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Research Article

Certain New Class of Harmonic Functions Involving Quantum Calculus

Mohammad Faisal Khan

Department of Basic Science, College of Science and Theoretical Studies, Saudi Electronic University, Riyadh 11673, Saudi Arabia

Correspondence should be addressed to Mohammad Faisal Khan; f.khan@seu.edu.sa

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Here in this paper, we are using the concepts of q-calculus operator theory associated with harmonic functions and define the q-Noor integral operator for harmonic functions $f \in \mathscr{H}^0$. We investigate a new class $\mathcal{S}^0_{\mathscr{H}}(m, q, \alpha)$ of harmonic functions $f \in \mathscr{H}^0$. In this class, we prove a necessary and sufficient convolution condition for the functions $f \in \mathscr{H}^0$ and also we proved that this sufficient coefficient condition is sense preserving and univalent in the class $\mathcal{S}^0_{\mathscr{H}}(m, q, \alpha)$. It is proved that this coefficient condition is necessary for the functions in its subclass $\mathcal{T}\mathcal{S}^0_{\mathscr{H}}(m, q, \alpha)$. By using this necessary and sufficient coefficient condition, we obtained results based on the convexity and compactness and results on the radii of q-starlikeness and q-convexity of order α in the class $\mathcal{T}\mathcal{S}^0_{\mathscr{H}}(m, q, \alpha)$. Also we obtained extreme points for the functions in the class $\mathcal{T}\mathcal{S}^0_{\mathscr{H}}(m, q, \alpha)$.

1. Introduction and Definitions

A complex-valued function $f = u + iv$ is said to be harmonic in in open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ if both u and v are real valued harmonic functions in U. Also the complex-valued harmonic function $f = u + iv$ can also be expressed as $f = h + \bar{g}$, where h and g are analytic in U. In particular, h is called analytic part, and g is called coanalytic part of f. The Jacobian of the function $f = u + iv$ is given by

$$\mathcal{J}_f(z) = \left|h'(z)\right|^2 - \left|g'(z)\right|^2. \tag{1}$$

It is known (see [1]) that every harmonic function $f = h + \bar{g}$ to be locally univalent and sense preserving in U if and only if $\mathcal{J}_f(z) > 0$ in U which is equivalent to $u(z) = (g'(z))/(h'(z))$ in U such that

$$|u(z)| < 1, \text{for all } z \in U. \tag{2}$$

For detail (see [2]). Let $\mathscr{H}$ indicates the class of harmonic functions in U. Also let $\mathscr{H}^0$ denoted by the family of harmonic functions $f = h + \bar{g} \in \mathscr{H}$ which have the series expansion of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n}, (z \in U), \tag{3}$$

where h and g are analytic functions with the following series expansion:

$$\begin{aligned} h(z) &= z + \sum_{n=2}^{\infty} a_n z^n, \\ g(z) &= \sum_{n=1}^{\infty} b_n z^n, \\ |b_1| &< 1, (z \in U). \end{aligned} \tag{4}$$

The series defined in (3) and (4) are convergent in the open unit disc U. Also let $\mathcal{S}$ represents all functions (say $f \in \mathcal{S}$) which are univalent analytic in U and satisfy the condition

$$f(0) = f'(0) - 1 = 0. \tag{5}$$

Further, let $\mathcal{S}_{\mathcal{H}}$ denotes the class of all harmonic functions $f = h + \bar{g} \in \mathcal{H}^0$ which are sense preserving and univalent in U. The class $\mathcal{S}_{\mathcal{H}}$ reduces to the class $\mathcal{S}$ if coanalytic part of f is zero.

Clunie and Small [3] and Small [4] studied the class $\mathcal{S}_{\mathcal{H}}$ along with some of their subfamilies. Particularly, they explored and studied the families of starlike harmonic and convex harmonic functions in U, which are given as follows:

$$\begin{aligned} \mathcal{S}_{\mathcal{H}}^* &= \left\{ f \in \mathcal{S}_{\mathcal{H}} : \frac{\mathcal{D}_{\mathcal{H}} f(z)}{f(z)} \prec \frac{1+z}{1-z}, (z \in U) \right\}, \\ \mathcal{S}_{\mathcal{H}}^c &= \{ f \in \mathcal{S}_{\mathcal{H}} : \mathcal{D}_{\mathcal{H}} f(z) \in \mathcal{S}_{\mathcal{H}}^* (z \in U) \}, \end{aligned} \tag{6}$$

where

$$\mathcal{D}_{\mathcal{H}} f(z) = z h'(z) - \bar{z} g'(z). \tag{7}$$

In [5], Dziok introduced a new family $\mathcal{S}_{\mathcal{H}}^*(L, M)$, $L, M \in \mathbb{C}$, and $L \neq M$ of Janowski harmonic functions and defined by

$$\mathcal{S}_{\mathcal{H}}^*(L, M) = \left\{ f \in \mathcal{S}_{\mathcal{H}} : \frac{\mathcal{D}_{\mathcal{H}} f(z)}{f(z)} \prec \frac{1+Lz}{1+Mz}, (z \in U) \right\}, \tag{8}$$

where $\mathcal{D}_{\mathcal{H}} f(z)$ is given by (5). We can see that

$$\mathcal{S}_{\mathcal{H}}^*(1, -1) = \mathcal{S}_{\mathcal{H}}^*. \tag{9}$$

The convolution of two functions $h, g \in \mathcal{A}$, is defined by

$$(h * g)z = \sum_{n=1}^{\infty} a_n b_n z^n, \tag{10}$$

where

$$\begin{aligned} h(z) &= \sum_{n=1}^{\infty} a_n z^n, \\ g(z) &= \sum_{n=1}^{\infty} b_n z^n. \end{aligned} \tag{11}$$

