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Research Article

Unifications of Continuous and Discrete Fractional Inequalities of the Hermite–Hadamard–Jensen–Mercer Type via Majorization

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The main objective of the paper is to develop an innovative idea of bringing continuous and discrete inequalities into a unified form. The desired objective is thus obtained by embedding majorization theory with the existing notion of continuous inequalities. These notions are applied to the latest generalized form of the inequalities, popularly known as the Hermite–Hadamard–Jensen–Mercer inequalities. Moreover, the frequently-used Caputo fractional operators are employed, which are rightly considered critical, especially for applied problems. Both weighted and unweighted forms of the developed results are discussed. In addition to this, some bounds are also provided for the absolute difference between the left- and right-sides of the main results.

1. Introduction

The field of mathematical inequalities contributes to a wide area of research in mathematics. With the passage of time, this field has emerged as a separate discipline, despite the fact that it was being used as a tool earlier. The addition of the notion of convexity enriched its literature and stimulated a new trend among researchers. As a result, many new inequalities came to the surface. These inequalities are (but not limited to) Ostrowski inequalities [1], Jensen's inequalities [2], the Jensen–Mercer inequalities [3], Fejér inequalities [4], Hermite–Hadamard inequalities [5], and their various variants. The Hermite–Hadamard inequality is believed to be the most widespread inequality in the literature and has received much attention in the last few years. This inequality is defined as follows:

If $\phi\colon I \longrightarrow \mathbb{R}$ is a convex function with $\vartheta, \theta \in I$ such that $\vartheta < \theta$ then

$$\phi\left(\frac{\vartheta+\theta}{2}\right) \le \frac{1}{\theta-\vartheta}\int_{\vartheta}^{\theta}\phi(u)du \le \frac{\phi(\vartheta)+\phi(\theta)}{2}. \quad (1)$$

The direction of the inequality given in (1) reverses whenever the function ϕ is concave. This inequality has been established for different generalized convex functions, for example, s–convex [6], η–convex [7], strongly convex [8], and coordinate convex function [9]. Research works in this field have also been extended to the theory of fractional calculus. As there are multiple numbers of fractional operators but due to our interest, we limit ourselves to the well-known Caputo fractional operators. Their definition is given as follows:

Definition 1 (Caputo fractional derivative operators). Consider a function $\phi \in C^n[\vartheta, \theta]$ (the space of functions whose n^{th}-derivative exist and continuous on

$[\vartheta, \theta])$, $\alpha > 0$, such that $\alpha \notin \{1, 2, 3, \ldots\}$ and $n = [\alpha] + 1$. Then the α ordered, Caputo fractional derivative operators are defined as [10]

$$\begin{aligned} {}^{c}D^{\alpha}_{\vartheta^+}\phi(z) &= \frac{1}{\Gamma(n-\alpha)}\int_{\vartheta}^{z}\frac{\phi^{(n)}(u)}{(z-u)^{\alpha-n+1}}du, \quad z > \vartheta, \\ {}^{c}D^{\alpha}_{\theta^-}\phi(z) &= \frac{(-1)^n}{\Gamma(n-\alpha)}\int_{z}^{\theta}\frac{\phi^{(n)}(u)}{(u-z)^{\alpha-n+1}}du, \quad z < \theta. \end{aligned} \tag{2}$$

Where ${}^{c}D^{\alpha}_{\vartheta^+}\phi(z)$ and ${}^{c}D^{\alpha}_{\theta^-}\phi(z)$ stand for the left- and right-sided Caputo fractional derivative operators, respectively.

It may be noted that if the usual derivative of the function ϕ of order n exists, then it coincides with ${}^{c}D^{\alpha}_{\vartheta^+}\phi(z)$ for $\alpha = n \in \{1, 2, 3, \ldots\}$. Also, for $n = 1$ and $\alpha = 0$, we have

$$\left({}^{c}D^{\alpha}_{\vartheta^+}\phi\right)(z) = \left({}^{c}D^{\alpha}_{\theta^-}\phi\right)(z) = \phi(z). \tag{3}$$

Some research work related to the Caputo fractional operators can be found in [11–13] and the references therein.

The associated Hermite–Jensen–Mercer inequality in terms of Caputo fractional operators is defined as follows [14]:

Theorem 1 (Hermite–Jensen–Mercer inequality). *Consider a function ϕ defined on the interval $[\vartheta, \theta]$, such that $\phi \in C^n[\vartheta, \theta]$ and $\phi^{(n)}$ is convex on $[\vartheta, \theta]$ with $[x_1, y_1] \subset [\vartheta, \theta]$, $\alpha > 0$, then we have*

$$\begin{aligned} \phi^{(n)}\left(\vartheta + \theta - \frac{x_1 + y_1}{2}\right) &\leq \frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(y_1 - x_1)^{n-\alpha}}\left\{\left({}^{c}D^{\alpha}_{(\vartheta+\theta-(x_1+y_1/2))^+}\phi\right)(\vartheta+\theta-x_1) + (-1)^n\left({}^{c}D^{\alpha}_{(\vartheta+\theta-(x_1+y_1/2))^-}\phi\right)(\vartheta+\theta-y_1)\right\} \\ &\leq \phi^{(n)}(\vartheta) + \phi^{(n)}(\theta) - \frac{\phi^{(n)}(x_1) + \phi^{(n)}(y_1)}{2}. \end{aligned} \tag{4}$$

The inequality (1) can be obtained from (4) when $n = 1$, $\alpha = 0$, $x_1 = \vartheta$ and $y_1 = \theta$. Some more work related to Hermite–Jensen–Mercer inequalities via fractional operators can be traced in [15–20]. Now, we state the definition of majorization in terms of which we want to present our results [21].

Definition 2 (Majorization). Let us consider $\mathbf{a} = (a_1, \ldots, a_l)$ and $\mathbf{b} = (b_1, \ldots, b_l)$ are two l–tuples of real numbers arranged in order $a_{[l]} \leq a_{[l-1]} \leq \cdots \leq a_{[1]}$, and $b_{[l]} \leq b_{[l-1]} \leq \cdots \leq b_{[1]}$, then $\mathbf{a}$ is said to majorize $\mathbf{b}$ (or $\mathbf{b}$ is said to be majorized by $\mathbf{a}$), if for $k = 1, 2, \ldots, l-1$, we have

$$\sum_{s=1}^{k} b_{[s]} \leq \sum_{s=1}^{k} a_{[s]}, \quad \text{and} \sum_{s=1}^{l} a_s = \sum_{s=1}^{l} b_s. \tag{5}$$

If $\mathbf{a}$ majorizes $\mathbf{b}$, then symbolically it is written as $\mathbf{b} \prec \mathbf{a}$.

Niezgoda [22] has used the concept of majorization and extended the Jensen–Mercer inequality given as follows:

Theorem 2 (Majorized discrete Jensen–Mercer inequality). *Let us consider a convex function ϕ defined on the interval I, $r \times l$ real matrix (x_{is}), and $l-$ tuple $\delta = (\delta_1, \ldots, \delta_l)$ such that $\delta_s, x_{is} \in I$ for all $i = 1, 2, \ldots, r$, $s \in \{1, \ldots, l\}$ with $\sigma_i \geq 0$, $\sum_{i=1}^{r}\sigma_i = 1$. If δ majorizes every row of (x_{is}), then we have*

$$\phi\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\sum_{i=1}^{r}\sigma_i x_{is}\right) \leq \sum_{s=1}^{l}\phi(\delta_s) - \sum_{s=1}^{l-1}\sum_{i=1}^{r}\sigma_i\phi(x_{is}). \tag{6}$$

The following lemmas will help us to prove our main results [23].

Lemma 1. *Let us consider a convex function ϕ defined on the interval I, $r \times l$ real matrix (x_{is}), and two $l-$ tuples $\delta = (\delta_1, \ldots, \delta_l)$, $\mathbf{p} = (p_1, \ldots, p_l)$, such that $\delta_s, x_{is} \in I$, $\sigma_i \geq 0$, $\sum_{i=1}^{r}\sigma_i = 1$, $p_s \geq 0$, with $p_l \neq 0$, $\eta = 1/p_l$, for all $i = 1, 2, \ldots, r, s \in \{1, \ldots, l\}$. If for each $i = 1, 2, \ldots, r$, $(x_{i1}, \ldots, x_{im})$ is a decreasing l–tuple and satisfying*

$$\begin{aligned} \sum_{s=1}^{k} p_s x_{is} &\leq \sum_{s=1}^{k} p_s\delta_s, \quad \text{for } k = 1, 2, \ldots, l-1, \sum_{s=1}^{l} p_s\delta_s \\ &= \sum_{s=1}^{l} p_s x_{is}, \end{aligned} \tag{7}$$

Then

$$\begin{aligned} &\phi\left(\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\sum_{i=1}^{r}\eta\sigma_i p_s x_{is}\right) \\ &\leq \sum_{s=1}^{l}\eta p_s\phi(\delta_s) - \sum_{s=1}^{l-1}\sum_{i=1}^{r}\eta\sigma_i p_s\phi(x_{is}). \end{aligned} \tag{8}$$

Lemma 2. *Let us consider a convex function ϕ defined on the interval I, $r \times l$ real matrix (x_{is}), and two $l-$ tuples $\delta = (\delta_1, \ldots, \delta_l)$, $\mathbf{p} = (p_1, \ldots, p_l)$, such that $\delta_s, x_{is} \in I$, $\sigma_i \geq 0$, $\sum_{i=1}^{r}\sigma_i = 1$, $p_s \geq 0$, with $p_l \neq 0$, $\eta = 1/p_l$, for all $i = 1, 2, \ldots, r, s \in \{1, \ldots, l\}$. If for each $i = 1, \ldots, r$, $(\delta_s - x_{is})$ and x_{is} are monotonically in the same sense and*

$$\sum_{s=1}^{l} p_s\delta_s = \sum_{s=1}^{l} p_s x_{is}, \tag{9}$$

Then

$$\begin{aligned}\phi\left(\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\sum_{i=1}^{r}\eta\sigma_i p_s x_{is}\right) \\ \leq \sum_{s=1}^{l}\eta p_s\phi(\delta_s) - \sum_{s=1}^{l-1}\sum_{i=1}^{r}\eta\sigma_i p_s\phi(x_{is}).\end{aligned} \tag{10}$$

The theory of majorization has been successful in drawing the attention of researchers working in various fields. It has been used as a key element in solving complicated optimization problems [24, 25]. Some more recent applications of majorization theory in signal processing and communication can be seen in [26, 27]. For further successive work carried out via the concept of majorization, one is referred to [28–34] and the references therein. In the present era, despite the existence of various diverse research fields, the shrinking of more than one research field into one is direly needed. The reason is that new ideas grow fast when they attract the attention of a maximum number of researchers. In our case, since inequalities have two main aspects, one is that of continuous inequalities and the other is of discrete inequalities. Both subfields have been absorbing the attention of many researchers at the same time. The fault is that the majority of the results are based only on simple conversions from discrete to continuous or vice versa. The concept of adding new ideas or strengthening an existing one is rarely utilized. In such a situation, there is a need for the provision of such a platform which can play the role of bringing researchers from the abovementioned subfields together and utilize their energies and efforts in one direction. The present attempt may be considered one of the endeavors in this regard.

