mapping, we have

$$\begin{aligned}
\zeta\left(\alpha(w_n, w_0)\rho(w_{n+1}, w_2), (1-\varepsilon)\begin{pmatrix} E_I(w_{n-1}, w_0) \\ -\phi(E_I(w_{n-1}, w_0)) \end{pmatrix} \right. \\
\left. + \Lambda\varepsilon^{\xi}\psi(\varepsilon) P_I(w_{n-1}, w_0)\right) \geq 0,
\end{aligned} \tag{34}$$

$$\begin{aligned}
&\big((1-\varepsilon)(E_I(w_{n-1}, w_0) - \phi(E_I(w_{n-1}, w_0))) + \Lambda\varepsilon^{\xi}\psi(\varepsilon) P_I(w_{n-1}, w_0)\big) \\
&\quad - \alpha(w_n, w_0)\rho(w_{n+1}, w_2) \geq 0.
\end{aligned} \tag{35}$$

Together with (35), we obtain

$$\begin{aligned}
\rho(w_{n+1}, w_2) &\leq \alpha(w_n, w_0)\rho(w_{n+1}, w_2) \\
&\leq (1-\varepsilon)(E_I(w_{n-1}, w_0) - \phi(E_I(w_{n-1}, w_0))) \\
&\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon) P_I(w_{n-1}, w_0),
\end{aligned} \tag{36}$$

where

$$\begin{aligned}
E_I(w_{n-1}, w_0) &= \rho(w_n, w_1) + |\rho(w_n, w_{n+1}) - \rho(w_1, w_2)| \\
&\leq \rho(w_n, w_0) + \rho(w_1, w_0) \\
&\quad + |\rho(w_n, w_{n+1}) - \rho(w_1, w_2)| \leq \|w_n\| + \|w_1\| \\
&\quad + |\rho_{n+1} - \rho_2| = \|w_n\| + \|w_1\| + \rho_2 - \rho_{n+1} \\
&\leq \|w_n\| + 2\|w_1\| + \|w_2\| - \rho_{n+1} \leq \|w_n\| \\
&\quad + 2\|w_1\| + \|w_2\|, \\
P_I(w_{n-1}, w_0) &= [1 + \|w_{n-1}\| + \|w_0\| + \|w_n\| + \|w_1\| + \|w_{n+1}\| \\
&\quad + \|w_2\|]^{\vartheta} \leq [1 + \|w_1\| + \|w_2\| + \|w_n\| + \|w_n\| \\
&\quad + \|w_1\| + \|w_1\| + \|w_2\| + \|w_n\| + \|w_2\|]^{\vartheta} \\
&= [1 + 3\|w_1\| + 3\|w_2\| + 3\|w_n\|]^{\vartheta}.
\end{aligned} \tag{37}$$

Now, we derive that

$$\begin{aligned}
\|w_n\| &< \|w_1\| + 2\|w_2\| + (1-\varepsilon)(\|w_n\| + 2\|w_1\| + \|w_2\| \\
&\quad - \phi(\|w_n\| + 2\|w_1\| + \|w_2\|)) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)[1 + 3\|w_1\| \\
&\quad + 3\|w_2\| + 3\|w_n\|]^{\vartheta}.
\end{aligned} \tag{38}$$

Using $\vartheta \leq \xi$, we get

$$\begin{aligned}
\varepsilon\|w_n\| &< (3-2\varepsilon)\|w_1\| + (3-\varepsilon)\|w_2\| + \Lambda\varepsilon^{\xi}\psi(\varepsilon)[1 + 3\|w_1\| \\
&\quad + 3\|w_2\| + 3\|w_n\|]^{\vartheta} \leq (3-2\varepsilon)\|w_1\| + (3-\varepsilon)\|w_2\| \\
&\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)[1 + 3\|w_1\| + 3\|w_2\| + 3\|w_n\|]^{\xi} \\
&= (3-2\varepsilon)\|w_1\| + (3-\varepsilon)\|w_2\| + \Lambda\varepsilon^{\xi}\psi(\varepsilon)(1 + 3\|w_n\|)^{\xi} \\
&\quad \cdot \left(\frac{1 + 3\|w_1\| + 3\|w_2\|}{1 + 3\|w_n\|}\right)^{\xi} \leq 3\|w_1\| + 3\|w_2\| + \Lambda\varepsilon^{\xi}\psi \\
&\quad \cdot (\varepsilon) 3^{\xi}\|w_n\|^{\xi}\left(\frac{1}{3\|w_n\|} + 1\right)^{\xi}(1 + 3\|w_1\| + 3\|w_2\|)^{\xi}.
\end{aligned} \tag{39}$$

Conversely, we assume that $\{\|w_n\|\}$ is not bounded sequence. So, there exists a subsequence $\{\|w_{n_j}\|\}$ of $\{\|w_n\|\}$ such that $\lim_{j\longrightarrow\infty} w_{n_j}=\infty$. If we take $\varepsilon=\varepsilon_j=(1+3\|w_1\|+3\|w_2\|)/\|w_{n_j}\|$ in (39) inequality; then we have

$$\begin{aligned}1 &\le \Lambda 3^{\xi}\left(\varepsilon^{\xi}\|w_n\|^{\xi}\right)(1+3\|w_1\|+3\|w_2\|)^{\xi}\left(\frac{1}{3\left\|w_{n_j}\right\|}+1\right)^{\xi} \\ &\cdot \psi\left(\varepsilon_j\right) \le \Lambda 3^{\xi}(1+3\|w_1\|+3\|w_2\|)^{\xi}(1+3\|w_1\|+3\|w_2\|)^{\xi} \\ &\cdot \left(\frac{1}{3\left\|w_{n_j}\right\|}+1\right)^{\xi}\psi\left(\varepsilon_j\right) \le \Lambda 3^{\xi}(1+3\|w_1\|+3\|w_2\|)^{2\xi} \\ &\cdot \left(\frac{1}{3\left\|w_{n_j}\right\|}+1\right)^{\xi}\psi\left(\varepsilon_j\right).\end{aligned} \tag{40}$$

If we take limit in (40) inequality as $j\longrightarrow\infty$, then we get

$$\Lambda 3^{\xi}(1+3\|w_1\|+3\|w_2\|)^{2\xi}\left(\frac{1}{3\left\|w_{n_j}\right\|}+1\right)^{\xi}\psi\left(\varepsilon_j\right)\longrightarrow 0, \tag{41}$$

which is a contradiction. Therefore, we demonstrate that the sequence$\{\|w_n\|\}$ is bounded. So, there exists $A>0$ such that $\|w_n\|\le A$ for all $n\in\mathbb{N}$. Following this line of work, we demonstrate that $\delta=0$. Since S is a weak E-Pata convex contractive mapping, we have

$$\begin{aligned}&\zeta(\alpha(w_{n-1},w_n)\rho(w_{n+1},w_{n+2}),(1-\varepsilon)(E_I(w_{n-1},w_n) \\ &\quad-\phi(E_I(w_{n-1},w_n)))+\Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w_{n-1},w_n))\ge 0, \\ &(1-\varepsilon)(E_I(w_{n-1},w_n)-\phi(E_I(w_{n-1},w_n))) \\ &\quad+\Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w_{n-1},w_n)-\alpha(w_{n-1},w_n)\rho(w_{n+1},w_{n+2})\ge 0.\end{aligned} \tag{42}$$

Since $\rho_{n+1}\le\rho_n$ for all $n\in\mathbb{N}$, we have

$$\begin{aligned}E_I(w_{n-1},w_n)&=\rho(w_n,w_{n+1})+|\rho(w_n,w_{n+1})-\rho(w_{n+1},w_{n+2})| \\ &=2\rho(w_n,w_{n+1})-\rho(w_{n+1},w_{n+2})=2\rho_{n+1}-\rho_{n+2}.\end{aligned} \tag{43}$$