Similarly, the convolution of two harmonic functions $f = h + \bar{g}$ and $f_1 = h_1 + \bar{g}_1$ is defined by

$$(f * f_1)(z) = (h * h_1)(z) + (g * \bar{g}_1)(z). \tag{12}$$

The function h subordinate to a function g and write $h(z) \prec g(z)$,$z \in U$, if there exists a complex-valued function v which map U into itself such that $v(0) = 0$ and $h(z) = g(v(z))$. In particular, if g is univalent in U, then we have the following equivalence:

$$\begin{aligned} &h(z) \prec g(z), \\ &z \in U \Leftrightarrow h(0) = g(0), \\ &h(U) \subset g(U). \end{aligned} \tag{13}$$

In the nineteen century, several mathematician has been using q-calculus operator theory in various area of science, such that fractional calculus, q-difference equation, optimal control, q-integral equations, and geometric function theory (GFT). In 1908, Jackson [6] introduced the q -derivative and q-integral operator and discussed some of their applications. In the year 1990, Ismail et al. [7] gave the idea of q-extension of class of q-starlike functions by implementing the q-calculus theory. Kanas and Raducanu [8] used q-calculus operator theory and introduced the q -Ruscheweyh differential operator for analytic functions. Zhang et. al [9] introduced a generalized conic domain $\Omega_{k,\alpha,q}$ by using the basic concepts of q-calculus and studied new subclass of q-starlike functions. Arif et al. defined q-Noor integral operator [10] by using the concept of convolution and used it to investigated some new subclasses of analytic functions. Further, in article [11], Khan et al. discussed some applications of q-derivative operator for multivalent functions, while coefficient estimates for a certain family of analytic functions involving a q-derivative operator were discussed by Raza et al. [12]. Recently, Srivastava et. al published few articles in which they implemented basic concepts of q-calculus operator theory and studied class of q-starlike functions from different aspects (see [13–16]). Additionally, a recently published article by Srivastava [17] is very suitable for researchers to work on this topic. For more recently, Khan et al. [18, 19] used the concepts of q-calculus operator theory to define some new subclasses of analytic functions. Also for more detail, we may refer to [20–25].

For, $q \in (0, 1)$, the q-derivative operator (∂_q) of f is defined as follows:

$$\begin{aligned} \partial_q f(z) &= \frac{f(z) - f(qz)}{(1-q)z}, z \neq 0, 0 < q < 1, \\ &= 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}. \end{aligned} \tag{14}$$

Making use of (3) and (14), and for $n \in \mathbb{N}$, we have

$$\begin{aligned} \partial_q f(z) &= \partial_q h(z) + \overline{\partial_q g(z)}, \\ &= 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1} + \sum_{n=2}^{\infty} [n]_q \overline{b_n z^{n1}}, [n]_q \\ &= (1-q)^{-1}(1-q^n). \end{aligned} \tag{15}$$

For more detail (see [26, 27]).

Definition 1 (see [10, 28]). The q-Noor integral operator for the analytic function h is defined by

$$I_q^m h(z) = h(z) * \left(T_{q,m+1}(z)\right)^{-1}, z \in U, m > -1, \tag{16}$$

where

$$\left(T_{q,m+1}(z)\right)^{-1} = z + \sum_{n=2}^{\infty} \frac{[n]_q! \Gamma_q(1+m)}{\Gamma_q(n+m)} z^n,$$
$$\left(T_{q,m+1}(z)\right)^{-1} * T_{q,m+1}(z) = z\partial_q h(z). \tag{17}$$

Thus, we have

$$\begin{aligned} I_q^m h(z) &= z + \sum_{n=2}^{\infty} \frac{[n]_q! \Gamma_q(1+m)}{\Gamma_q(n+m)} a_n z^n \\ &= z + \sum_{n=2}^{\infty} \frac{[n]_q!}{[m+1]_{n-1}} a_n z^n \\ &= z + \sum_{n=2}^{\infty} \Psi_n^q a_n z^n, \end{aligned} \tag{18}$$

where

$$\Psi_n^q = \frac{[n]_q!}{[m+1]_{n-1}}. \tag{19}$$

Clearly,

$$I_q^0 h(z) = z\partial_q h(z) = \frac{z}{(1-z)(1-qz)},$$
$$I_q^1 h(z) = \frac{z}{1-z}. \tag{20}$$

Remark 2. When $q \longrightarrow 1-$, then q -Noor integral operator reduces to Noor integral operator (see [29]).

