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Research Article

On Analytical Solution of Time-Fractional Biological Population Model by means of Generalized Integral Transform with Their Uniqueness and Convergence Analysis

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This research utilizes the generalized integral transform and the Adomian decomposition method to derive a fascinating explicit pattern for outcomes of the biological population model (BPM). It assists us in comprehending the dynamical technique of demographic variations in BPMs and generates significant projections. Besides that, generalized integral transforms are the unification of other existing transforms. To investigate the closed form solutions, we employed a fractional complex transform to deal with a partial differential equation of fractional order and a generalized decomposition method was applied to analyze the nonlinear equation. Several aspects of the Caputo and Atangana–Baleanu fractional derivative operators are discussed with the aid of a generalized integral transform. In mathematical terms, the variety of equations and their solutions have been discovered and identified with various novel features of the projected model. To provide additional context for these ideas, numerous sorts of illustrations and tabulations are presented. The precision and efficacy of the proposed technique suggest that it can be used for a variety of nonlinear evolutionary problems.

1. Introduction

Historically, a framework of nonlinear developmental equations was designed to model the proportion of a demographic in particular domains [1, 2]. Several scholars have examined analytical, semiautomatic, and numerical solutions to fractional systems in a variety of fields, including virology, chaos, bifurcation, thermodynamics, neural networks, random walks, image processing, aquifer and anomalous spreading, and so on [3–5]. These behaviors have been described by expounding fractional numerical simulations compared with experimental findings in order to assert their nonlocal features, when this sort of characteristic [6–10] cannot be articulated using nonlinear PDEs of integer order.

Numerous efficient and comprehensive approaches, such as the tan-cot function method [11], the Adomian decomposition method [12], the homotopy perturbation method [13], the homotopy analysis method [14], the wavelet method [15], and the Lie symmetry analysis [16], have been constructed determined by the flexibility to form complex nonlinear phenomena in diversified disciplines such as diseases, optical fibers, fluid flow, thermodynamics, electrostatics, reaction-diffusion, and plasma physics.

Population dynamics simulations are being used to comprehend, interpret, and forecast the movements and permanence of biodiversity. Such frameworks are used to evaluate a population's welfare, evaluate the reasons for population decreases or rapid expansion, recommend corporate agendas, and assess the forecasting of a population's anticipated reactions to diverse organizational processes from a managerial perspective.

Leading up to delving into the step-by-step methodologies for formulating and interpreting continuously BPMs, we will take a look at population genetic sculpting in the past, granting us an overview of the key figures in the field of ecology and evolution as well as the techniques they formed to comprehend population systems from a physics perspective.

In this study, we investigate a fundamental model in biology. The degenerate parabolic equation appears in the spatial diffusion of biological populations [17, 18]:

$$\mathcal{Q}_{\varsigma} = \mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1} + \mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2} + \sigma(\mathcal{Q}), \quad \varsigma \geq 0, \mathbf{w}_1, \mathbf{w}_2 \in \mathbb{R}, \tag{1}$$

subject to initial condition (IC) $\mathcal{Q}(\mathbf{w}_1, \mathbf{w}_2, 0)$, where $\mathcal{Q}$ signifies the population density and σ denotes the population supply due to births and deaths.

Certain specific features of (1), for instance, Hölder estimates of its solutions, are investigated in [19]. Furthermore, two basic examples of constitutive formulations for σ are the Malthusian law [17],

$$\sigma = \nu\mathcal{Q}, \quad (\nu = \text{constant}), \tag{2}$$

and the Verhulst law [19],

$$\sigma = \nu\mathcal{Q} - \gamma\mathcal{Q}^2, \quad (\nu, \gamma = \text{constant}). \tag{3}$$

Thus, we assume the more generic version of σ as $\sigma(\mathcal{Q}) = \hbar\mathcal{Q}^{\alpha}(1 - r_1\mathcal{Q}^{\beta})$ which is held for porous media [20, 21] as follows:

$$\mathcal{Q}_{\varsigma} = \mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1} + \mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2} + \hbar\mathcal{Q}^{\alpha}\left(1 - r_1\mathcal{Q}^{\beta}\right), \quad \varsigma \geq 0, \mathbf{w}_1, \mathbf{w}_2 \in \mathbb{R}, \tag{4}$$

where $\alpha, \beta, \hbar$, and r_1 are real constants.

It is noted that Malthusian and Verhulst laws are the particular cases that can be attained by inserting $\hbar = \nu, \alpha = 1$, and $r_1 = 0$ and $\hbar = \nu, \alpha = \beta = 1$, and $r_1 = \gamma/\nu$, respectively.

FDEs are remarkably appropriate for simulating biochemical mechanisms because they are specifically apprehensive about biological analytical models' memory, which appears to be a substantial advancement over traditional integer-order mathematical methodologies, and it is associated with chaotic systems, which are prevalent in BPMs. Rashid et al. [22] recently considered the fractional spatial diffusion of a BPM via a new integral transform in the singular and nonsingular kernel settings. Zellal and Belghaba [23] reported the variational iteration method to find an accurate algorithm for solving BPM. Singh [24] expounded the analysis of the fractional blood alcohol model with a composite fractional derivative. An epidemic model SEI_1I_2R for the transmission of HIV epidemics by the mean value theorem was investigated by Naik et al. [25]. A homotopy decomposition method was employed by investigating the HIV infection of $CD4^+$ by Atangana and Alabaraoye [26]. For more details on BPM, see [27, 28] and the citations therein.

Amidst George Adomian's massive boost in 1980, the Adomian decomposition method introduced a well-noted terminology. It has been intensively implemented for a diverse set of nonlinear PDEs, for instance, the Korteweg–De Vries model [29], Fisher's model [30], Zakharov–Kuznetsov equation [31], and so on. The ADM was determined to be significantly related to a variety of integral transforms, including Laplace, Swai, Mohand, Aboodh, Elzaki, and others. Very recently, Jafari [32] propounded a well-known integral transform which is known to generalized integral transform. The dominant feature of this transformation is that it has the ability to recapture several existing transformations (see Remark 1).

Motivated by the above propensity, we aim to establish a semianalytical approach by mingling the generalized integral transform with the Adomian decomposition method. With the assistance of fractional derivative operators, we constructed the approximate analytical solution to BPM. Furthermore, the convergence and uniqueness analysis is carried out in the Caputo fractional derivative framework. The proposed findings are in close harmony with the exact solutions. Sketching and comparison analysis solutions are drafted with a powerful and pragmatic approach. Both operators consistently behave according to the projected method.

2. Preliminaries

In this section, we evoke some essential concepts, notions, and definitions concerning fractional derivative operators depending on power and Mittag-Leffler as a kernel, along with the detailed consequences of the generalized integral transform.

Definition 1 (see [6]). The Caputo fractional derivative (CFD) is described as follows:

$${}_0^c\mathbf{D}_{\varsigma}^{\chi} = \begin{cases} \dfrac{1}{\Gamma(r-\chi)} \displaystyle\int_0^{\varsigma} \dfrac{\mathcal{Q}^{(r)}(w_1)}{(\varsigma - \mathbf{w}_1)^{\chi+1-r}} d\mathbf{w}_1, & r - 1 < \chi < r, \\ \dfrac{d^r}{d\varsigma^r}\mathcal{Q}(\varsigma), & \chi = r. \end{cases} \tag{5}$$

Definition 2 (see [7]). The Atangana–Baleanu fractional derivative operator in the Caputo form (ABC) is stated as follows:

$${}^{ABC}_{\eta_1}\mathbf{D}_{\varsigma}^{\chi}(\mathcal{Q}(\varsigma)) = \frac{\mathbb{A}(\chi)}{1-\chi}\int_{\eta_1}^{\varsigma} \mathcal{Q}'(\varsigma) E_{\chi}\left[-\frac{\chi(\varsigma - w_1)^{\chi}}{1-\chi}\right] d\mathbf{w}_1, \tag{6}$$

where $\mathcal{Q} \in \mathcal{H}^1(a_1, a_2)$ (Sobolev space), $a_1 < a_2, \chi \in [0, 1]$, and $\mathbb{A}(\chi)$ signifies a normalization function as $\mathbb{A}(\chi) = \mathbb{A}(0) = \mathbb{A}(1) = 1$.

Definition 3 (see [7]). The fractional integral of the ABC operator is described as follows:

$${}^{ABC}_{\eta_1}\mathcal{I}_{\varsigma}^{\chi}(\mathcal{Q}(\varsigma)) = \frac{1-\chi}{\mathbb{A}(\chi)}\mathcal{Q}(\varsigma) + \frac{\chi}{\Gamma(\chi)\mathbb{A}(\chi)}\int_{\eta_1}^{\varsigma} \mathcal{Q}(w_1)(\varsigma - w_1)^{\chi-1} d\mathbf{w}_1. \tag{7}$$

Definition 4 (see [32]). Consider an integrable mapping $\mathcal{Q}(\varsigma)$ defined on a set $\mathcal{P}$; then,

$$\mathcal{P} = \{\mathcal{Q}(\varsigma) \colon \exists M > 0, \kappa > 0, |\mathcal{Q}(\varsigma)| < M \exp(\kappa\varsigma), \quad \text{if } \varsigma \geq 0\}. \tag{8}$$

Definition 5 (see [32]). Suppose the mappings $\phi(\mathfrak{s}), \psi(\mathfrak{s})\colon \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that $\varphi(\mathfrak{s}) \neq 0 \,\forall \mathfrak{s} \in \mathbb{R}^+$. The generalized integral transform of the mapping $\mathcal{Q}(\varsigma)$ presented by $\mathbf{Q}(\mathfrak{s})$ is described as

$$\mathbb{J}\{\mathcal{Q}(\varsigma), \mathfrak{s}\} = \mathbf{Q}(\mathfrak{s}) = \phi(s_1)\int_0^\infty \mathcal{Q}(\varsigma)\exp(-\psi(\mathfrak{s})\varsigma)d\varsigma. \tag{9}$$

Theorem 1 (see [32]) (convolution property). *For generalized integral transform, the subsequent holds true:*

$$\mathbb{J}\{\mathcal{Q}_1 * \mathcal{Q}_2\} = \frac{1}{\phi(\mathfrak{s})}\mathbf{Q}_1(\mathfrak{s}) * \mathbf{Q}_2(\mathfrak{s}). \tag{10}$$

Definition 6. The generalized integral transform of the CFD operator is stated as follows:

$$\mathbb{J}\left\{{}_0^c\mathbf{D}_\varsigma^\chi(\mathcal{Q}(\varsigma)), \mathfrak{s}\right\} = \psi^\chi(\mathfrak{s})\mathbf{Q}(s_1) - \phi(\mathfrak{s})\sum_{\kappa=0}^{\chi-1}\psi^{\chi-\kappa-1}(s_1)\mathcal{Q}^{(\kappa)}(0),$$
$$r-1 < \chi < r, \phi, \psi > 0. \tag{11}$$

Remark 1. Definition 6 leads to the following conclusions:

(1) Taking $\phi(\mathfrak{s}) = 1$ and $\psi(\mathfrak{s}) = \mathfrak{s}$, then we acquire the Laplace transform [33]
(2) Taking $\phi(\mathfrak{s})(1/\mathfrak{s})$ and $\psi(\mathfrak{s}) = (1/\mathfrak{s})$, then we acquire the α-Laplace transform [34]
(3) Taking $\phi(\mathfrak{s}) = (1/\mathfrak{s})$ and $\psi(\mathfrak{s}) = (1/\mathfrak{s})$, then we acquire the Sumudu transform [35]
(4) Taking $\phi(\mathfrak{s}) = (1/\mathfrak{s})$ and $\psi(\mathfrak{s}) = 1$, then we acquire the Aboodh transform [36]
(5) Taking $\phi(\mathfrak{s}) = \mathfrak{s}$ and $\psi(\mathfrak{s}) = \mathfrak{s}^2$, then we acquire the Pourreza transform [37, 38]
(6) Taking $\phi(\mathfrak{s}) = \mathfrak{s}$ and $\psi(\mathfrak{s}) = (1/\mathfrak{s})$, then we acquire the Elzaki transform [39]
(7) Taking $\phi(\mathfrak{s}) = \mathbf{w}_2$ and $\psi(\mathfrak{s}) = (\mathfrak{s}/\mathbf{w}_2)$, then we acquire the natural transform [40]
(8) Taking $\phi(\mathfrak{s}) = \mathfrak{s}^2$ and $\psi(\mathfrak{s}) = \mathfrak{s}$, then we acquire the Mohand transform [41]
(9) Taking $\phi(\mathfrak{s}) = (1/\mathfrak{s}^2)$ and $\psi(\mathfrak{s}) = (1/\mathfrak{s})$, then we acquire the Swai transform [42]
(10) Taking $\phi(\mathfrak{s}) = 1$ and $\psi(\mathfrak{s}) = (1/\mathfrak{s})$, then we get the Kamal transform [43]
(11) Taking $\phi(\mathfrak{s}) = \mathfrak{s}^\alpha$ and $\psi(\mathfrak{s}) = (1/\mathfrak{s})$, then we acquire the $G\_$ transform [44, 45]

Definition 7 (see [46]). The generalized integral transform of the ABC fractional derivative operator is described as

$$\mathbb{J}\left\{{}_{\eta_1}^{ABC}\mathbf{D}_\varsigma^\chi(\mathcal{Q}(\varsigma)), \mathfrak{s}\right\}(\chi) = \frac{\mathbb{A}(\chi)\psi^\chi(\mathfrak{s})}{\chi + (1-\chi)\psi^\chi(\mathfrak{s})}\left(\mathbf{Q}(\mathfrak{s}) - \frac{\phi(\mathfrak{s})}{\psi(\mathfrak{s})}\mathcal{Q}(0)\right). \tag{12}$$

Remark 2. Definition 7 leads to the following conclusions:

(1) Taking $\phi(\mathfrak{s}) = 1$ and $\psi(\mathfrak{s}) = \mathfrak{s}$, then we acquire the Laplace transform of ABC fractional derivative operator [7, 47]
(2) Taking $\phi(\mathfrak{s}) = \mathfrak{s}$ and $\psi(\mathfrak{s}) = (1/\mathfrak{s})$, then we acquire the Elzaki transform of ABC fractional derivative operator [48]
(3) Taking $\phi(\mathfrak{s}) = \psi(\mathfrak{s}) = (1/\mathfrak{s})$, then we get the Sumudu transform of ABC fractional derivative operator [49]
(4) Taking $\phi(\mathfrak{s}) = 1$ and $\psi(\mathfrak{s}) = \mathfrak{s}/\mathbf{w}_2$, then we get the Shehu transform of ABC fractional derivative operator [49]

Definition 8 (see [50]). The Mittag-Leffler function for single parameter is described as

$$E_\chi(z) = \sum_{\kappa=0}^{\infty}\frac{z_1^\kappa}{\Gamma(\kappa\chi + 1)}, \quad \chi, z_1 \in \mathbb{C}, \mathfrak{R}(\chi) \geq 0. \tag{13}$$

3. Description of the Generalized Decomposition Method

Consider the generic fractional form of PDE:

$$\mathbf{D}_\varsigma^\chi\mathcal{Q}(w_1,\varsigma) + \mathcal{L}\mathcal{Q}(\mathbf{w}_1,\varsigma) + \tilde{N}\mathcal{Q}(\mathbf{w}_1,\varsigma) = \mathcal{F}(\mathbf{w}_1,\varsigma), \quad \varsigma > 0, 0 < \chi \leq 1, \tag{14}$$

with ICs

$$\mathcal{Q}(\mathbf{w}_1, 0) = \mathcal{G}(\mathbf{w}_1), \tag{15}$$

where $\mathbf{D}_\varsigma^\chi = (\partial^\chi\mathcal{Q}(\mathbf{w}_1,\varsigma)/\partial\varsigma^\chi)$ symbolizes the Caputo and ABC fractional derivative of order $\chi \in (0,1]$, while $\mathfrak{L}$ and $\tilde{N}$ denote the linear and nonlinear factors, respectively. Also, $\mathcal{F}(\mathbf{w}_1,\varsigma)$ represents the source term.