Since the sequence$\{\|w_n\|\}$ is bounded, we have

$$\begin{aligned}P_I(w_{n-1},w_n)&=\Lambda\varepsilon^{\xi}\psi(\varepsilon)[1+\|w_{n-1}\|+\|w_n\|+\|w_n\|+\|w_{n+1}\| \\ &\quad+\|w_{n+2}\|+\|w_{n+3}\|]^{\vartheta}\le\Lambda\varepsilon^{\xi}\psi(\varepsilon)(1+6A)^{\vartheta}.\end{aligned} \tag{44}$$

Now, we can write

$$\begin{aligned}\rho_{n+2}&=\rho(w_{n+1},w_{n+2})\le\alpha(w_{n-1},w_n)\rho(w_{n+1},w_{n+2}) \\ &\le(1-\varepsilon)(E_I(w_{n-1},w_n)-\phi(E_I(w_{n-1},w_n))) \\ &\quad+\Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w_{n-1},w_n)\le(1-\varepsilon) \\ &\quad\cdot(2\rho_{n+1}-\rho_{n+2}-\phi(2\rho_{n+1}-\rho_{n+2})) \\ &\quad+\Lambda\varepsilon^{\xi}\psi(\varepsilon)(1+6A)^{\vartheta}.\end{aligned} \tag{45}$$

If we take the limit as $n\longrightarrow\infty$ in (45) inequality, then we obtain

$$\begin{aligned}\delta&\le(1-\varepsilon)(\delta-\phi(\delta))+\Lambda\varepsilon^{\xi}\psi(\varepsilon)(1+6A)^{\vartheta}\delta \\ &\le\Lambda\varepsilon^{\xi-1}\psi(\varepsilon)(1+6A)^{\vartheta}.\end{aligned} \tag{46}$$

$\delta\le 0$ as $\varepsilon\longrightarrow 0$, that is $\lim_{n\longrightarrow\infty}\rho(w_{n+1},w_{n+2})=\delta=0$. Now, we demonstrate that $\{w_n\}$ is a Cauchy sequence. On the contrary, assume that the sequence $\{w_n\}$ is not a Cauchy. From Lemma 10, there exist subsequence $\{w_{m_j}\}$and $\{w_{n_j}\}$ with $n_j>m_j>j$ such that $\lim_{k\longrightarrow\infty}\rho(x_{m_k-1},x_{n_k+1})=\varsigma$, $\lim_{k\longrightarrow\infty}\rho(x_{m_k-1},x_{n_k})=\varsigma$, $\lim_{k\longrightarrow\infty}\rho(x_{m_k},x_{n_k})=\varsigma$, $\lim_{k\longrightarrow\infty}\rho(x_{m_k+1},x_{n_k+1})=\varsigma$, and $\lim_{k\longrightarrow\infty}\rho(x_{m_k},x_{n_k-1})=\varsigma$. Since S is a weak $E-$Pata convex contractive mapping, we have

$$\begin{aligned}&\zeta\left(\alpha\left(w_{n_j-1},w_{m_j-1}\right)\rho\left(w_{n_j+1},w_{m_j+1}\right),(1-\varepsilon)\right. \\ &\quad\left.\cdot\left(E_I\left(w_{n_j-1},w_{m_j-1}\right)-\phi\left(E_I\left(w_{n_j-1},w_{m_j-1}\right)\right)\right)\right)\ge 0, \\ &(1-\varepsilon)\left(E_I\left(w_{n_j-1},w_{m_j-1}\right)-\phi\left(E_I\left(w_{n_j-1},w_{m_j-1}\right)\right)\right) \\ &\quad-\alpha\left(w_{n_j-1},w_{m_j-1}\right)\rho\left(w_{n_j+1},w_{m_j+1}\right)\ge 0,\end{aligned} \tag{47}$$

where

$$\begin{aligned}E_I\left(w_{n_j-1},w_{m_j-1}\right)&=\rho\left(w_{n_j},w_{m_j}\right) \\ &\quad+\left|\rho\left(w_{n_j},w_{n_j+1}\right)-\rho\left(w_{m_j},w_{m_j+1}\right)\right|, \\ P_I\left(w_{n_j-1},w_{m_j-1}\right)&=\Lambda\varepsilon^{\xi}\psi(\varepsilon)\left[1+\left\|w_{n_j-1}\right\|+\left\|w_{m_j-1}\right\|\right. \\ &\quad\left.+\left\|w_{n_j}\right\|+\left\|w_{m_j}\right\|+\left\|w_{n_j+1}\right\|+\left\|w_{m_j+1}\right\|\right]^{\vartheta} \\ &=\Lambda\varepsilon^{\xi}\psi(\varepsilon)[1+6A]^{\vartheta}.\end{aligned} \tag{48}$$

Now, we have

$$\begin{aligned}
\varsigma &\le \rho\left(w_{n_j+1}, w_{m_j+1}\right) \le \alpha\left(w_{n_j-1}, w_{m_j-1}\right)\rho\left(w_{n_j+1}, w_{m_j+1}\right) \\
&\le (1-\varepsilon)\left(E_I\left(w_{n_j-1}, w_{m_j-1}\right) - \phi\left(E_I\left(w_{n_j-1}, w_{m_j-1}\right)\right)\right) \\
&\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I\left(w_{n_j-1}, w_{m_j-1}\right) \\
&\le (1-\varepsilon)\begin{pmatrix} \rho\left(w_{n_j}, w_{m_j}\right) + \left|\rho\left(w_{n_j}, w_{n_j+1}\right) - \rho\left(w_{m_j}, w_{m_j+1}\right)\right| \\ -\phi\left(\rho\left(w_{n_j}, w_{m_j}\right) + \left|\rho\left(w_{n_j}, w_{n_j+1}\right) - \rho\left(w_{m_j}, w_{m_j+1}\right)\right|\right) \end{pmatrix} \\
&\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)[1+6A]^{\vartheta} \\
&\le (1-\varepsilon)\left(\rho\left(w_{n_j}, w_{m_j}\right) + \left|\rho\left(w_{n_j}, w_{n_j+1}\right) - \rho\left(w_{m_j}, w_{m_j+1}\right)\right|\right) \\
&\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)[1+6A]^{\vartheta}.
\end{aligned} \tag{49}$$

If we take the limit as $j \longrightarrow \infty$, then we obtain

$$\varsigma \le (1-\varepsilon)\varsigma + K\varepsilon\psi(\varepsilon), \tag{50}$$

and so, we have

$$\varsigma \le K\psi(\varepsilon), \tag{51}$$

and thus, we get that $\varsigma = 0$, which is a contradiction. Therefore, we concluded that $\{w_n\}$ is a Cauchy sequence in (W, ρ). By the completeness of W, the sequence $\{w_n\}$ is convergent to some $\omega \in W$ that is $w_n \longrightarrow \omega$ as $n \longrightarrow +\infty$. Since S is continuous, $Sw_n \longrightarrow S\omega$ as $n \longrightarrow +\infty$. By the uniqueness of the limit, we obtain $\omega = S\omega$ that is ω is a fixed point of S.

Next, we will demonstrate the uniqueness of the fixed point. Suppose that T and ω are two fixed points of S. Since S satisfies the hypothesis (iv) of Theorem 13, S is an weak E-Pata convex contractive mapping; we have

$$\begin{aligned}
\rho(\omega, T) &\le \alpha(\omega, T)\rho\left(S^2\omega, S^2T\right) \\
&\le (1-\varepsilon)(E_I(\omega, T) - \phi(E_I(\omega, T))) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(\omega, T) \\
&\le (1-\varepsilon)\left(\begin{matrix} \rho(S\omega, ST) + \left|\rho\left(S\omega, S^2\omega\right) - \rho\left(ST, S^2T\right)\right| \\ -\phi(\rho(S\omega, ST) + \left|\rho\left(S\omega, S^2\omega\right) - \rho\left(ST, S^2T\right)\right| \end{matrix}\right)\Bigg) \\
&\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)\left[1 + \|\omega\| + \|T\| + \|S\omega\| + \|ST\| + \left\|S^2\omega\right\|\right. \\
&\quad \left. + \left\|S^2T\right\|\right]^{\vartheta} \le (1-\varepsilon)\rho(\omega, T) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)[1 + 3\|\omega\| + 3\|T\|]^{\vartheta}.
\end{aligned} \tag{52}$$

We obtain that $\rho(\omega, T) < K\psi(\varepsilon)$ for some $K \ge 0$, and so, we get $\omega = T$. Hence, S has a unique fixed point in W, that is $\omega = S\omega$, $\omega \in W$.