First of all Jahangiri [30] applied certain q-calculus operators to complex harmonic functions and obtained some useful results, while Porwal and Gupta discussed some application of q-calculus to harmonic univalent functions in [31]. Recently Arif et al. [27] introduced some new families of harmonic functions associated with the symmetric circular region. For some more recent investigation about harmonic univalent functions, we may refer to [32, 33]. By taking the motivation from the article Arif et al. [27], we define the q-Noor integral operator for the harmonic function $f = h + \bar{g}$.

Definition 3. Let the q-Noor integral operator I_q^m of order $m > -1$, for the harmonic function $f = h + \bar{g}$ be defined as

$$I_q^m f(z) = I_q^m h(z) + (-1)^m I_q^m \bar{g}(z), \tag{21}$$

where $h(z)$ and $g(z)$ is given by (4).

In this paper, by using the concepts of q-calculus operator theory and q-Noor integral operator for harmonic functions f, we define a new class $\mathcal{S}_{\mathcal{H}}^0(m, q, \alpha)$ of harmonic functions $f \in \mathcal{H}^0$. In this class, we prove a necessary and sufficient convolution condition for the functions $f \in \mathcal{H}^0$ and prove that this sufficient coefficient condition is sense preserving and univalent in the class $\mathcal{S}_{\mathcal{H}}^0(m, q, \alpha)$. It is proved that this coefficient condition is necessary for the functions in its subclass $\mathcal{T}\mathcal{S}_{\mathcal{H}}^0(m, q, \alpha)$. By using this necessary and sufficient coefficient condition, we obtained results based on the convexity and compactness and results on the radii of q-starlikeness and q-convexity of order α and extreme points for the functions in the class $\mathcal{T}\mathcal{S}_{\mathcal{H}}^0(m, q, \alpha)$. This research work will motivate future research to work in the area of q-calculus operators together with harmonic functions.

Definition 4. Let $\mathcal{S}_{\mathcal{H}}^0(m, q, \alpha)$ be the family of harmonic functions $f \in \mathcal{H}^0$ that satisfy the subordination condition

$$\frac{I_q^{m+1} f(z)}{I_q^m f(z)} \prec \frac{1 + \mathcal{Q}z}{1 + \mathcal{T}z}, (q \in (0,1), 0 \le \alpha < 1, z \in U), \tag{22}$$

where

$$\mathcal{Q} = \alpha(1+q) - 1,$$
$$\mathcal{T} = q. \tag{23}$$

Inequalities (22) is equivalent to the condition

$$\left| \frac{I_q^{m+1} f(z) - I_q^m f(z)}{\mathcal{T}\left(I_q^{m+1} f(z)\right) - \mathcal{Q} I_q^m f(z)} \right| < 1. \tag{24}$$

We denote by $\mathcal{T}\mathcal{S}_{\mathcal{H}}^0(m, q, \alpha)$ a subclass of harmonic functions $f = h + \bar{g} \in \mathcal{S}_{\mathcal{H}}^0(m, q, \alpha)$, where for m, functions h and g are of the form:

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, g(z) = (-1)^m \sum_{n=2}^{\infty} |b_n| z^n, z \in U. \tag{25}$$

2. Main Results

Theorem 5. *Let $f \in \mathcal{H}^0$. Then the function $f \in \mathcal{S}_{\mathcal{H}}^0(m, q, \alpha)$ if and only if*

$$I_q^m f(z) * \phi(z, \gamma) \neq 0, (\gamma \in \mathbb{C}, |\gamma| = 1, z \in U \setminus \{0\}), \tag{26}$$

where

$$\begin{aligned} \phi(z, \gamma) = &\frac{(\mathcal{T} - \mathcal{Q})\zeta z - (1 + \mathcal{T}\zeta) q z^2}{(1-z)(1-qz)} \\ &- \left(\frac{2z + (\mathcal{Q} + \mathcal{T})\bar{\zeta} z (1 + \mathcal{T}\zeta) q z^2}{(1z)(1qz)} \right). \end{aligned} \tag{27}$$

Proof. Let $f = h + \bar{g} \in \mathcal{H}^0$ be of the form (3). Then the function $f \in \mathcal{S}_{\mathcal{H}}^0(m, q, \alpha)$ if and only if (22) holds or equivalently

$$\frac{I_q^{m+1} f(z)}{I_q^m f(z)} \neq \frac{1 + \mathcal{Q}\zeta}{1 + \mathcal{T}\zeta}, (\zeta \in \mathbb{C}, |\zeta| = 1, z \in E\{0\}), \tag{28}$$

which by (21) is given by

$$\begin{aligned}&(1+\mathcal{T}\zeta)\left[I_q^m\left(I_q h(z)\right)+(-1)^{m+1}I_q^m\left(I_q\bar{g}(z)\right)\right]\\&-(1+\mathcal{Q}\zeta)\left[I_q^m h(z)+(-1)^m I_q^m\bar{g}(z)\right]\neq 0.\end{aligned}\tag{29}$$