Taking into account the generalized integral transform to (14), we acquire

$$\mathbb{J}\left[\mathbf{D}_\varsigma^\chi\mathcal{Q}(\mathbf{w}_1,\varsigma) + \mathcal{L}\mathcal{Q}(\mathbf{w}_1,\varsigma) + \tilde{N}\mathcal{Q}(\mathbf{w}_1,\varsigma)\right] = \mathbb{J}[\mathcal{F}(\mathbf{w}_1,\varsigma)]. \tag{16}$$

First, by applying the differentiation rule of generalized integral transform with respect to CFD, we apply the ABC fractional derivative operator as follows:

$$\psi^{\chi}(\mathfrak{s})\mathscr{U}(\mathbf{w}_1,\mathfrak{s}) = \phi(\mathfrak{s})\sum_{\kappa=0}^{\ell-1}\psi^{\chi-1-\kappa}(\mathfrak{s})\mathcal{Q}^{(\kappa)}(0) + \mathbb{J}[\mathscr{L}\mathcal{Q}(\mathbf{w}_1,\varsigma) + \tilde{N}\mathcal{Q}(\mathbf{w}_1,\varsigma)] + \mathbb{J}[\mathscr{F}(\mathbf{w}_1,\varsigma)], \tag{17}$$

$$\frac{\psi^{\chi}(\mathfrak{s})\mathbb{A}(\chi)}{\chi+(1-\chi)\psi^{\chi}(\mathfrak{s})}\mathscr{U}(\mathbf{w}_1,\mathfrak{s}) = \frac{\phi(\mathfrak{s})}{\psi(\mathfrak{s})}\frac{\psi^{\chi}(\mathfrak{s})\mathbb{A}(\chi)}{\chi+(1-\chi)\psi^{\chi}(\mathfrak{s})}\mathcal{Q}(0) + \mathbb{J}[\mathscr{L}\mathcal{Q}(\mathbf{w}_1,\varsigma) + \tilde{N}\mathcal{Q}(\mathbf{w}_1,\varsigma)] + \mathbb{J}[\mathscr{F}(\mathbf{w}_1,\varsigma)]. \tag{18}$$

The inverse generalized integral transform of (17) and (18) yields

$$\mathcal{Q}(\mathbf{w}_1,\varsigma) = \mathbb{J}^{-1}\left[\phi(\mathfrak{s})\sum_{\kappa=0}^{\ell-1}\psi(\mathfrak{s})^{\chi-\kappa-1}\mathcal{Q}^{(\kappa)}(0) + \frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}[\mathscr{F}(\mathbf{w}_1,\varsigma)]\right] - \mathbb{J}^{-1}\left[\frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}[\mathscr{L}\mathcal{Q}(\mathbf{w}_1,\varsigma) + \tilde{N}\mathcal{Q}(\mathbf{w}_1,\varsigma)]\right], \tag{19}$$

$$\mathcal{Q}(\mathbf{w}_1,\varsigma) = \mathbb{J}^{-1}\left[\frac{\phi(\mathfrak{s})}{\psi(\mathfrak{s})}\mathcal{Q}(0) + \frac{\chi+(1-\chi)\psi^{\chi}(\mathfrak{s})}{\psi^{\chi}(\mathfrak{s})\mathbb{A}(\chi)}\mathbb{J}[\mathscr{F}(\mathbf{w}_1,\varsigma)]\right] - \mathbb{J}^{-1}\left[\frac{\chi+(1-\chi)\psi^{\chi}(\mathfrak{s})}{\psi^{\chi}(\mathfrak{s})\mathbb{A}(\chi)}\mathbb{J}[\mathscr{L}\mathcal{Q}(\mathbf{w}_1,\varsigma) + \tilde{N}\mathcal{Q}(\mathbf{w}_1,\varsigma)]\right]. \tag{20}$$

The generalized decomposition method solution $\mathcal{Q}(\mathbf{w}_1,\varsigma)$ is represented by the following infinite series:

$$\mathcal{Q}(\mathbf{w}_1,\varsigma) = \sum_{\ell=0}^{\infty}\mathcal{Q}_{\ell}(\mathbf{w}_1,\varsigma). \tag{21}$$

Thus, the nonlinear term $\tilde{N}(\mathbf{w}_1,\varsigma)$ can be evaluated by the Adomian decomposition method prescribed as

$$\tilde{N}\mathcal{Q}(\mathbf{w}_1,\varsigma) = \sum_{\ell=0}^{\infty}\tilde{A}_{\ell}(\mathcal{Q}_0,\mathcal{Q}_1,\ldots), \quad \ell = 0,1,\ldots, \tag{22}$$

where

$$\tilde{A}_{\ell}(\mathcal{Q}_0,\mathcal{Q}_1,\ldots) = \frac{1}{\ell!}\left[\frac{d^{\ell}}{d\chi^{\ell}}\tilde{N}\left(\sum_{j=0}^{\infty}\chi^{j}\mathcal{Q}_j\right)\right]_{\chi=0}, \quad \ell > 0. \tag{23}$$

Inserting (21) and (22) into (19) and (20), respectively, we have

$$\sum_{\ell=0}^{\infty}\mathcal{Q}_{\ell}(\mathbf{w}_1,\varsigma) = \mathscr{G}(\mathbf{w}_1) + \tilde{\mathscr{G}}(\mathbf{w}_1) - \mathbb{J}^{-1}\left[\frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left[\mathscr{L}\mathcal{Q}(\mathbf{w}_1,\varsigma) + \sum_{\ell=0}^{\infty}\tilde{A}_{\ell}\right]\right], \tag{24}$$

$$\sum_{\ell=0}^{\infty}\mathcal{Q}_{\ell}(\mathbf{w}_1,\varsigma) = \mathscr{G}(\mathbf{w}_1) + \tilde{\mathscr{G}}(\mathbf{w}_1) - \mathbb{J}^{-1}\left[\frac{\chi+(1-\chi)\psi^{\chi}(\mathfrak{s})}{\mathbb{A}(\chi)\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left[\mathscr{L}\mathcal{Q}(\mathbf{w}_1,\varsigma) + \sum_{\ell=0}^{\infty}\tilde{A}_{\ell}\right]\right]. \tag{25}$$

Consequently, the recursive technique for (24) and (25) is established as follows:

$$\begin{aligned}
\mathcal{Q}_0(\mathbf{w}_1,\varsigma) &= \mathscr{G}(\mathbf{w}_1) + \tilde{\mathscr{G}}(\mathbf{w}_1), \quad \ell = 0,\\
\mathcal{Q}_{\ell+1}(\mathbf{w}_1,\varsigma) &= -\mathbb{J}^{-1}\left[\frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left[\mathfrak{L}(\mathcal{Q}_{\ell}(\mathbf{w}_1,\varsigma)) + \sum_{\ell=0}^{\infty}\tilde{A}_{\ell}\right]\right], \quad \ell \geq 1,\\
\mathcal{Q}_{\ell+1}(\mathbf{w}_1,\varsigma) &= -\mathbb{J}^{-1}\left[\frac{\chi+(1-\chi)\psi^{\chi}(\mathfrak{s})}{\mathbb{A}(\chi)\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left[\mathfrak{L}(\mathcal{Q}_{\ell}(\mathbf{w}_1,\varsigma)) + \sum_{\ell=0}^{\infty}\tilde{A}_{\ell}\right]\right], \quad \ell \geq 1.
\end{aligned} \tag{26}$$

4. Some New Mathematical Aspects of Generalized Decomposition Method

The subsequent subsections will highlight how the sufficient requirements guarantee the emergence of a unique solution. Our anticipated existence of solutions in the case of GDM is followed by [51].

Theorem 2 (uniqueness theorem). *Equation (26) has a unique solution whenever* $0<\epsilon<1$, *where* $\epsilon=((\mathscr{K}_1+\mathscr{K}_2+\mathscr{K}_3))\varsigma^{(\chi)}/(\Gamma(\chi+1))$.

Proof. Assume all continuous functions on the Banach space are denoted by $\Omega=(\mathbb{C}[\mathscr{I}],\|\cdot\|)$. Also, suppose that $\mathscr{I}=[0,\mathscr{T}]$ has the norm $\|\cdot\|$. Now, we define a function $\mathscr{U}\colon\Omega\mapsto\Omega$ such that

$$\mathcal{Q}_{\ell+1}(\mathbf{w}_1,\varsigma)=\mathcal{Q}(\mathbf{w}_1,\varsigma)+\mathbb{J}^{-1}\left[\frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left[\mathfrak{L}\left[\mathcal{Q}_{\ell}(\mathbf{w}_1,\varsigma)\right]+\overline{P}\left[\mathcal{Q}_{\ell}(\mathbf{w}_1,\varsigma)\right]+\tilde{N}\left[\mathcal{Q}_{\ell}(\mathbf{w}_1,\varsigma)\right]\right]\right],\quad \ell\geq 0, \tag{27}$$

where $\mathfrak{L}[\mathcal{Q}(\mathbf{w}_1,\varsigma)]\equiv\partial^3\mathcal{Q}(\mathbf{w}_1,\varsigma)/\partial\mathbf{w}_1^2$ and $\overline{P}[\mathcal{Q}(\mathbf{w}_1,\varsigma)]\equiv\partial\mathcal{Q}(\mathbf{w}_1,\varsigma)/\partial\mathbf{w}_1$. Here, suppose that $\mathfrak{L}[\mathcal{Q}(\mathbf{w}_1,\varsigma)]$ and $\mathfrak{M}[\mathcal{Q}(\mathbf{w}_1,\varsigma)]$ are also Lipschitzian with $|\overline{P}\mathcal{Q}-\overline{P}\widehat{\mathcal{Q}}|<\mathscr{K}_1|\mathcal{Q}-\widehat{\mathcal{Q}}|$ and $|\mathscr{L}\mathcal{Q}-\mathfrak{L}\widehat{\mathcal{Q}}|<\mathscr{K}_2|\mathcal{Q}-\widehat{\mathcal{Q}}|$, where $\mathscr{K}_1$ and $\mathscr{K}_2$ are Lipschitz constant, respectively, and $\mathcal{Q}$ and $\widehat{\mathcal{Q}}$ are distinct functional values.

$$\begin{aligned}
\|\mathscr{U}\mathcal{Q}-\mathscr{U}\widehat{\mathcal{Q}}\| &= \max_{\varsigma\in\mathscr{I}}\left|\begin{matrix}\mathbb{J}^{-1}\left[\frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left[\mathfrak{L}\left[\mathcal{Q}(\mathbf{w}_1,\varsigma)\right]+\overline{P}\left[\mathcal{Q}(\mathbf{w}_1,\varsigma)\right]+\tilde{N}\left[\mathcal{Q}(\mathbf{w}_1,\varsigma)\right]\right]\right]\\ -\mathbb{J}^{-1}\left[\frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left[\mathfrak{L}\left[\widehat{\mathcal{Q}}(\mathbf{w}_1,\varsigma)\right]+\overline{P}\left[\widehat{\mathcal{Q}}(\mathbf{w}_1,\varsigma)\right]+\tilde{N}\left[\widehat{\mathcal{Q}}(\mathbf{w}_1,\varsigma)\right]\right]\right]\end{matrix}\right|\\
&\leq \max_{\varsigma\in\mathscr{I}}\left|\begin{matrix}\mathbb{J}^{-1}\left[\frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left[\mathfrak{L}\left[\mathcal{Q}(\mathbf{w}_1,\varsigma)\right]-\mathfrak{L}\left[\widehat{\mathcal{Q}}(\mathbf{w}_1,\varsigma)\right]\right]\right]\\ +\mathbb{J}^{-1}\left[\frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left[\overline{P}\left[\mathcal{Q}(\mathbf{w}_1,\varsigma)\right]-\overline{P}\left[\widehat{\mathcal{Q}}(\mathbf{w}_1,\varsigma)\right]\right]\right]\\ +\mathbb{J}^{-1}\left[\frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left[\tilde{N}\left[\mathcal{Q}(\mathbf{w}_1,\varsigma)\right]-\tilde{N}\left[\widehat{\mathcal{Q}}(\mathbf{w}_1,\varsigma)\right]\right]\right]\end{matrix}\right|\\
&\leq \max_{\varsigma\in\mathscr{I}}\left[\begin{matrix}\mathscr{K}_1\mathbb{J}^{-1}\left[\frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left|\mathcal{Q}(\mathbf{w}_1,\varsigma)-\widehat{\mathcal{Q}}(\mathbf{w}_1,\varsigma)\right|\right]\\ +\mathscr{K}_2\mathbb{J}^{-1}\left[\frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left|\mathcal{Q}(\mathbf{w}_1,\varsigma)-\widehat{\mathcal{Q}}(\mathbf{w}_1,\varsigma)\right|\right]\\ +\mathscr{K}_3\mathbb{J}^{-1}\left[\frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left|\mathcal{Q}(\mathbf{w}_1,\varsigma)-\widehat{\mathcal{Q}}(\mathbf{w}_1,\varsigma)\right|\right]\end{matrix}\right]\\
&\leq \max_{\varsigma\in\mathscr{I}}(\mathscr{K}_1+\mathscr{K}_2+\mathscr{K}_3)\mathbb{J}^{-1}\left[\frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left|\mathcal{Q}(\mathbf{w}_1,\varsigma)-\widehat{\mathcal{Q}}(\mathbf{w}_1,\varsigma)\right|\right]\\
&\leq (\mathscr{K}_1+\mathscr{K}_2+\mathscr{K}_3)\mathbb{J}^{-1}\left[\frac{1}{\psi^{\chi+1}(\mathfrak{s})}\mathbb{J}\left\|\mathcal{Q}(\mathbf{w}_1,\varsigma)-\widehat{\mathcal{Q}}(\mathbf{w}_1,\varsigma)\right\|\right]\\
&= (\mathscr{K}_1+\mathscr{K}_2+\mathscr{K}_3)\mathbb{J}^{-1}\left[\frac{\phi(\mathfrak{s})}{\psi^{\chi}(\mathfrak{s})}\left\|\mathcal{Q}(\mathbf{w}_1,\varsigma)-\widehat{\mathcal{Q}}(\mathbf{w}_1,\varsigma)\right\|\right]\\
&= \frac{((\mathscr{K}_1+\mathscr{K}_2+\mathscr{K}_3))\varsigma^{(\chi)}}{\Gamma(\chi+1)}\left\|\mathcal{Q}(\mathbf{w}_1,\varsigma)-\widehat{\mathcal{Q}}(\mathbf{w}_1,\varsigma)\right\|.
\end{aligned} \tag{28}$$

Since $0 < \epsilon < 1$, the mapping is contraction. Consequently, by Banach contraction fixed point theorem, (14) has a unique solution. This gives the desired result. □

Theorem 3 (convergence analysis). *The general form solution of (14) will be convergent.*

Proof. Assume that $\widehat{W}_\ell$ be the nth partial sum, i.e., $\widehat{W}_\ell = \sum_{m=0}^{\ell} \mathcal{Q}_\ell(\mathbf{w}_1, \varsigma)$. Here, we prove a Cauchy sequence $\{\widehat{W}_\ell\}$ in Banach space U.