Following this line of work, Theorem 14 does not require the continuity of S. □

Theorem 14. *Let (W, ρ) be a complete metric space, $\alpha : W \times W \longrightarrow [0,+\infty)$ and $S : W \longrightarrow W$ be a weak Pata-convex mapping. Suppose that*

(i) S is triangular $\alpha-$admissible

(ii) there exists $w_0 \in W$ such that $\alpha(w_0, Sw_0) \ge 1$

(iii) S^2 is continuous and for all $\omega \in FixS^2$, $\alpha(S\omega, \omega) \ge 1$

(iv) for all $w, \omega \in FixS^2$, $\alpha(w, \omega) \ge 1$

Then, S has a unique fixed point in W.

Proof. Following the proof of Theorem 13, we have already proved that $\{w_n\}$ is a Cauchy sequence in W. Since W is complete, we have $w_n \longrightarrow \omega \in W$ as $n \longrightarrow +\infty$. Taking into account hypothesis (iii) Theorem 14, we have $\lim_{n\longrightarrow\infty} \rho(w_n, S^2\omega) = \lim_{n\longrightarrow\infty}\rho(S^2w_{n-2}, S^2\omega) = 0$. In the uniqueness of the limit, we obtain that $S^2\omega = \omega$. Next, we will prove that $\omega = S\omega$. On the contrary, we assume that ω is not fixed point of S. So, we have

$$\begin{aligned}
0 < \rho(S\omega, \omega) &= \rho\left(S^2(S\omega), S^2\omega\right) \le \alpha(S\omega, \omega)\rho\left(S^2(S\omega), S^2\omega\right) \\
&\le (1-\varepsilon)(E_I(S\omega, \omega) - \phi(E_I(S\omega, \omega))) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(S\omega, \omega) \\
&\le (1-\varepsilon)\begin{pmatrix} \rho\left(S\omega, S^2\omega\right) + \left|\rho\left(S\omega, S^2\omega\right) - \rho\left(S^2\omega, S^3\omega\right)\right| \\ -\phi\left(\rho\left(S\omega, S^2\omega\right) + \left|\rho\left(S\omega, S^2\omega\right) - \rho\left(S^2\omega, S^3\omega\right)\right|\right) \end{pmatrix} \\
&\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)\left[1 + \|S\omega\| + \|\omega\| + \|S\omega\| + \left\|S^2\omega\right\| + \left\|S^3\omega\right\| + \left\|S^2\omega\right\|\right]^{\vartheta} \\
&\le (1-\varepsilon)\rho(S\omega, \omega) - \phi(\rho(S\omega, \omega)) + K\varepsilon^{\xi}\psi(\varepsilon),
\end{aligned} \tag{53}$$

for some $K > 0$. We obtain

$$\rho(S\omega, \omega) < (1-\varepsilon)\rho(S\omega, \omega) + K\varepsilon^{\xi}\psi(\varepsilon). \tag{54}$$

For $\varepsilon = 0$ in (54) which is a contradiction. Thus, we make an inference that $S\omega = \omega$, and so, ω is a fixed point of S. Following the proof of Theorem 13, the uniqueness of fixed point of S can be obtained.

Theorem 15 is other fundamental result of our work. □

Theorem 15. *Let (W, ρ) be a complete metric space, $\alpha : W \times W \longrightarrow [0,+\infty)$ and $S : W \longrightarrow W$ be a weak Pata convex contractive mapping via simulation function. On the assumption that all of the Theorem 13 hypotheses are satisfied, then h has a unique fixed point.*

Proof. In the proof of Theorem 13, we have got that

$$\alpha\left(S^n w_0, S^{n+1}w_0\right) \ge 1 \text{ for all } n \in \mathbb{N} \text{ and } \alpha(S^n w_0, S^m w_0) \ge 1 \text{ for all } m > n \ge 0. \tag{55}$$

Setting $\ell = \min\{\rho(w_0, Sw_0), \rho(Sw_0, S^2w_0)\}$ and now, we demonstrate that

$\{\rho(S^n w_0, S^{n+1}w_0)\}$ is a nonincreasing sequence. Since S is a weak Pata convex contractive mapping via simulation function, we have

$$\begin{aligned}
&\zeta\left(\alpha(w_0, Sw_0)\rho\left(S^2w_0, S^3w_0\right), (1-\varepsilon)(M_I(w_0, Sw_0)\right. \\
&\quad \left. - \phi(M_I(w_0, Sw_0))) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w_0, Sw_0)\right) \ge 0.
\end{aligned} \tag{56}$$

Using hypothesis (ii) of the Theorem 15, we get

$$\begin{aligned}
\rho(S^2w_0, S^3w_0) &\le \alpha(w_0, Sw_0)\rho(S^2w_0, S^3w_0) \le (1-\varepsilon) \\
&\quad \cdot (M_I(w_0, Sw_0) - \phi(M_I(w_0, Sw_0)) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w_0, Sw_0) \\
&= (1-\varepsilon)\left(\begin{array}{c} \max\left\{\begin{array}{c} \rho(w_0, Sw_0), \rho(Sw_0, S^2w_0), \rho(w_0, Sw_0), \\ \rho(Sw_0, S^2w_0), \rho(Sw_0, S^2w_0), \rho(S^2w_0, S^3w_0) \end{array}\right\} \\ -\phi\left(\max\left\{\begin{array}{c} \rho(w_0, Sw_0), \rho(Sw_0, S^2w_0), \\ \rho(w_0, Sw_0), \rho(Sw_0, S^2w_0), \\ \rho(Sw_0, S^2w_0), \rho(S^2w_0, S^3w_0) \end{array}\right\}\right) \end{array}\right) \\
&\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)\left[1 + \|w_0\| + 2\|Sw_0\| + 2\|S^2w_0\| + \|S^3w_0\|\right]^{\vartheta} \\
&\le (1-\varepsilon)\left(\begin{array}{c} \max\{\rho(w_0, Sw_0), \rho(Sw_0, S^2w_0), \rho(S^2w_0, S^3w_0)\} \\ -\phi(\max\{\rho(w_0, Sw_0), \rho(Sw_0, S^2w_0), \rho(S^2w_0, S^3w_0)\}) \end{array}\right) \\
&\quad + K\varepsilon^{\xi}\psi(\varepsilon),
\end{aligned} \tag{57}$$

for some $K > 0$. Assuming that $\max\{\ell, \rho(S^2w_0, S^3w_0)\} = \rho(S^2w_0, S^3w_0)$, then we have $\rho(Sw_0, S^2w_0) < \rho(S^2w_0, S^3w_0)$. Thus, we have

$$\begin{aligned}
\rho(S^2w_0, S^3w_0) &\le (1-\varepsilon)\left(\rho(S^2w_0, S^3w_0) - \phi\left(\rho(S^2w_0, S^3w_0)\right)\right) \\
&\quad + K\varepsilon^{\xi}\psi(\varepsilon),
\end{aligned} \tag{58}$$

and since $\rho(S^2w_0, S^3w_0) \ge \rho(S^2w_0, S^3w_0) - \phi(\rho(S^2w_0, S^3w_0))$, we have

$$\rho(S^2w_0, S^3w_0) < (1-\varepsilon)\rho(S^2w_0, S^3w_0) + K\varepsilon^{\xi}\psi(\varepsilon). \tag{59}$$