On using (20), the condition (29) may also be given by

$$\begin{aligned}&I_q^m h(z)*\left[(1+\mathcal{T}\zeta)\frac{z}{1-z}-(1+\mathcal{Q}\zeta)\frac{z}{(1-z)(1-qz)}\right]\\&-(-1)^m I_q^m\bar{g}(z)*\left[(1+\mathcal{T}\zeta)\frac{\bar{z}}{1-\bar{z}}\right.\\&\left.+(1+\mathcal{Q}\zeta)\frac{\bar{z}}{(1-\bar{z})(1-q\bar{z})}\right]\neq 0.\end{aligned}\tag{30}$$

Which on using the convolution $*$ between two harmonic functions, we get

$$I_q^m f(z)*\phi(z,\gamma)\neq 0,\tag{31}$$

where the harmonic function $\phi(z,\gamma)$ is given by (27). □

Theorem 6. *Let $f=h+\bar{g}\in\mathcal{H}^0$ be of the form (3) and $q\in(0,1)$, $0\le\alpha<1$. If*

$$\sum_{n=2}^{\infty}L_n|a_n|+M_n|b_n|\le\mathcal{T}-\mathcal{Q},\tag{32}$$

where

$$L_n=(\Psi_n^q)^m\{\Psi_n^q(1+\mathcal{T})-(1+\mathcal{Q})\},\tag{33}$$

$$M_n=(\Psi_n^q)^m\{\Psi_n^q(1+\mathcal{T})+(1+\mathcal{Q})\},\tag{34}$$

where Ψ_n^q is given by (19), then.

(i) the function f is locally univalent and sense-preserving as $q\longrightarrow 1-$

(ii) the function $f\in\mathcal{S}_{\mathcal{H}}^0(m,q,\alpha)$

Equality occurs for the function

$$\begin{aligned}&f(z)=z+\sum_{n=2}^{\infty}\frac{\mathcal{T}-\mathcal{Q}}{L_n}\gamma_n z^n+\sum_{n=2}^{\infty}\frac{\mathcal{T}-\mathcal{Q}}{M_n}\bar{\beta}_n\bar{z}^n,\\&\sum_{n=2}^{\infty}(|\gamma_n|+|\beta_n|)=1.\end{aligned}\tag{35}$$

Proof. For part (i), it is clear that the theorem is true for the function $f(z)\equiv z$. Let $f=h+\bar{g}$ and assume that there exist $n\ge 2$ such that $a_n\neq 0$ or $b_n\neq 0$. Since $\Psi_n^q>1$, we observe from (33) and (34) that $L_n\ge M_n>\Psi_n^q(\mathcal{T}-\mathcal{Q})$, by which the condition (32) implies the condition

$$\begin{aligned}&\sum_{n=2}^{\infty}\Psi_n^q(|a_n|+|b_n|)<1,\\&\left|\partial_q h(z)-\partial_q g(z)\right|\ge 1-\sum_{n=2}^{\infty}\Psi_n^q|a_n||z|^{n-1}-\sum_{n=2}^{\infty}\Psi_n^q|b_n||z|^{n-1},\\&>1-|z|\sum_{n=2}^{\infty}\Psi_n^q(|a_n|+|a_n|)\ge 1-|z|>0,\end{aligned}\tag{36}$$

in D which implies as $q\longrightarrow 1-$ that $|h'(z)|>|g'(z)|$ in D that is a function f is locally univalent and sense-preserving in D.

For part (i), to prove that $f\in\mathcal{S}_{\mathcal{H}}^0(m,q,\alpha)$, we only need to show that f satisfy the condition (24). Consider for $f=h+\bar{g}$ and for $|z|=r(0<r<1)$, we can write (24) as

$$\begin{aligned}&\left|I_q^{m+1}f(z)-I_q^m f(z)\right|-\left|\mathcal{T}\left(I_q^{m+1}f(z)\right)-\mathcal{Q}I_q^m f(z)\right|\\&=\left|\sum_{n=2}^{\infty}(\Psi_n^q)^m(\Psi_n^q-1)a_n z^n-(-1)^m\sum_{n=2}^{\infty}(\Psi_n^q)^m(\Psi_n^q+1)\bar{b}_n\bar{z}^n\right|\\&-\left|(\mathcal{T}-\mathcal{Q})z+\sum_{n=2}^{\infty}(\Psi_n^q)^m(\mathcal{T}(\Psi_n^q)-\mathcal{Q})a_n z^n\right|\\&-(-1)^m\sum_{n=2}^{\infty}(\Psi_n^q)^m(\mathcal{T}(\Psi_n^q)+\mathcal{Q})\bar{b}_n\bar{z}^n\\&\le\sum_{n=2}^{\infty}(\Psi_n^q)^m(\Psi_n^q-1)a_n r^n-(-1)^m\sum_{n=2}^{\infty}(\Psi_n^q)^m(\Psi_n^q+1)\bar{b}_n\bar{r}^n\\&+(\mathcal{T}-\mathcal{Q})r+\sum_{n=2}^{\infty}(\Psi_n^q)^m(\mathcal{T}(\Psi_n^q)-\mathcal{Q})a_n r^n\\&+\sum_{n=2}^{\infty}(\Psi_n^q)^m(\mathcal{T}(\Psi_n^q)+\mathcal{Q})\bar{b}_n\bar{r}^n<\sum_{n=2}^{\infty}(L_n|a_n|+M_n|b_n|)r^n\\&-(\mathcal{T}-\mathcal{Q})\le\sum_{n=2}^{\infty}(L_n|a_n|+M_n|b_n|)r^n-(\mathcal{T}-\mathcal{Q})\le 0,\end{aligned}\tag{37}$$

if the condition (32) holds. This proves the condition (24). This completes the proof of Theorem 6. □

