Research Article

Certain New Class of Harmonic Functions Involving Quantum Calculus

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Here in this paper, we are using the concepts of q-calculus operator theory associated with harmonic functions and define the q-Noor integral operator for harmonic functions \( f \in \mathcal{H}^0 \). We investigate a new class \( \mathcal{S}^q_{\alpha}(m, q, \alpha) \) of harmonic functions \( f \in \mathcal{H}^0 \). In this class, we prove a necessary and sufficient convolution condition for the functions \( f \in \mathcal{H}^0 \) and also we proved that this sufficient coefficient condition is sense preserving and univalent in the class \( \mathcal{S}^q_{\alpha}(m, q, \alpha) \). It is proved that this coefficient condition is necessary for the functions in its subclass \( \mathcal{T} \mathcal{S}^q_{\alpha}(m, q, \alpha) \). By using this necessary and sufficient coefficient condition, we obtained results based on the convexity and compactness and results on the radii of q-starlikeness and q-convexity of order \( \alpha \) in the class \( \mathcal{T} \mathcal{S}^q_{\alpha}(m, q, \alpha) \). Also we obtained extreme points for the functions in the class \( \mathcal{T} \mathcal{S}^q_{\alpha}(m, q, \alpha) \).

1. Introduction and Definitions

A complex-valued function \( f = u + iv \) is said to be harmonic in in open unit disc \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) if both \( u \) and \( v \) are real valued harmonic functions in \( U \). Also the complex-valued harmonic function \( f = u + iv \) can also be expressed as \( f = h + \bar{g} \), where \( h \) and \( g \) are analytic in \( U \). In particular, \( h \) is called analytic part, and \( g \) is called coanalytic part of \( f \). The Jacobian of the function \( f = u + iv \) is given by

\[
\mathcal{J}_f(z) = |h'(z)|^2 - |g'(z)|^2.
\]

(1)

It is known (see [1]) that every harmonic function \( f = h + \bar{g} \) to be locally univalent and sense preserving in \( U \) if and only if \( \mathcal{J}_f(z) > 0 \) in \( U \) which is equivalent to \( u(z) = (g'(z))/h'(z) \) in \( U \) such that

\[
|u(z)| < 1, \text{ for all } z \in U.
\]

(2)

For detail (see [2]). Let \( \mathcal{H} \) indicates the class of harmonic functions in \( U \). Also let \( \mathcal{H}^0 \) denoted by the family of harmonic functions \( f = h + \bar{g} \in \mathcal{H} \) which have the series expansion of the form:

\[
f(z) = z + \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n, \quad (z \in U),
\]

(3)

where \( h \) and \( g \) are analytic functions with the following series expansion:

\[
h(z) = z + \sum_{n=1}^{\infty} a_n z^n,
\]

(4)

\[
g(z) = \sum_{n=1}^{\infty} b_n z^n,
\]

\[
|b_n| < 1, \quad (z \in U).
\]

The series defined in (3) and (4) are convergent in the open unit disc \( U \). Also let \( \mathcal{S} \) represents all functions (say \( f \in \mathcal{S} \)) which are univalent analytic in \( U \) and satisfy the condition

\[
f(0) = f'(0) - 1 = 0.
\]

(5)
Further, let $\mathcal{S}_H^*$ denote the class of all harmonic functions $f = h + \bar{g} \in H^0$ which are sense preserving and univalent in $U$. The class $\mathcal{S}_H^*$ reduces to the class $\mathcal{S}$ if coanalytic part of $f$ is zero.