The present paper is summarized as follows: first of all, Theorem 3 is devoted to the establishment of a new unified form of Hermite–Hadamard–Jensen–Mercer inequality. This objective is achieved by utilizing the majorized l−tuples in the context of Caputo fractional operators. A slightly different variant of Theorem 3 is presented in the form of Theorem 4. In order to verify and provide proof of the fact that the newly-obtained results are the unifications and generalizations of those already existing results, Remark 1 and Remark 2 are presented. In addition to this, weighted versions of the obtained results are also provided, taking the weighted generalized Mercer's inequality into account. These weighted results can be traced to Theorem 5 and Theorem 6. Moreover, two new identities, connected with the right- and left-sides of Theorem 3 and Theorem 4, respectively, are discovered. Employing these lemmas, various bounds associated with the absolute difference of the two right- and left-most terms in the main results are obtained. These results are discussed in Theorem 7, Theorem 8, Theorem 9, Theorem 10, and Theorem 11. Remark 5, Remark 6, and Remark 7 show that the newly-derived identities also generalize those previously-defined identities, while Remark 8 discusses the previous version of Theorem 10. Corollary 1 gives details about a previous bound while Corollary 2, and Corollary 3 provide information about the classical integral versions of Theorem 9 and Theorem 11. At the end, conclusion of the overall attempt is presented.

2. Main Results

The following theorem presents the Hermite–Hadamard–Jensen–Mercer fractional inequality for the Caputo fractional operators.

Theorem 3. *Let us consider a function $\phi \in C^n(I)$ and $\delta = (\delta_1, \ldots, \delta_l)$, $\mathbf{x} = (x_1, \ldots, x_l)$, $\mathbf{y} = (y_1, \ldots, y_l)$ are three l-tuples, such that $\delta_s, x_s, y_s \in I$, for all $s \in \{1, \ldots, l\}$, $x_l > y_l$, $\alpha > 0$. If $\phi^{(n)}$ is a convex function on I, $\mathbf{x} \prec \delta$, and $\mathbf{y} \prec \delta$, then*

$$\begin{aligned}\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right) &\leq \frac{\Gamma(n-\alpha+1)}{2\left(\sum_{s=1}^{l-1}(y_s-x_s)\right)^{n-\alpha}} \\ &\quad\cdot\left\{\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)^{+}}\phi\right)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)+(-1)^n\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)^{-}}\phi\right)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)\right\} \\ &\leq \frac{\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)+\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)}{2} \\ &\leq \sum_{s=1}^{l}\phi^{(n)}(\delta_s) - \frac{\sum_{s=1}^{l-1}\phi^{(n)}(x_s)+\sum_{s=1}^{l-1}\phi^{(n)}(y_s)}{2}.\end{aligned} \tag{11}$$

Proof. It can be written as

$$\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)=\phi^{(n)}\left\{\frac{1}{2}\left\{\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s+\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right\}\right\},$$
$$=\phi^{(n)}\left\{\frac{1}{2}\left\{t\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)+(1-t)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)\right\}\right.$$
$$\left.+\left\{t\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)+(1-t)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)\right\}\right\}. \tag{12}$$

Since $\phi^{(n)}$ is a convex function, therefore (12) gives the following inequality:

$$\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)\le\frac{1}{2}\left\{\phi^{(n)}\left\{t\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)+(1-t)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)\right\}\right.$$
$$\left.+\phi^{(n)}\left\{t\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)+(1-t)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)\right\}\right\}. \tag{13}$$

By multiplying both sides of (13) by $t^{n-\alpha-1}$ and then integrating over $t\in[0,1]$, we get

$$\frac{1}{n-\alpha}\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)$$
$$\le\frac{1}{2}\left\{\int_0^1 t^{n-\alpha-1}\phi^{(n)}\left\{t\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)+(1-t)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)\right\}dt\right.$$
$$\left.+\int_0^1 t^{n-\alpha-1}\phi^{(n)}\left\{t\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)+(1-t)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)\right\}dt\right\}$$
$$=\frac{1}{2\left(\sum_{s=1}^{l-1}(y_s-x_s)\right)^{n-\alpha}}\left\{\int_{\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s}^{\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s}\frac{\phi^{(n)}(u)}{\left(u-\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)\right)^{\alpha-n+1}}du\right.$$
$$\left.+\int_{\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s}^{\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s}\frac{\phi^{(n)}(u)}{\left(\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)-u\right)^{\alpha-n+1}}du\right\}. \tag{14}$$

In order to apply the definition of the Caputo fractional operators in (14), first, we show that

$$\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s<\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s. \tag{15}$$

By the hypotheses, we have $\mathbf{x}\prec\delta$ and $\mathbf{y}\prec\delta$, therefore

$$\sum_{s=1}^{l-1}y_s-\sum_{s=1}^{l-1}x_s=x_l-y_l. \tag{16}$$

Also,

$$x_l > y_l \Rightarrow x_l - y_l > 0. \tag{17}$$

By substituting (17) in (16), and adding $\sum_{s=1}^{l}\delta_s$ to both sides, we get

$$\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1} y_s < \sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1} x_s. \tag{18}$$

Now (14) implies

$$\begin{aligned}\frac{1}{n-\alpha}\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right) \le \frac{\Gamma(n-\alpha)}{2\left(\sum_{s=1}^{l-1}(y_s-x_s)\right)^{n-\alpha}} &\left\{\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{\delta_s}^{l-1}y_s\right)^{+}}\phi\right)\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}x_s\right)\right.\\ &\left.+(-1)^{n}\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)^{-}}\phi\right)\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}y_s\right)\right\},\end{aligned} \tag{19}$$

and so

$$\begin{aligned}\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right) \le \frac{\Gamma(n-\alpha+1)}{2\left(\sum_{s=1}^{l-1}(y_s-x_s)\right)^{n-\alpha}} &\left\{\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)^{+}}\phi\right)\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}x_s\right)\right.\\ &\left.+(-1)^{n}\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)^{-}}\phi\right)\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}y_s\right)\right\}.\end{aligned} \tag{20}$$

Thus, the first inequality of (11) is completed. Now, using the convexity of $\phi^{(n)}$, we obtain the second inequality in the following manner:

$$\phi^{(n)}\left(t\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}x_s\right) + (1-t)\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}y_s\right)\right) \le t\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}x_s\right) + (1-t)\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}y_s\right), \tag{21}$$

$$\phi^{(n)}\left(t\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}y_s\right) + (1-t)\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}x_s\right)\right) \le t\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}y_s\right) + (1-t)\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}x_s\right). \tag{22}$$

Adding (21) and (22), and then applying Theorem 2 for $r = 1$ and $\sigma_1 = 1$, we obtain

$$\begin{aligned}&\phi^{(n)}\left(t\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}x_s\right) + (1-t)\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}y_s\right)\right) + \phi^{(n)}\left(t\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}y_s\right) + (1-t)\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}x_s\right)\right)\\ &\le \phi^{(n)}\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}x_s\right) + \phi^{(n)}\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}y_s\right)\\ &\le 2\sum_{s=1}^{l}\phi^{(n)}(\delta_s) - \left\{\sum_{s=1}^{l-1}\phi^{(n)}(x_s) + \sum_{s=1}^{l-1}\phi^{(n)}(y_s)\right\}.\end{aligned} \tag{23}$$

By multiplying both sides of (23) by $t^{n-\alpha-1}$ and then integrating over $t \in [0,1]$, we get the second and third inequality in (11). □

Remark 1. For the hypothesis of Theorem 3, if $l = 2$, then we get the following inequality:

$$\begin{aligned}\phi^{(n)}\left(\delta_1+\delta_2-\frac{x_1+y_1}{2}\right)&\leq\frac{\Gamma(n-\alpha+1)}{2(y_1-x_1)^{n-\alpha}}\left\{\left({}^{c}D^{\alpha}_{(\delta_1+\delta_2-y_1)^{+}}\phi\right)(\delta_1+\delta_2-x_1)+(-1)^n\left({}^{c}D^{\alpha}_{(\delta_1+\delta_2-x_1)^{-}}\phi\right)(\delta_1+\delta_2-y_1)\right\},\\&\leq\frac{\phi^{(n)}(\delta_1+\delta_2-x_1)+\phi^{(n)}(\delta_1+\delta_2-y_1)}{2}\\&\leq\phi^{(n)}(\delta_1)+\phi^{(n)}(\delta_2)-\frac{\phi^{(n)}(x_1)+\phi^{(n)}(y_1)}{2}.\end{aligned}\tag{24}$$

Moreover, for $n=1$ and $\alpha=0$, we obtain the result of Kian and Moslehian [35].