The inequality (59) is true for all $\varepsilon \in [0, 1]$. For $\varepsilon = 0$, we obtain $\rho(S^2w_0, S^3w_0) < \rho(S^2w_0, S^3w_0)$ which is a contradiction. Hence, we obtain

$$\rho(S^2w_0, S^3w_0) \le \ell. \tag{60}$$

Analogously, since S is a weak Pata convex *c*ontractive mapping via simulation function, we have

$$\begin{aligned}
&\zeta\left(\begin{array}{c} \alpha(Sw_0, S^2w_0)\rho(S^3w_0, S^4w_0), \\ (1-\varepsilon)\left(M_I(Sw_0, S^2w_0) - \phi(M_I(Sw_0, S^2w_0))\right) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(Sw_0, S^2w_0) \end{array}\right) \ge 0, \\
&(1-\varepsilon)\left(\begin{array}{c} M_I(Sw_0, S^2w_0) \\ -\phi(M_I(Sw_0, S^2w_0)) \end{array}\right) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(Sw_0, S^2w_0) - \alpha(Sw_0, S^2w_0)\rho(S^3w_0, S^4w_0) \ge 0,
\end{aligned} \tag{61}$$

and we can write that

$$\begin{aligned}
\rho(S^3w_0, S^4w_0) &\le \alpha(Sw_0, S^2w_0)\rho(S^3w_0, S^4w_0) \\
&\le (1-\varepsilon)\left(\begin{array}{c} \max\left\{\begin{array}{c} \rho(Sw_0, S^2w_0), \rho(S^2w_0, S^3w_0), \rho(Sw_0, S^2w_0), \\ \rho(S^2w_0, S^3w_0), \rho(S^2w_0, S^3w_0), \rho(S^3w_0, S^4w_0) \end{array}\right\} \\ -\phi\left(\max\left\{\begin{array}{c} \rho(Sw_0, S^2w_0), \rho(S^2w_0, S^3w_0), \\ \rho(Sw_0, S^2w_0), \rho(S^2w_0, S^3w_0), \\ \rho(S^2w_0, S^3w_0), \rho(S^3w_0, S^4w_0) \end{array}\right\}\right) \end{array}\right) \\
&\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)\left[1 + \|Sw_0\| + \|S^2w_0\| + \|S^2w_0\| + \|S^3w_0\| + \|S^3w_0\| + \|S^4w_0\|\right]^{\vartheta} \\
&\le (1-\varepsilon)\left(\max\{\rho(Sw_0, S^2w_0), \rho(S^2w_0, S^3w_0), \rho(S^3w_0, S^4w_0)\}\right) + K\varepsilon^{\xi}\psi(\varepsilon)
\end{aligned} \tag{62}$$

for some $K > 0$. In case that

$$\begin{aligned}
&\max\{\rho(Sw_0, S^2w_0), \rho(S^2w_0, S^3w_0), \rho(S^3w_0, S^4w_0)\} \\
&\quad = \rho(S^3w_0, S^4w_0),
\end{aligned} \tag{63}$$

then we have

$$\rho(S^3w_0, S^4w_0) < (1-\varepsilon)\rho(S^3w_0, S^4w_0) + K\varepsilon^{\xi}\psi(\varepsilon). \tag{64}$$

The inequality (64) is true for all $\varepsilon \in [0, 1]$. For $\varepsilon = 0$, we obtain $\rho(S^3w_0, S^4w_0) < \rho(S^3w_0, S^4w_0)$ is again a contradiction. Therefore, we obtain

$$\rho(S^3w_0, S^4w_0) \le \rho(S^2w_0, S^3w_0) \le \ell. \tag{65}$$

Again, by induction, since S is a weak Pata convex *c*ontractive mapping via simulation function, we have

$$\zeta\left(\begin{array}{c} \alpha(S^{n-2}w_0, S^{n-1}w_0)\rho(S^nw_0, S^{n+1}w_0), (1-\varepsilon) \\ \cdot\left(\begin{array}{c} M_I(S^{n-2}w_0, S^{n-1}w_0) \\ -\phi(M_I(S^{n-2}w_0, S^{n-1}w_0)) \end{array}\right) \\ + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(S^{n-2}w_0, S^{n-1}w_0) \end{array}\right) \ge 0,$$

$$\left(\begin{array}{c} (1-\varepsilon)\left(\begin{array}{c} M_I(S^{n-2}w_0, S^{n-1}w_0) \\ -\phi(M_I(S^{n-2}w_0, S^{n-1}w_0)) \end{array}\right) \\ + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(S^{n-2}w_0, S^{n-1}w_0) \\ - \alpha(S^{n-2}w_0, S^{n-1}w_0)\rho(S^nw_0, S^{n+1}w_0) \end{array}\right) \ge 0, \tag{66}$$

and we have that

$$\begin{aligned}
\rho\left(S^n w_0, S^{n+1} w_0\right) &\leq \alpha\left(S^{n-2} w_0, S^{n-1} w_0\right)\rho\left(S^n w_0, S^{n+1} w_0\right) \\
&\leq (1-\varepsilon)\left(\begin{array}{c} \max\left\{\begin{array}{c} \rho\left(S^{n-2}w_0, S^{n-1}w_0\right), \rho\left(S^{n-1}w_0, S^n w_0\right), \\ \rho\left(S^{n-2}w_0, S^{n-1}w_0\right), \rho\left(S^{n-1}w_0, S^n w_0\right), \\ \rho\left(S^{n-1}w_0, S^n w_0\right), \rho\left(S^n w_0, S^{n+1}w_0\right) \end{array}\right\} \\ \phi\left(\max\left\{\begin{array}{c} \rho\left(S^{n-2}w_0, S^{n-1}w_0\right), \rho\left(S^{n-1}w_0, S^n w_0\right), \\ \rho\left(S^{n-2}w_0, S^{n-1}w_0\right), \rho\left(S^{n-1}w_0, S^n w_0\right), \\ \rho\left(S^{n-1}w_0, S^n w_0\right), \rho\left(S^n w_0, S^{n+1}w_0\right) \end{array}\right\}\right) \end{array}\right) \\
&\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)\left[1 + \left\|S^{n-2}w_0\right\| + \left\|S^{n-1}w_0\right\| + \left\|S^{n-1}w_0\right\| + \left\|S^n w_0\right\| + \left\|S^n w_0\right\| + \left\|S^{n+1}w_0\right\|\right]^{\vartheta} \\
&< (1-\varepsilon)\left(\begin{array}{c} \max\left\{\begin{array}{c} \rho\left(S^{n-2}w_0, S^{n-1}w_0\right), \rho\left(S^{n-1}w_0, S^n w_0\right), \\ \rho\left(S^n w_0, S^{n+1}w_0\right) \end{array}\right\} \\ -\phi\left(\max\left\{\begin{array}{c} \rho\left(S^{n-2}w_0, S^{n-1}w_0\right), \rho\left(S^{n-1}w_0, S^n w_0\right), \\ \rho\left(S^n w_0, S^{n+1}w_0\right) \end{array}\right\}\right) \end{array}\right) + K\varepsilon^{\xi}\psi(\varepsilon),
\end{aligned} \tag{67}$$

for some $K > 0$. In case that $\max\{\rho(S^{n-2}w_0, S^{n-1}w_0), \rho(S^{n-1}w_0, S^n w_0), \rho(S^n w_0, S^{n+1}w_0)\} = \rho(S^n w_0, S^{n+1}w_0)$, then we have

$$\rho\left(S^n w_0, S^{n+1}w_0\right) < (1-\varepsilon)\rho\left(S^n w_0, S^{n+1}w_0\right) + K\varepsilon^{\xi}\psi(\varepsilon). \tag{68}$$

Again, the inequality (68) is true for all $\varepsilon \in [0, 1]$ and for $\varepsilon = 0$, we obtain $\rho(S^n w_0, S^{n+1}w_0) < \rho(S^n w_0, S^{n+1}w_0)$ is again a contradiction. Consequently, we can find that

$$\begin{aligned}
\rho\left(S^n w_0, S^{n+1}w_0\right) &\leq \rho\left(S^{n-1}w_0, S^n w_0\right) \leq \cdots \leq \rho\left(S^3 w_0, S^4 w_0\right) \\
&\leq \rho\left(S^2 w_0, S^3 w_0\right) \leq \rho\left(S w_0, S^2 w_0\right).
\end{aligned} \tag{69}$$