Clunie and Small [3] and Small [4] studied the class $\mathcal{S}_H^*$ along with some of their subfamilies. Particularly, they explored and studied the families of starlike harmonic and convex harmonic functions in $U$, which are given as follows:

$$\mathcal{S}_H^* = \left\{ f \in \mathcal{S}_H : \frac{\partial \mathcal{H}_f(z)}{f(z)} < \frac{1+z}{1-z}, (z \in U) \right\},$$

$$\mathcal{S}_H^* = \left\{ f \in \mathcal{S}_H : \frac{\partial \mathcal{H}_f(z)}{f(z)} \in \mathcal{S}_H^*(z \in U) \right\},$$

where

$$\partial \mathcal{H}_f(z) = z h'(z) - z g'(z).$$

In [5], Dziok introduced a new family $\mathcal{S}_H^*(L, M)$, $L, M \in \mathbb{C}$, and $L \neq M$ of Janowski harmonic functions and defined by

$$\mathcal{S}_H^*(L, M) = \left\{ f \in \mathcal{S}_H : \frac{\partial \mathcal{H}_f(z)}{f(z)} < \frac{1+Lz}{1+Mz}, (z \in U) \right\},$$

where $\partial \mathcal{H}_f(z)$ is given by (5). We can see that

$$\mathcal{S}_H^*(1, -1) = \mathcal{S}_H^*.$$  

The convolution of two functions $h, g \in \mathcal{S}_H^*$ is defined by

$$(h \ast g)z = \sum_{n=1}^{\infty} a_n b_n z^n,$$

where

$$h(z) = \sum_{n=1}^{\infty} a_n z^n,$$

$$g(z) = \sum_{n=1}^{\infty} b_n z^n.$$

Similarly, the convolution of two harmonic functions $f = h + g$ and $f_1 = h_1 + g_1$ is defined by

$$(f \ast f_1)(z) = (h \ast h_1)(z) + (g \ast g_1)(z).$$

The function $h$ subordinate to a function $g$ and write $h(z) \prec g(z), z \in U$, if there exists a complex-valued function $v$ which map $U$ into itself such that $v(0) = 0$ and $h(z) = g(v(z))$. In particular, if $g$ is univalent in $U$, then we have the following equivalence:

$$h(z) \prec g(z),$$

$$z \in U \Leftrightarrow h(0) = g(0),$$

$$h(U) \subset g(U).$$

In the nineteenth century, several mathematician has been using $q$-calculus operator theory in various area of science, such that fractional calculus, $q$-difference equation, optimal control, $q$-integral equations, and geometric function theory (GFT). In 1908, Jackson [6] introduced the $q$-derivative and $q$-integral operator and discussed some of their applications. In the year 1990, Ismail et al. [7] gave the idea of $q$-extension of class of $q$-starlike functions by implementing the $q$-calculus theory. Kanasi and Raducanu [8] used $q$-calculus operator theory and introduced the $q$-Ruscheweyh differential operator for analytic functions. Zhang et al. [9] introduced a generalized conic domain $\Omega_k(q, \gamma)$ by using the basic concepts of $q$-calculus and studied new subclass of $q$-starlike functions. Arif et al. defined $q$-Noor integral operator [10] by using the concept of convolution and used it to investigated some new subclasses of analytic functions. Further, in article [11], Khan et al. discussed some applications of $q$-derivative operator for multivalent functions, while coefficient estimates for a certain family of analytic functions involving a $q$-derivative operator were discussed by Raza et al. [12]. Recently, Srivastava et al. published few articles in which they implemented basic concepts of $q$-calculus operator theory and studied class of $q$-starlike functions from different aspects (see [13–16]). Additionally, a recently published article by Srivastava [17] is very suitable for researchers to work on this topic. For more recently, Khan et al. [18, 19] used the concepts of $q$-calculus operator theory to define some new subclasses of analytic functions. Also for more detail, we may refer to [20–25].