We acquire by considering a new form of Adomian polynomials:

$$\begin{aligned} \overline{R}\left(\widehat{W}_\ell\right) &= \widetilde{\mathscr{H}}_\ell + \sum_{p=0}^{\ell-1} \widetilde{\mathscr{H}}_p, \\ \widetilde{N}\left(\widehat{W}_\ell\right) &= \widetilde{\mathscr{H}}_\ell + \sum_{c=0}^{\ell-1} \widetilde{\mathscr{H}}_c. \end{aligned} \tag{29}$$

Now,

$$\begin{aligned}
\left\|\widehat{W}_\ell - \widehat{W}_q\right\| &= \max_{\varsigma\in\mathcal{J}}\left|\widehat{W}_\ell - \widehat{W}_q\right| \\
&= \max_{\varsigma\in\mathcal{J}}\left|\sum_{m=q+1}^{\ell} \widehat{\mathcal{Q}}(\mathbf{w}_1, \varsigma)\right|, \quad (m = 1, 2, 3, \dots) \\
&\le \max_{\varsigma\in\mathcal{J}}\left|\begin{gathered} \mathbb{J}^{-1}\left[\frac{1}{\psi^\chi(\mathfrak{s})}\mathbb{J}\left[\sum_{m=q+1}^{\ell} \mathfrak{L}[\mathcal{Q}_{\ell-1}(\mathbf{w}_1, \varsigma)]\right]\right] \\ +\mathbb{J}^{-1}\left[\frac{1}{\psi^\chi(\mathfrak{s})}\mathbb{J}\left[\sum_{m=q+1}^{\ell} \overline{P}[\mathcal{Q}_{\ell-1}(\mathbf{w}_1, \varsigma)]\right]\right] \\ +\mathbb{J}^{-1}\left[\frac{1}{\psi^\chi(\mathfrak{s})}\mathbb{J}\left[\sum_{m=q+1}^{\ell} \widetilde{\mathscr{H}}_{\ell-1}(\mathbf{w}_1, \varsigma)\right]\right] \end{gathered}\right| \\
&= \max_{\varsigma\in\mathcal{J}}\left|\begin{gathered} \mathbb{J}^{-1}\left[\frac{1}{\psi^\chi(\mathfrak{s})}\mathbb{J}\left[\sum_{m=q}^{\ell-1} \mathfrak{L}[\mathcal{Q}_{\ell}(\mathbf{w}_1, \varsigma)]\right]\right] \\ +\mathbb{J}^{-1}\left[\frac{1}{\psi^\chi(\mathfrak{s})}\mathbb{J}\left[\sum_{m=q}^{\ell-1} \overline{P}[\mathcal{Q}_{\ell}(\mathbf{w}_1, \varsigma)]\right]\right] \\ +\mathbb{J}^{-1}\left[\frac{1}{\psi^\chi(\mathfrak{s})}\mathbb{J}\left[\sum_{m=q}^{\ell-1} \widetilde{\mathscr{H}}_{\ell}(\mathbf{w}_1, \varsigma)\right]\right] \end{gathered}\right| \\
&\le \max_{\varsigma\in\mathcal{J}}\left|\begin{gathered} \mathbb{J}^{-1}\left[\frac{1}{\psi^\chi(\mathfrak{s})}\mathbb{J}\left[\sum_{m=q}^{\ell-1} \mathfrak{L}\left(\widehat{W}_{\ell-1}\right) - \mathfrak{L}\left(\widehat{W}_{q-1}\right)\right]\right] \\ +\mathbb{J}^{-1}\left[\frac{1}{\psi^\chi(\mathfrak{s})}\mathbb{J}\left[\sum_{m=q}^{\ell-1} \overline{P}\left(\widehat{W}_{\ell-1}\right) - \overline{P}\left(\widehat{W}_{q-1}\right)\right]\right] \\ +\mathbb{J}^{-1}\left[\frac{1}{\psi^\chi(\mathfrak{s})}\mathbb{J}\left[\sum_{m=q}^{\ell-1} \widetilde{N}\left(\widehat{W}_{\ell-1}\right) - \widetilde{N}\left(\widehat{W}_{q-1}\right)\right]\right] \end{gathered}\right| \\
&\le \max_{\varsigma\in\mathcal{J}}\left|\begin{gathered} \mathbb{J}^{-1}\left[\frac{1}{\psi^\chi(\mathfrak{s})}\mathbb{J}\left[\mathfrak{L}\left(\widehat{W}_{\ell-1}\right) - \mathfrak{L}\left(\widehat{W}_{q-1}\right)\right]\right] \\ +\mathbb{J}^{-1}\left[\frac{1}{\psi^\chi(\mathfrak{s})}\mathbb{J}\left[\overline{P}\left(\widehat{W}_{\ell-1}\right) - \overline{P}\left(\widehat{W}_{q-1}\right)\right]\right] \\ +\mathbb{J}^{-1}\left[\frac{1}{\psi^\chi(\mathfrak{s})}\mathbb{J}\left[\widetilde{N}\left(\widehat{W}_{\ell-1}\right) - \widetilde{N}\left(\widehat{W}_{q-1}\right)\right]\right] \end{gathered}\right| \\
&\le \mathscr{K}_1 \max_{\varsigma\in\mathcal{J}} \mathbb{J}^{-1}\left|\left[\frac{1}{\psi^\chi(\mathfrak{s})}\mathbb{J}\left[\left(\widehat{W}_{\ell-1}\right) - \left(\widehat{W}_{q-1}\right)\right]\right]\right| \\
&\quad + \mathscr{K}_2 \max_{\varsigma\in\mathcal{J}}\left|\mathbb{J}^{-1}\left[\frac{1}{\psi^\chi(\mathfrak{s})}\mathbb{J}\left[\left(\widehat{W}_{\ell-1}\right) - \left(\widehat{W}_{q-1}\right)\right]\right]\right| \\
&\quad + \mathscr{K}_3 \max_{\varsigma\in\mathcal{J}}\left|\mathbb{J}^{-1}\left[\frac{1}{\psi^\chi(\mathfrak{s})}\mathbb{J}\left[\left(\widehat{W}_{\ell-1}\right) - \left(\widehat{W}_{q-1}\right)\right]\right]\right| \\
&= (\mathscr{K}_1 + \mathscr{K}_2 + \mathscr{K}_3)\mathbb{J}^{-1}\left[\frac{\phi(\mathfrak{s})}{\psi^{\chi+1}(\mathfrak{s})}\left\|\widehat{W}_{\ell-1} - \widehat{W}_{q-1}\right\|\right] \\
&= \frac{(\mathscr{K}_1 + \mathscr{K}_2 + \mathscr{K}_3)\varsigma^{(\chi)}}{\Gamma(\chi+1)}\left\|\widehat{W}_{\ell-1} - \widehat{W}_{q-1}\right\|.
\end{aligned} \tag{30}$$

Consider $n = q + 1$; then,

$$\left\|\widehat{W}_{q+1} - \widehat{W}_q\right\| \leq \epsilon\left\|\widehat{W}_q - \widehat{W}_{q-1}\right\| \leq \widehat{W}^2\left\|\widehat{W}_{q-1} - \widehat{W}_{q-2}\right\| \leq \cdots \leq \epsilon^q\left\|\widehat{W}_1 - \widehat{W}_0\right\|, \tag{31}$$

where $(\mathscr{K}_1 + \mathscr{K}_2 + \mathscr{K}_3)\varsigma^{(\chi)}/(\Gamma(\chi+1))$. Now, from triangular inequality, we have

$$\begin{aligned}\left\|\widehat{W}_\ell - \widehat{W}_q\right\| &\leq \left\|\widehat{W}_{q+1} - \widehat{W}_q\right\| + \left\|\widehat{W}_{q+2} - \widehat{W}_{q+1}\right\| + \cdots + \left\|\widehat{W}_\ell - \widehat{W}_{\ell-1}\right\| \\ &\leq \left[\epsilon^q + \epsilon^{q+1} + \cdots + \epsilon^{\ell-1}\right]\left\|\widehat{W}_1 - \widehat{W}_0\right\| \\ &\leq \epsilon^q\left(\frac{1-\epsilon^{\ell-q}}{\epsilon}\right)\left\|\mathcal{Q}_1\right\|.\end{aligned} \tag{32}$$

Since $0 < \epsilon < 1$, we have $(1 - \epsilon^{\ell-q}) < 1$; then,

$$\left\|\widehat{W}_\ell - \widehat{W}_q\right\| \leq \frac{\epsilon^q}{1-\epsilon}\max_{\varsigma\in\mathcal{J}}\left\|\mathcal{Q}_1\right\|. \tag{33}$$

Therefore, $|\mathcal{Q}_1| < \infty$ (since $\mathcal{Q}(\mathbf{w}_1, \varsigma)$ is bounded). Furthermore, as $q \mapsto \infty$, then $\|\widehat{W}_\ell - \widehat{W}_q\| \mapsto 0$. Thus, $\{\widehat{W}_1\}$ is a Cauchy sequence in K. Consequently, the series $\sum_{n=0}^{\infty}\mathcal{Q}_\ell$ is convergent and this yields the immediate consequence. □

Theorem 4 (see [51]) (error estimate). *The absolute error of the series solution (14)–(26) is calculated as*

$$\max_{\varsigma\in\mathcal{J}}\left|\mathcal{Q}(\mathbf{w}_1, \varsigma) - \sum_{\ell=1}^{q}\mathcal{Q}_\ell(\mathbf{w}_1, \varsigma)\right| \leq \frac{\epsilon^q}{1-\epsilon}\max_{\varsigma\in\mathcal{J}}\left\|\mathcal{Q}_1\right\|. \tag{34}$$

5. Application of Generalized Integral Transform for Biological Population Model

In what follows, we illustrate the technique described in Section 3 by considering three numerical tests to validate the supremacy and efficacy of the generalized decomposition method.

Example 1 (Malthusian law [17]). Assume the time-fractional BPM (4) having $\alpha = 1$ and $r_1 = 0$; then,

$$\mathbf{D}_\varsigma^\alpha\mathcal{Q}(\mathbf{w}_1, \mathbf{w}_2, \varsigma) = \mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1}(\mathbf{w}_1, \mathbf{w}_2, \varsigma) + \mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2}(\mathbf{w}_1, \mathbf{w}_2, \varsigma) + \hbar\mathcal{Q}(\mathbf{w}_1, \mathbf{w}_2, \varsigma), \tag{35}$$

with IC

$$\mathcal{Q}_0(\mathbf{w}_1, \mathbf{w}_2, 0) = \sqrt{\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_1\mathbf{w}_2}. \tag{36}$$

Proof. Foremost, we provide the solution of (35) in two general cases.

*Case 1*First, we apply the Caputo fractional derivative operator coupled with the generalized integral transform and Adomian decomposition method. Applying the generalized integral transform for Example 1,

$$\psi^\chi(\mathfrak{s})\mathcal{U}(\mathbf{w}_1, \mathfrak{s}) - \phi(\mathfrak{s})\sum_{\kappa=0}^{m-1}\psi^{\chi-\kappa-1}(\mathfrak{s})\mathcal{Q}^{(\kappa)}(0) = \mathbb{J}\left[\mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1}(\mathbf{w}_1, \mathbf{w}_2, \varsigma) + \mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2}(\mathbf{w}_1, \mathbf{w}_2, \varsigma) + \hbar\mathcal{Q}(\mathbf{w}_1, \mathbf{w}_2, \varsigma)\right]. \tag{37}$$

Taking into consideration the IC given in (36), we have

$$\mathscr{U}(\mathbf{w}_1,\mathfrak{s}) = \frac{\psi(\mathfrak{s})}{\phi(\mathfrak{s})}\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,0) + \frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left[\mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) + \mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) + \hbar\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)\right]. \tag{38}$$

Employing the inverse generalized integral transform, we obtain

$$\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) = \mathbb{J}^{-1}\left[\frac{\psi(\mathfrak{s})}{\phi(\mathfrak{s})}\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,0) + \frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left[\mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) + \mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) + \hbar\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)\right]\right]. \tag{39}$$

Thanks to the generalized decomposition method, we find

$$\begin{aligned}\mathcal{Q}_0(\mathbf{w}_1,\mathbf{w}_2,\varsigma) &= \mathbb{J}^{-1}\left[\frac{\psi(\mathfrak{s})}{\phi(\mathfrak{s})}\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,0)\right] = \mathbb{J}^{-1}\left[\frac{\psi(\mathfrak{s})}{\phi(\mathfrak{s})}\sqrt{\mathbf{w}_1+\mathbf{w}_2+\mathbf{w}_1\mathbf{w}_2}\right] \\ &= \sqrt{\mathbf{w}_1+\mathbf{w}_2+\mathbf{w}_1\mathbf{w}_2}.\end{aligned} \tag{40}$$

Here, we surmise that the unknown function $\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)$ can be written by an infinite series of the form

$$\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) = \sum_{\ell=0}^{\infty}\mathcal{Q}_\ell(\mathbf{w}_1,\mathbf{w}_2,\varsigma). \tag{41}$$

Also, the nonlinearity $\mathscr{F}(\mathcal{Q})$ can be decomposed by an infinite series of polynomials represented by

$$\begin{aligned}\mathscr{F}(\mathcal{Q}) &= \mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1} = \sum_{\ell=0}^{\infty}\mathscr{A}_\ell, \\ \mathscr{F}(\mathcal{Q}) &= \mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2} = \sum_{\ell=0}^{\infty}\mathscr{B}_\ell,\end{aligned} \tag{42}$$

where $\mathcal{Q}_\ell(\mathbf{w}_1,\mathbf{w}_2,\varsigma)$ will be evaluated recurrently and $\mathscr{A}_\ell$ and $\mathscr{B}_\ell$ are the so-called polynomials of $\mathcal{Q}_0,\mathcal{Q}_1,\ldots,\mathcal{Q}_\ell$ established by [52].

$$\sum_{\ell=0}^{\infty}\mathcal{Q}_{\ell+1}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) = \mathbb{J}^{-1}\left[\frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left[\sum_{\ell=0}^{\infty}(\mathscr{A})_\ell + \sum_{\ell=0}^{\infty}(\mathscr{B})_\ell + \hbar\sum_{\ell=0}^{\infty}(\mathcal{Q})_\ell\right]\right], \quad \ell = 0,1,2,\ldots. \tag{43}$$

The first few Adomian polynomials are presented as follows:

$$\begin{aligned}\mathscr{A}_\ell\left(\mathcal{Q}^2\right)_{\mathbf{w}_1\mathbf{w}_1} &= \begin{cases}\left(\mathcal{Q}_0^2\right)_{\mathbf{w}_1\mathbf{w}_1}, & \ell = 0, \\ \left(2\mathcal{Q}_0\mathcal{Q}_1\right)_{\mathbf{w}_1\mathbf{w}_1}, & \ell = 1, \\ \left(\mathcal{Q}_1^2 + 2\mathcal{Q}_0\mathcal{Q}_1\right)_{\mathbf{w}_1\mathbf{w}_1}, & \ell = 2,\end{cases} \\ \mathscr{B}_\ell\left(\mathcal{Q}^2\right)_{\mathbf{w}_2\mathbf{w}_2} &= \begin{cases}\left(\mathcal{Q}_0^2\right)_{\mathbf{w}_2\mathbf{w}_2}, & \ell = 0, \\ \left(2\mathcal{Q}_0\mathcal{Q}_1\right)_{\mathbf{w}_2\mathbf{w}_2}, & \ell = 1, \\ \left(\mathcal{Q}_1^2 + 2\mathcal{Q}_0\mathcal{Q}_1\right)_{\mathbf{w}_2\mathbf{w}_2}, & \ell = 2.\end{cases}\end{aligned} \tag{44}$$

For $\ell = 0,1,2,3,\ldots$,

$$\begin{aligned}\mathcal{Q}_1(\mathbf{w}_1,\mathbf{w}_2,\varsigma) &= \mathbb{J}^{-1}\left[\frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left[\mathscr{A}_0 + \mathscr{B}_0 + \hbar\mathcal{Q}_0\right]\right] \\ &= \hbar\sqrt{\mathbf{w}_1+\mathbf{w}_2+\mathbf{w}_1\mathbf{w}_2}\,\frac{\varsigma^{\chi}}{\Gamma(\chi+1)}, \\ \mathcal{Q}_2(\mathbf{w}_1,\mathbf{w}_2,\varsigma) &= \mathbb{J}^{-1}\left[\frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left[\mathscr{A}_1 + \mathscr{B}_1 + \hbar\mathcal{Q}_1\right]\right] \\ &= \hbar^2\sqrt{\mathbf{w}_1+\mathbf{w}_2+\mathbf{w}_1\mathbf{w}_2}\,\frac{\varsigma^{2\chi}}{\Gamma(2\chi+1)}, \\ \mathcal{Q}_3(\mathbf{w}_1,\mathbf{w}_2,\varsigma) &= \mathbb{J}^{-1}\left[\frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left[\mathscr{A}_2 + \mathscr{B}_2 + \hbar\mathcal{Q}_2\right]\right] \\ &= \hbar^3\sqrt{\mathbf{w}_1+\mathbf{w}_2+\mathbf{w}_1\mathbf{w}_2}\,\frac{\varsigma^{3\chi}}{\Gamma(3\chi+1)}, \\ &\vdots.\end{aligned} \tag{45}$$