Starting at the point $w_0 \in W$, the sequence $\{w_n\}$ is constructed by $w_n = Sw_{n-1} = S^n w_0$, $n \geq 1$. If $w_{n_0+1} = w_{n_0}$ for any $n_0 \in \mathbb{N}$, then w_{n_0} is a fixed point of S. Hereby, assume that $w_{n_0+1} \neq w_{n_0}$ for all $n_0 \in \mathbb{N}$ and let $\rho_n = \rho(w_{n-1}, w_n)$. Therefore, we get that $\{\rho_n\}$ is a nonincreasing sequence. Thereupon, there exists a $\delta \geq 0$ such that

$$\lim_{n\longrightarrow\infty} \rho(w_{n-1}, w_n) = \lim_{n\longrightarrow\infty} \rho_n = \delta. \tag{70}$$

We will demostrate that $\delta = 0$. For this, we should demostrate that the sequence $\{\|w_n\|\}$ is bounded. Since $\{\rho_n\}$ is a nonincreasing sequence, we have

$$\begin{aligned}
\rho_{n+1} = \rho(w_n, w_{n+1}) &\leq \rho(w_{n-1}, w_n) \leq \cdots \leq \rho(w_3, w_4) \\
&\leq \rho(w_2, w_3) \leq \rho(w_1, w_2) = \rho_2 \leq \|w_1\| + \|w_2\|.
\end{aligned} \tag{71}$$

From the triangle inequality, we can write

$$\begin{aligned}
\|w_n\| = \rho(w_n, w_0) &\leq \rho(w_n, w_{n+1}) + \rho(w_{n+1}, w_2) + \rho(w_2, w_0) \\
&= \rho_{n+1} + \rho(w_{n+1}, w_2) + \|w_2\| \leq \rho_2 + \rho(w_{n+1}, w_2) + \|w_2\| \\
&\leq \|w_1\| + 2\|w_2\| + \rho(w_{n+1}, w_2).
\end{aligned} \tag{72}$$

Since S is a weak Pata convex contractive mapping via simulation function, we have

$$\begin{aligned}
&\zeta(\alpha(w_n, w_0)\rho(w_{n+1}, w_2), (1-\varepsilon)(M_I(w_{n-1}, w_0) \\
&\quad - \phi(M_I(w_{n-1}, w_0))) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w_{n-1}, w_0)) \geq 0, \\
&\left((1-\varepsilon)\left(\begin{array}{c} M_I(w_{n-1}, w_0) \\ -\phi(M_I(w_{n-1}, w_0)) \end{array}\right) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w_{n-1}, w_0)\right. \\
&\quad \left. - \alpha(w_n, w_0)\rho(w_{n+1}, w_2)\right) \geq 0.
\end{aligned} \tag{73}$$

Together with (71), we obtain that

$$\begin{aligned}
\rho(w_{n+1}, w_2) \leq \alpha(w_n, w_0)\rho(w_{n+1}, w_2) &\leq (1-\varepsilon) \\
&\cdot (M_I(w_{n-1}, w_0) - \phi(M_I(w_{n-1}, w_0))) \\
&+ \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w_{n-1}, w_0).
\end{aligned} \tag{74}$$

From (71) and $\rho_2 \leq \|w_1\| + \|w_2\|$, we have

$$\begin{aligned}
M_I(w_{n-1}, w_0) &= \max\left\{\begin{array}{c} \rho(w_{n-1}, w_0), \rho(w_n, w_1), \rho(w_{n-1}, w_n), \\ \rho(w_0, w_1), \rho(w_n, w_{n+1}), \rho(w_1, w_2) \end{array}\right\} \\
&= \max\{\rho(w_{n-1}, w_0), \rho(w_n, w_1), \rho_n, \rho_1, \rho_{n+1}, \rho_2\} \\
&\leq \|w_1\| + \|w_2\| + \|w_n\|,
\end{aligned}$$

$$\begin{aligned}
P_I(w_{n-1}, w_0) &= [1 + \|w_{n-1}\| + \|w_0\| + \|w_n\| + \|w_1\| + \|w_{n+1}\| + \|w_2\|]^{\vartheta} \\
&\leq [1 + 3\|w_1\| + 3\|w_2\| + 3\|w_n\|]^{\vartheta}.
\end{aligned} \tag{75}$$

Now, we derive that

$$\begin{aligned}
\varepsilon\|w_n\| &< (2-\varepsilon)\|w_1\| + (3-\varepsilon)\|w_2\| + \Lambda\varepsilon^{\xi}\psi(\varepsilon)[1 + 3\|w_1\| \\
&\quad + 3\|w_2\| + 3\|w_n\|]^{\vartheta} \leq (2-\varepsilon)\|w_1\| + (3-\varepsilon)\|w_2\| \\
&\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)(1 + 3\|w_n\|)^{\xi}\left(\frac{1 + 3\|w_1\| + 3\|w_2\|}{1 + 3\|w_n\|}\right)^{\xi} \\
&\leq 2\|w_1\| + 3\|w_2\| + \Lambda\varepsilon^{\xi}\psi(\varepsilon)3^{\xi}\|w_n\|^{\xi}\left(\frac{1}{3\|w_n\|} + 1\right)^{\xi} \\
&\quad \cdot (1 + 3\|w_1\| + 3\|w_2\|)^{\xi}.
\end{aligned} \tag{76}$$

Contrarily, supposing that $\{\|w_n\|\}$ is not bounded sequence. Thence, there exists a subsequence $\{\|w_{n_j}\|\}$ of $\{\|w_n\|\}$ such that $\lim_{j\longrightarrow\infty} w_{n_j} = \infty$. If we take $\varepsilon = \varepsilon_j = (1 + 3\|w_1\| + 3\|w_2\|)/\|w_{n_j}\|$ in (76) inequality, then we have

$$1 \le \Lambda 3^{\xi}\left(\varepsilon^{\xi}\|w_n\|^{\xi}\right)(1+3\|w_1\|+3\|w_2\|)^{\xi}\left(\frac{1}{3\left\|w_{n_j}\right\|}+1\right)^{\xi} \cdot \psi\left(\varepsilon_j\right) \le \Lambda 3^{\xi}(1+3\|w_1\|+3\|w_2\|)^{\xi}(1+3\|w_1\|+3\|w_2\|)^{\xi} \cdot \left(\frac{1}{3\left\|w_{n_j}\right\|}+1\right)^{\xi}\psi\left(\varepsilon_j\right) \le \Lambda 3^{\xi}(1+3\|w_1\|+3\|w_2\|)^{2\xi} \cdot \left(\frac{1}{3\left\|w_{n_j}\right\|}+1\right)^{\xi}\psi\left(\varepsilon_j\right). \tag{77}$$

If we take limit in (77) inequality as $j \longrightarrow \infty$, then we get that

$$\Lambda 3^{\xi}(1+3\|w_1\|+3\|w_2\|)^{2\xi}\left(\frac{1}{3\left\|w_{n_j}\right\|}+1\right)^{\xi}\psi\left(\varepsilon_j\right) \longrightarrow 0 \tag{78}$$

is a contradiction. Next, we show that the sequence$\{\|w_n\|\}$ is bounded. So, there exists $A>0$ such that $\|w_n\| \le A$ for all $n \in \mathbb{N}$. Following this line of work, we will demonstrate that $\delta = 0$. Since S is a weak Pata convex *c*ontractive mapping via simulation function, we have

$$\begin{aligned}
&\zeta\left(\alpha(w_{n-1},w_n)\rho(w_{n+1},w_{n+2}),(1-\varepsilon)\begin{pmatrix} M_I(w_{n-1},w_n) \\ -\phi(M_I(w_{n-1},w_n)) \end{pmatrix} + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w_{n-1},w_n)\right) \ge 0,\\
&\left((1-\varepsilon)\begin{pmatrix} M_I(w_{n-1},w_n) \\ -\phi(M_I(w_{n-1},w_n)) \end{pmatrix} + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w_{n-1},w_n) - \alpha(w_{n-1},w_n)\rho(w_{n+1},w_{n+2})\right) \ge 0,
\end{aligned} \tag{79}$$

where

$$\begin{aligned}
M_I(w_{n-1},w_n) &= \max\begin{Bmatrix} \rho(w_{n-1},w_n),\rho(w_n,w_{n+1}),\rho(w_{n-1},w_n), \\ \rho(w_n,w_{n+1}),\rho(w_n,w_{n+1}),\rho(w_{n+1},w_{n+2}) \end{Bmatrix}\\
&= \max\{\rho(w_{n-1},w_n),\rho(w_n,w_{n+1}),\rho(w_{n+1},w_{n+2})\}\\
&\le \|w_1\|+\|w_2\|+\|w_n\|.
\end{aligned} \tag{80}$$