For, $q \in (0, 1)$, the $q$-derivative operator ($\partial_q$) of $f$ is defined as follows:

$$\partial_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, z \neq 0, 0 < q < 1,$$

$$= 1 + \sum_{n=1}^{\infty} [n]_q a_n z^{n-1}.$$  

Making use of (3) and (14), and for $n \in \mathbb{N}$, we have

$$\partial_q f(z) = \partial_q h(z) + \partial_q g(z),$$

$$= 1 + \sum_{n=1}^{\infty} [n]_q a_n z^{n-1} + \sum_{n=2}^{\infty} [n]_q \frac{[n]_q}{[n]_q} z^{n-1}, [n]_q$$

$$= (1-q)^{-1}(1-q^n).$$

For more detail (see [26, 27]).

**Definition 1** (see [10, 28]). The $q$-Noor integral operator for the analytic function $h$ is defined by

$$I_q^m h(z) = h(z) * (T_{q,m+1}(z))^{-1}, z \in U, m > -1,$$
where

\[
(T_{q, m+1}(z))^{-1} = z + \sum_{n=2}^{\infty} \frac{[n]_q}{[n]_q^{(1+m)}} a_n z^n,
\]

\[
(T_{q, m+1}(z))^{-1} = z \partial_q h(z).
\]

Thus, we have

\[
I_q^m h(z) = z + \sum_{n=2}^{\infty} \frac{[n]_q}{[n]_q^{(1+m)}} a_n z^n
\]

\[
= z + \sum_{n=2}^{\infty} \frac{[n]_q}{[n]_q^{(m+1)}} a_n z^n,
\]

where

\[
\psi_n = \frac{[n]_q}{[n]_q^{(m+1)}}.
\]

Clearly,

\[
I_q^0 h(z) = z \partial_q h(z) = \frac{z}{(1-z)(1-qz)},
\]

\[
I_q^1 h(z) = \frac{z}{1-z}.
\]

Remark 2. When \(q \rightarrow 1\), then \(q\)-Noor integral operator reduces to Noor integral operator (see [29]).

First of all Jahangiri [30] applied certain \(q\)-calculus operators to complex harmonic functions and obtained some useful results, while Porwal and Gupta discussed some application of \(q\)-calculus to harmonic univalent functions in [31]. Recently Arif et al. [27] introduced some new families of harmonic functions associated with the symmetric circular region. For some more recent investigation about harmonic univalent functions, we may refer to [32, 33]. By taking the motivation from the article Arif et al. [27], we define the \(q\)-Noor integral operator for the harmonic function \(f = h + \bar{g}\).

Definition 3. Let the \(q\)-Noor integral operator \(I_q^m\) of order \(m > 1\), for the harmonic function \(f = h + \bar{g}\) be defined as

\[
I_q^m f(z) = I_q^m h(z) + (-1)^m I_q^m g(z),
\]

where \(h(z)\) and \(g(z)\) is given by (4).

In this paper, by using the concepts of \(q\)-calculus operator theory and \(q\)-Noor integral operator for harmonic functions \(f\), we define a new class \(\Delta_q^0(m, q, \alpha)\) of harmonic functions \(f \in \mathcal{H}^0\). In this class, we prove a necessary and sufficient convolution condition for the functions \(f \in \mathcal{H}^0\) and prove that this sufficient coefficient condition is sense preserving and univalent in the class \(\Delta_q^0(m, q, \alpha)\). It is proved that this coefficient condition is necessary for the functions in its subclass \(\mathcal{F} \Delta_q^0(m, q, \alpha)\). By using this necessary and sufficient coefficient condition, we obtained results based on the convexity and compactness and results on the radii of \(q\)-starlikeness and \(q\)-convexity of order \(\alpha\) and extreme points for the functions in the class \(\mathcal{F} \Delta_q^0(m, q, \alpha)\). This research work will motivate future research to work in the area of \(q\)-calculus operators together with harmonic functions.