Theorem 7. *Let $f=h+\bar{g}\in\mathcal{H}^0$ where h and g are given by (25). Then $f\in\mathcal{T}\mathcal{S}_{\mathcal{H}}^0(m,q,\alpha)$ if and only if the condition (32) holds that is*

$$\sum_{n=2}^{\infty}L_n|a_n|+M_n|b_n|\le\mathcal{T}-\mathcal{Q},\tag{38}$$

where L_n and M_n are given by (33) and (34).

Proof. If part is proved in Theorem 6. To prove only if part, let $f\in\mathcal{T}\mathcal{S}_{\mathcal{H}}^0(m,q,\alpha)$. Then by the class condition (22), we have from (24) that for any $z\in E$.

$$\left|\frac{A(\mathcal{T},\mathcal{Q})a_n z^n + B(\mathcal{T},\mathcal{Q})\bar{b_n}\bar{z}^n}{(\mathcal{T}-\mathcal{Q})z - C(\mathcal{T},\mathcal{Q})|a_n|z^n - D(\mathcal{T},\mathcal{Q})|\bar{b_n}|\bar{z}^n}\right| < 1, \quad (39)$$

where

$$\begin{aligned} A(\mathcal{T},\mathcal{Q}) &= \sum_{n=2}^{\infty} (\Psi_n^q)^m (\Psi_n^q - 1), \\ B(\mathcal{T},\mathcal{Q}) &= \sum_{n=2}^{\infty} (\Psi_n^q)^m (\Psi_n^q + 1), \\ C(\mathcal{T},\mathcal{Q}) &\ll \sum_{n=2}^{\infty} (\Psi_n^q)^m (\mathcal{T}(\Psi_n^q) - \mathcal{Q}), \\ D(\mathcal{T},\mathcal{Q}) &\ll \sum_{n=2}^{\infty} (\Psi_n^q)^m (\mathcal{T}(\Psi_n^q) + \mathcal{Q}). \end{aligned} \quad (40)$$

For $z = r(0 \le r < 1)$, we obtain

$$\frac{A(\mathcal{T},\mathcal{Q})|a_n|r^{n-1} + B(\mathcal{T},\mathcal{Q})|\bar{b_n}|r^{n1}}{(\mathcal{T}-\mathcal{Q}) - C(\mathcal{T},\mathcal{Q})|a_n|r^{n-1} - D(\mathcal{T},\mathcal{Q})|\bar{b_n}|r^{n1}} < 1, \quad (41)$$

which proves for L_n and M_n defined by (33) and (34) that

$$\sum_{n=2}^{\infty}(L_n|a_n| + M_n|b_n|)r^{n-1} < \mathcal{T} - \mathcal{Q}. \quad (42)$$

Let σ_n be the sequence of partial sums of the series

$$\sum_{n=2}^{\infty}(L_n|a_n| + M_n|b_n|). \quad (43)$$

Then σ_n is a nondecreasing sequence, and by (42), it is bounded above. Thus, as $r \longrightarrow 1^-$, it is convergent and

$$\sum_{n=2}^{\infty}(L_n|a_n| + M_n|b_n|) = \lim_{n\longrightarrow\infty}\sigma_n \le \mathcal{T} - \mathcal{Q}. \quad (44)$$

This gives the condition (32). □

Remark 8. Theorem 7 gives a necessary and sufficient condition for the functions $f = h + \bar{g} \in \mathcal{H}^0$, where h and g are given by (25) to be q-starlike and q-convex of order α in E if we put $m = 0$ and $m = 1$, respectively, in (38) and are given by

$$\sum_{n=2}^{\infty}\{(\Psi_n^q - \alpha)|a_n| + (\Psi_n^q + \alpha)\}|b_n| \le 1 - \alpha, \quad (45)$$

$$\sum_{n=2}^{\infty}\Psi_n^q\{(\Psi_n^q - \alpha)|a_n| + (\Psi_n^q + \alpha)\}|b_n| \le 1 - \alpha. \quad (46)$$

Theorem 9. *The class $\mathcal{T}\mathcal{S}_{\mathcal{H}}^0(m,q,\alpha)$ is a convex and compact subclass of the class of functions $f = h + \bar{g} \in \mathcal{H}^0$, where h and g are given by (25).*