Remark 2. If we take $x_1=\delta_1$, $y_1=\delta_2$, then inequality (24) reduces to inequality (2.2) in [15].

Another result for the Hermite–Hadamard–Jensen–Mercer fractional inequality is given as follows:

Theorem 4. *Let all the conditions in the hypothesis of Theorem 3 hold. Then,*

$$\begin{aligned}\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)&\leq\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{\left(\sum_{s=1}^{l-1}(y_s-x_s)\right)^{n-\alpha}}\left\{(-1)^n\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}(x_s+y_s/2)\right)^{-}}\phi\right)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)\right.\\&\left.+\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}(x_s+y_s/2)\right)^{+}}\phi\right)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)\right\}\leq\sum_{s=1}^{l}\phi^{(n)}(\delta_s)-\frac{\sum_{s=1}^{l-1}\phi^{(n)}(x_s)+\sum_{s=1}^{l-1}\phi^{(n)}(y_s)}{2}.\end{aligned}\tag{25}$$

Proof. Let us consider $t\in[0,1]$. To prove the required result, we proceed as follows:

$$\begin{aligned}\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)&=\phi^{(n)}\left\{\frac{1}{2}\left\{\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s+\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right\}\right\},\\&=\phi^{(n)}\left\{\frac{1}{2}\left\{\sum_{s=1}^{l}\delta_s-\left(\frac{t}{2}\sum_{s=1}^{l-1}x_s+\frac{2-t}{2}\sum_{s=1}^{l-1}y_s\right)+\sum_{s=1}^{l}\delta_s-\left(\frac{t}{2}\sum_{s=1}^{l-1}y_s+\frac{2-t}{2}\sum_{s=1}^{l-1}x_s\right)\right\}\right\}.\end{aligned}\tag{26}$$

Since $\phi^{(n)}$ is a convex function, therefore (26) gives the following inequality:

$$\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)\leq\frac{1}{2}\left\{\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s-\left(\frac{t}{2}\sum_{s=1}^{l-1}x_s+\frac{2-t}{2}\sum_{s=1}^{l-1}y_s\right)\right)+\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s-\left(\frac{t}{2}\sum_{s=1}^{l-1}y_s+\frac{2-t}{2}\sum_{s=1}^{l-1}x_s\right)\right)\right\}.\tag{27}$$

By multiplying both sides of (27) by $t^{n-\alpha-1}$ and then integrating over $t\in[0,1]$, we obtain

$$\frac{1}{n-\alpha}\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)\leq\frac{1}{2}\left\{\int_0^1 t^{n-\alpha-1}\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s-\left(\frac{t}{2}\sum_{s=1}^{l-1}x_s+\frac{2-t}{2}\sum_{s=1}^{l-1}y_s\right)\right)\mathrm{d}t\right.$$
$$\left.+\int_0^1 t^{n-\alpha-1}\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s-\left(\frac{t}{2}\sum_{s=1}^{l-1}y_s+\frac{2-t}{2}\sum_{s=1}^{l-1}x_s\right)\right)\mathrm{d}t\right\}$$
$$=\frac{1}{2\left(\sum_{s=1}^{l-1}y_s-x_s/2\right)^{n-\alpha}}\left\{\int_{\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s}^{\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}(x_s+y_s/2)}\frac{\phi^{(n)}(u)}{\left(u-\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)\right)^{\alpha-n+1}}\mathrm{d}u\right.$$
$$\left.+\int_{\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}(x_s+y_s/2)}^{\sum_{s=1}^{l}\delta\sum_{s=1}^{l}\delta_{ss}-\sum_{s=1}^{l-1}x_s}\frac{\phi^{(n)}(u)}{\left(\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)-u\right)^{\alpha-n+1}}\mathrm{d}u\right\}. \tag{28}$$

Following the same procedure, as given in the proof of Theorem 3, we can show that

$$\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)<\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s,\quad\text{and}\ \sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)>\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s. \tag{29}$$

Now, from (28), we deduce

$$\frac{1}{n-\alpha}\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)=\frac{2^{n-\alpha-1}\Gamma(n-\alpha)}{\left(\sum_{s=1}^{l-1}(y_s-x_s)\right)^{n-\alpha}}$$
$$\times\left\{(-1)^n\left({}^cD^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}(x_s+y_s/2)\right)^-}\phi\right)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)\right.$$
$$\left.+\left({}^cD^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}(x_s+y_s/2)\right)^+}\phi\right)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)\right\}. \tag{30}$$

So, we have

$$\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)\leq\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{\left(\sum_{s=1}^{l-1}(y_s-x_s)\right)^{n-\alpha}}$$
$$\times\left\{(-1)^n\left({}^cD^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}(x_s+y_s/2)\right)^-}\phi\right)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)\right.$$
$$\left.+\left({}^cD^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}(x_s+y_s/2)\right)^+}\phi\right)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)\right\}. \tag{31}$$

This proves the first inequality in (25).

In order to prove the second inequality of (25), we use Theorem 2 for $r = 2$, $\sigma_1 = t/2$, and $\sigma_2 = 2 - t/2$ as follows:

$$\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s - \left(\frac{t}{2}\sum_{s=1}^{l-1}x_s + \frac{2-t}{2}\sum_{s=1}^{l-1}y_s\right)\right) \le \sum_{s=1}^{l}\phi^{(n)}(\delta_s) - \left(\frac{t}{2}\sum_{s=1}^{l-1}\phi^{(n)}(x_s) + \frac{2-t}{2}\sum_{s=1}^{l-1}\phi^{(n)}(y_s)\right), \quad (32)$$

$$\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s - \left(\frac{t}{2}\sum_{s=1}^{l-1}y_s + \frac{2-t}{2}\sum_{s=1}^{l-1}x_s\right)\right) \le \sum_{s=1}^{l}\phi^{(n)}(\delta_s) - \left(\frac{t}{2}\sum_{s=1}^{l-1}\phi^{(n)}(y_s) + \frac{2-t}{2}\sum_{s=1}^{l-1}\phi^{(n)}(x_s)\right). \quad (33)$$

Adding (32) and (33), we get

$$\begin{aligned}&\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s - \left(\frac{t}{2}\sum_{s=1}^{l-1}x_s + \frac{2-t}{2}\sum_{s=1}^{l-1}y_s\right)\right) + \phi^{(n)}\left(\sum_{s=1}^{l}\delta_s - \left(\frac{t}{2}\sum_{s=1}^{l-1}y_s + \frac{2-t}{2}\sum_{s=1}^{l-1}x_s\right)\right)\\ &\quad\le 2\sum_{s=1}^{l}\phi^{(n)}(\delta_s) - \left(\sum_{s=1}^{l-1}\phi^{(n)}(x_s) + \sum_{s=1}^{l-1}\phi^{(n)}(y_s)\right).\end{aligned} \quad (34)$$

By multiplying both sides of (34) by $t^{n-\alpha-1}$ and then integrating over $t \in [0,1]$, we obtain the second inequality of (25). □

We establish the following result for the Caputo fractional operators on the basis of Lemma 1.

Theorem 5. *Let us consider a function $\phi \in C^n(I)$, such that $\phi^{(n)}$ is a convex function on I and $\delta = (\delta_1, \ldots, \delta_l)$, $\mathbf{x} = (x_1, \ldots, x_l)$, $\mathbf{y} = (y_1, \ldots, y_l)$, and $\mathbf{p} = (p_1, \ldots, p_l)$ are four l-tuples, such that $\delta_s, x_s, y_s \in I$, $p_s \ge 0$ with $p_l \ne 0$ for all $s \in \{1, \ldots, l\}$, $\eta = 1/p_l$, $x_l > y_l$, $\alpha > 0$. If $\mathbf{x}$ and $\mathbf{y}$ are decreasing l-tuples and*

$$\begin{aligned}&\sum_{s=1}^{k}p_s x_s \le \sum_{s=1}^{k}p_s\delta_s, \sum_{s=1}^{k}p_s y_s \le \sum_{s=1}^{k}p_s\delta_s, \quad \text{for } k = 1, \ldots, l-1,\\ &\sum_{s=1}^{l}p_s\delta_s = \sum_{s=1}^{l}p_s x_s, \ \sum_{s=1}^{l}p_s\delta_s = \sum_{s=1}^{l}p_s y_s,\end{aligned} \quad (35)$$

then

$$\begin{aligned}&\phi^{(n)}\left(\sum_{s=1}^{l}\eta p_s\delta_s - \eta\sum_{s=1}^{l-1}\left(\frac{p_s x_s + p_s y_s}{2}\right)\right) \le \frac{\Gamma(n-\alpha+1)}{2\left(\sum_{s=1}^{l-1}(\eta p_s y_s - \eta p_s x_s)\right)^{n-\alpha}}\left\{\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s y_s\right)^{+}}\phi\right)\left(\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s x_s\right)\right.\\ &\qquad\left. +(-1)^n\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s x_s\right)^{-}}\phi\right)\left(\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s y_s\right)\right\}\\ &\qquad\le \frac{\phi^{(n)}\left(\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s y_s\right) + \phi^{(n)}\left(\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s x_s\right)}{2}\\ &\qquad\le \sum_{s=1}^{l}\eta p_s\phi^{(n)}(\delta_s) - \frac{\sum_{s=1}^{l-1}\eta p_s\phi^{(n)}(x_s) + \sum_{s=1}^{l-1}\eta p_s\phi^{(n)}(y_s)}{2}.\end{aligned} \quad (36)$$