Definition 4. Let \(\Delta_q^0(m, q, \alpha)\) be the family of harmonic functions \(f \in \mathcal{H}^0\) that satisfy the subordination condition

\[
\frac{I_q^{m+1} f(z)}{I_q^m f(z)} < \frac{1 + \alpha z}{1 + \mathcal{F} z}, (q \in (0, 1), 0 \leq \alpha < 1, z \in U),
\]

where

\[
\mathcal{F} = \alpha (1 + q) - 1,
\]

\[
\mathcal{F} = q.
\]

Inequalities (22) is equivalent to the condition

\[
\left| \frac{I_q^{m+1} f(z)}{I_q^m f(z)} - Q \right| < 1.
\]

We denote by \(\mathcal{F} \Delta_q^0(m, q, \alpha)\) a subclass of harmonic functions \(f = h + \bar{g} \in \Delta_q^0(m, q, \alpha)\), where for \(m\), functions \(h\) and \(g\) are of the form:

\[
h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, g(z) = (-1)^m \sum_{n=2}^{\infty} |b_n| z^n, z \in U.
\]

2. Main Results

Theorem 5. Let \(f \in \mathcal{H}^0\). Then the function \(f \in \Delta_q^0(m, q, \alpha)\) if and only if

\[
I_q^m f(z) \neq 0, (\gamma \in \mathcal{C}, |\gamma| = 1, z \in U \setminus \{0\}),
\]

where

\[
\phi(z, \gamma) = \frac{(\mathcal{F} - Q)z - (1 + \mathcal{F} \xi)z^2}{(1-z)(1-qz)}
\]

\[
- \left( \frac{2z + (\mathcal{F} + Q)z(1 + \mathcal{F} \xi)z^2}{1z/1qz} \right).
\]

Proof. Let \(f = h + \bar{g} \in \mathcal{H}^0\) be of the form (3). Then the function \(f \in \Delta_q^0(m, q, \alpha)\) if and only if (22) holds or equivalently

\[
\frac{I_q^{m+1} f(z)}{I_q^m f(z) \neq 1 + \mathcal{F} \xi, (\xi \in \mathcal{C}, |\xi| = 1, z \in E(\{0\}),}
\]
which by \((21)\) is given by
\[
(1 + \mathcal{F} \zeta) \left[ I^m_q \left( I_q^m h(z) \right) + (-1)^{m+1} I^m_q \left( I_q^m g(z) \right) \right] \\
- (1 + \partial \zeta) \left[ I^m_q h(z) + (-1)^m I^m_q g(z) \right] \neq 0.
\]  

(29)

On using \((20)\), the condition \((29)\) may also be given by
\[
I^m_q h(z) \left[ \begin{array}{l}
\frac{z}{1 - z} - (1 + \partial \zeta) \frac{z}{(1 - z)(1 - qz)} \\
- (-1)^m I^m_q g(z) \left[ \begin{array}{l}
\frac{z}{1 - z} \\
+ (1 + \partial \zeta) \frac{z}{(1 - z)(1 - qz)} \\
\neq 0.
\end{array}\right]
\end{array}\right]
\]

(30)

Which on using the convolution * between two harmonic functions, we get
\[
I^m_q f(z) * \phi(z, \gamma) \neq 0,
\]

(31)

where the harmonic function \(\phi(z, \gamma)\) is given by \((27)\).

\(\square\)

**Theorem 6.** Let \(f = h + \bar{g} \in \mathcal{H}^\mathcal{O}\) be of the form \((3)\) and \(q \in \(0, 1\), 0 \leq \alpha < 1\). If
\[
\sum_{n=2}^{\infty} L_n |a_n| + M_n |b_n| \leq \mathcal{F} - \mathcal{O},
\]

(32)

where
\[
L_n = (\Psi^q_n)^m (\Psi^q_n (1 + \mathcal{F}) - (1 + \mathcal{O})),
\]

(33)

\[
M_n = (\Psi^q_n)^m (\Psi^q_n (1 + \mathcal{F}) + (1 + \mathcal{O})),
\]

(34)

where \(\Psi^q_n\) is given by \((19)\), then.