Since the sequence $\{\|w_n\|\}$ is bounded, we have

$$\begin{aligned}
P_I(w_{n-1},w_n) &= \Lambda\varepsilon^{\xi}\psi(\varepsilon)[1+\|w_{n-1}\|+\|w_n\|+\|w_n\| + \|w_{n+1}\|+\|w_{n+2}\|+\|w_{n+3}\|]^{\vartheta}\\
&\le \Lambda\varepsilon^{\xi}\psi(\varepsilon)(1+6A)^{\vartheta}.
\end{aligned} \tag{81}$$

Therefore, we have

$$\begin{aligned}
\rho_{n+2} &= \rho(w_{n+1},w_{n+2}) \le \alpha(w_{n-1},w_n)\rho(w_{n+1},w_{n+2})\\
&\le (1-\varepsilon)(M_I(w_{n-1},w_n)-\phi(M_I(w_{n-1},w_n)))\\
&\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w_{n-1},w_n) \le (1-\varepsilon)\\
&\quad \cdot \begin{pmatrix} \max\{\rho(w_{n-1},w_n),\rho(w_n,w_{n+1}),\rho(w_{n+1},w_{n+2})\} \\ -\phi(\max\{\rho(w_{n-1},w_n),\rho(w_n,w_{n+1}),\rho(w_{n+1},w_{n+2})\}) \end{pmatrix}\\
&\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)(1+6A)^{\vartheta}.
\end{aligned} \tag{82}$$

If the limit is taken as $n \longrightarrow \infty$ in (82) inequality, then we get

$$\begin{aligned}
\delta &\le (1-\varepsilon)(\delta-\phi(\delta)) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)(1+6A)^{\vartheta}\\
\delta &\le \Lambda\varepsilon^{\xi-1}\psi(\varepsilon)(1+6A)^{\vartheta}.
\end{aligned} \tag{83}$$

$\delta \le 0$ as $\varepsilon \longrightarrow 0$, that is $\lim_{n\longrightarrow\infty}\rho(w_{n+1},w_{n+2}) = \delta = 0$. Now, we demonstrate that $\{w_n\}$ is a Cauchy sequence. Contrarily, supposing that the sequence $\{w_n\}$ is not a Cauchy. From Lemma 10, we say that there exist subsequence $\{w_{m_j}\}$ and $\{w_{n_j}\}$ with $n_j > m_j > j$ such that $\lim_{k\longrightarrow\infty}\rho(x_{m_k-1},x_{n_k+1}) = \varsigma$, $\lim_{k\longrightarrow\infty}\rho(x_{m_k},x_{n_k}) = \varsigma$, $\lim_{k\longrightarrow\infty}\rho(x_{m_k-1},x_{n_k}) = \varsigma$, $\lim_{k\longrightarrow\infty}\rho(x_{m_k+1},x_{n_k+1}) = \varsigma$, and $\lim_{k\longrightarrow\infty}\rho(x_{m_k},x_{n_k-1}) = \varsigma$. Since S is a weak Pata convex contractive mapping, we have

$$\begin{aligned}
&\zeta\Big(\alpha\left(w_{n_j-1},w_{m_j-1}\right)\rho\left(w_{n_j+1},w_{m_j+1}\right),(1-\varepsilon) \cdot \left(M_I\left(w_{n_j-1},w_{m_j-1}\right)-\phi\left(M_I\left(w_{n_j-1},w_{m_j-1}\right)\right)\right) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I\left(w_{n_j-1},w_{m_j-1}\right)\Big) \ge 0,\\
&\Big((1-\varepsilon)\left(M_I\left(w_{n_j-1},w_{m_j-1}\right)-\phi\left(M_I\left(w_{n_j-1},w_{m_j-1}\right)\right)\right) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I\left(w_{n_j-1},w_{m_j-1}\right) - \alpha\left(w_{n_j-1},w_{m_j-1}\right) \cdot \rho\left(w_{n_j+1},w_{m_j+1}\right)\Big) \ge 0
\end{aligned} \tag{84}$$

where

$$M_I\left(w_{n_j-1},w_{m_j-1}\right) = \max\begin{Bmatrix} \rho\left(w_{n_j-1},w_{m_j-1}\right),\rho\left(w_{n_j},w_{m_j}\right),\rho\left(w_{n_j-1},w_{n_j}\right), \\ \rho\left(w_{m_j-1},w_{m_j}\right),\rho\left(w_{n_j},w_{n_j+1}\right),\rho\left(w_{m_j},w_{m_j+1}\right) \end{Bmatrix},$$

$$
\begin{aligned}
P_I\left(w_{n_j-1}, w_{m_j-1}\right) &= \Lambda\varepsilon^{\xi}\psi(\varepsilon)\left[1 + \left\|w_{n_j-1}\right\| + \left\|w_{m_j-1}\right\|\right. \\
&\quad \left. + \left\|w_{n_j}\right\| + \left\|w_{m_j}\right\| + \left\|w_{n_j+1}\right\| + \left\|w_{m_j+1}\right\|\right]^{\vartheta} \\
&= \Lambda\varepsilon^{\xi}\psi(\varepsilon)[1 + 6A]^{\vartheta}.
\end{aligned} \tag{85}
$$

Now, we can write

$$
\begin{aligned}
\varsigma &\le \rho\left(w_{n_j+1}, w_{m_j+1}\right) \le \alpha\left(w_{n_j-1}, w_{m_j-1}\right)\rho\left(w_{n_j+1}, w_{m_j+1}\right) \\
&\le (1-\varepsilon)\left(M_I\left(w_{n_j-1}, w_{m_j-1}\right) - \phi\left(M_I\left(w_{n_j-1}, w_{m_j-1}\right)\right)\right) \\
&\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I\left(w_{n_j-1}, w_{m_j-1}\right) \\
&\le (1-\varepsilon)\left(\begin{array}{c}
\max\left\{\begin{array}{l}
\rho\left(w_{n_j-1}, w_{m_j-1}\right), \rho\left(w_{n_j}, w_{m_j}\right), \\
\rho\left(w_{n_j-1}, w_{n_j}\right), \rho\left(w_{m_j-1}, w_{m_j}\right), \\
\rho\left(w_{n_j}, w_{n_j+1}\right), \rho\left(w_{m_j}, w_{m_j+1}\right)
\end{array}\right\} \\
-\phi\left(\max\left\{\begin{array}{l}
\rho\left(w_{n_j-1}, w_{m_j-1}\right), \rho\left(w_{n_j}, w_{m_j}\right), \\
\rho\left(w_{n_j-1}, w_{n_j}\right), \rho\left(w_{m_j-1}, w_{m_j}\right), \\
\rho\left(w_{n_j}, w_{n_j+1}\right), \rho\left(w_{m_j}, w_{m_j+1}\right)
\end{array}\right\}\right)
\end{array}\right) \\
&\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)[1 + 6A]^{\vartheta}.
\end{aligned} \tag{86}
$$

If we take the limit as $j \longrightarrow \infty$, we get

$$
\varsigma \le (1-\varepsilon)(\varsigma - \phi(\varsigma)) + K\varepsilon\psi(\varepsilon) \le (1-\varepsilon)\varsigma + K\varepsilon\psi(\varepsilon), \tag{87}
$$

and so, we have

$$
\varsigma \le K\psi(\varepsilon), \tag{88}
$$

that is, we get $\varsigma = 0$ which is a contradiction. Therefore, we concluded that $\{w_n\}$ is a Cauchy sequence in (W, ρ). By the completeness of W, the sequence $\{w_n\}$ is convergent to some $\omega \in W$ that is $w_n \longrightarrow \omega$ as $n \longrightarrow +\infty$. Since S is continuous, $Sw_n \longrightarrow S\omega$ as $n \longrightarrow +\infty$. By the uniqueness of the limit, we obtain $\omega = S\omega$ that is ω is a fixed point of S.