Proof. It can be written as

$$\begin{aligned}\phi^{(n)}\left(\sum_{s=1}^{l}\eta p_s\delta_s-\eta\sum_{s=1}^{l-1}p_s\left(\frac{x_s+y_s}{2}\right)\right)&=\phi^{(n)}\left\{\frac{1}{2}\left\{\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s+\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s\right\}\right\}\\&=\phi^{(n)}\left\{\frac{1}{2}\left\{t\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s\right)+(1-t)\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s\right)\right.\right.\\&\quad\left.\left.+t\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s\right)+(1-t)\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s\right)\right\}\right\}.\end{aligned}\tag{37}$$

Since $\phi^{(n)}$ is a convex function, therefore (37) gives the following inequality:

$$\begin{aligned}\phi^{(n)}\left(\sum_{s=1}^{l}\eta p_s\delta_s-\eta\sum_{s=1}^{l-1}p_s\left(\frac{x_s+y_s}{2}\right)\right)&\leq\frac{1}{2}\left\{\phi^{(n)}\left\{t\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s\right)+(1-t)\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s\right)\right\}\right.\\&\quad\left.+\phi^{(n)}\left\{t\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s\right)+(1-t)\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s\right)\right\}\right\}.\end{aligned}\tag{38}$$

By multiplying both sides of (38) by $t^{n-\alpha-1}$ and then integrating over $t\in[0,1]$, we get

$$\begin{aligned}\frac{1}{n-\alpha}\phi^{(n)}\left(\sum_{s=1}^{l}\eta p_s\delta_s-\eta\sum_{s=1}^{l-1}p_s\left(\frac{x_s+y_s}{2}\right)\right)&\leq\frac{1}{2}\left\{\int_0^1 t^{n-\alpha-1}\phi^{(n)}\left\{t\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s\right)+(1-t)\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s\right)\right\}\mathrm{d}t\right.\\&\quad\left.+\int_0^1 t^{n-\alpha-1}\phi^{(n)}\left\{t\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s\right)+(1-t)\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s\right)\right\}\mathrm{d}t\right\},\\&=\frac{1}{2\left(\sum_{s=1}^{l-1}(\eta p_s y_s-\eta p_s x_s)\right)^{n-\alpha}}\left\{\int_{\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s}^{\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s}\frac{\phi^{(n)}(u)}{\left(u-\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s\right)\right)^{\alpha-n+1}}\mathrm{d}u\right.\\&\quad\left.+\int_{\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s}^{\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s}\frac{\phi^{(n)}(u)}{\left(\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s\right)-u\right)^{\alpha-n+1}}\mathrm{d}u\right\}.\end{aligned}\tag{39}$$

In order to apply the definition of the Caputo fractional operators in (39), first, we show that

$$\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s<\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s.\tag{40}$$

Given that $\sum_{s=1}^{l}p_s\delta_s=\sum_{s=1}^{l}p_s x_s$ and $\sum_{s=1}^{l}p_s\delta_s=\sum_{s=1}^{l}p_s y_s$, we have

$$\sum_{s=1}^{l-1}p_s y_s-\sum_{s=1}^{l-1}p_s x_s=p_l x_l-p_l y_l.\tag{41}$$

Also,

$$x_l>y_l\Rightarrow p_l x_l>p_l y_l\Rightarrow p_l x_l-p_l y_l>0.\tag{42}$$

By substituting (42) in (41), and adding $\sum_{s=1}^{l}\eta p_s\delta_s$ to both sides, we get

$$\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s<\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s. \tag{43}$$

Now (39) implies

$$\begin{aligned}\frac{1}{n-\alpha}\phi^{(n)}\left(\sum_{s=1}^{l}\eta p_s\delta_s-\eta\sum_{s=1}^{l-1}p_s\left(\frac{x_s+y_s}{2}\right)\right)\leq&\frac{\Gamma(n-\alpha)}{2\left(\sum_{s=1}^{l-1}(\eta p_s y_s-\eta p_s x_s)\right)^{n-\alpha}}\\&\times\left\{\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s\right)^{+}}\phi\right)\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s\right)\right.\\&\left.+(-1)^n\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s\right)^{-}}\phi\right)\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s\right)\right\},\end{aligned}\tag{44}$$

and so

$$\begin{aligned}\phi^{(n)}\left(\sum_{s=1}^{l}\eta p_s\delta_s-\eta\sum_{s=1}^{l-1}p_s\left(\frac{x_s+y_s}{2}\right)\right)\leq&\frac{\Gamma(n-\alpha+1)}{2\left(\sum_{s=1}^{l-1}(\eta p_s y_s-\eta p_s x_s)\right)^{n-\alpha}}\times\left\{\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s\right)^{+}}\phi\right)\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s\right)\right.\\&\left.+(-1)^n\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s\right)^{-}}\phi\right)\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s\right)\right\}.\end{aligned}\tag{45}$$

Thus, we achieved the first inequality of (36).

To prove the second inequality, from the convexity of $\phi^{(n)}$ we may write that

$$\phi^{(n)}\left(t\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s\right)+(1-t)\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s\right)\right)\leq t\phi^{(n)}\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s\right)+(1-t)\phi^{(n)}\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s\right), \tag{46}$$

$$\phi^{(n)}\left(t\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s\right)+(1-t)\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s\right)\right)\leq t\phi^{(n)}\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s\right)+(1-t)\phi^{(n)}\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s\right). \tag{47}$$

Adding (46) and (47) and then using Lemma 1 for $r=2$, $\sigma_1=t$, and $\sigma_2=1-t$, we obtain

$$\begin{aligned}&\phi^{(n)}\left(t\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s\right)+(1-t)\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s\right)\right)+\phi^{(n)}\left(t\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s\right)+(1-t)\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s\right)\right)\\&\leq\phi^{(n)}\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s x_s\right)+\phi^{(n)}\left(\sum_{s=1}^{l}\eta p_s\delta_s-\sum_{s=1}^{l-1}\eta p_s y_s\right)\\&\leq 2\sum_{s=1}^{l}\eta p_s\phi^{(n)}(\delta_s)-\left\{\sum_{s=1}^{l-1}\eta p_s\phi^{(n)}(x_s)+\sum_{s=1}^{l-1}\eta p_s\phi^{(n)}(y_s)\right\}.\end{aligned}\tag{48}$$

By multiplying both sides of (48) by $t^{n-\alpha-1}$ and then integrating over $t \in [0,1]$, we get the second and third inequality in (36). □

We establish the following result for the Caputo fractional operators on the basis of Lemma 2.

Theorem 6. *Let us consider a function $\phi \in C^n(I)$, such that $\phi^{(n)}$ is a convex function on I and $\delta = (\delta_1, \ldots, \delta_l)$, $\mathbf{x} = (x_1, \ldots, x_l)$, $\mathbf{y} = (y_1, \ldots, y_l)$, and $\mathbf{p} = (p_1, \ldots, p_l)$ are four l-tuples, such that $\delta_s, x_s, y_s \in I$, $p_s \geq 0$ with $p_l \neq 0$ for all $s \in \{1, \ldots, l\}$, $\eta = 1/p_l$, $x_l > y_l$, $\alpha > 0$. If $\delta - \mathbf{x}$, $\mathbf{x}$, $\delta - \mathbf{y}$, and $\mathbf{y}$ are monotonically in the same sense and*

$$\sum_{s=1}^{l} p_s\delta_s = \sum_{s=1}^{l} p_s x_s, \quad \sum_{s=1}^{l} p_s\delta_s = \sum_{s=1}^{l} p_s y_s, \tag{49}$$

then

$$\begin{aligned}
\phi^{(n)}\left(\sum_{s=1}^{l}\eta p_s\delta_s - \eta\sum_{s=1}^{l-1}\left(\frac{p_s x_s + p_s y_s}{2}\right)\right) \leq{}& \frac{\Gamma(n-\alpha+1)}{2\left(\sum_{s=1}^{l-1}(\eta p_s y_s - \eta p_s x_s)\right)^{n-\alpha}} \\
&\cdot\left\{\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s y_s\right)^{+}}\phi\right)\left(\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s x_s\right)\right. \\
&\left.+(-1)^n\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s x_s\right)^{-}}\phi\right)\left(\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s y_s\right)\right\} \\
\leq{}& \frac{\phi^{(n)}\left(\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s y_s\right) + \phi^{(n)}\left(\sum_{s=1}^{l}\eta p_s\delta_s - \sum_{s=1}^{l-1}\eta p_s x_s\right)}{2} \\
\leq{}& \sum_{s=1}^{l}\eta p_s\phi^{(n)}(\delta_s) - \frac{\sum_{s=1}^{l-1}\eta p_s\phi^{(n)}(x_s) + \sum_{s=1}^{l-1}\eta p_s\phi^{(n)}(y_s)}{2}.
\end{aligned} \tag{50}$$

Proof. By using Lemma 2 and following the procedure given in the proof of Theorem 5, we can obtain (50). □

Remark 3. Theorem 5 and Theorem 6 provide weighted forms of Theorem 3.

Remark 4. The weighted versions of Theorem 4 can be obtained in a similar fashion.

3. Bounds Associated with the Main Results

In this section first, we discover two new identities associated with the right- and left-sides of the main results. Then utilizing these identities, we establish bounds for the absolute difference of the two right- and left-most terms of the main results.