(i) the function \(f\) is locally univalent and sense-preserving as \(q \rightarrow 1\).

(ii) the function \(f \in \mathcal{D}_\mathcal{P}(m, q, \alpha)\)

Equality occurs for the function
\[
f(z) = z + \sum_{n=2}^{\infty} \mathcal{F} - \mathcal{O} \gamma_n z^n + \sum_{n=2}^{\infty} \mathcal{F} - \mathcal{O} \beta_n z^n,
\]

(35)

\[\sum_{n=2}^{\infty} (|\gamma_n| + |\beta_n|) = 1.\]

**Proof.** For part (i), it is clear that the theorem is true for the function \(f(z) = z\). Let \(f = h + \bar{g}\) and assume that there exist \(n \geq 2\) such that \(a_n \neq 0\) or \(b_n \neq 0\). Since \(\Psi^q_n > 1\), we observe from \((33)\) and \((34)\) that \(L_n \geq M_n > \Psi^q_n (\mathcal{F} - \mathcal{O})\), by which the condition \((32)\) implies the condition
\[
\sum_{n=2}^{\infty} |\Psi^q_n (|a_n| + |b_n|) < 1,
\]

\[
|\partial_q h(z) - \partial_q g(z)| \geq 1 - \sum_{n=2}^{\infty} |\Psi^q_n |^n - \sum_{n=2}^{\infty} |\Psi^q_n |^n z^n - 1 - |z| \sum_{n=2}^{\infty} |\Psi^q_n (|a_n| + |a_n|) \geq 1 - |z| > 0,
\]

(36)

in \(D\) which implies as \(q \rightarrow 1 -\) that \(|h'(z)| > |g'(z)|\) in \(D\) that is a function \(f\) is locally univalent and sense-preserving in \(D\).

For part (ii), to prove that \(f \in \mathcal{D}_\mathcal{P}(m, q, \alpha)\), we only need to show that \(f\) satisfy the condition \((24)\). Consider for \(f = h + \bar{g}\) and for \(|z| = r (0 < r < 1)\), we can write \((24)\) as
\[
|I^m_q f(z) - I^m_q f(z)| - \mathcal{F} (I^m_q f(z) - I^m_q f(z)) = \sum_{n=2}^{\infty} (\Psi^q_n)^m (\Psi^q_n - 1) a_n \beta_n - (-1)^m \sum_{n=2}^{\infty} (\Psi^q_n)^m (\Psi^q_n + 1) b_n \gamma_n
\]

\[
- (\mathcal{F} - \mathcal{O}) z + \sum_{n=2}^{\infty} (\Psi^q_n)^m (\mathcal{F} (\Psi^q_n) - \mathcal{O}) a_n \gamma_n
\]

\[
- (-1)^m \sum_{n=2}^{\infty} (\Psi^q_n)^m (\mathcal{F} (\Psi^q_n) + \mathcal{O}) b_n \gamma_n
\]

\[
\leq \sum_{n=2}^{\infty} (\Psi^q_n)^m (\Psi^q_n - 1) a_n r^n - (-1)^m \sum_{n=2}^{\infty} (\Psi^q_n)^m (\Psi^q_n + 1) b_n r^n
\]

\[
+ \mathcal{F} - \mathcal{O} r + \sum_{n=2}^{\infty} (\Psi^q_n)^m (\mathcal{F} (\Psi^q_n) - \mathcal{O}) a_n r^n
\]

\[
+ \sum_{n=2}^{\infty} (\Psi^q_n)^m (\mathcal{F} (\Psi^q_n) + \mathcal{O}) b_n r^n - \sum_{n=2}^{\infty} (L_n |a_n| + M_n |b_n|) r^n - (\mathcal{F} - \mathcal{O}) \leq 0,
\]

(37)

if the condition \((32)\) holds. This proves the condition \((24)\). This completes the proof of Theorem 6. \(\square\)

**Theorem 7.** Let \(f = h + \bar{g} \in \mathcal{H}^\mathcal{O}\) where \(h\) and \(g\) are given by \((25)\). Then \(f \in \mathcal{D}_\mathcal{P}(m, q, \alpha)\) if and only if the condition \((32)\) holds that is
\[
\sum_{n=2}^{\infty} L_n |a_n| + M_n |b_n| \leq \mathcal{F} - \mathcal{O},
\]

(38)

where \(L_n\) and \(M_n\) are given by \((33)\) and \((34)\).