Proof. Let $i = 1, 2, f_i \in \mathcal{T}\mathcal{S}_{\mathcal{H}}^0(m,q,\alpha)$; and let this m is of the form

$$f_i(z) = z - \sum_{n=2}^{\infty}|a_{i,n}|z^n + (-1)^m\sum_{n=2}^{\infty}|b_{i,n}|\bar{z}^n, z \in U. \quad (47)$$

Then for $0 \le \rho \le 1$

$$\begin{aligned} F(z) &= \rho f_1(z) + (1-\rho)f_2(z) \\ &= z - \sum_{n=2}^{\infty}(\rho|a_{1,n}| + (1-\rho)|a_{2,n}|)z^n \\ &\quad + (-1)^m\sum_{n=2}^{\infty}(\rho|b_{1,n}| + (1-\rho)|b_{2,n}|)\bar{z}^n. \end{aligned} \quad (48)$$

and by Theorem 7, we get L_n and M_n given by (33) and (34) that

$$\begin{aligned} &\sum_{n=2}^{\infty}\{L_n(\rho|a_{1,n}| + (1-\rho)|a_{2,n}|) + M_n(\rho|b_{1,n}| + (1-\rho)|b_{2,n}|)\} \\ &= \rho\sum_{n=2}^{\infty}\{L_n|a_{1,n}| + M_n|b_{1,n}|\} + (1-\rho)\sum_{n=2}^{\infty}\{L_n|a_{2,n}| \\ &\quad + M_n|b_{2,n}|\} \le \rho(\mathcal{T}-\mathcal{Q}) + (1-\rho)(\mathcal{T}-\mathcal{Q}) = \mathcal{T}-\mathcal{Q}. \end{aligned} \quad (49)$$

Therefore, $F \in \mathcal{T}\mathcal{S}_{\mathcal{H}}^0(m,q,\alpha)$. Hence, the class $\mathcal{T}\mathcal{S}_{\mathcal{H}}^0(m,q,\alpha)$ is convex.

On the other hand, if we consider $f_i \in \mathcal{T}\mathcal{S}_{\mathcal{H}}^0(m,q,\alpha)$, $i \in N = \{1, 2, 3 \cdots\}$ of the form (47), and then by Theorem 7, we get for L_n and M_n defined by (33) and (34).

$$\sum_{n=2}^{\infty}(L_n|a_{i,n}| + M_n|b_{i,n}|) \le \mathcal{T} - \mathcal{Q}. \quad (50)$$

Hence, for $|z| \le r(0 < r < 1)$

$$\begin{aligned} |f_i(z)| &\le r + \sum_{n=2}^{\infty}(|a_{i,n}| + |b_{i,n}|)r^n \le \mathcal{T} - \mathcal{Q} \\ &\le r + \frac{\sum_{n=2}^{\infty}(L_n|a_{i,n}| + M_n|b_{i,n}|)r^n}{(\Psi_2^q)^m\{\Psi_2^q(1+\mathcal{T}) - (1+\mathcal{Q})\}} \\ &< r + \frac{\mathcal{T} - \mathcal{Q}}{(\Psi_2^q))^m\{\Psi_2^q(1+\mathcal{T}) - (1+\mathcal{Q})\}}r^2. \end{aligned} \quad (51)$$

Similarly, we get for $|z| \le r(0 < r < 1)$,

$$|f_i(z)| > r - \frac{\mathcal{T} - \mathcal{Q}}{(\Psi_2^q)^m\{\Psi_2^q(1+\mathcal{T}) - (1+\mathcal{Q})\}}r^2. \quad (52)$$

Therefore, class $\mathcal{T}\mathcal{S}_{\mathcal{H}}^0(m,q,\alpha)$ is locally uniformly bounded.

If we assume that $f_i \longrightarrow f$, then we conclude that $|a_{i,n}| \longrightarrow |a_n|$ and $|b_{i,n}| \longrightarrow |b_n|$ as $i \longrightarrow \infty$ for any $n = 2, 3 \cdots$. Hence, from (50), we get

$$\sum_{n=2}^{\infty}(L_n|a_n| + M_n|b_n|) \le \mathcal{T} - \mathcal{Q}, \tag{53}$$

which proves that $f \in \mathcal{T}\mathcal{S}^0_{\mathcal{H}}(m, q, \alpha)$. Therefore, the class $\mathcal{T}\mathcal{S}^0_{\mathcal{H}}(m, q, \alpha)$ is closed. This proves that class $\mathcal{T}\mathcal{S}^0_{\mathcal{H}}(m, q, \alpha)$ is compact. □

Corollary 10. *Let $f \in \mathcal{T}\mathcal{S}^0_{\mathcal{H}}(m, q, \alpha)$. Then for $|z| = r(r < 1)$*

$$\begin{aligned} & r - \frac{\mathcal{T} - \mathcal{Q}}{\left(\Psi_2^q\right)^m\left\{\Psi_2^q(1+\mathcal{T}) - (1+\mathcal{Q})\right\}} r^2, \\ & < |f(z)| < r + \frac{\mathcal{T} - \mathcal{Q}}{\left(\Psi_2^q\right)^m\left\{\Psi_2^q(1+\mathcal{T}) - (1+\mathcal{Q})\right\}} r^2. \end{aligned} \tag{54}$$