Lemma 3. *Let us consider a differentiable function ϕ defined on I, such that $\phi \in C^{n+1}(I)$ and $\delta = (\delta_1, \ldots, \delta_l)$, $\mathbf{x} = (x_1, \ldots, x_l)$, and $\mathbf{y} = (y_1, \ldots, y_l)$ are three $l-$ tuples, such that $\delta_s, x_s, y_s \in I$, for all $s \in \{1, \ldots, l\}$, $\alpha > 0$, $t \in [0,1]$. If $\phi^{(n+1)} \in L(I)$, then*

$$\begin{aligned}
&\frac{\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}y_s\right) + \phi^{(n)}\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}x_s\right)}{2} - \frac{\Gamma(n-\alpha+1)}{2\left(\sum_{s=1}^{l-1}(y_s - x_s)\right)^{n-\alpha}} \times \left\{\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}y_s\right)^{+}}\phi\right)\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}x_s\right)\right. \\
&\left.+(-1)^n\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}x_s\right)^{-}}\phi\right)\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}y_s\right)\right\} \\
&= \frac{\sum_{s=1}^{l-1}(y_s - x_s)}{2}\int_0^1\left(t^{n-\alpha} - (1-t)^{n-\alpha}\right)\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}(t x_s + (1-t)y_s)\right)dt.
\end{aligned} \tag{51}$$

Proof. To prove our required result, we consider that

$$\begin{aligned} I &= \int_0^1 \left(t^{n-\alpha} - (1-t)^{n-\alpha}\right)\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(tx_s + (1-t)y_s\right)\right)\mathrm{d}t,\\ &= \int_0^1 t^{n-\alpha}\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(tx_s + (1-t)y_s\right)\right)\mathrm{d}t - \int_0^1 (1-t)^{n-\alpha}\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(tx_s + (1-t)y_s\right)\right)\mathrm{d}t\\ &= I_1 - I_2. \end{aligned} \tag{52}$$

Assuming that $\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1} y_s < \sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1} x_s$ and using integration by parts formula, we obtain

$$\begin{aligned} I_1 &= \int_0^1 t^{n-\alpha}\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(tx_s + (1-t)y_s\right)\right)\mathrm{d}t = \frac{t^{n-\alpha}\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(tx_s + (1-t)y_s\right)\right)}{\sum_{s=1}^{l-1}\left(y_s - x_s\right)}\Bigg|_0^1 - \frac{n-\alpha}{\sum_{s=1}^{l-1}\left(y_s - x_s\right)}\\ &\quad\times \int_0^1 t^{n-\alpha-1}\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(tx_s + (1-t)y_s\right)\right)\mathrm{d}t,\\ &= \frac{\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}x_s\right)}{\sum_{s=1}^{l-1}\left(y_s - x_s\right)} - \frac{\Gamma(n-\alpha+1)}{\left(\sum_{s=1}^{l-1}\left(y_s - x_s\right)\right)^{n-\alpha+1}}(-1)^n\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}x_s\right)^-}\phi\right)\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}y_s\right). \end{aligned} \tag{53}$$

Similarly,

$$\begin{aligned} I_2 &= \int_0^1 (1-t)^{n-\alpha}\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(tx_s + (1-t)y_s\right)\right)\mathrm{d}t\\ &= \frac{(1-t)^{n-\alpha}\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(tx_s + (1-t)y_s\right)\right)}{\sum_{s=1}^{l-1}\left(y_s - x_s\right)}\Bigg|_0^1 + \frac{n-\alpha}{\sum_{s=1}^{l-1}\left(y_s - x_s\right)}\\ &\quad\times \int_0^1 (1-t)^{n-\alpha-1}\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(tx_s + (1-t)y_s\right)\right)\mathrm{d}t\\ &= -\frac{\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}y_s\right)}{\sum_{s=1}^{l-1}\left(y_s - x_s\right)} + \frac{\Gamma(n-\alpha+1)}{\left(\sum_{s=1}^{l-1}\left(y_s - x_s\right)\right)^{n-\alpha+1}}\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}y_s\right)^+}\phi\right)\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}x_s\right). \end{aligned} \tag{54}$$

Now, we have

$$I=\frac{\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)+\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)}{\sum_{s=1}^{l-1}(y_s-x_s)}-\frac{\Gamma(n-\alpha+1)}{\left(\sum_{s=1}^{l-1}(y_s-x_s)\right)^{n-\alpha+1}}\times\left\{\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)^{+}}\phi\right)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)+(-1)^{n}\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)^{-}}\phi\right)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)\right\}. \tag{55}$$

Multiplying both sides by $\sum_{s=1}^{l-1}(y_s-x_s)/2$, we get (51). □

Remark 5. Lemma 3 gives the following equality for $x_1=\delta_1$, $y_1=\delta_2$, and $l=2$ proved by Farid et al. [15].

$$\frac{\phi^{(n)}(\delta_1)+\phi^{(n)}(\delta_2)}{2}-\frac{\Gamma(n-\alpha+1)}{2(\delta_2-\delta_1)^{n-\alpha}}\left\{\left({}^{c}D^{\alpha}_{\delta_1^{+}}\phi\right)(\delta_2)+(-1)^{n}\left({}^{c}D^{\alpha}_{\delta_2^{-}}\phi\right)(\delta_1)\right\}=\frac{\delta_2-\delta_1}{2}\int_0^1\left(t^{n-\alpha}-(1-t)^{n-\alpha}\right)\phi^{(n+1)}(t\delta_2+(1-t)\delta_1)dt. \tag{56}$$

Remark 6. If we take $\alpha=0$ and $n=1$ in Remark 5, then the equality (56) gives

$$\frac{\phi(\delta_1)+\phi(\delta_2)}{2}-\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\phi(u)du=\frac{\delta_2-\delta_1}{2}\int_0^1(2t-1)\phi'(t\delta_2+(1-t)\delta_1)dt. \tag{57}$$

The equality (57) has been proved by Dragomir and Agarwal [5].

The following results have been established on the basis of Lemma 3:

Theorem 7. *Let us consider a differentiable function ϕ defined on I, such that $\phi\in C^{n+1}(I)$ and $\delta=(\delta_1,\ldots,\delta_l)$, $\mathbf{x}=(x_1,\ldots,x_l)$, $\mathbf{y}=(y_1,\ldots,y_l)$ are three $l-$ tuples, such that $\delta_s,x_s,y_s\in I$, for all $s\in\{1,\ldots,l\}$, $x_l>y_l$, $\alpha>0$. If δ majorizes $\mathbf{x}$, $\mathbf{y}$, and $|\phi^{(n+1)}|$ is convex on I, then*

$$\left|\frac{\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)+\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)}{2}-\frac{\Gamma(n-\alpha+1)}{2\left(\sum_{s=1}^{l-1}(y_s-x_s)\right)^{n-\alpha}}\times\left\{\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)^{+}}\phi\right)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)+(-1)^{n}\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)^{-}}\phi\right)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)\right\}\right|\leq\frac{\sum_{s=1}^{l-1}|y_s-x_s|}{n-\alpha+1}\left(1-\frac{1}{2^{n-\alpha}}\right)\left\{\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right|-\frac{\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right|+\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|}{2}\right\}. \tag{58}$$

Proof. From Lemma 3, it follows that

$$\begin{aligned}
&\left|\frac{\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)+\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)}{2}-\frac{\Gamma(n-\alpha+1)}{2\left(\sum_{s=1}^{l-1}(y_s-x_s)\right)^{n-\alpha}}\right.\\
&\left.\times\left\{\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)^{+}}\phi\right)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)+(-1)^{n}\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)^{-}}\phi\right)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)\right\}\right|\\
&=\left|\frac{\sum_{s=1}^{l-1}(y_s-x_s)}{2}\int_0^1\left(t^{n-\alpha}-(1-t)^{n-\alpha}\right)\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(tx_s+(1-t)y_s\right)\right)dt\right|\\
&\leq\frac{\sum_{s=1}^{l-1}(y_s-x_s)}{2}\int_0^1\left|\left(t^{n-\alpha}-(1-t)^{n-\alpha}\right)\right|\left|\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(tx_s+(1-t)y_s\right)\right)\right|dt.
\end{aligned}\tag{59}$$

Using Theorem 2 for $r=2$, $\sigma_1=t$, and $\sigma_2=1-t$ in (59) as a consequence of the convexity of $|\phi^{(n+1)}|$, we obtain

$$\begin{aligned}
&\leq\frac{\sum_{s=1}^{l-1}|y_s-x_s|}{2}\int_0^1\left|\left(t^{n-\alpha}-(1-t)^{n-\alpha}\right)\right|\times\left\{\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right|-\left(t\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right|+(1-t)\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|\right)\right\}dt.\\
&=\frac{\sum_{s=1}^{l-1}|y_s-x_s|}{2}\left[\int_0^{1/2}\left((1-t)^{n-\alpha}-t^{n-\alpha}\right)\left\{\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right|\right.\right.\\
&\left.-\left(t\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right|+(1-t)\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|\right)\right\}dt+\int_{1/2}^{1}\left(t^{n-\alpha}-(1-t)^{n-\alpha}\right)\\
&\left.\times\left\{\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right|-\left(t\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right|+(1-t)\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|\right)\right\}dt\right].\\
&=\frac{\sum_{s=1}^{l-1}|y_s-x_s|}{2}\left(C_1+C_2\right).
\end{aligned}\tag{60}$$