The approximate solution for Example 1 is expressed as follows:

$$\begin{aligned}\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) &= \Phi_0(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\Phi_1(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\Phi_2(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\Phi_3(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\cdots\\ &= \sqrt{\mathbf{w}_1+\mathbf{w}_2+\mathbf{w}_1\mathbf{w}_2}\left(1+\frac{\hbar\varsigma^{\chi}}{\Gamma(\chi+1)}+\frac{\hbar^2\varsigma^{2\chi}}{\Gamma(2\chi+1)}+\frac{\hbar^3\varsigma^{3\chi}}{\Gamma(3\chi+1)}+\cdots\right).\end{aligned} \tag{46}$$

Case 2. Here, we surmise ABC fractional derivative operator coupled with the generalized integral transform and Adomian decomposition method. Applying the generalized integral transform for Example 1,

$$\frac{\psi^{\chi}(\mathfrak{s})\mathbb{A}(\chi)}{\chi+(1-\chi)\psi^{\chi}(\mathfrak{s})}\mathcal{U}(\mathbf{w}_1,\mathfrak{s})-\phi(\mathfrak{s})\sum_{\kappa=0}^{m-1}\psi^{\chi-\kappa-1}(\mathfrak{s})\mathcal{Q}^{(\kappa)}(0)=\mathbb{J}\left[\mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\hbar\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)\right]. \tag{47}$$

Taking into consideration the IC given in (36), we have

$$\mathcal{U}(\mathbf{w}_1,\mathfrak{s})=\frac{\psi(\mathfrak{s})}{\phi(\mathfrak{s})}\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,0)+\frac{\chi+(1-\chi)\psi^{\chi}(\mathfrak{s})}{\psi^{\chi}(\mathfrak{s})\mathbb{A}(\chi)}\mathbb{J}\left[\mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\hbar\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)\right]. \tag{48}$$

Employing the inverse generalized integral transform, we obtain

$$\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)=\mathbb{J}^{-1}\left[\frac{\psi(\mathfrak{s})}{\phi(\mathfrak{s})}\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,0)+\frac{\chi+(1-\chi)\psi^{\chi}(\mathfrak{s})}{\psi^{\chi}(\mathfrak{s})\mathbb{A}(\chi)}\mathbb{J}\left[\mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\hbar\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)\right]\right]. \tag{49}$$

Thanks to the generalized decomposition method, we find

$$\begin{aligned}\mathcal{Q}_0(\mathbf{w}_1,\mathbf{w}_2,\varsigma) &= \mathbb{J}^{-1}\left[\frac{\psi(\mathfrak{s})}{\phi(\mathfrak{s})}\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,0)\right]=\mathbb{J}^{-1}\left[\frac{\psi(s_1)}{\phi(\mathfrak{s})}\sqrt{\mathbf{w}_1+\mathbf{w}_2+\mathbf{w}_1\mathbf{w}_2}\right]\\ &= \sqrt{\mathbf{w}_1+\mathbf{w}_2+\mathbf{w}_1\mathbf{w}_2}.\end{aligned} \tag{50}$$

Here, we surmise that the unknown function $\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)$ can be written by an infinite series of the form

$$\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)=\sum_{\ell=0}^{\infty}\mathcal{Q}_{\ell}(\mathbf{w}_1,\mathbf{w}_2,\varsigma). \tag{51}$$

Also, the nonlinearity $\mathcal{F}_j(\mathcal{Q})$, $j=1,2$, can be decomposed by an infinite series of polynomials represented by

$$\begin{aligned}\mathcal{F}_1(\mathcal{Q}) &= \mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1}=\sum_{\ell=0}^{\infty}\mathcal{A}_{\ell},\\ \mathcal{F}_2(\mathcal{Q}) &= \mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2}=\sum_{\ell=0}^{\infty}\mathcal{B}_{\ell},\end{aligned} \tag{52}$$

where $\mathcal{Q}_{\ell}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)$ will be evaluated recurrently and $\mathcal{A}_{\ell}$ and $\mathcal{B}_{\ell}$ are the so-called polynomials of $\mathcal{Q}_0,\mathcal{Q}_1,\ldots,\mathcal{Q}_{\ell}$ defined in (44).

For $\ell = 0, 1, 2, 3, \ldots,$

$$\begin{aligned}
\mathcal{Q}_1(\mathbf{w}_1, \mathbf{w}_2, \varsigma) &= \mathbb{J}^{-1}\left[\frac{\chi + (1-\chi)\psi^{\chi}(\mathfrak{s})}{\psi^{\chi}(\mathfrak{s})\mathbb{A}(\chi)}\mathbb{J}[\mathcal{A}_0 + \mathcal{B}_0 + \hbar\mathcal{Q}_0]\right] \\
&= \frac{\hbar\sqrt{\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_1\mathbf{w}_2}}{\mathbb{A}(\chi)}\left[\frac{\chi\varsigma^{\chi}}{\Gamma(\chi+1)} + (1-\chi)\right], \\
\mathcal{Q}_2(\mathbf{w}_1, \mathbf{w}_2, \varsigma) &= \mathbb{J}^{-1}\left[\frac{\chi + (1-\chi)\psi^{\chi}(\mathfrak{s})}{\psi^{\chi}(\mathfrak{s})\mathbb{A}(\chi)}\mathbb{J}[\mathcal{A}_1 + \mathcal{B}_1 + \hbar\mathcal{Q}_1]\right] \\
&= \frac{\hbar^2\sqrt{\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_1\mathbf{w}_2}}{\mathbb{A}^2(\chi)}\left[\frac{\chi^2\varsigma^{2\chi}}{\Gamma(2\chi+1)} + 2\chi(1-\chi)\frac{\varsigma^{\chi}}{\Gamma(\chi+1)} + (1-\chi)^2\right], \\
\mathcal{Q}_3(\mathbf{w}_1, \mathbf{w}_2, \varsigma) &= \mathbb{J}^{-1}\left[\frac{\chi + (1-\chi)\psi^{\chi}(\mathfrak{s})}{\psi^{\chi}(\mathfrak{s})\mathbb{A}(\chi)}\mathbb{J}[\mathcal{A}_2 + \mathcal{B}_2 + \hbar\mathcal{Q}_2]\right] \\
&= \frac{\hbar^3\sqrt{\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_1\mathbf{w}_2}}{\mathbb{A}^3(\chi)}\left[\frac{\chi^3\varsigma^{3\chi}}{\Gamma(3\chi+1)} + 3\chi^2(1-\chi)\frac{\varsigma^{2\chi}}{\Gamma(2\chi+1)} + 3\chi(1-\chi)^2\frac{\varsigma^{\chi}}{\Gamma(\chi+1)} + (1-\chi)^3\right], \\
&\vdots .
\end{aligned} \tag{53}$$

The approximate solution for Example 1 is expressed as follows:

$$\begin{aligned}
\mathcal{Q}(\mathbf{w}_1, \mathbf{w}_2, \varsigma) &= \Phi_0(\mathbf{w}_1, \mathbf{w}_2, \varsigma) + \Phi_1(\mathbf{w}_1, \mathbf{w}_2, \varsigma) + \Phi_2(\mathbf{w}_1, \mathbf{w}_2, \varsigma) + \Phi_3(\mathbf{w}_1, \mathbf{w}_2, \varsigma) + \cdots \\
&= \sqrt{\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_1\mathbf{w}_2}\left(1 + \frac{\hbar}{\mathbb{A}(\chi)}\left(\frac{\varsigma^{\chi}}{\Gamma(\chi+1)} + (1-\chi)\right)\right. \\
&\quad + \frac{\hbar^2}{\mathbb{A}^2(\chi)}\left(\frac{\chi^2\varsigma^{2\chi}}{\Gamma(2\chi+1)} + 2\chi(1-\chi)\frac{\varsigma^{\chi}}{\Gamma(\chi+1)} + (1-\chi)^2\right) \\
&\quad \left. + \frac{\hbar^3}{\mathbb{A}^3(\chi)}\left(\frac{\chi^3\varsigma^{3\chi}}{\Gamma(3\chi+1)} + 3\chi^2(1-\chi)\frac{\varsigma^{2\chi}}{\Gamma(2\chi+1)} + 3\chi(1-\chi)^2\frac{\varsigma^{\chi}}{\Gamma(\chi+1)} + (1-\chi)^3\right) + \cdots\right).
\end{aligned} \tag{54}$$

For $\chi = 1$, we obtained the exact solution of Example 1 as

$$\mathcal{Q}(\mathbf{w}_1, \mathbf{w}_2, \varsigma) = \sqrt{\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_1\mathbf{w}_2}\exp(\hbar\varsigma). \tag{55}$$

The analytical approximate solutions including certain random initialization produced by the proposed methodology are shown in Table 1. The VIMHP is employed to perform the comparative analysis, which forecasts the precision of the proposed methodology based on its lower error. The findings in this study are tremendously helpful in comprehending the internal components of natural disasters. We will describe the scientific clarification of the solutions for the BP model in this paragraph of the article. The exact and numerical solution for $\alpha = 1, r_1 = 0, \varsigma = 0.01$, and $\chi = 1$ is shown in Figure 1.

Furthermore, the absolute error for the aforementioned assumptions is depicted in Figure 2(a). Figure 2(b) demonstrates the behavior of the findings in three-dimensional simulation for different fractional orders by employing the Caputo fractional derivative operator.

Finally, Figure 3 represents the two-dimensional behavior of exact, approximate by Caputo, and approximate solutions derived by ABC fractional operators with varying fractional orders. These solutions have a distinctive characteristic that allows them to interact with other solutions derived by [23]. The proposed findings have particle-like geometries in their solutions. The synthesized trajectory is either a success or a descent from one asymptotic state to the next. The accuracy of the proposed method can be enhanced by increasing the recursive terms. □

TABLE 1: Exact $\mathcal{Q}_E$ and approximate solutions $\mathcal{Q}_{\text{CFD}}$ and $\mathcal{Q}_{\text{ABC}}$ of $\mathcal{Q}(\mathbf{w}_1, \mathbf{w}_2, \varsigma)$ of Example 1 having absolute errors $E_1 = \|\mathcal{Q}_E - \mathcal{Q}_{\text{CFD}}\|$ and $E_2 = \|\mathcal{Q}_E - \mathcal{Q}_{\text{ABC}}\|$ when $\chi = 1, \varsigma = 0.2, r_1 = 0$, and $\hbar = 1/2$ for different values of $\mathbf{w}_1$ and $\mathbf{w}_2$.

$(\mathbf{w}_1, \mathbf{w}_2)$	$\mathcal{Q}_E$ sol.	$\mathcal{Q}_{\text{CFD}}$ sol.	$\mathcal{Q}_{\text{ABC}}$ sol.	VIMHP sol. [23]	Error = E_1	Error = E_2
(−10, 10)	11.051709	11.051666	11.051666	11.051344	$4.2510e-5$	$4.2510e-5$
(−8, 8)	8.841367	8.841333	8.841333	8.841600	$3.4008e-5$	$3.4008e-5$
(−6, 6)	6.631025	6.631000	6.631000	6.631555	$2.5506e-5$	$2.5506e-5$
(−4, 4)	6.631025	6.631000	6.631000	6.667890	$2.5506e-5$	$2.5506e-5$
(−2, 2)	2.210341	2.210333	2.210333	2.226699	$8.5020e-6$	$8.5020e-6$
(0, 0)	0.000000	0.000000	0.000000	1.008975	0.000000	0.000000
(2, 2)	3.125800	3.125800	3.125800	3.998500	$1.2024e-5$	$1.2024e-5$
(4, 4)	5.414100	5.414100	5.414100	6.889200	$2.0826e-5$	$2.0826e-5$
(6, 6)	7.657100	7.657100	7.657100	8.100650	$2.9453e-5$	$2.9453e-5$
(8, 8)	9.885100	9.885100	9.885100	9.987890	$3.8023e-5$	$3.8023e-5$
(10, 10)	1.210600	1.210600	1.210600	1.9823	$4.6567e-5$	$4.6567e-5$

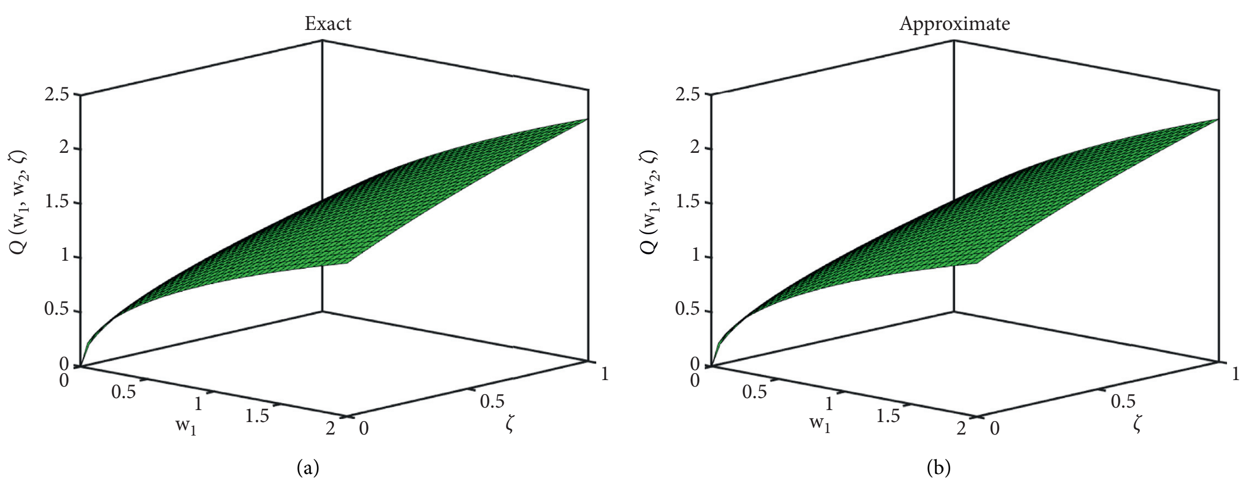


FIGURE 1: Three-dimensional illustration of the exact and approximate solution of Example 1 when $\alpha = 1, r_1 = 0, \varsigma = 0.01$, and $\chi = 1$.

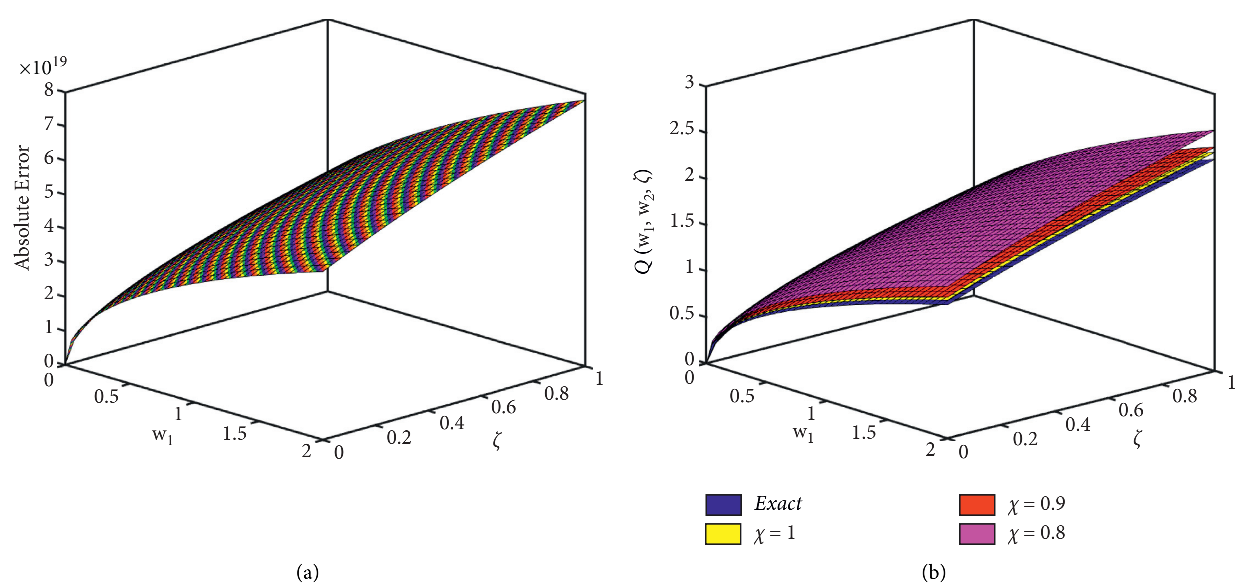


FIGURE 2: Three-dimensional illustration of the absolute error and multiple surface of various fractional orders for Example 1 when $\alpha = 1, r_1 = 0$, and $\varsigma = 0.01$.