**Proof.** If part is proved in Theorem 6. To prove only if part, let \(f \in \mathcal{D}_\mathcal{P}(m, q, \alpha)\). Then by the class condition \((22)\), we have from \((24)\) that for any \(z \in E\).
Theorem 9. The class \( \mathcal{F} \mathcal{D}_\mathcal{F}^0(m, q, \alpha) \) is a convex and compact subclass of the class of functions \( f = h + g \in \mathcal{H}^0 \), where \( h \) and \( g \) are given by (25) to be \( q \)-starlike and \( q \)-convex of order \( \alpha \) in \( E \) if we put \( m = 0 \) and \( m = 1 \), respectively, in (38) and are given by

\[
\sum_{n=2}^{\infty} \left\{ (\Psi_n^q - \alpha)|a_n| + (\Psi_n^q + \alpha) \right\}|b_n| \leq 1 - \alpha,
\]

and

\[
\sum_{n=2}^{\infty} \Psi_n^q \left( (\Psi_n^q - \alpha)|a_n| + (\Psi_n^q + \alpha) \right) |b_n| \leq 1 - \alpha.
\]

Proof. Let \( i = 1, 2, f_i \in \mathcal{F} \mathcal{D}_\mathcal{F}^0(m, q, \alpha) \); and let this \( m \) is of the form

\[
f_i(z) = z - \sum_{n=2}^{\infty} |a_{i,n}| z^n + (1 - m) \sum_{n=2}^{\infty} |b_{i,n}| z^n, z \in U.
\]

Then for \( 0 \leq \rho \leq 1 \)

\[
F(z) = \rho f_1(z) + (1 - \rho)f_2(z) = z - \sum_{n=2}^{\infty} (\rho|a_{1,n}| + (1 - \rho)|a_{2,n}|) z^n + (1 - m) \sum_{n=2}^{\infty} (\rho|b_{1,n}| + (1 - \rho)|b_{2,n}|) z^n.
\]

and by Theorem 7, we get \( L_n \) and \( M_n \) given by (33) and (34) that

\[
\sum_{n=2}^{\infty} \left\{ L_n |a_{i,n}| + M_n |b_{i,n}| \right\} \leq \rho(\mathcal{F} - \mathcal{Q}) + (1 - \rho)(\mathcal{F} - \mathcal{Q}) = \mathcal{F} - \mathcal{Q}.
\]

Therefore, \( F \in \mathcal{F} \mathcal{D}_\mathcal{F}^0(m, q, \alpha) \). Hence, the class \( \mathcal{F} \mathcal{D}_\mathcal{F}^0(m, q, \alpha) \) is convex.

On the other hand, if we consider \( f_i \in \mathcal{F} \mathcal{D}_\mathcal{F}^0(m, q, \alpha) \), \( i \in N = \{1, 2, 3 \cdots \} \) of the form (47), and then by Theorem 7, we get for \( L_n \) and \( M_n \) defined by (33) and (34).

\[
\sum_{n=2}^{\infty} \left\{ L_n |a_{i,n}| + M_n |b_{i,n}| \right\} \leq \mathcal{F} - \mathcal{Q}.
\]
which proves that \( f \in \mathcal{T} \mathcal{S}_R^0(m, q, \alpha) \). Therefore, the class \( \mathcal{T} \mathcal{S}_R^0(m, q, \alpha) \) is closed. This proves that class \( \mathcal{T} \mathcal{S}_R^0(m, q, \alpha) \) is compact.