Now finding C_1 and C_2, we have

$$\begin{aligned}
C_1&=\int_0^{1/2}\left((1-t)^{n-\alpha}-t^{n-\alpha}\right)\left\{\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right|-\left(t\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right|+(1-t)\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|\right)\right\}dt\\
&=\left(\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right|\right)\left(\int_0^{\frac{1}{2}}\left((1-t)^{n-\alpha}-t^{n-\alpha}\right)dt\right)-\left\{\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right|\right.\\
&\left.\times\int_0^{\frac{1}{2}}t\left((1-t)^{n-\alpha}-t^{n-\alpha}\right)dt+\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|\int_0^{1/2}\left((1-t)^{n-\alpha}-t^{n-\alpha}\right)(1-t)dt\right\}\\
&=\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right|\left(\frac{1-2^{\alpha-n}}{n-\alpha+1}\right)-\left\{\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right|\left(\int_0^{1/2}t(1-t)^{n-\alpha}dt-\int_0^{1/2}t^{n-\alpha+1}dt\right)\right.\\
&\left.+\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|\left(\int_0^{1/2}(1-t)^{n-\alpha+1}dt-\int_0^{1/2}(1-t)t^{n-\alpha}dt\right)\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right|\left(\frac{1-2^{\alpha-n}}{n-\alpha+1}\right) - \left\{\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right|\right.\\
&\quad\times\left(\frac{1}{(n-\alpha+1)(n-\alpha+2)} - \frac{2^{\alpha-n-1}}{n-\alpha+2}\right) + \left.\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|\left(\frac{1}{n-\alpha+2} - \frac{2^{\alpha-n-1}}{n-\alpha+1}\right)\right\},\\
C_2 &= \int_{1/2}^{1}\left(t^{n-\alpha} - (1-t)^{n-\alpha}\right)\left\{\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right| - \left(t\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right| + (1-t)\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|\right)\right\}dt\\
&= \left(\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right|\right)\left(\int_{1/2}^{1}(t^{n-\alpha} - (1-t)^{n-\alpha})dt\right) - \left\{\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right|\right.\\
&\quad\times\int_{1/2}^{1} t\left(t^{n-\alpha} - (1-t)^{n-\alpha}\right)dt + \left.\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|\int_{1/2}^{1}\left(t^{n-\alpha} - (1-t)^{n-\alpha}\right)(1-t)dt\right\}\\
&= \sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right|\left(\frac{1-2^{\alpha-n}}{n-\alpha+1}\right) - \left\{\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right|\left(\int_{1/2}^{1} t^{n-\alpha+1}dt - \int_{1/2}^{1} t(1-t)^{n-\alpha}dt\right)\right.\\
&\quad+\left.\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|\left(\int_{\frac{1}{2}}^{1}(1-t)t^{n-\alpha}dt - \int_{1/2}^{1}(1-t)^{n-\alpha+1}dt\right)\right\}\\
&= \sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right|\left(\frac{1-2^{\alpha-n}}{n-\alpha+1}\right) - \left\{\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right|\left(\frac{1}{n-\alpha+2} - \frac{2^{\alpha-n-1}}{n-\alpha+1}\right)\right.\\
&\quad+\left.\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|\left(\frac{1}{(n-\alpha+1)(n-\alpha+2)} - \frac{2^{\alpha-n-1}}{n-\alpha+2}\right)\right\}.
\end{aligned}
\tag{61}
$$

Adding C_1 and C_2, we get

$$
C_1 + C_2 = 2\left(\frac{1-2^{\alpha-n}}{n-\alpha+1}\right)\left\{\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right| - \frac{\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right| + \sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|}{2}\right\}. \tag{62}
$$

Inserting (62) in (60), we achieve (58). □

Corollary 1. *If we take $l = 2$, $x_1 = \delta_1$, and $y_1 = \delta_2$ in Theorem 7, then inequality (58) reduces to*

$$
\left|\frac{\phi^{(n)}(\delta_1) + \phi^{(n)}(\delta_2)}{2} - \frac{\Gamma(n-\alpha+1)}{2(\delta_2-\delta_1)^{n-\alpha}}\left\{{}^{c}D_{\delta_1^+}^{\alpha}\phi(\delta_2) + (-1)^{nc}D_{\delta_2^-}^{\alpha}\phi(\delta_1)\right\}\right| \le \frac{|\delta_2-\delta_1|}{n-\alpha+1}\left(1 - \frac{1}{2^{n-\alpha}}\right)\left\{\frac{\left|\phi^{(n+1)}(\delta_1)\right| + \left|\phi^{(n+1)}(\delta_2)\right|}{2}\right\}, \tag{63}
$$

which is proved in [15].

Theorem 8. *Let us consider a differentiable function ϕ defined on I, such that $\phi \in C^{n+1}(I)$ and $\delta = (\delta_1, \ldots, \delta_l)$, $\mathbf{x} = (x_1, \ldots, x_l)$, $\mathbf{y} = (y_1, \ldots, y_l)$ are three $l-$ tuples, such that $\delta_s, x_s, y_s \in I$, for all $s \in \{1, \ldots, l\}$, $x_l > y_l$, $\alpha > 0$. If $q > 1$, δ majorizes $\mathbf{x}$, $\mathbf{y}$, and $|\phi^{(n+1)}|^q$ is convex on I, then*

$$\left|\frac{\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)+\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)}{2}-\frac{\Gamma(n-\alpha+1)}{2\left(\sum_{s=1}^{l-1}(y_s-x_s)\right)^{n-\alpha}}\right.$$
$$\left.\times\left\{\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)^{+}}\phi\right)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)+(-1)^{n}\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)^{-}}\phi\right)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)\right\}\right| \tag{64}$$
$$\leq\frac{\sum_{s=1}^{l-1}|y_s-x_s|}{n-\alpha+1}\left(1-\frac{1}{2^{n-\alpha}}\right)\left\{\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right|^{q}-\frac{\sum_{s=1}^{l-1}|\phi^{(n+1)}(x_s)|^{q}+\sum_{s=1}^{l-1}|\phi^{(n+1)}(y_s)|^{q}}{2}\right\}^{1/q}.$$

Proof. From Lemma 3, it follows that

$$\left|\frac{\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)+\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)}{2}-\frac{\Gamma(n-\alpha+1)}{2\left(\sum_{s=1}^{l-1}(y_s-x_s)\right)^{n-\alpha}}\right.$$
$$\left.\times\left\{\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)^{+}}\phi\right)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)+(-1)^{n}\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)^{-}}\phi\right)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)\right\}\right|$$
$$=\left|\frac{\sum_{s=1}^{l-1}(y_s-x_s)}{2}\int_0^1\left(t^{n-\alpha}-(1-t)^{n-\alpha}\right)\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}(tx_s+(1-t)y_s)\right)\mathrm{d}t\right| \tag{65}$$
$$\leq\frac{\sum_{s=1}^{l-1}|y_s-x_s|}{2}\int_0^1\left|t^{n-\alpha}-(1-t)^{n-\alpha}\right|\left|\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}(tx_s+(1-t)y_s)\right)\right|\mathrm{d}t.$$

By applying power mean inequality to the above integral, we obtain

$$\leq\frac{\sum_{s=1}^{l-1}|y_s-x_s|}{2}\left(\int_0^1\left|t^{n-\alpha}-(1-t)^{n-\alpha}\right|\mathrm{d}t\right)^{1-1/q}$$
$$\times\left(\int_0^1\left|t^{n-\alpha}-(1-t)^{n-\alpha}\right|\times\left|\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}(tx_s+(1-t)y_s)\right)\right|^{q}\mathrm{d}t\right)^{1/q}$$
$$=\frac{\sum_{s=1}^{l-1}|y_s-x_s|}{2}\left(\int_0^{1/2}\left((1-t)^{n-\alpha}-t^{n-\alpha}\right)\mathrm{d}t+\int_{1/2}^{1}\left(t^{n-\alpha}-(1-t)^{n-\alpha}\right)dt\right)^{1-1/q} \tag{66}$$
$$\times\left(\int_0^1\left|t^{n-\alpha}-(1-t)^{n-\alpha}\right|\left|\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}(tx_s+(1-t)y_s)\right)\right|^{q}\mathrm{d}t\right)^{1/q}.$$

Since $|\phi^{(n+1)}|^q$ is convex, therefore using Theorem 2 for $r=2$, $\sigma_1=t$, and $\sigma_2=1-t$ in (66), we obtain

$$= \frac{\sum_{s=1}^{l-1}|y_s - x_s|}{2}\left(\int_0^{1/2}((1-t)^{n-\alpha} - t^{n-\alpha})dt + \int_{1/2}^{1}(t^{n-\alpha} - (1-t)^{n-\alpha})dt\right)^{1-1/q}$$

$$\times \int_0^1 \left|t^{n-\alpha} - (1-t)^{n-\alpha}\right|\left(\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right|^q - \left(t\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right|^q + (1-t)\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|^q dt\right)\right)^{1/q}$$

$$= \frac{\sum_{s=1}^{l-1}|y_s - x_s|}{2}\left(\int_0^{1/2}((1-t)^{n-\alpha} - t^{n-\alpha})dt + \int_{1/2}^{1}(t^{n-\alpha} - (1-t)^{n-\alpha})dt\right)^{1-1/q} \tag{67}$$

$$\times \left\{\int_0^{1/2}((1-t)^{n-\alpha} - t^{n-\alpha})\left(\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right|^q - \left(t\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right|^q + (1-t)\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|^q\right)\right)dt\right.$$

$$\left.+ \int_{1/2}^{1}(t^{n-\alpha} - (1-t)^{n-\alpha})\left(\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right|^q - \left(t\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right|^q + (1-t)\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|^q\right)\right)dt\right\}^{1/q}.$$

By calculating these simple integrals, we get (64). □

Another lemma is established as follows:

Lemma 4. *Let all the conditions in the hypothesis of Lemma 3 hold. Then,*

$$\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{\left(\sum_{s=1}^{l-1}(y_s - x_s)\right)^{n-\alpha}}\left\{\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}(x_s+y_s/2)\right)^{+}}\phi\right)\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}x_s\right)\right.$$

$$\left.+ (-1)^n\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}(x_s+y_s/2)\right)^{-}}\phi\right)\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}y_s\right)\right\} - \phi^{(n)}\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(\frac{x_s + y_s}{2}\right)\right)$$

$$= \frac{\sum_{s=1}^{l-1}(y_s - x_s)}{4}\left\{\int_0^1 t^{n-\alpha}\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s - \left(\frac{2-t}{2}\sum_{s=1}^{l-1}x_s + \frac{t}{2}\sum_{s=1}^{l-1}y_s\right)\right)dt\right. \tag{68}$$

$$\left.- \int_0^1 t^{n-\alpha}\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s - \left(\frac{2-t}{2}\sum_{s=1}^{l-1}y_s + \frac{t}{2}\sum_{s=1}^{l-1}x_s\right)\right)dt\right\}.$$

Proof. It can be easily proved by following the procedure given in the proof of Lemma 3. □

Remark 7. When we take $l = 2$, $x_1 = \delta_1$, and $y_1 = \delta_2$ in Lemma 4, then it reduces to the equality (3.1) in [16].

The following results have been established on the basis of Lemma 4.