Now, we will demonstrate that the fixed point is unique. Assuming that T and ω are two fixed points of S. From hypothesis (iv) of Theorem 15 and since S is an a weak Pata convex contractive mapping via simulation function, we have

$$
\begin{aligned}
&\zeta\left(\alpha(\omega, T)\rho\left(S^2\omega, S^2T\right), (1-\varepsilon)(M_I(\omega, T) - \phi(M_I(\omega, T)))\right. \\
&\quad \left. + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(\omega, T)\right) \ge 0, \\
&(1-\varepsilon)(M_I(\omega, T) - \phi(M_I(\omega, T))) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(\omega, T) \\
&\quad - \alpha(\omega, T)\rho\left(S^2\omega, S^2T\right) \ge 0,
\end{aligned} \tag{89}
$$

and so, we have

$$
\begin{aligned}
\rho(\omega, T) &\le \alpha(\omega, T)\rho\left(S^2\omega, S^2T\right) \\
&\le (1-\varepsilon)(M_I(\omega, T) - \phi(M_I(\omega, T))) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(\omega, T) \\
&\le (1-\varepsilon)\left(\begin{array}{c}
\max\left\{\begin{array}{l}
\rho(\omega, T), \rho(S\omega, ST), \rho(\omega, S\omega), \\
\rho(T, ST), \rho\left(S\omega, S^2\omega\right), \rho\left(ST, S^2T\right)
\end{array}\right\} \\
-\phi\left(\max\left\{\rho(\omega, T), \rho(S\omega, ST), \rho(\omega, S\omega), \rho(T, ST), \rho\left(S\omega, S^2\omega\right), \rho\left(ST, S^2T\right)\right\}\right)
\end{array}\right) \\
&\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)\left[1 + \|\omega\| + \|T\| + \|S\omega\| + \|ST\| + \left\|S^2\omega\right\| + \left\|S^2T\right\|\right]^{\vartheta} \\
&\le (1-\varepsilon)\rho(\omega, T) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)[1 + 3\|\omega\| + 3\|T\|]^{\vartheta}.
\end{aligned} \tag{90}
$$

We obtain that $\rho(\omega, T) < K\psi(\varepsilon)$ for some $K \ge 0$, and thus, we get $\omega = T$. Hence, S has a unique fixed point in W. □

Example 16. Let $(W, |.|)$ the usual metric space where $W = [0, (3/2)]$. Let define the mappings $S : W \longrightarrow W$ by

$$
S(w) = \begin{cases} \dfrac{w^2 + 1}{3}, & w \in [0, 1), \\ \dfrac{1}{2}, & w \in \left[1, \dfrac{3}{2}\right], \end{cases} \tag{91}
$$

$\phi : [0,+\infty) \longrightarrow 0, +\infty)$ by $\phi(w) = w/10$ and $\alpha : W \times W \longrightarrow [0,+\infty)$ by

$$
\alpha(w, v) = \begin{cases} 1, & w, v \in [0, 1], \\ 0, & w, v \notin [0, 1]. \end{cases} \tag{92}
$$

It is easily seen that S is a triangular α -admissible mapping, and also, $S^2w = (w^4 + 2w^2 + 10)/27, w \in [0, (3/2)]$. Though the mapping, S is discontinuous in $x = 1$ and S^2 is continuous on $W = [0, (3/2)]$. Now, we want to demonstrate that S satisfies (11). For $w, v \in [0, 1]$, we have

$$
\begin{aligned}
\rho\left(S^2w, S^2v\right) &= \left|\frac{w^4 + 2w^2}{27} - \frac{v^4 + 2v^2}{27}\right| \le \frac{2}{9}|w - v| + \frac{1}{2}|Sw - Sv| \\
&= \frac{3}{4}\left(\frac{8}{27}|w - v| + \frac{2}{3}|Sw - Sv|\right) \\
&\le \frac{3}{4}\max\{|w - v|, |Sw - hv|\} \le \frac{3}{4}M_I(w, v).
\end{aligned} \tag{93}
$$

Since $\phi(w) = w/10$ and $\alpha(w, v) = 1$, for $w, v \in [0, 1]$, we get that

$$
\begin{aligned}
\alpha(w, v)\rho\left(S^2w, S^2v\right) &\le \frac{5}{6}\frac{9}{10}(M_I(w, v)) \\
&= \frac{5}{6}(M_I(w, v) - \phi(M_I(w, v))).
\end{aligned} \tag{94}
$$

For arbitrary $\varepsilon \in [0, 1]$, as one can see, the above inequality turns into the following inequality,

$$\alpha(w,v)\rho\left(S^2w,S^2v\right)\le(1-\varepsilon)(M_I(w,v)-\phi(M_I(w,v))) + \left(\frac{3}{4}+\varepsilon-1\right)M_I(w,v)\le(1-\varepsilon)(M_I(w,v)-\phi(M_I(w,v))) + \left(\frac{3}{4}+\varepsilon-1\right)\left[1+\|w\|+\|v\|+\|Sw\|+\|Sv\|+\left\|S^2w\right\|+\left\|S^2v\right\|\right]. \tag{95}$$

Now, our goal is to show that $\gamma\ge0$ and $\Lambda\ge0$ such that

$$\left(\frac{3}{4}+\varepsilon-1\right)\left[1+\|w\|+\|v\|+\|Sw\|+\|Sv\|+\left\|S^2w\right\|+\left\|S^2v\right\|\right] \le\Lambda\varepsilon^{\gamma+1}\left[1+\|w\|+\|v\|+\|Sw\|+\|Sv\|+\left\|S^2w\right\|+\left\|S^2v\right\|\right] \tag{96}$$

holds for all $w,v\in[0,1]$, and every $0\le\varepsilon\le1$. We can find $\Lambda\ge0$ such that

$$\Lambda=\frac{((3/4)+\varepsilon-1)}{\varepsilon^{\gamma+1}} \tag{97}$$

holds for each $0\le\varepsilon\le1$ and some $\gamma\ge0$. If we choose γ such that $(\gamma/(\gamma+1))>1-(3/4)$, then

$$\Lambda=\frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}(1-(3/4))^\gamma}. \tag{98}$$

Hence, we have that

$$\alpha(w,v)\rho\left(S^2w,S^2v\right)\le(1-\varepsilon)(M_I(w,v)-\phi(M_I(w,v)))+\Lambda\varepsilon^{\gamma+1} \cdot\left[1+\|w\|+\|v\|+\|Sw\|+\|Sv\|+\left\|S^2w\right\|+\left\|S^2v\right\|\right]. \tag{99}$$

Now, we can write

$$\left((1-\varepsilon)(M_I(w,v)-\phi(M_I(w,v)))+\Lambda\varepsilon^{\gamma+1} \cdot\left[1+\|w\|+\|v\|+\|Sw\|+\|Sv\|+\left\|S^2w\right\|+\left\|S^2v\right\|\right] -\alpha(w,v)\rho\left(S^2w,S^2v\right)\right)\ge0, \tag{100}$$

and for $\zeta\in Z'$, we have

$$\zeta\left(\begin{array}{c}\alpha(w,v)\rho\left(S^2w,S^2v\right),\\ (1-\varepsilon)(M_I(w,v)-\phi(M_I(w,v)))+\Lambda\varepsilon^{\gamma+1}\left[1+\|w\|+\|v\|+\|Sw\|+\|Sv\|+\left\|S^2w\right\|+\left\|S^2v\right\|\right]\end{array}\right)\ge0, \tag{101}$$

which satisfies for each $\varepsilon>0$ and all $w,v\in[0,1]$. If $\varepsilon=0$, it can be seen that (11) is satisfied. Hence, all conditions of Theorem 15 are satisfied with $\xi=\vartheta=1$ and $\psi(\varepsilon)=\varepsilon^\gamma$. By an application of Theorem 15, S has a unique fixed point in $W=[0,(3/2)]$.