Furthermore,

$$\left\{ w \in \mathbb{C} : |w| < 1 - \frac{\mathcal{T} - \mathcal{Q}}{\left(\Psi_2^q\right)^m\left\{\Psi_2^q(1+\mathcal{T}) - (1+\mathcal{Q})\right\}} \right\} \subset f(U). \tag{55}$$

Remark 11. The minimum of all values of the radius $r \in (0, 1)$ for functions $f \in \mathcal{T}\mathcal{S}^0_{\mathcal{H}}(m, q, \alpha)$ such that

$$\frac{f(rz)}{r} \in \mathcal{H}^*_q(\alpha) \tag{56}$$

is called the radius of q-starlikeness of order α and is denoted by $r\mathcal{H}^*_q(\alpha)\mathcal{T}\mathcal{S}^0_{\mathcal{H}}(m, q, \alpha)$.

Now in next theorem, we obtain the radius of q-starlikeness of order α for functions $f \in \mathcal{T}\mathcal{S}^0_{\mathcal{H}}(m, q, \alpha)$.

Theorem 12. *Let $0 \le \alpha < 1$, and L_n and M_n are defined by (33) and (34). Then*

$$\begin{aligned} & r\mathcal{H}^*_q(\alpha)\left(\mathcal{T}\mathcal{S}^0_{\mathcal{H}}(m, q, \alpha)\right) \\ & = \inf_{n\ge 2}\left[\frac{1-\alpha}{\mathcal{T} - \mathcal{Q}} \min\left\{\frac{L_n}{\Psi_n^q - \alpha}, \frac{M_n}{\Psi_n^q + \alpha}\right\}\right]^{1/n-1}, \end{aligned} \tag{57}$$

where Ψ_n^q defined by (19).

Proof. Let $f = h + \bar{g} \in \mathcal{T}\mathcal{S}^0_{\mathcal{H}}(m, q, \alpha)$, then by Theorem 7, we have

$$\sum_{n=2}^{\infty} L_n|a_n| + M_n|b_n| \le \mathcal{T} - \mathcal{Q}, \tag{58}$$

where L_n and M_n are defined, respectively, by (33) and (34). Let r_0 be the radius of q-starlikeness of order α. Then $f(r_0 z)/r_0 \in \mathcal{H}^*_q(\alpha)$ if and only if from (45) that

$$\sum_{n=2}^{\infty}\{\{(\Psi_n^q - \alpha)|a_n| + (\Psi_n^q + \alpha)\}|b_n|\} r_0^{k-1} \le 1 - \alpha, \tag{59}$$

which is true if

$$\begin{aligned} & \frac{\Psi_n^q - \alpha}{1-\alpha} r_0^{k-1} \le \frac{L_n}{\mathcal{T} - \mathcal{Q}}, n = 2, 3 \cdots, \\ & \frac{\Psi_n^q + \alpha}{1-\alpha} r_0^{k-1} \le \frac{M_n}{\mathcal{T} - \mathcal{Q}}, n = 2, 3 \cdots. \end{aligned} \tag{60}$$

Or if

$$r_0 \le \left[\frac{1-\alpha}{\mathcal{T} - \mathcal{Q}} \min\left\{\frac{L_n}{\Psi_n^q - \alpha}, \frac{M_n}{\Psi_n^q + \alpha}\right\}\right]^{1/(n-1)}. \tag{61}$$

□

It follows that the radius $r\mathcal{H}^*_q(\alpha)(\mathcal{T}\mathcal{S}^0_{\mathcal{H}}(m, q, \alpha))$ in (57).

Similarly, we may find the radius of q-convexity of order α for functions $f = h + \bar{g} \in \mathcal{T}\mathcal{S}^0_{\mathcal{H}}(m, q, \alpha)$, which is as below:

Theorem 13. *Let $0 \le \alpha < 1$, and L_n and M_n are defined by (33) and (34). Then*

$$\begin{aligned} & r\mathcal{H}^c_q(\alpha)\left(\mathcal{T}\mathcal{S}^0_{\mathcal{H}}(m, q, \alpha)\right) \\ & = \inf_{n\ge 2}\left[\frac{1-\alpha}{(\mathcal{T} - \mathcal{Q})\Psi_n^q} \min\left\{\frac{L_n}{\Psi_n^q - \alpha}, \frac{M_n}{\Psi_n^q + \alpha}\right\}\right]^{1/(n-1)}, \end{aligned} \tag{62}$$

where Ψ_n^q is defined by (19).