Theorem 9. *Let all the conditions in the hypothesis of Theorem 7 hold. Then,*

$$\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{\left(\sum_{s=1}^{l-1}(y_s-x_s)\right)^{n-\alpha}}\left\{\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}(x_s+y_s/2)\right)^{+}}\phi\right)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)\right.\right.$$
$$\left.\left.+(-1)^n\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}(x_s+y_s/2)\right)^{-}}\phi\right)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)\right\}-\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)\right|$$
$$\leq\frac{\sum_{s=1}^{l-1}|y_s-x_s|}{2(n-\alpha+1)}\left\{\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right|-\frac{\sum_{s=1}^{l-1}\phi^{(n+1)}(x_s)+\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|}{2}\right\}. \tag{69}$$

Proof. From Lemma 4, it follows that

$$\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{\left(\sum_{s=1}^{l-1}(y_s-x_s)\right)^{n-\alpha}}\left\{\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}(x_s+y_s/2)\right)^{+}}\phi\right)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)\right.\right.$$
$$\left.\left.+(-1)^n\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}(x_s+y_s/2)\right)^{-}}\phi\right)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)\right\}-\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)\right|$$
$$=\left|\frac{\sum_{s=1}^{l-1}(y_s-x_s)}{4}\left\{\int_0^1 t^{n-\alpha}\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}x_s+\frac{t}{2}\sum_{s=1}^{l-1}y_s\right)\right)dt\right.\right.$$
$$\left.\left.-\int_0^1 t^{n-\alpha}\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}y_s+\frac{t}{2}\sum_{s=1}^{l-1}x_s\right)\right)dt\right\}\right|,$$
$$\leq\frac{\sum_{s=1}^{l-1}|y_s-x_s|}{4}\left\{\int_0^1 t^{n-\alpha}\left|\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}x_s+\frac{t}{2}\sum_{s=1}^{l-1}y_s\right)\right)\right|dt\right.$$
$$\left.+\int_0^1 t^{n-\alpha}\left|\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}y_s+\frac{t}{2}\sum_{s=1}^{l-1}x_s\right)\right)\right|dt\right\}. \tag{70}$$

By utilizing Theorem 2 for $r=2$, $\sigma_1=2-t/2$, and $\sigma_2=t/2$ in (70), we obtain

$$\leq\frac{\sum_{s=1}^{l-1}|y_s-x_s|}{4}\left\{\int_0^1 t^{n-\alpha}\left(\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right|-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right|+\frac{t}{2}\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|\right)\right)dt\right.$$
$$\left.+\int_0^1 t^{n-\alpha}\left(\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right|-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|+\frac{t}{2}\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right|\right)\right)|dt\right\}$$
$$=\frac{\sum_{s=1}^{l-1}|y_s-x_s|}{4}\left\{\frac{\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right|}{n-\alpha+1}-\frac{\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right|}{n-\alpha+1}+\frac{\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right|}{2(n-\alpha+2)}-\frac{\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|}{2(n-\alpha+2)}\right.$$
$$\left.+\frac{\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right|}{n-\alpha+1}-\frac{\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|}{n-\alpha+1}+\frac{\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|}{2(n-\alpha+2)}-\frac{\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right|}{2(n-\alpha+2)}\right\}$$
$$=\frac{\sum_{s=1}^{l-1}|y_s-x_s|}{2(n-\alpha+1)}\left\{\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right|-\frac{\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right|+\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|}{2}\right\}. \tag{71}$$

This finishes the proof. □

Corollary 2. *If we take $l = 2$, $n = 1$, and $\alpha = 0$ in Theorem 9, then we get the following inequality:*

$$\left|\frac{1}{y_1 - x_1}\int_{\delta_1+\delta_2-y_1}^{\delta_1+\delta_2-x_1}\phi(u)\mathrm{d}u - \phi\left(\delta_1+\delta_2-\frac{x_1+y_1}{2}\right)\right| \le \frac{|y_1 - x_1|}{4}\left\{|\phi'(\delta_1)| + |\phi'(\delta_2)| - \frac{|\phi\prime(x_1)| + |\phi\prime(y_1)|}{2}\right\}. \tag{72}$$

Theorem 10. *Let us consider a differentiable function ϕ defined on I, such that $\phi \in C^{n+1}(I)$ and $\delta = (\delta_1, \ldots, \delta_l)$, $\mathbf{x} = (x_1, \ldots, x_l)$, $\mathbf{y} = (y_1, \ldots, y_l)$ are three $l-$ tuples, such that $\delta_s, x_s, y_s \in I$, for all $s \in \{1, \ldots, l\}$, $x_l > y_l$, $\alpha > 0$. If $q > 1$ such that $1/p + 1/q = 1$, δ majorizes $\mathbf{x}$, $\mathbf{y}$, and $|\phi^{(n+1)}|^q$ is convex on I, then*

$$\begin{aligned}&\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{\left(\sum_{s=1}^{l-1}(y_s - x_s)\right)^{n-\alpha}}\left\{\left({}^cD^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}(x_s+y_s/2)\right)^+}\phi\right)\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}x_s\right)\right.\right.\\&\left.\left.+(-1)^n\left({}^cD^{\alpha}_{\left(\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}(x_s+y_s/2)\right)\right)^-}\phi\right)\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}y_s\right)\right\} - \phi^{(n)}\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)\right|\\&\le \frac{\sum_{s=1}^{l-1}|y_s - x_s|}{16}\left(\frac{4}{np - \alpha p + 1}\right)^{1/p}\left\{4^{1/q}\cdot 2\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right| - \left(3^{1/q}+1\right)\left(\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right| + \sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|\right)\right\}.\end{aligned} \tag{73}$$

Proof. From Lemma 4, it follows that

$$\begin{aligned}&\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{\left(\sum_{s=1}^{l-1}(y_s - x_s)\right)^{n-\alpha}}\left\{\left({}^cD^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}(x_s+y_s/2)\right)^+}\phi\right)\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}x_s\right)\right.\right.\\&\left.\left.+(-1)^n\left({}^cD^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}(x_s+y_s/2)\right)^-}\phi\right)\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}y_s\right)\right\} - \phi^{(n)}\left(\sum_{s=1}^{l}\delta_s - \sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)\right|\\&= \left|\frac{\sum_{s=1}^{l-1}(y_s - x_s)}{4}\left\{\int_0^1 t^{n-\alpha}\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s - \left(\frac{2-t}{2}\sum_{s=1}^{l-1}x_s + \frac{t}{2}\sum_{s=1}^{l-1}y_s\right)\right)\mathrm{d}t\right.\right.\\&\left.\left.-\int_0^1 t^{n-\alpha}\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s - \left(\frac{2-t}{2}\sum_{s=1}^{l-1}y_s + \frac{t}{2}\sum_{s=1}^{l-1}x_s\right)\right)\mathrm{d}t\right\}\right|,\\&\le \frac{\sum_{s=1}^{l-1}|y_s - x_s|}{4}\left\{\int_0^1 t^{n-\alpha}\left|\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s - \left(\frac{2-t}{2}\sum_{s=1}^{l-1}x_s + \frac{t}{2}\sum_{s=1}^{l-1}y_s\right)\right)\right|\mathrm{d}t\right.\\&\left.+\int_0^1 t^{n-\alpha}\left|\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s - \left(\frac{2-t}{2}\sum_{s=1}^{l-1}y_s + \frac{t}{2}\sum_{s=1}^{l-1}x_s\right)\right)\right|\mathrm{d}t\right\}.\end{aligned} \tag{74}$$

By applying Hölder's inequality to the above integral, we have

$$\begin{aligned}
&\leq \frac{\sum_{s=1}^{l-1}|y_s - x_s|}{4}\left\{\left(\int_0^1 t^{(n-\alpha)p}\mathrm{d}t\right)^{1/p}\left(\int_0^1\left|\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s - \left(\frac{2-t}{2}\sum_{s=1}^{l-1}x_s + \frac{t}{2}\sum_{s=1}^{l-1}y_s\right)\right)\right|^q \mathrm{d}t\right)^{1/q}\right.\\
&\quad\left.+\left(\int_0^1 t^{(n-\alpha)p}\mathrm{d}t\right)^{1/p}\left(\int_0^1\left|\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s - \left(\frac{2-t}{2}\sum_{s=1}^{l-1}y_s + \frac{t}{2}\sum_{s=1}^{l-1}x_s\right)\right)^q\right|\mathrm{d}t\right)^{1/q}\right\}\\
&= \frac{\sum_{s=1}^{l-1}|y_s - x_s|}{4}\left(\int_0^1 t^{(n-\alpha)p}\mathrm{d}t\right)^{1/p}\left\{\left(\int_0^1\left|\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s - \left(\frac{2-t}{2}\sum_{s=1}^{l-1}x_s + \frac{t}{2}\sum_{s=1}^{l-1}y_s\right)\right)\right|^q \mathrm{d}t\right)^{1/q}\right.\\
&\quad\left.+\left(\int_0^1\left|\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s - \left(\frac{2-t}{2}\sum_{s=1}^{l-1}y_s + \frac{t}{2}\sum_{s=1}^{l-1}x_s\right)\right)\right|^q \mathrm{d}t\right)^{1/q}\right\}.
\end{aligned} \tag{75}$$

Since $|\phi^{(n+1)}|^q$ is convex, therefore using Theorem 2 for $r = 2$, $\sigma_1 = 2 - t/2$, and $\sigma_2 = t/2$ in (75), we obtain

$$\begin{aligned}
&= \frac{\sum_{s=1}^{l-1}|y_s - x_s|}{4}\left(\frac{1}{np-\alpha p+1}\right)^{1/p}\left\{\left(\int_0^1\left(\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right|^q - \left(\frac{2-t}{2}\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right|^q + \frac{t}{2}\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|^q\right)\right)\mathrm{d}t\right)^{1/q}\right.\\
&\quad\left.+\left(\int_0^1\left(\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right|^q - \left(\frac{2-t}{2}\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|^q + \frac{t}{2}\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right|^q\right)\right)\mathrm{d}t\right)^{1/q}\right\}\\
&= \frac{\sum_{s=1}^{l-1}|y_s - x_s|}{4}\left(\frac{1}{np-\alpha p+1}\right)^{1/p}\left\{\left\{\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right|^q - \frac{1}{4}\left(3\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right|^q + \sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|^q\right)\right\}^{1/q}\right.\\
&\quad\left.+\left\{\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right|^q - \frac{1}{4}\left(3\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|^q + \sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right|^q\right)\right\}^{1/q}\right\}.
\end{aligned} \tag{76}$$

By using Minkowski's inequality, we get

$$= \frac{\sum_{s=1}^{l-1}|y_s - x_s|}{16}\left(\frac{4}{np-\alpha p+1}\right)^{1/p}\left\{4^{1/q}\cdot 2\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right| - \left(3^{1/q}+1\right)\left(\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right| + \sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|\right)\right\}. \tag{77}$$

This completes the proof. □

Remark 8. If we choose $l = 2$ in Theorem 10, then we get inequality (36) in [14].