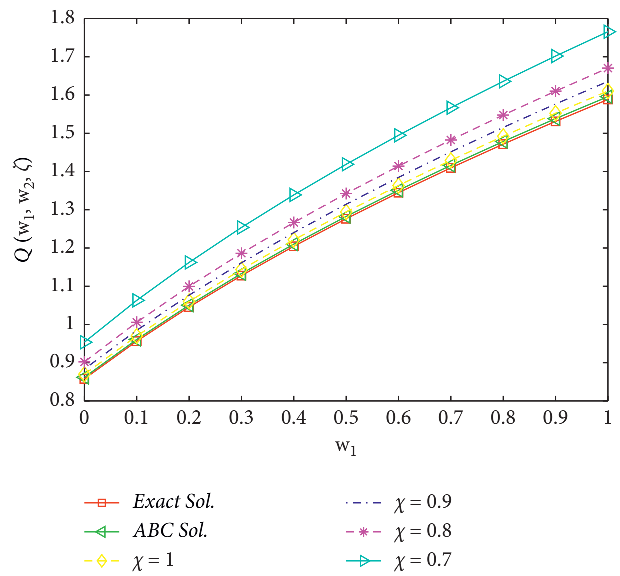


Figure 3: Two-dimensional illustration of Example 1 for various fractional orders when $\alpha = 1, r_1 = 0$, and $\varsigma = 0.01$.

Remark 3. Example 1 leads to the conclusion that

(i) If we replace the IC $\mathcal{Q}(\mathbf{w}_1, \mathbf{w}_2, \varsigma) = \sqrt{\alpha_1 \mathbf{w}_1 + \beta_1 \mathbf{w}_2 + \gamma_1 \mathbf{w}_1 \mathbf{w}_2}$ in Example 1, then the approximate solution can be achieved as

$$\mathcal{Q}(\mathbf{w}_1, \mathbf{w}_2, \varsigma) = \sqrt{\alpha_1 \mathbf{w}_1 + \beta_1 \mathbf{w}_2 + \gamma_1 \mathbf{w}_1 \mathbf{w}_2}\left(1 + \frac{\hbar \varsigma^{\chi}}{\Gamma(\chi+1)} + \frac{\hbar^2 \varsigma^{2\chi}}{\Gamma(2\chi+1)} + \frac{\hbar^3 \varsigma^{3\chi}}{\Gamma(3\chi+1)} + \cdots\right). \quad (56)$$

(ii) The closed form solution in the stated case will be $\mathcal{Q}(\mathbf{w}_1, \mathbf{w}_2, \varsigma) = \sqrt{\alpha_1 \mathbf{w}_1 + \beta_1 \mathbf{w}_2 + \gamma_1 \mathbf{w}_1 \mathbf{w}_2} \exp(\hbar \varsigma)$, where α_1 and β_1 are the real constants.

(iii) Letting $\alpha_1 = \beta_1 = 0$ and $\gamma_1 = 1$ along with $\hbar = 1/2$, we get the result proposed independently by Roul [53] and Shakeri and Dehgan [54], respectively.

Example 2. Assume the time-fractional BPM (4) having $\alpha = 1, r_1 = 0$, and $\hbar = 1$; then,

$$\mathbf{D}_{\varsigma}^{\alpha} \mathcal{Q}(\mathbf{w}_1, \mathbf{w}_2, \varsigma) = \mathcal{Q}^2_{\mathbf{w}_1 \mathbf{w}_1}(\mathbf{w}_1, \mathbf{w}_2, \varsigma) + \mathcal{Q}^2_{\mathbf{w}_2 \mathbf{w}_2}(\mathbf{w}_1, \mathbf{w}_2, \varsigma) + \mathcal{Q}(\mathbf{w}_1, \mathbf{w}_2, \varsigma), \quad (57)$$

with IC

$$\mathcal{Q}_0(\mathbf{w}_1, \mathbf{w}_2, 0) = \sqrt{\sin(\theta \mathbf{w}_1)\cosh(\theta \mathbf{w}_2)}, \quad \theta \in \mathbb{R}. \quad (58)$$

Proof. . Foremost, we provide the solution of (57) in two general cases.

Case 1. First, we apply the Caputo fractional derivative operator coupled with the generalized integral transform and Adomian decomposition method. Applying the generalized integral transform for Example 2,

$$\psi^{\chi}(\mathfrak{s})\mathcal{U}(\mathbf{w}_1, \mathfrak{s}) - \phi(\mathfrak{s}) \sum_{\kappa=0}^{m-1} \psi^{\chi-\kappa-1}(\mathfrak{s})\mathcal{Q}^{(\kappa)}(0) = \mathbb{J}\left[\mathcal{Q}^2_{\mathbf{w}_1 \mathbf{w}_1}(\mathbf{w}_1, \mathbf{w}_2, \varsigma) + \mathcal{Q}^2_{\mathbf{w}_2 \mathbf{w}_2}(\mathbf{w}_1, \mathbf{w}_2, \varsigma) + \mathcal{Q}(\mathbf{w}_1, \mathbf{w}_2, \varsigma)\right]. \quad (59)$$

Taking into consideration the IC given in (58), we have

$$\mathscr{U}(\mathbf{w}_1, \mathfrak{s}) = \frac{\psi(\mathfrak{s})}{\phi(\mathfrak{s})}\mathscr{Q}(\mathbf{w}_1, \mathbf{w}_2, 0) + \frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left[\mathscr{Q}^2_{\mathbf{w}_1\mathbf{w}_1}(\mathbf{w}_1, \mathbf{w}_2, \varsigma) + \mathscr{Q}^2_{\mathbf{w}_2\mathbf{w}_2}(\mathbf{w}_1, \mathbf{w}_2, \varsigma) + \mathscr{Q}(\mathbf{w}_1, \mathbf{w}_2, \varsigma)\right]. \tag{60}$$

Employing the inverse generalized integral transform, we obtain

$$\mathscr{Q}(\mathbf{w}_1, \mathbf{w}_2, \varsigma) = \mathbb{J}^{-1}\left[\frac{\psi(\mathfrak{s})}{\phi(\mathfrak{s})}\mathscr{Q}(\mathbf{w}_1, \mathbf{w}_2, 0) + \frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left[\mathscr{Q}^2_{\mathbf{w}_1\mathbf{w}_1}(\mathbf{w}_1, \mathbf{w}_2, \varsigma) + \mathscr{Q}^2_{\mathbf{w}_2\mathbf{w}_2}(\mathbf{w}_1, \mathbf{w}_2, \varsigma) + \mathscr{Q}(\mathbf{w}_1, \mathbf{w}_2, \varsigma)\right]\right]. \tag{61}$$

Thanks to the generalized decomposition method, we find

$$\begin{aligned}\mathscr{Q}_0(\mathbf{w}_1, \mathbf{w}_2, \varsigma) &= \mathbb{J}^{-1}\left[\frac{\psi(\mathfrak{s})}{\phi(\mathfrak{s})}\mathscr{Q}(\mathbf{w}_1, \mathbf{w}_2, 0)\right] = \mathbb{J}^{-1}\left[\frac{\psi(\mathfrak{s})}{\phi(\mathfrak{s})}\sqrt{\sin(\theta\mathbf{w}_1)\cosh(\theta\mathbf{w}_2)}\right] \\ &= \sqrt{\sin(\theta\mathbf{w}_1)\cosh(\theta\mathbf{w}_2)}.\end{aligned} \tag{62}$$

Here, we surmise that the unknown function $\mathscr{Q}(\mathbf{w}_1, \mathbf{w}_2, \varsigma)$ can be written by an infinite series of the form

$$\mathscr{Q}(\mathbf{w}_1, \mathbf{w}_2, \varsigma) = \sum_{\ell=0}^{\infty}\mathscr{Q}_{\ell}(\mathbf{w}_1, \mathbf{w}_2, \varsigma). \tag{63}$$

Also, the nonlinearity $\mathscr{F}_j(\mathscr{Q})$, $j = 1, 2$, can be decomposed by an infinite series of polynomials represented by $\mathscr{F}_1(\mathscr{Q}) = \mathscr{Q}^2_{\mathbf{w}_1\mathbf{w}_1} = \sum_{\ell=0}^{\infty}\mathscr{A}_{\ell}$ and $\mathscr{F}_2(\mathscr{Q}) = \mathscr{Q}^2_{\mathbf{w}_2\mathbf{w}_2} = \sum_{\ell=0}^{\infty}\mathscr{B}_{\ell}$ defined in (44).

$$\sum_{\ell=0}^{\infty}\mathscr{Q}_{\ell+1}(\mathbf{w}_1, \mathbf{w}_2, \varsigma) = \mathbb{J}^{-1}\left[\frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left[\sum_{\ell=0}^{\infty}(\mathscr{A})_{\ell} + \sum_{\ell=0}^{\infty}(\mathscr{B})_{\ell} + \sum_{\ell=0}^{\infty}(\mathscr{Q})_{\ell}\right]\right], \quad \ell = 0, 1, 2, \ldots. \tag{64}$$

For $\ell = 0, 1, 2, 3, \ldots$,

$$\begin{aligned}\mathscr{Q}_1(\mathbf{w}_1, \mathbf{w}_2, \varsigma) &= \mathbb{J}^{-1}\left[\frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left[\mathscr{A}_0 + \mathscr{B}_0 + \hbar\mathscr{Q}_0\right]\right] \\ &= \sqrt{\sin(\theta\mathbf{w}_1)\cosh(\theta\mathbf{w}_2)}\,\frac{\varsigma^{\chi}}{\Gamma(\chi+1)}, \\ \mathscr{Q}_2(\mathbf{w}_1, \mathbf{w}_2, \varsigma) &= \mathbb{J}^{-1}\left[\frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left[\mathscr{A}_1 + \mathscr{B}_1 + \hbar\mathscr{Q}_1\right]\right] \\ &= \sqrt{\sin(\theta\mathbf{w}_1)\cosh(\theta\mathbf{w}_2)}\,\frac{\varsigma^{2\chi}}{\Gamma(2\chi+1)}, \\ \mathscr{Q}_3(\mathbf{w}_1, \mathbf{w}_2, \varsigma) &= \mathbb{J}^{-1}\left[\frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left[\mathscr{A}_2 + \mathscr{B}_2 + \hbar\mathscr{Q}_2\right]\right] \\ &= \sqrt{\sin(\theta\mathbf{w}_1)\cosh(\theta\mathbf{w}_2)}\,\frac{\varsigma^{3\chi}}{\Gamma(3\chi+1)}, \\ &\vdots.\end{aligned} \tag{65}$$

The approximate solution for Example 2 is expressed as follows:

$$\begin{aligned}\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) &= \Phi_0(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\Phi_1(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\Phi_2(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\Phi_3(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\cdots \\ &= \sqrt{\sin(\theta\mathbf{w}_1)\cosh(\theta\mathbf{w}_2)}\left(1+\frac{\varsigma^{\chi}}{\Gamma(\chi+1)}+\frac{\varsigma^{2\chi}}{\Gamma(2\chi+1)}+\frac{\varsigma^{3\chi}}{\Gamma(3\chi+1)}+\cdots\right).\end{aligned} \tag{66}$$

Case 2. Here, we surmise ABC fractional derivative operator coupled with the generalized integral transform and Adomian decomposition method. Applying the generalized integral transform for Example 2,

$$\frac{\psi^{\chi}(\mathfrak{s})\mathbb{A}(\chi)}{\chi+(1-\chi)\psi^{\chi}(\mathfrak{s})}\mathcal{U}(\mathbf{w}_1,\mathfrak{s})-\phi(\mathfrak{s})\sum_{\kappa=0}^{m-1}\psi^{\chi-\kappa-1}(\mathfrak{s})\mathcal{Q}^{(\kappa)}(0)=\mathbb{J}\left[\mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)\right]. \tag{67}$$

Taking into consideration the IC given in (58), we have

$$\mathcal{U}(\mathbf{w}_1,\mathfrak{s})=\frac{\psi(\mathfrak{s})}{\phi(\mathfrak{s})}\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,0)+\frac{\chi+(1-\chi)\psi^{\chi}(\mathfrak{s})}{\psi^{\chi}(\mathfrak{s})\mathbb{A}(\chi)}\mathbb{J}\left[\mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)\right]. \tag{68}$$

Employing the inverse generalized integral transform, we obtain

$$\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)=\mathbb{J}^{-1}\left[\frac{\psi(\mathfrak{s})}{\phi(\mathfrak{s})}\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,0)+\frac{\chi+(1-\chi)\psi^{\chi}(\mathfrak{s})}{\psi^{\chi}(\mathfrak{s})\mathbb{A}(\chi)}\mathbb{J}\left[\mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)\right]\right]. \tag{69}$$

Thanks to the generalized decomposition method, we find

$$\begin{aligned}\mathcal{Q}_0(\mathbf{w}_1,\mathbf{w}_2,\varsigma) &= \mathbb{J}^{-1}\left[\frac{\psi(\mathfrak{s})}{\phi(\mathfrak{s})}\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,0)\right]=\mathbb{J}^{-1}\left[\frac{\psi(\mathfrak{s})}{\phi(\mathfrak{s})}\sqrt{\sin(\theta\mathbf{w}_1)\cosh(\theta\mathbf{w}_2)}\right] \\ &= \sqrt{\sin(\theta\mathbf{w}_1)\cosh(\theta\mathbf{w}_2)}.\end{aligned} \tag{70}$$

Here, we surmise that the unknown function $\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)$ can be written by an infinite series of the form

$$\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)=\sum_{\ell=0}^{\infty}\mathcal{Q}_{\ell}(\mathbf{w}_1,\mathbf{w}_2,\varsigma). \tag{71}$$

Also, the nonlinearity $\mathcal{F}_j(\mathcal{Q})$, $j=1,2$, can be decomposed by an infinite series of polynomials represented by

$$\begin{aligned}\mathcal{F}_1(\mathcal{Q}) &= \mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1}=\sum_{\ell=0}^{\infty}\mathcal{A}_{\ell}, \\ \mathcal{F}_2(\mathcal{Q}) &= \mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2}=\sum_{\ell=0}^{\infty}\mathcal{B}_{\ell},\end{aligned} \tag{72}$$

where $\mathcal{Q}_{\ell}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)$ will be evaluated recurrently and $\mathcal{A}_{\ell}$ and $\mathcal{B}_{\ell}$ are the so-called polynomials of $\mathcal{Q}_0,\mathcal{Q}_1,\ldots,\mathcal{Q}_{\ell}$ defined in (44).