Theorem 14. *$f = h + \bar{g} \in \mathcal{T}\mathcal{S}^0_{\mathcal{H}}(m, q, \alpha)$ be of the form (25). Then if and only if*

$$f(z) = \sum_{n=1}^{\infty}\{x_n h_n(z) + y_n g_n(z)\}, \tag{63}$$

where

$$\begin{aligned} & h_1(z) = z, \\ & h_n(z) = z - \frac{\mathcal{T} - \mathcal{Q}}{L_n} z^n, \\ & g_1(z) = z, \\ & g_n(z) = z - \frac{\mathcal{T} - \mathcal{Q}}{M_n} \bar{z}^n, \text{for } n = 2, 3, \cdots, \\ & x_n, y_n \ge 0, \\ & x_1 = 1 - \sum_{n=2}^{\infty} x_n - \sum_{n=2}^{\infty} y_n. \end{aligned} \tag{64}$$

In particular the points h_n and g_n are called the extreme points of the closed convex hull of the class $\mathcal{T}\mathcal{S}^0_{\mathcal{H}}(m, q, \alpha)$, denoted by clco $\mathcal{T}\mathcal{S}^0_{\mathcal{H}}(m, q, \alpha)$.

Proof. Let f be given by (63). Then from (64), it is of the form

$$f(z)=z-\sum_{n=2}^{\infty}x_n\left(\frac{\mathcal{T}-\mathcal{Q}}{L_n}\right)z^n+(-1)^m\sum_{n=2}^{\infty}y_n\left(\frac{\mathcal{T}-\mathcal{Q}}{M_n}\right)\bar{z}^n, \tag{65}$$

which by Theorem 7 proves that $f\in\mathcal{T}\mathcal{S}_{\mathcal{H}}^{0}(m,q,\alpha)$. Since for this function

$$\begin{aligned}\sum_{n=2}^{\infty}&\left(L_nx_n\left(\frac{\mathcal{T}-\mathcal{Q}}{L_n}\right)+M_ny_n\left(\frac{\mathcal{T}-\mathcal{Q}}{M_n}\right)\right)\\&=(\mathcal{T}-\mathcal{Q})\sum_{n=2}^{\infty}\{x_n+y_n\}\\&=(\mathcal{T}-\mathcal{Q})(1-x_1-y_1)\le\mathcal{T}-\mathcal{Q}.\end{aligned} \tag{66}$$

Conversely, let $f=h+\bar{g}\in\mathcal{T}\mathcal{S}_{\mathcal{H}}^{0}(m,q,\alpha)$ and set

$$\begin{aligned}x_n&=\frac{L_n}{\mathcal{T}-\mathcal{Q}}|a_n|,\\x_n&=\frac{M_n}{\mathcal{T}-\mathcal{Q}}|b_n|.\end{aligned} \tag{67}$$

Then on using (64), we obtain

$$\begin{aligned}f(z)&=z-\sum_{n=2}^{\infty}|a_n|z^n+(-1)^m\sum_{n=2}^{\infty}|b_n|\bar{z}^n\\&=z-\sum_{n=2}^{\infty}x_n\left(\frac{\mathcal{T}-\mathcal{Q}}{L_n}\right)z^n+(-1)^m\sum_{n=2}^{\infty}y_n\left(\frac{\mathcal{T}-\mathcal{Q}}{M_n}\right)\bar{z}^n\\&=z-\sum_{n=2}^{\infty}x_n(z-h_n(z))+\sum_{n=2}^{\infty}y_n(g_n(z)-z)\\&=\left\{1-\sum_{n=2}^{\infty}(x_n+y_n)\right\}z+\sum_{n=2}^{\infty}\{x_nh_n(z)+y_ng_n(z)\},\end{aligned} \tag{68}$$

which is of the form (63). This proofs Theorem 14. □

Corollary 15. *Let $f\in\mathcal{T}\mathcal{S}_{\mathcal{H}}^{0}(m,q,\alpha)$ be of the form (25). Then*

$$\begin{aligned}|a_n|&\le\frac{\mathcal{T}-\mathcal{Q}}{L_n},\\|b_n|&\le\frac{\mathcal{T}-\mathcal{Q}}{M_n},\\n&=2,3,4\cdots,\end{aligned} \tag{69}$$

where L_n and M_n are defined, respectively, by (33) and (34). Equality in the inequalities (69) occurs for the extremal functions $h_n(z)$ and $g_n(z)$ given in (64) for $n=1,2,3$.

3. Conclusion

In this paper, we defined a new class $\mathcal{S}_{\mathcal{H}}^{0}(m,q,\alpha)$ of harmonic functions $f\in\mathcal{H}^0$ associated with newly defined q-Noor integral operator for harmonic functions f. In this class, we proved necessary and sufficient convolution condition for the functions $f\in\mathcal{H}^0$. We proved that, sufficient coefficient condition for the functions $f\in\mathcal{H}^0$ to be sense preserving and univalent and also this coefficient condition is necessary for subclass $\mathcal{T}\mathcal{S}_{\mathcal{H}}^{0}(m,q,\alpha)$. By using the necessary and sufficient coefficient condition, we obtained results based on the convexity and compactness and results on the radii of q-starlikeness and q-convexity of order α in the class $\mathcal{T}\mathcal{S}_{\mathcal{H}}^{0}(m,q,\alpha)$. Also we investigated extreme points for the functions $f\in\mathcal{T}\mathcal{S}_{\mathcal{H}}^{0}(m,q,\alpha)$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that they have no competing interests.

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