Suppose that $\varepsilon=0$ in Theorems 13 and 15; then we obtain the following corollaries.

Corollary 17. *Let (W,ρ) be a complete metric space and $\zeta\in Z'$ and $S:W\longrightarrow W$ be two functions. If for all $w,v\in W$, there exists a function, $\alpha:W\times W\longrightarrow[0,+\infty)$ such that S satisfies the inequality either*

$$\zeta\left(\alpha(w,v)\rho\left(S^2w,S^2v\right),E_I(w,v)-\phi(E_I(w,v))\right)\ge0$$
$$\text{or }\zeta\left(\alpha(w,v)\rho\left(S^2w,S^2v\right),M_I(w,v)-\phi(M_I(w,v))\right)\ge0, \tag{102}$$

where $\phi:[0,+\infty)\longrightarrow[0,+\infty)$ is a continuous and nondecreasing function with $\phi(0)=0$ and $\phi(w)>0$, for all $w>0$, and assuming that all of the hypotheses of Theorem 13 are satisfied, then S has a unique fixed point.

Karapinar's contractive conditions [10, 32, 45] are a special case of ours, and also, Corollary 17 generalizes the results of Samet [17] and Istratescu [26–28].

Corollary 18. *Let (W,ρ) be a complete metric space and $S:W\longrightarrow W$ be a function. If for all $w,v\in W$, there exist two functions, $\alpha:W\times W\longrightarrow[0,+\infty)$ such that S satisfies the inequality either*

$$\alpha(w,v)\rho\left(S^2w,S^2v\right)\le E_I(w,v)-\phi(E_I(w,v))$$
$$\text{or }\alpha(w,v)\rho\left(S^2w,S^2v\right)\le M_I(w,v)-\phi(M_I(w,v)), \tag{103}$$

where $\phi:[0,+\infty)\longrightarrow[0,+\infty)$ is a continuous and nondecreasing function with $\phi(0)=0$ and $\phi(w)>0$, for all $w>0$, and assuming that all of the hypotheses of Theorem 13 are satisfied, then S has a unique fixed point.

In comparison with recent results such as Alber et al. [14] and Zhang [16], our results are a generalization of them.

Corollary 19. *Let (W,ρ) be a complete metric space and $S:W\longrightarrow W$ be a function. If for all $w,v\in W$, there exists a function $\alpha:W\times W\longrightarrow[0,+\infty)$ such that S satisfies the inequality either*

$$\alpha(w,v)\rho\left(S^2w,S^2v\right)\le E_I(w,v)$$
$$\text{or }\alpha(w,v)\rho\left(S^2w,S^2v\right)\le M_I(w,v), \tag{104}$$

and assuming that all of the hypotheses of Theorem 13 are satisfied, then h has a unique fixed point.

Putting $\alpha(w,v)=1$ in Theorems 13 and 15, we can see the following results.

Corollary 20. *Let (W,ρ) be a complete metric space and $\zeta\in Z'$ and $S:W\longrightarrow W$ be two functions. If for all $w,v\in W$, and $\varepsilon\in[0,1]$, there exists a function $\psi\in\Psi$, such that S satisfies the inequality either*

$$\zeta\left(\rho\left(S^2w, S^2v\right), (1-\varepsilon)(E_I(w,v) - \phi(E_I(w,v))) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w,v)\right) \geq 0$$
$$\text{or } \zeta\left(\rho\left(S^2w, S^2v\right), (1-\varepsilon)(M_I(w,v) - \phi(M_I(w,v))) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w,v)\right) \geq 0, \quad (105)$$

where $\phi : [0,+\infty) \longrightarrow [0,+\infty)$ *is a continuous and nondecreasing function with* $\phi(0) = 0$ *and* $\phi(w) > 0$*, for all* $w > 0$*, and assuming that all of the hypotheses of Theorem 13 are satisfied, then S has a unique fixed point.*

Corollary 21. *Let* (W, ρ) *be a complete metric space and* $S : W \longrightarrow W$ *be a function. If for all* $w, v \in W$*, and* $\varepsilon \in [0, 1]$*, there exists a function* $\psi \in \Psi$*, such that S satisfies the inequality either*

$$\rho\left(S^2w, S^2v\right) \leq (1-\varepsilon)(E_I(w,v) - \phi(E_I(w,v))) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w,v)$$
$$\text{or } \rho\left(S^2w, S^2v\right) \leq (1-\varepsilon)(M_I(w,v) - \phi(M_I(w,v))) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w,v), \quad (106)$$

where $\phi : [0,+\infty) \longrightarrow [0,+\infty)$ *is a continuous and nondecreasing function with* $\phi(0) = 0$ *and* $\phi(w) > 0$*, for all* $w > 0$*, and assuming that all of the Theorem 13 hypotheses are satisfied, then S has a unique fixed point.*

Assume now that $\alpha(w, v) = 1$ and $\varepsilon = 0$ in Theorem 13 and Theorem 15; then we get the following corollaries.

Corollary 22. *Let* (W, ρ) *be a complete metric space and* $\zeta \in Z'$*, and* $S : W \longrightarrow W$ *be two functions. If for all* $w, v \in W$*, S satisfies the inequality either*

$$\zeta\left(\rho\left(S^2w, S^2v\right), E_I(w,v) - \phi(E_I(w,v))\right) \geq 0$$
$$\text{or } \zeta\left(\rho\left(S^2w, S^2v\right), M_I(w,v) - \phi(M_I(w,v))\right) \geq 0, \quad (107)$$

where $\phi : [0,+\infty) \longrightarrow [0,+\infty)$ *is a continuous and nondecreasing function with* $\phi(0) = 0$ *and* $\phi(s) > 0$*, for all* $s > 0$ *and assume that S is continuous or* S^2 *is continuous. Then, S has a unique fixed point that is* $\omega = S\omega$, $\omega \in W$.

Corollary 23. *Let* (W, ρ) *be a complete metric space and* $S : W \longrightarrow W$ *be a function. If for all* $w, v \in W$*, S satisfies the inequality either*

$$\rho\left(S^2w, S^2v\right) \leq E_I(w,v) - \phi(E_I(w,v))$$
$$\text{or } \rho\left(S^2w, S^2v\right) \leq M_I(w,v) - \phi(M_I(w,v)), \quad (108)$$

where $\phi : [0,+\infty) \longrightarrow [0,+\infty)$ *is a continuous and nondecreasing function with* $\phi(0) = 0$ *and* $\phi(w) > 0$*, for all* $w > 0$ *and assume that S is continuous or* S^2 *is continuous. Then, S has a unique fixed point that is* $\omega = S\omega$, $\omega \in W$.

We derive that the main result of Pata [2] and Banach [1] can be expressed as a corollary of our main result.

3. Conclusion

We present the concept of weak E-Pata convex contractions and weak Pata convex contractions in metric spaces in this paper. After that, we investigate the existence of a fixed point for our novel type contraction and we state some consequences. Our results generalize and merge the results derived by Istratescu [26] and Pata [2] and some other related results in the literature. Besides the corollaries in this paper, to underline the novelty of our given results, we give an example that shows that Theorem 15 is a genuine generalization of Istratescu's results [26]. Our novel concept allows for further studies and applications.

Data Availability

The data used to support the findings of this study are included in the references within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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