Theorem 11. *Let all the conditions in the hypothesis of Theorem 8 hold. Then,*

$$\begin{aligned}
&\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{\left(\sum_{s=1}^{l-1}(y_s-x_s)\right)^{n-\alpha}}\left\{\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}(x_s+y_s/2)\right)^{+}}\phi\right)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)\right.\right.\\
&\quad\left.\left.+(-1)^{n}\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}(x_s+y_s/2)\right)^{-}}\phi\right)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)\right\}-\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)\right|\\
&\quad\le\frac{\sum_{s=1}^{l-1}|y_s-x_s|}{4}\left(\frac{1}{n-\alpha+1}\right)^{1-1/q}\left\{\left(\frac{1}{n-\alpha+1}\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right|^{q}\right.\right.\\
&\quad\left.-\left(\frac{n-\alpha+3}{2(n-\alpha+1)(n-\alpha+2)}\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right|^{q}+\frac{1}{2(n-\alpha+2)}\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|^{q}\right)\right)^{1/q}\\
&\quad\left.+\left(\frac{1}{n-\alpha+1}\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right|^{q}-\left(\frac{n-\alpha+3}{2(n-\alpha+1)(n-\alpha+2)}\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|^{q}+\frac{1}{2(n-\alpha+2)}\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right|^{q}\right)\right)^{1/q}\right\}.
\end{aligned}\tag{78}$$

Proof. From Lemma 4, it follows that

$$\begin{aligned}
&\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{\left(\sum_{s=1}^{l-1}(y_s-x_s)\right)^{n-\alpha}}\left\{\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}(x_s+y_s/2)\right)^{+}}\phi\right)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}x_s\right)\right.\right.\\
&\quad\left.\left.+(-1)^{n}\left({}^{c}D^{\alpha}_{\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}(x_s+y_s/2)\right)^{-}}\phi\right)\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}y_s\right)\right\}-\phi^{(n)}\left(\sum_{s=1}^{l}\delta_s-\sum_{s=1}^{l-1}\left(\frac{x_s+y_s}{2}\right)\right)\right|\\
&\quad=\left|\frac{\sum_{s=1}^{l-1}(y_s-x_s)}{4}\left\{\int_0^1 t^{n-\alpha}\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}x_s+\frac{t}{2}\sum_{s=1}^{l-1}y_s\right)\right)\mathrm{d}t\right.\right.\\
&\quad\left.\left.-\int_0^1 t^{n-\alpha}\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}y_s+\frac{t}{2}\sum_{s=1}^{l-1}x_s\right)\right)\mathrm{d}t\right\}\right|.\\
&\quad\le\frac{\sum_{s=1}^{l-1}|y_s-x_s|}{4}\left\{\int_0^1 t^{n-\alpha}\left|\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}x_s+\frac{t}{2}\sum_{s=1}^{l-1}y_s\right)\right)\right|\mathrm{d}t\right.\\
&\quad\left.+\int_0^1 t^{n-\alpha}\left|\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s-\left(\frac{2-t}{2}\sum_{s=1}^{l-1}y_s+\frac{t}{2}\sum_{s=1}^{l-1}x_s\right)\right)\right|\mathrm{d}t\right\}.
\end{aligned}\tag{79}$$

By applying power mean inequality to the above integral, we get

$$\leq \frac{\sum_{s=1}^{l-1}|y_s - x_s|}{4}\left\{\left(\left(\int_0^1 t^{n-\alpha}dt\right)^{1-1/q}\left(\int_0^1 t^{n-\alpha}\left|\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s - \left(\frac{2-t}{2}\sum_{s=1}^{l-1}x_s + \frac{t}{2}\sum_{s=1}^{l-1}y_s\right)\right)\right|^q \mathrm{d}t\right)^{1/q}\right.\right.$$
$$\left.+\left(\int_0^1 t^{n-\alpha}dt\right)^{1-1/q}\left(\int_0^1 t^{n-\alpha}\left|\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s - \left(\frac{2-t}{2}\sum_{s=1}^{l-1}y_s + \frac{t}{2}\sum_{s=1}^{l-1}x_s\right)\right)\right|^q \mathrm{d}t\right)^{1/q}\right\}$$
$$= \frac{\sum_{s=1}^{l-1}|y_s - x_s|}{4}\left(\frac{1}{n-\alpha+1}\right)^{1-1/q}\left\{\left(\left(\int_0^1 t^{n-\alpha}\left|\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s - \left(\frac{2-t}{2}\sum_{s=1}^{l-1}x_s + \frac{t}{2}\sum_{s=1}^{l-1}y_s\right)\right)\right|^q \mathrm{d}t\right)\right.\right.$$
$$\left.+\left(\int_0^1 t^{n-\alpha}\left|\phi^{(n+1)}\left(\sum_{s=1}^{l}\delta_s - \left(\frac{2-t}{2}\sum_{s=1}^{l-1}y_s + \frac{t}{2}\sum_{s=1}^{l-1}x_s\right)\right)\right|^q\right)\mathrm{d}t^{1/q}\right\}. \tag{80}$$

Due to the convexity of $|\phi^{(n+1)}|^q$, using Theorem 2 for $r = 2$, $\sigma_1 = 2 - t/2$, and $\sigma_2 = t/2$ in (80), we have

$$= \frac{\sum_{s=1}^{l-1}|y_s - x_s|}{4}\left(\frac{1}{n-\alpha+1}\right)^{1-1/q}\left\{\left(\left(\int_0^1 t^{n-\alpha}\left(\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right|^q - \left(\frac{2-t}{2}\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right|^q\right.\right.\right.\right.\right.$$
$$\left.\left.\left.+\frac{t}{2}\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|^q\right)\right)\mathrm{d}t\right)^{1/q} + \left(\int_0^1 t^{n-\alpha}\left(\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right|^q - \left(\frac{2-t}{2}\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|^q + \frac{t}{2}\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right|^q\right)\right)\right)\mathrm{d}t^{1/q}\right\}$$
$$= \frac{\sum_{s=1}^{l-1}|y_s - x_s|}{4}\left(\frac{1}{n-\alpha+1}\right)^{1-1/q}\left\{\left(\frac{1}{n-\alpha+1}\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right|^q\right.\right.$$
$$\left.-\left(\frac{n-\alpha+3}{2(n-\alpha+1)(n-\alpha+2)}\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right|^q + \frac{1}{2(n-\alpha+2)}\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|^q\right)\right)^{1/q}$$
$$\left.+\left(\frac{1}{n-\alpha+1}\sum_{s=1}^{l}\left|\phi^{(n+1)}(\delta_s)\right|^q - \left(\frac{n-\alpha+3}{2(n-\alpha+1)(n-\alpha+2)}\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(y_s)\right|^q + \frac{1}{2(n-\alpha+2)}\sum_{s=1}^{l-1}\left|\phi^{(n+1)}(x_s)\right|^q\right)\right)^{1/q}\right\}. \tag{81}$$

□

Hence, the proof is completed.

Corollary 3. *For $l = 2$, $n = 1$, and $\alpha = 0$, Theorem 11 gives the following inequality:*

$$\left|\frac{1}{y_1 - x_1}\int_{\delta_1+\delta_2-y_1}^{\delta_1+\delta_2-x_1}\phi(u)\mathrm{d}u - \phi\left(\delta_1 + \delta_2 - \frac{x_1+y_1}{2}\right)\right| \leq \frac{|y_1 - x_1|}{2^{((3q-1)/q)}}\left\{\left(\frac{|\phi\prime(\delta_1)| + |\phi\prime(\delta_2)|}{2} - \frac{2|\phi\prime(x_1)|^q + |\phi\prime(y_1)|^q}{2}\right)^{1/q}\right.$$
$$\left.+\left(\frac{|\phi\prime(\delta_1)| + |\phi\prime(\delta_2)|}{2} - \frac{2|\phi\prime(y_1)|^q + |\phi\prime(x_1)|^q}{2}\right)^{1/q}\right\}. \tag{82}$$

Remark 9. We can also obtain weighted versions for all the results derived in this section.

4. Conclusion

A new idea in the form of unified inequalities has been put forward. Tools that helped during the development of the main results are the notions of some existing inequalities, majorization theory, and various forms of convex functions. The results have been put up in the context of Hermite–Hadamard–Jensen–Mercer inequalities. The selection of the present areas of inequalities has been made on the basis of their consistent attraction for researchers and their vast applicability in enormous fields. Both the weighted and unweighted versions of the obtained results have been presented. Moreover, some new identities for differentiable functions have been derived. Using these identities and considering the convexity of $|\phi^{(n+1)}|$ and $|\phi^{(n+1)}|^q\,(q>1)$, bounds for the absolute difference of the right- and left-sides of the main results have been provided.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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