For $\ell = 0, 1, 2, 3, \ldots$,

$$\begin{aligned}
\mathcal{Q}_1(\mathbf{w}_1, \mathbf{w}_2, \varsigma) &= \mathbb{J}^{-1}\left[\frac{\chi + (1-\chi)\psi^{\chi}(\mathfrak{s})}{\psi^{\chi}(\mathfrak{s})\mathbb{A}(\chi)}\mathbb{J}[\mathcal{A}_0 + \mathcal{B}_0 + \mathcal{Q}_0]\right] \\
&= \frac{\sqrt{\sin(\theta\mathbf{w}_1)\cosh(\theta\mathbf{w}_2)}}{\mathbb{A}(\chi)}\left[\frac{\chi\varsigma^{\chi}}{\Gamma(\chi+1)} + (1-\chi)\right], \\
\mathcal{Q}_2(\mathbf{w}_1, \mathbf{w}_2, \varsigma) &= \mathbb{J}^{-1}\left[\frac{\chi + (1-\chi)\psi^{\chi}(\mathfrak{s})}{\psi^{\chi}(\mathfrak{s})\mathbb{A}(\chi)}\mathbb{J}[\mathcal{A}_1 + \mathcal{B}_1 + \mathcal{Q}_1]\right] \\
&= \frac{\sqrt{\sin(\theta\mathbf{w}_1)\cosh(\theta\mathbf{w}_2)}}{\mathbb{A}^2(\chi)}\left[\frac{\chi^2\varsigma^{2\chi}}{\Gamma(2\chi+1)} + 2\chi(1-\chi)\frac{\varsigma^{\chi}}{\Gamma(\chi+1)} + (1-\chi)^2\right], \\
\mathcal{Q}_3(\mathbf{w}_1, \mathbf{w}_2, \varsigma) &= \mathbb{J}^{-1}\left[\frac{\chi + (1-\chi)\psi^{\chi}(\mathfrak{s})}{\psi^{\chi}(\mathfrak{s})\mathbb{A}(\chi)}\mathbb{J}[\mathcal{A}_2 + \mathcal{B}_2 + \mathcal{Q}_2]\right] \\
&= \frac{\sqrt{\sin(\theta\mathbf{w}_1)\cosh(\theta\mathbf{w}_2)}}{\mathbb{A}^3(\chi)}\left[\frac{\chi^3\varsigma^{3\chi}}{\Gamma(3\chi+1)} + 3\chi^2(1-\chi)\frac{\varsigma^{2\chi}}{\Gamma(2\chi+1)} + 3\chi(1-\chi)^2\frac{\varsigma^{\chi}}{\Gamma(\chi+1)} + (1-\chi)^3\right], \\
&\vdots .
\end{aligned} \tag{73}$$

The approximate solution for Example 2 is expressed as follows:

$$\begin{aligned}
\mathcal{Q}(\mathbf{w}_1, \mathbf{w}_2, \varsigma) &= \Phi_0(\mathbf{w}_1, \mathbf{w}_2, \varsigma) + \Phi_1(\mathbf{w}_1, \mathbf{w}_2, \varsigma) + \Phi_2(\mathbf{w}_1, \mathbf{w}_2, \varsigma) + \Phi_3(\mathbf{w}_1, \mathbf{w}_2, \varsigma) + \cdots \\
&= \sqrt{\sin(\theta\mathbf{w}_1)\cosh(\theta\mathbf{w}_2)}\left(1 + \frac{1}{\mathbb{A}(\chi)}\left(\frac{\varsigma^{\chi}}{\Gamma(\chi+1)} + (1-\chi)\right)\right. \\
&\quad + \frac{1}{\mathbb{A}^2(\chi)}\left(\frac{\chi^2\varsigma^{2\chi}}{\Gamma(2\chi+1)} + 2\chi(1-\chi)\frac{\varsigma^{\chi}}{\Gamma(\chi+1)} + (1-\chi)^2\right) \\
&\quad \left. + \frac{1}{\mathbb{A}^3(\chi)}\left(\frac{\chi^3\varsigma^{3\chi}}{\Gamma(3\chi+1)} + 3\chi^2(1-\chi)\frac{\varsigma^{2\chi}}{\Gamma(2\chi+1)} + 3\chi(1-\chi)^2\frac{\varsigma^{\chi}}{\Gamma(\chi+1)} + (1-\chi)^3\right) + \cdots\right).
\end{aligned} \tag{74}$$

For $\chi = 1$, we obtained the exact solution of Example 2 as

$$\mathcal{Q}(\mathbf{w}_1, \mathbf{w}_2, \varsigma) = \sqrt{\sin(\theta\mathbf{w}_1)\cosh(\theta\mathbf{w}_2)}\exp(\varsigma). \tag{75}$$

The analytical approximate solutions including certain random initialization produced by the proposed methodology are shown in Table 2. The VIMHP is employed to perform the comparative analysis, which forecasts the precision of the proposed methodology based on its lower error. The findings in this study are tremendously helpful in comprehending the internal components of natural disasters. We will describe the scientific clarification of the solutions for the BP model in this paragraph of the article. The exact and numerical solution for $\alpha = 1, r_1 = 0, \varsigma = 0.01$, and $\hbar = \chi = 1$ is shown in Figure 4.

Furthermore, the absolute error for the aforementioned assumptions is depicted in Figure 5(a). Figure 5(b) demonstrates the behavior of the findings in three-dimensional simulation for different fractional orders by employing the Caputo fractional derivative operator.

Finally, Figure 6 represents the two-dimensional behavior of exact, approximate by Caputo, and approximate solutions derived by ABC fractional operators with varying fractional orders. These solutions have a distinctive characteristic that allows them to interact with other solutions derived by [23]. The proposed findings have particle-like geometries in their solutions. The synthesized trajectory is either a success or a descent from one asymptotic state to the next. The accuracy of the proposed method can be enhanced by increasing the recursive terms. □

Example 3 (Verhulst law [19]). Assume the time-fractional BPM (4) having $\alpha = \beta = 1$; then,

$$\mathbf{D}_{\varsigma}^{\alpha}\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) = \mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) + \mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) + \hbar\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)(1 - r_1\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)), \tag{76}$$

with IC

$$\mathcal{Q}_0(\mathbf{w}_1,\mathbf{w}_2,0) = \exp\left(\sqrt{\frac{\hbar r_1}{8}}(\mathbf{w}_1 + \mathbf{w}_2)\right). \tag{77}$$

Proof. Foremost, we provide the solution of (76) in two general cases.

Case 1. First, we apply the Caputo fractional derivative operator coupled with the generalized integral transform and Adomian decomposition method. Applying the generalized integral transform for Example 3,

$$\begin{aligned}\psi^{\chi}(\mathfrak{s})\mathcal{U}(\mathbf{w}_1,\mathfrak{s}) - \phi(\mathfrak{s})\sum_{\kappa=0}^{m-1}\psi^{\chi-\kappa-1}(\mathfrak{s})\mathcal{Q}^{(\kappa)}(0) = \mathbb{J}\big[&\mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) + \mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) \\ &+ \hbar\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)(1 - r_1\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma))].\end{aligned} \tag{78}$$

Taking into consideration the IC given in (77), we have

$$\mathcal{U}(\mathbf{w}_1,\mathfrak{s}) = \frac{\psi(\mathfrak{s})}{\phi(\mathfrak{s})}\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,0) + \frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left[\mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) + \mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) + \hbar\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)(1 - r_1\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma))\right]. \tag{79}$$

Employing the inverse generalized integral transform, we obtain

$$\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) = \mathbb{J}^{-1}\left[\frac{\psi(\mathfrak{s})}{\phi(\mathfrak{s})}\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,0) + \frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left[\mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) + \mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) + \hbar\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)(1 - r_1\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma))\right]\right]. \tag{80}$$

Thanks to the generalized decomposition method, we find

$$\begin{aligned}\mathcal{Q}_0(\mathbf{w}_1,\mathbf{w}_2,\varsigma) = \mathbb{J}^{-1}\left[\frac{\psi(\mathfrak{s})}{\phi(\mathfrak{s})}\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,0)\right] &= \mathbb{J}^{-1}\left[\frac{\psi(\mathfrak{s})}{\phi(\mathfrak{s})}\exp\left(\sqrt{\frac{\hbar r_1}{8}}(\mathbf{w}_1 + \mathbf{w}_2)\right)\right] \\ &= \exp\left(\sqrt{\frac{\hbar r_1}{8}}(\mathbf{w}_1 + \mathbf{w}_2)\right).\end{aligned} \tag{81}$$

Here, we surmise that the unknown function $\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)$ can be written by an infinite series of the form

$$\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) = \sum_{\ell=0}^{\infty}\mathcal{Q}_{\ell}(\mathbf{w}_1,\mathbf{w}_2,\varsigma). \tag{82}$$

TABLE 2: Exact $\mathcal{Q}_E$ and approximate solution $\mathcal{Q}_A$ of $\mathcal{Q}(\mathbf{w}_1, \mathbf{w}_2, \varsigma)$ of Example 2 having absolute errors $E_1 = \|\mathcal{Q}_E - \mathcal{Q}_{\text{CFD}}\|$ and $E_2 = \|\mathcal{Q}_E - \mathcal{Q}_{\text{ABC}}\|$ when $\chi = 1, \varsigma = 0.2, r_1 = -8/9$, and $\hbar = -1$ for different values of $\mathbf{w}_1$ and $\mathbf{w}_2$.

$(\mathbf{w}_1, \mathbf{w}_2)$	$\mathcal{Q}_E$ sol.	$\mathcal{Q}_{\text{CFD}}$ sol.	$\mathcal{Q}_{\text{ABC}}$ sol.	VIMHP sol. [23]	Error = E_1	Error = E_2
(−10, 10)	10.303436	10.303000	10.302851	10.993451	$5.8565e-4$	$5.8565e-4$
(−8, 8)	5.552607	5.552292	5.552292	5.990091	$3.1561e-4$	$3.1561e-4$
(−6, 6)	1.455853	1.455770	1.455770	1.995099	$8.2751e-5$	$8.2751e-5$
(−4, 4)	2.259085	2.258957	2.258957	2.879342	$1.2841e-4$	$1.2841e-4$
(−2, 2)	1.391787	1.391708	1.391708	2.600834	$7.9109e-5$	$7.9109e-5$
(0, 0)	0.000000	0.000000	0.000000	0.990087	0.000000	0.000000
(2, 2)	1.391787	1.391700	1.391700	1.990980	$7.9109e-5$	$7.9109e-5$
(4, 4)	2.259085	2.258957	2.258957	2.998456	$1.2841e-4$	$1.2841e-4$
(6, 6)	1.455853	1.455770	1.455770	2.009987	$8.2751e-5$	$8.2751e-5$
(8, 8)	5.552607	5.5523292	5.5523292	5.968901	$3.1561e-4$	$3.1561e-4$
(10, 10)	10.303000	10.303044	10.303044	10.990234	$5.8565e-4$	$5.8565e-4$

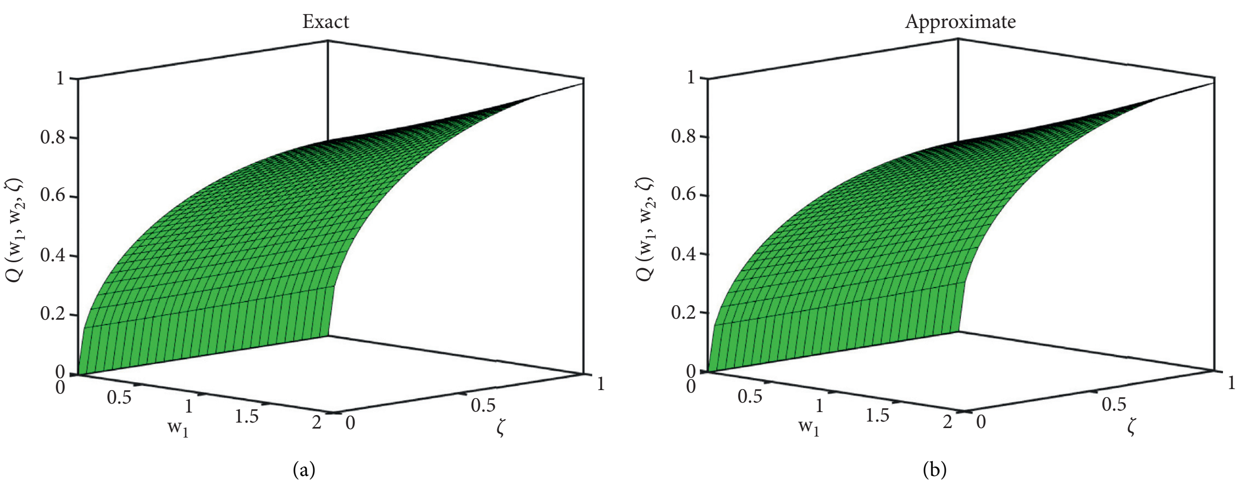


FIGURE 4: Three-dimensional illustration of exact and approximate solution of Example 2 when $\alpha = 1, r_1 = 0, \varsigma = 0.01, \hbar = 1$, and $\chi = 1$.

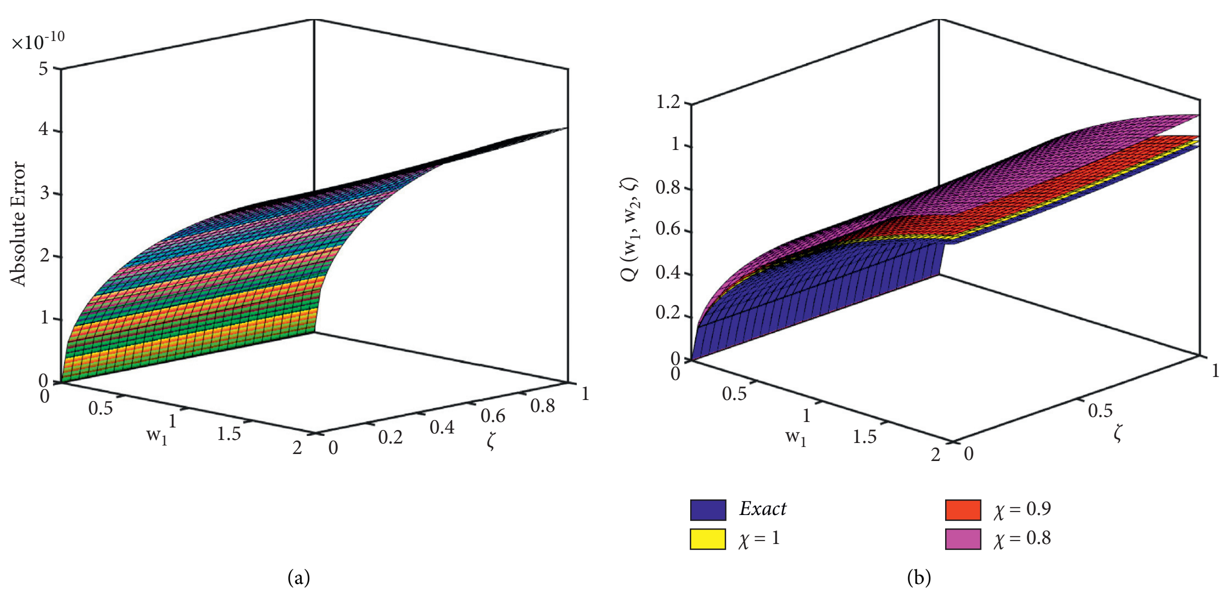


FIGURE 5: Three-dimensional illustration of the absolute error and multiple surface of various fractional orders for Example 2 when $\alpha = 1, r_1 = 0, \hbar = 1$, and $\varsigma = 0.01$.

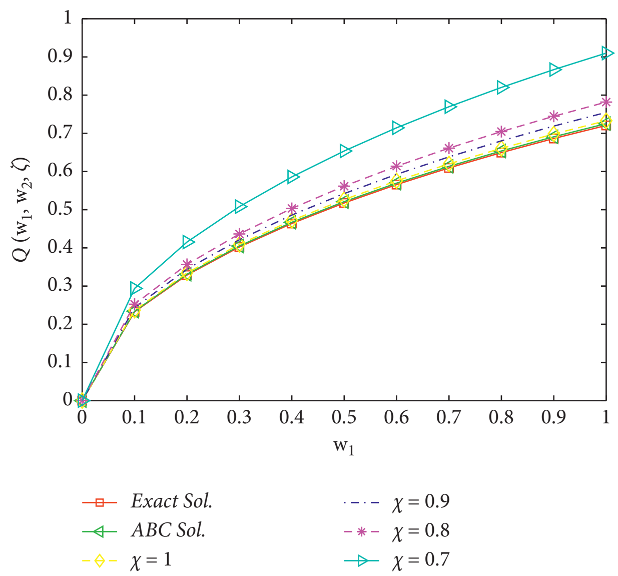


FIGURE 6: Two-dimensional illustration of Example 2 for various fractional orders when $\alpha = 1, r_1 = 0, \hbar = 1$, and $\varsigma = 0.01$.