\[ r - \left( \frac{\mathcal{T} - \mathcal{Q}}{(\Psi^2)^m(\{\Psi^2(1 + \mathcal{T}) - (1 + \mathcal{Q})\})} \right)^2, \]

\[ \langle f(z) \rangle < r + \left( \frac{\mathcal{T} - \mathcal{Q}}{(\Psi^2)^m(\{\Psi^2(1 + \mathcal{T}) - (1 + \mathcal{Q})\})} \right)^2. \]

Furthermore,

\[ \{ w \in \mathbb{C} : |w| < 1 - \left( \frac{\mathcal{T} - \mathcal{Q}}{(\Psi^2)^m(\{\Psi^2(1 + \mathcal{T}) - (1 + \mathcal{Q})\})} \right) \} \subset f(U). \]

**Corollary 10.** Let \( f \in \mathcal{T} \mathcal{S}_R^0(m, q, \alpha). \) Then for \( |z| = r < 1 \)

\[ r - \left( \frac{\mathcal{T} - \mathcal{Q}}{(\Psi^2)^m(\{\Psi^2(1 + \mathcal{T}) - (1 + \mathcal{Q})\})} \right)^2, \]

\[ |f(z)| < r + \left( \frac{\mathcal{T} - \mathcal{Q}}{(\Psi^2)^m(\{\Psi^2(1 + \mathcal{T}) - (1 + \mathcal{Q})\})} \right)^2. \]

**Remark 11.** The minimum of all values of the radius \( r \in (0, 1) \) for functions \( f \in \mathcal{T} \mathcal{S}_R^0(m, q, \alpha) \) such that

\[ \frac{f(rz)}{r} \in \mathcal{H}^*(\alpha) \]

is called the radius of \( q \)-starlikeness of order \( \alpha \) and is denoted by \( \mathcal{H}^*(\alpha) \mathcal{T} \mathcal{S}_R^0(m, q, \alpha) \).

Now in next theorem, we obtain the radius of \( q \)-starlikeness of order \( \alpha \) for functions \( f \in \mathcal{T} \mathcal{S}_R^0(m, q, \alpha) \).

**Theorem 12.** Let \( 0 \leq \alpha < 1 \), and \( L_n \) and \( M_n \) are defined by (33) and (34). Then

\[ r_{0} \leq \frac{1 - \alpha}{\mathcal{T} - \mathcal{Q}} \min \left\{ \frac{L_n}{\Psi^q_n - \alpha}, \frac{M_n}{\Psi^q_n + \alpha} \right\}^{1/(n-1)}. \]

**Theorem 13.** Let \( 0 \leq \alpha < 1 \), and \( L_n \) and \( M_n \) are defined by (33) and (34). Then

\[ r_{0} \leq \inf_{n \geq 2} \left\{ \frac{1 - \alpha}{\mathcal{T} - \mathcal{Q}} \min \left\{ \frac{L_n}{\Psi^q_n - \alpha}, \frac{M_n}{\Psi^q_n + \alpha} \right\}^{1/(n-1)} \right\}, \]

where \( \Psi^q_n \) is defined by (19).

**Theorem 14.** \( f = h + \bar{g} \in \mathcal{T} \mathcal{S}_R^0(m, q, \alpha) \) be of the form (25). Then if and only if

\[ f(z) = \sum_{n=1}^{\infty} \left\{ x_n h_n(z) + y_n g_n(z) \right\}, \]

where

\[ h_n(z) = z, \quad g_n(z) = z - \frac{\mathcal{T} - \mathcal{Q}}{M_n} z^n, \]

for \( n = 2, 3, \ldots \).

In particular the points \( h_n \) and \( g_n \) are called the extreme points of the closed convex hull of the class \( \mathcal{T} \mathcal{S}_R^0(m, q, \alpha) \), denoted by \( \text{clco} \mathcal{T} \mathcal{S}_R^0(m, q, \alpha) \).
Proof. Let \( f \) be given by (63). Then from (64), it is of the form

\[
f(z) = z - \sum_{n=2}^{\infty} x_n \left( \frac{T_n^\infty}{L_n} \right) z^n + (-1)^m \sum_{n=2}^{\infty} y_n \left( \frac{T_n^\infty}{M_n} \right) z^n,
\]

which by Theorem 7 proves that \( f \in \mathcal{D}^0_{\mathcal{D}}(m, q, \alpha) \). Since for this function

\[
\sum_{n=2}^{\infty} \left( \frac{T_n^\infty}{L_n} \right) + M_n y_n \left( \frac{T_n^\infty}{M_n} \right) = (T - \alpha) \sum_{n=2}^{\infty} \{ x_n + y_n \}
\]

\[
= (T - \alpha) (1 - x_1 - y_1) \leq T - \alpha.
\]

Conversely, let \( f = h + g \in \mathcal{D}^0_{\mathcal{D}}(m, q, \alpha) \) and set

\[
x_n = \frac{L_n}{T - \alpha} |a_n|, \quad x_n = \frac{M_n}{T - \alpha} |b_n|.
\]

Then on using (64), we obtain

\[
f(z) = z - \sum_{n=2}^{\infty} x_n z^n + (-1)^m \sum_{n=2}^{\infty} y_n z^n
\]

\[
= z - \sum_{n=2}^{\infty} x_n \left( \frac{T_n^\infty}{L_n} \right) z^n + (-1)^m \sum_{n=2}^{\infty} y_n \left( \frac{T_n^\infty}{M_n} \right) z^n
\]

\[
= z - \sum_{n=2}^{\infty} x_n (z - h_n(z)) + \sum_{n=2}^{\infty} y_n (g_n(z) - z)
\]

\[
= \left\{ 1 - \sum_{n=2}^{\infty} (x_n + y_n) \right\} z + \sum_{n=2}^{\infty} \left\{ x_n h_n(z) + y_n g_n(z) \right\},
\]

which is of the form (63). This proofs Theorem 14. \( \square \)

Corollary 15. Let \( f \in \mathcal{D}^0_{\mathcal{D}}(m, q, \alpha) \) be of the form (25). Then

\[
|a_n| \leq \frac{T_n^\infty}{L_n},
\]

\[
|b_n| \leq \frac{T_n^\infty}{M_n},
\]

\[
n = 2, 3, 4 \cdots,
\]

where \( L_n \) and \( M_n \) are defined, respectively, by (33) and (34). Equality in the inequalities (69) occurs for the extremal functions \( h_n(z) \) and \( g_n(z) \) given in (64) for \( n = 1, 2, 3 \).

3. Conclusion

In this paper, we defined a new class \( \mathcal{D}^0_{\mathcal{D}}(m, q, \alpha) \) of harmonic functions \( f \in \mathcal{H}^0 \) associated with newly defined \( q \)-Noor integral operator for harmonic functions \( f \). In this class, we proved necessary and sufficient convolution condition for the functions \( f \in \mathcal{H}^0 \). We proved that, sufficient coefficient condition for the functions \( f \in \mathcal{H}^0 \) to be sense preserving and univalent and also this coefficient condition is necessary for subclass \( \mathcal{D}^0_{\mathcal{D}}(m, q, \alpha) \). By using the necessary and sufficient coefficient condition, we obtained results based on the convexity and compactness and results on the radii of \( q \)-starlikeness and \( q \)-convexity of order \( \alpha \) in the class \( \mathcal{D}^0_{\mathcal{D}}(m, q, \alpha) \). Also we investigated extreme points for the functions \( f \in \mathcal{D}^0_{\mathcal{D}}(m, q, \alpha) \).

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that they have no competing interests.

References