Also, the nonlinearity $\mathscr{F}_j(\mathcal{Q})$, $j = 1, 2, 3$, can be decomposed by an infinite series of polynomials represented by $\mathscr{F}_1(\mathcal{Q}) = \mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1} = \sum_{\ell=0}^{\infty} \mathscr{A}_\ell$ and $\mathscr{F}_2(\mathcal{Q}) = \mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2} = \sum_{\ell=0}^{\infty} \mathscr{B}_\ell$ defined in (44) along with $\mathscr{F}_3(\mathcal{Q}) = \mathcal{Q}(1 - r_1\mathcal{Q}) = \sum_{\ell=0}^{\infty} \mathscr{C}_\ell$. Thus, we have

$$\mathscr{C}_\ell\left(\mathcal{Q} - r_1\mathcal{Q}^2\right) = \begin{cases} \left(\mathcal{Q}_0 - r_1\mathcal{Q}_0^2\right), & \ell = 0, \\ \left(\mathcal{Q}_2 - r_1\mathcal{Q}_0\mathcal{Q}_1\right), & \ell = 1, \\ \left(\mathcal{Q}_2 - 2r_1\mathcal{Q}_0\mathcal{Q}_2 - r_1\mathcal{Q}_1^2\right), & \ell = 2, \end{cases}$$
$$\sum_{\ell=0}^{\infty} \mathcal{Q}_{\ell+1}(\mathbf{w}_1, \mathbf{w}_2, \varsigma) = \mathbb{J}^{-1}\left[\frac{1}{\psi^\chi(\mathfrak{s})}\mathbb{J}\left[\sum_{\ell=0}^{\infty}(\mathscr{A})_\ell + \sum_{\ell=0}^{\infty}(\mathscr{B})_\ell + \sum_{\ell=0}^{\infty}(\mathscr{C})_\ell\right]\right], \quad \ell = 0, 1, 2, \ldots. \tag{83}$$

For $\ell = 0, 1, 2, 3, \ldots$,

$$\begin{aligned} \mathcal{Q}_1(\mathbf{w}_1, \mathbf{w}_2, \varsigma) &= \mathbb{J}^{-1}\left[\frac{1}{\psi^\chi(\mathfrak{s})}\mathbb{J}\left[\mathscr{A}_0 + \mathscr{B}_0 + \mathscr{C}_0\right]\right] \\ &= \exp\left(\sqrt{\frac{\hbar r_1}{8}}(\mathbf{w}_1 + \mathbf{w}_2)\right)\frac{\varsigma^\chi}{\Gamma(\chi+1)}, \\ \mathcal{Q}_2(\mathbf{w}_1, \mathbf{w}_2, \varsigma) &= \mathbb{J}^{-1}\left[\frac{1}{\psi^\chi(\mathfrak{s})}\mathbb{J}\left[\mathscr{A}_1 + \mathscr{B}_1 + \mathscr{C}_1\right]\right] \\ &= \exp\left(\sqrt{\frac{\hbar r_1}{8}}(\mathbf{w}_1 + \mathbf{w}_2)\right)\frac{\varsigma^{2\chi}}{\Gamma(2\chi+1)}, \\ \mathcal{Q}_3(\mathbf{w}_1, \mathbf{w}_2, \varsigma) &= \mathbb{J}^{-1}\left[\frac{1}{\psi^\chi(\mathfrak{s})}\mathbb{J}\left[\mathscr{A}_2 + \mathscr{B}_2 + \mathscr{C}_2\right]\right] \\ &= \exp\left(\sqrt{\frac{\hbar r_1}{8}}(\mathbf{w}_1 + \mathbf{w}_2)\right)\frac{\varsigma^{3\chi}}{\Gamma(3\chi+1)}, \\ &\vdots. \end{aligned} \tag{84}$$

The approximate solution for Example 3 is expressed as follows:

$$\begin{aligned}\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) &= \Phi_0(\mathbf{w}_1,\mathbf{w}_2,\varsigma) + \Phi_1(\mathbf{w}_1,\mathbf{w}_2,\varsigma) + \Phi_2(\mathbf{w}_1,\mathbf{w}_2,\varsigma) + \Phi_3(\mathbf{w}_1,\mathbf{w}_2,\varsigma) + \cdots \\ &= \exp\left(\sqrt{\frac{\hbar r_1}{8}}(\mathbf{w}_1+\mathbf{w}_2)\right)\left(1 + \frac{\varsigma^{\chi}}{\Gamma(\chi+1)} + \frac{\varsigma^{2\chi}}{\Gamma(2\chi+1)} + \frac{\varsigma^{3\chi}}{\Gamma(3\chi+1)} + \cdots\right).\end{aligned} \tag{85}$$

Case 2. Here, we surmise ABC fractional derivative operator coupled with the generalized integral transform and Adomian decomposition method. Applying the generalized integral transform for Example 3,

$$\begin{aligned}&\frac{\psi^{\chi}(\mathfrak{s})\mathbb{A}(\chi)}{\chi+(1-\chi)\psi^{\chi}(\mathfrak{s})}\mathcal{U}(\mathbf{w}_1,\mathfrak{s}) - \phi(\mathfrak{s})\sum_{\kappa=0}^{m-1}\psi^{\chi-\kappa-1}(\mathfrak{s})\mathcal{Q}^{(\kappa)}(0) = \mathbb{J}\left[\mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) + \mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)\right. \\ &\qquad \left. + \hbar\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)(1 - r_1\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma))\right].\end{aligned} \tag{86}$$

Taking into consideration the IC given in (77), we have

$$\begin{aligned}\mathcal{U}(\mathbf{w}_1,\mathfrak{s}) = {}&\frac{\psi(\mathfrak{s})}{\phi(\mathfrak{s})}\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,0) + \frac{\chi+(1-\chi)\psi^{\chi}(\mathfrak{s})}{\psi^{\chi}(\mathfrak{s})\mathbb{A}(\chi)}\mathbb{J}\left[\mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) + \mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)\right. \\ &\left. + \hbar\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)(1 - r_1\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma))\right].\end{aligned} \tag{87}$$

Employing the inverse generalized integral transform, we obtain

$$\begin{aligned}\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) = {}&\mathbb{J}^{-1}\left[\frac{\psi(\mathfrak{s})}{\phi(\mathfrak{s})}\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,0) + \frac{\chi+(1-\chi)\psi^{\chi}(\mathfrak{s})}{\psi^{\chi}(\mathfrak{s})\mathbb{A}(\chi)}\mathbb{J}\left[\mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) + \mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)\right.\right. \\ &\left.\left. + \hbar\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)(1 - r_1\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma))\right]\right].\end{aligned} \tag{88}$$

Thanks to the generalized decomposition method, we find

$$\begin{aligned}\mathcal{Q}_0(\mathbf{w}_1,\mathbf{w}_2,\varsigma) &= \mathbb{J}^{-1}\left[\frac{\psi(\mathfrak{s})}{\phi(\mathfrak{s})}\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,0)\right] = \mathbb{J}^{-1}\left[\frac{\psi(\mathfrak{s})}{\phi(\mathfrak{s})}\exp\left(\sqrt{\frac{\hbar r_1}{8}}(\mathbf{w}_1+\mathbf{w}_2)\right)\right] \\ &= \exp\left(\sqrt{\frac{\hbar r_1}{8}}(\mathbf{w}_1+\mathbf{w}_2)\right).\end{aligned} \tag{89}$$

Here, we surmise that the unknown function $\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)$ can be written by an infinite series of the form

$$\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) = \sum_{\ell=0}^{\infty}\mathcal{Q}_{\ell}(\mathbf{w}_1,\mathbf{w}_2,\varsigma). \tag{90}$$

Also, the nonlinearity $\mathcal{F}_j(\mathcal{Q})$, $j = 1, 2, 3$, can be decomposed by an infinite series of polynomials represented by $\mathcal{F}_1(\mathcal{Q}) = \mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1} = \sum_{\ell=0}^{\infty} \mathcal{A}_\ell$ and $\mathcal{F}_2(\mathcal{Q}) = \mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2} = \sum_{\ell=0}^{\infty} \mathcal{B}_\ell$ along with $\mathcal{F}_3(\mathcal{Q}) = \mathcal{Q}(1 - r_1\mathcal{Q}) = \sum_{\ell=0}^{\infty} \mathcal{C}_\ell$ defined in (44) and (83), respectively.

For $\ell = 0, 1, 2, 3, \ldots$,

$$\begin{aligned}
\mathcal{Q}_1(\mathbf{w}_1, \mathbf{w}_2, \varsigma) &= \mathbb{J}^{-1}\left[\frac{\chi + (1-\chi)\psi^\chi(\mathfrak{s})}{\psi^\chi(\mathfrak{s})\mathbb{A}(\chi)}\mathbb{J}[\mathcal{A}_0 + \mathcal{B}_0 + \mathcal{C}_0]\right] \\
&= \frac{1}{\mathbb{A}(\chi)}\exp\left(\sqrt{\frac{\hbar r_1}{8}}(\mathbf{w}_1 + \mathbf{w}_2)\right)\left[\frac{\chi\varsigma^\chi}{\Gamma(\chi+1)} + (1-\chi)\right], \\
\mathcal{Q}_2(\mathbf{w}_1, \mathbf{w}_2, \varsigma) &= \mathbb{J}^{-1}\left[\frac{\chi + (1-\chi)\psi^\chi(\mathfrak{s})}{\psi^\chi(\mathfrak{s})\mathbb{A}(\chi)}\mathbb{J}[\mathcal{A}_1 + \mathcal{B}_1 + \mathcal{C}_1]\right] \\
&= \frac{1}{\mathbb{A}^2(\chi)}\exp\left(\sqrt{\frac{\hbar r_1}{8}}(\mathbf{w}_1 + \mathbf{w}_2)\right)\left[\frac{\chi^2\varsigma^{2\chi}}{\Gamma(2\chi+1)} + 2\chi(1-\chi)\frac{\varsigma^\chi}{\Gamma(\chi+1)} + (1-\chi)^2\right], \\
\mathcal{Q}_3(\mathbf{w}_1, \mathbf{w}_2, \varsigma) &= \mathbb{J}^{-1}\left[\frac{\chi + (1-\chi)\psi^\chi(\mathfrak{s})}{\psi^\chi(\mathfrak{s})\mathbb{A}(\chi)}\mathbb{J}[\mathcal{A}_2 + \mathcal{B}_2 + \mathcal{C}_2]\right] \\
&= \frac{1}{\mathbb{A}^3(\chi)}\exp\left(\sqrt{\frac{\hbar r_1}{8}}(\mathbf{w}_1 + \mathbf{w}_2)\right)\left[\frac{\chi^3\varsigma^{3\chi}}{\Gamma(3\chi+1)} + 3\chi^2(1-\chi)\frac{\varsigma^{2\chi}}{\Gamma(2\chi+1)} + 3\chi(1-\chi)^2\frac{\varsigma^\chi}{\Gamma(\chi+1)} + (1-\chi)^3\right], \\
&\vdots .
\end{aligned} \tag{91}$$

The approximate solution for Example 3 is expressed as follows:

$$\begin{aligned}
\mathcal{Q}(\mathbf{w}_1, \mathbf{w}_2, \varsigma) &= \Phi_0(\mathbf{w}_1, \mathbf{w}_2, \varsigma) + \Phi_1(\mathbf{w}_1, \mathbf{w}_2, \varsigma) + \Phi_2(\mathbf{w}_1, \mathbf{w}_2, \varsigma) + \Phi_3(\mathbf{w}_1, \mathbf{w}_2, \varsigma) + \cdots \\
&= \exp\left(\sqrt{\frac{\hbar r_1}{8}}(\mathbf{w}_1 + \mathbf{w}_2)\right)\left(1 + \frac{1}{\mathbb{A}(\chi)}\left(\frac{\varsigma^\chi}{\Gamma(\chi+1)} + (1-\chi)\right)\right. \\
&\quad + \frac{1}{\mathbb{A}^2(\chi)}\left(\frac{\chi^2\varsigma^{2\chi}}{\Gamma(2\chi+1)} + 2\chi(1-\chi)\frac{\varsigma^\chi}{\Gamma(\chi+1)} + (1-\chi)^2\right) \\
&\quad \left. + \frac{1}{\mathbb{A}^3(\chi)}\left(\frac{\chi^3\varsigma^{3\chi}}{\Gamma(3\chi+1)} + 3\chi^2(1-\chi)\frac{\varsigma^{2\chi}}{\Gamma(2\chi+1)} + 3\chi(1-\chi)^2\frac{\varsigma^\chi}{\Gamma(\chi+1)} + (1-\chi)^3\right) + \cdots\right).
\end{aligned} \tag{92}$$

For $\chi = 1$, we obtained the exact solution of Example 3 as

$$\mathcal{Q}(\mathbf{w}_1, \mathbf{w}_2, \varsigma) = \exp\left(\sqrt{\frac{\hbar r_1}{8}}(\mathbf{w}_1 + \mathbf{w}_2) + \varsigma\right). \tag{93}$$

Table 3 shows the analytical approximate solutions with some free parameters that are provided by the proposed technique. The comparison analysis is conducted with the VIMHP that predicts the preciseness of the suggested scheme due to their lower error. The analytical findings are extremely useful in deciphering the internal components of acts of nature. The exact and numerical solution for $\alpha = 1 = \beta, \varsigma = 0.01$, and $\chi = 1$ is shown in Figure 7.

Furthermore, the absolute error for the aforementioned assumptions is depicted in Figure 8(a). Figure 8(b) demonstrates the behavior of the findings in three-dimensional simulation for different fractional orders by employing the Caputo fractional derivative operator.

Finally, Figure 9 represents the two-dimensional behavior of exact, approximate by Caputo, and approximate solutions derived by ABC fractional operators with varying fractional orders. These solutions have a distinctive characteristic that allows them to interact with other solutions derived by [23]. The proposed findings have particle-like geometries in their solutions. The synthesized trajectory is either a success or a descent from one asymptotic state to the next. The accuracy of the proposed method can be enhanced by increasing the recursive terms. □

Example 4. Assume the time-fractional BPM (4) having $\alpha = -1, \beta = 1, \hbar = (1/96)$, and $r_1 = 48$; then,

$$\mathbf{D}_{\varsigma}^{\alpha}\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) = \mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) + \mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) + \frac{1}{96}\mathcal{Q}^{-1}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) - \frac{1}{2}, \tag{94}$$

with IC

$$\mathcal{Q}_0(\mathbf{w}_1,\mathbf{w}_2,0) = \frac{1}{4}\sqrt{2\left(\mathbf{w}_1^2+\mathbf{w}_2^2\right) + \mathbf{w}_2 + 5}. \tag{95}$$

Proof. Foremost, we provide the solution of (76) in two general cases.

Case 1. First, we apply the Caputo fractional derivative operator coupled with the generalized integral transform and Adomian decomposition method. Applying the generalized integral transform for Example 4,

$$\psi^{\chi}(\mathfrak{s})\mathcal{U}(\mathbf{w}_1,\mathfrak{s}) - \phi(\mathfrak{s})\sum_{\kappa=0}^{m-1}\psi^{\chi-\kappa-1}(\mathfrak{s})\mathcal{Q}^{(\kappa)}(0) = \mathbb{J}\left[\mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) + \mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) + \frac{1}{96\mathcal{Q}_0} - \frac{1}{2}\right]. \tag{96}$$

Taking into consideration the IC given in (95), we have

$$\mathcal{U}(\mathbf{w}_1,\mathfrak{s}) = \frac{\psi(\mathfrak{s})}{\phi(\mathfrak{s})}\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,0) + \frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left[\mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) + \mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) + \frac{1}{96\mathcal{Q}_0} - \frac{1}{2}\right]. \tag{97}$$

Employing the inverse generalized integral transform, we obtain

$$\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) = \mathbb{J}^{-1}\left[\frac{\psi(\mathfrak{s})}{\phi(\mathfrak{s})}\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,0) + \frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left[\mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) + \mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) + \frac{1}{96\mathcal{Q}_0} - \frac{1}{2}\right]\right]. \tag{98}$$

Thanks to the generalized decomposition method, we find

$$\begin{aligned}\mathcal{Q}_0(\mathbf{w}_1,\mathbf{w}_2,\varsigma) &= \mathbb{J}^{-1}\left[\frac{\psi(\mathfrak{s})}{\phi(\mathfrak{s})}\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,0)\right] = \mathbb{J}^{-1}\left[\frac{\psi(\mathfrak{s})}{\phi(\mathfrak{s})}\frac{1}{4}\sqrt{2\left(\mathbf{w}_1^2+\mathbf{w}_2^2\right)+\mathbf{w}_2+5}\right] \\ &= \frac{1}{4}\sqrt{2\left(\mathbf{w}_1^2+\mathbf{w}_2^2\right)+\mathbf{w}_2+5}.\end{aligned} \tag{99}$$

Here, we surmise that the unknown function $\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)$ can be written by an infinite series of the form

$$\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) = \sum_{\ell=0}^{\infty}\mathcal{Q}_{\ell}(\mathbf{w}_1,\mathbf{w}_2,\varsigma). \tag{100}$$

TABLE 3: Exact $\mathcal{Q}_E$ and approximate solution $\mathcal{Q}_A$ of $\mathcal{Q}(\mathbf{w}_1, \mathbf{w}_2, \varsigma)$ of Example 3 having absolute errors $E_1 = \|\mathcal{Q}_E - \mathcal{Q}_{\mathrm{CFD}}\|$ and $E_2 = \|\mathcal{Q}_E - \mathcal{Q}_{\mathrm{ABC}}\|$ when $\chi = 1, \varsigma = 0.2, r_1 = -8/9$, and $\hbar = -1$ for different values of $\mathbf{w}_1$ and $\mathbf{w}_2$.

$(\mathbf{w}_1, \mathbf{w}_2)$	$\mathcal{Q}_E$ sol.	$\mathcal{Q}_{\mathrm{CFD}}$ sol.	$\mathcal{Q}_{\mathrm{ABC}}$ sol.	VIMHP sol. [23]	Error = E_1	Error = E_2
(−10, 10)	$1.04194e-3$	$1.04185e-3$	$1.041824e-3$	$1.04186e-3$	$8.1559e-8$	$8.1599e-8$
(−8, 8)	$3.95279e-3$	$3.95249e-3$	$3.95243e-3$	$3.95248e-3$	$3.0941e-7$	$3.0939e-7$
(−6, 6)	$1.49956e-2$	$1.49940e-2$	$1.49939e-2$	$1.49944e-2$	$1.1732e-6$	$1.1730e-6$
(−4, 4)	$5.68882e-2$	$5.68832e-2$	$5.68830e-2$	$5.68838e-2$	$4.4529e-6$	$4.4527e-6$
(−2, 2)	$2.15815e-1$	$2.15792e-1$	$2.15790e-1$	$2.15798e-1$	$1.6893e-5$	$1.6890e-5$
(0, 0)	$8.18731e-1$	$8.186677e-1$	$8.186670e-1$	$8.186667e-1$	$6.4086e-5$	$6.4082e-5$
(2, 2)	3.10599000	3.10575	3.10570	3.10566	$2.4312e-4$	$2.4310e-4$
(4, 4)	11.7831000	11.78212	11.78210	11.78218	$9.2233e-4$	$9.2226e-4$
(6, 6)	44.7011800	44.69756	44.69750	44.69769	$3.4989e-3$	$3.4982e-3$
(8, 8)	169.581450	169.56803	169.56810	169.56817	$1.3274e-2$	$1.3278e-2$
(10, 10)	643.335670	643.28526	643.28519	643.28534	$5.0357e-2$	$5.0350e-2$

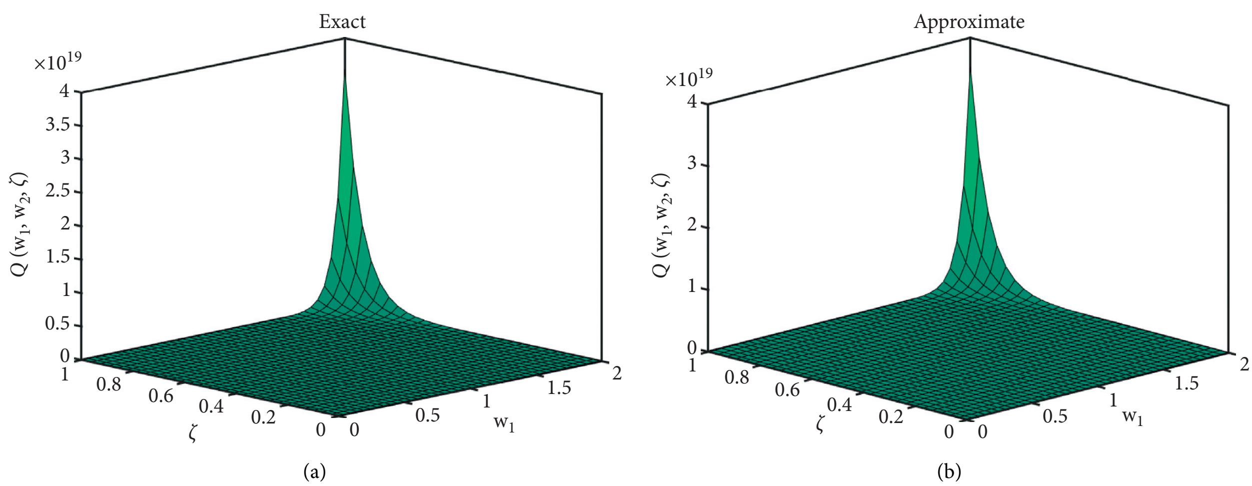


FIGURE 7: Three-dimensional illustration of exact and approximate solution of Example 3 when $\alpha = 1 = \beta, r_1 = 0, \varsigma = 0.01, \hbar = 1$, and $\chi = 1$.

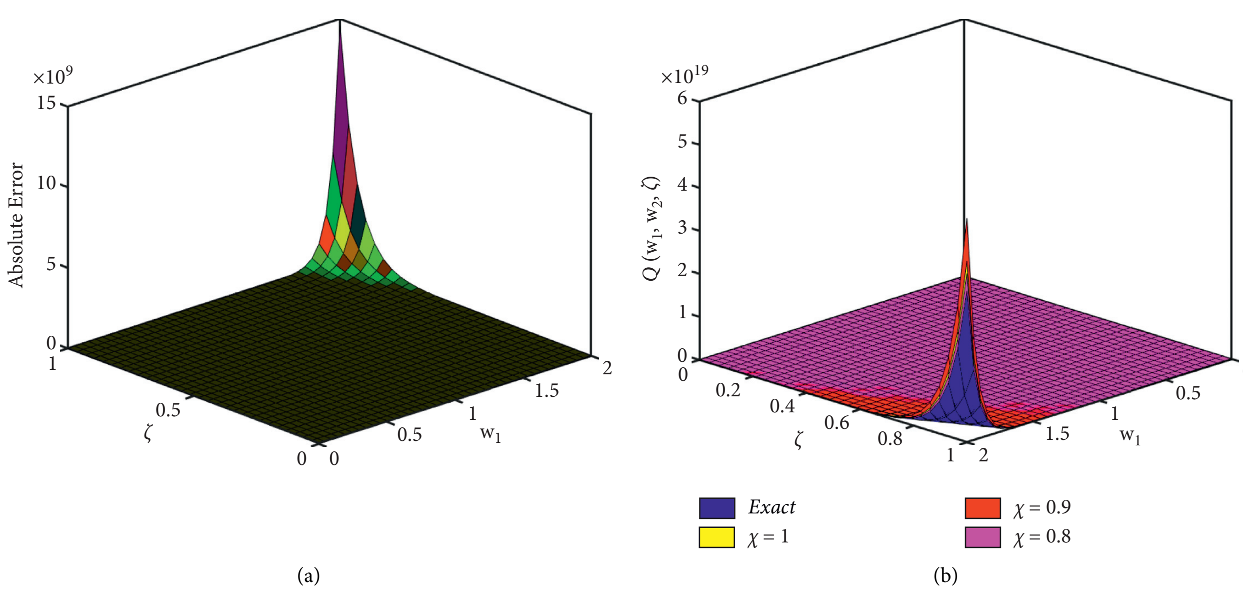


FIGURE 8: Three-dimensional illustration of the absolute error and multiple surface of various fractional orders for Example 3 when $\alpha = 1 = \beta, r_1 = 0, \hbar = 1$, and $\varsigma = 0.01$.

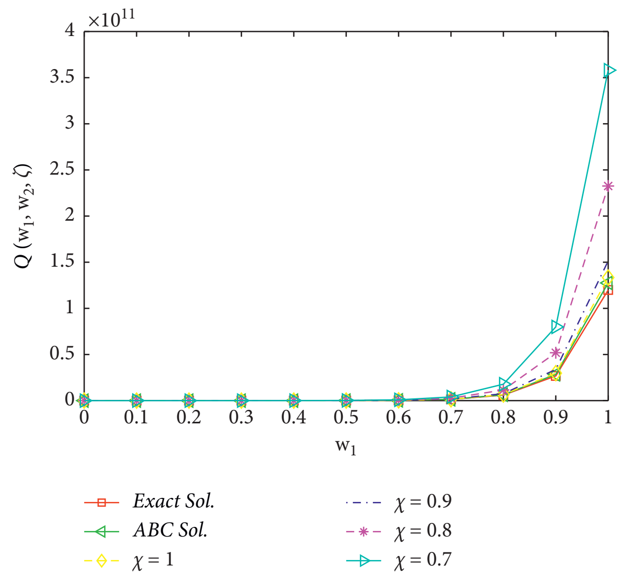


FIGURE 9: Two-dimensional illustration of Example 3 for various fractional orders when $\alpha = 1 = \beta, r_1 = 0, \hbar = 1$, and $\varsigma = 0.01$.

Also, the nonlinearity $\mathscr{F}_j(\mathcal{Q})$, $j = 1, 2, 3$, can be decomposed by an infinite series of polynomials represented by $\mathscr{F}_1(\mathcal{Q}) = \mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1} = \sum_{\ell=0}^{\infty} \mathscr{A}_\ell$ and $\mathscr{F}_2(\mathcal{Q}) = \mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2} = \sum_{\ell=0}^{\infty} \mathscr{B}_\ell$ defined in (44) along with $\mathscr{F}_3(\mathcal{Q}) = (1/96\mathcal{Q}_0) - (1/2) = \sum_{\ell=0}^{\infty} \mathscr{D}_\ell$. Thus, we have

$$\mathscr{D}_\ell\left(\frac{1}{96\mathcal{Q}} - \frac{1}{2}\right) = \begin{cases} \left(\dfrac{1}{96\mathcal{Q}_0} - \dfrac{1}{2}\right), & \ell = 0, \\[2ex] -\dfrac{1}{96}\left(\dfrac{\mathcal{Q}_1}{\mathcal{Q}_0^2}\right), & \ell = 1, \\[2ex] +\dfrac{1}{96}\left(\dfrac{-\mathcal{Q}_2}{\mathcal{Q}_0^2} + \dfrac{\mathcal{Q}_1^2}{\mathcal{Q}_0^3}\right), & \ell = 2, \end{cases} \tag{101}$$

$$\sum_{\ell=0}^{\infty} \mathcal{Q}_{\ell+1}(\mathbf{w}_1, \mathbf{w}_2, \varsigma) = \mathbb{J}^{-1}\left[\frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left[\sum_{\ell=0}^{\infty} (\mathscr{A})_\ell + \sum_{\ell=0}^{\infty} (\mathscr{B})_\ell + \sum_{\ell=0}^{\infty} (\mathscr{D})_\ell\right]\right], \quad \ell = 0, 1, 2, \ldots.$$

For $\ell = 0, 1, 2, 3, \ldots$,

$$\begin{aligned}
\mathcal{Q}_1(\mathbf{w}_1,\mathbf{w}_2,\varsigma) &= \mathbb{J}^{-1}\left[\frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left[\mathcal{A}_0+\mathcal{B}_0+\mathcal{D}_0\right]\right] \\
&= \frac{1}{24}\left(2\left(\mathbf{w}_1^2+\mathbf{w}_2^2+\mathbf{w}_2+5\right)\right)^{-1/2}\frac{\varsigma^{\chi}}{\Gamma(\chi+1)}, \\
\mathcal{Q}_2(\mathbf{w}_1,\mathbf{w}_2,\varsigma) &= \mathbb{J}^{-1}\left[\frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left[\mathcal{A}_1+\mathcal{B}_1+\mathcal{D}_1\right]\right] \\
&= -\frac{1}{144}\left(2\left(\mathbf{w}_1^2+\mathbf{w}_2^2+\mathbf{w}_2+5\right)\right)^{-3/2}\frac{\varsigma^{2\chi}}{\Gamma(2\chi+1)}, \\
\mathcal{Q}_3(\mathbf{w}_1,\mathbf{w}_2,\varsigma) &= \mathbb{J}^{-1}\left[\frac{1}{\psi^{\chi}(\mathfrak{s})}\mathbb{J}\left[\mathcal{A}_2+\mathcal{B}_2+\mathcal{D}_2\right]\right] \\
&= \frac{1}{288}\left(2\left(\mathbf{w}_1^2+\mathbf{w}_2^2+\mathbf{w}_2+5\right)\right)^{-5/2}\frac{\varsigma^{3\chi}}{\Gamma(3\chi+1)}, \\
&\vdots .
\end{aligned} \tag{102}$$

The approximate solution for Example 3 is expressed as follows:

$$\begin{aligned}
\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) &= \Phi_0(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\Phi_1(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\Phi_2(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\Phi_3(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\cdots \\
&= \frac{1}{4}\sqrt{2\left(\mathbf{w}_1^2+\mathbf{w}_2^2\right)+\mathbf{w}_2+5}+\frac{1}{24}\left(2\left(\mathbf{w}_1^2+\mathbf{w}_2^2+\mathbf{w}_2+5\right)\right)^{-1/2}\frac{\varsigma^{\chi}}{\Gamma(\chi+1)} \\
&\quad -\frac{1}{144}\left(2\left(\mathbf{w}_1^2+\mathbf{w}_2^2+\mathbf{w}_2+5\right)\right)^{-3/2}\frac{\varsigma^{2\chi}}{\Gamma(2\chi+1)}+\frac{1}{288}\left(2\left(\mathbf{w}_1^2+\mathbf{w}_2^2+\mathbf{w}_2+5\right)\right)^{-5/2}\frac{\varsigma^{3\chi}}{\Gamma(3\chi+1)}+\cdots .
\end{aligned} \tag{103}$$

Case 2. Here, we surmise ABC fractional derivative operator coupled with the generalized integral transform and Adomian decomposition method. Applying the generalized integral transform for Example 4,

$$\frac{\psi^{\chi}(\mathfrak{s})\mathbb{A}(\chi)}{\chi+(1-\chi)\psi^{\chi}(\mathfrak{s})}\mathcal{U}(\mathbf{w}_1,\mathfrak{s})-\phi(\mathfrak{s})\sum_{\kappa=0}^{m-1}\psi^{\chi-\kappa-1}(\mathfrak{s})\mathcal{Q}^{(\kappa)}(0) = \mathbb{J}\left[\mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\frac{1}{96\mathcal{Q}_0}-\frac{1}{2}\right]. \tag{104}$$

Taking into consideration the IC given in (95), we have

$$\mathcal{U}(\mathbf{w}_1,\mathfrak{s}) = \frac{\psi(\mathfrak{s})}{\phi(\mathfrak{s})}\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,0)+\frac{\chi+(1-\chi)\psi^{\chi}(\mathfrak{s})}{\psi^{\chi}(\mathfrak{s})\mathbb{A}(\chi)}\mathbb{J}\left[\mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\frac{1}{96\mathcal{Q}_0}-\frac{1}{2}\right]. \tag{105}$$

Employing the inverse generalized integral transform, we obtain

$$\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)=\mathbb{J}^{-1}\left[\frac{\psi(\mathfrak{s})}{\phi(\mathfrak{s})}\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,0)+\frac{\chi+(1-\chi)\psi^{\chi}(\mathfrak{s})}{\psi^{\chi}(\mathfrak{s})\mathbb{A}(\chi)}\mathbb{J}\left[\mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\frac{1}{96\mathcal{Q}_0}-\frac{1}{2}\right].\right. \quad (106)$$

Thanks to the generalized decomposition method, we find

$$\mathcal{Q}_0(\mathbf{w}_1,\mathbf{w}_2,\varsigma)=\mathbb{J}^{-1}\left[\frac{\psi(\mathfrak{s})}{\phi(\mathfrak{s})}\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,0)\right]=\mathbb{J}^{-1}\left[\frac{\psi(\mathfrak{s})}{\phi(\mathfrak{s})}\frac{1}{4}\sqrt{2\left(\mathbf{w}_1^2+\mathbf{w}_2^2\right)+\mathbf{w}_2+5}\right]$$
$$=\frac{1}{4}\sqrt{2\left(\mathbf{w}_1^2+\mathbf{w}_2^2\right)+\mathbf{w}_2+5}. \quad (107)$$

Here, we surmise that the unknown function $\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)$ can be written by an infinite series of the form

$$\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma)=\sum_{\ell=0}^{\infty}\mathcal{Q}_\ell(\mathbf{w}_1,\mathbf{w}_2,\varsigma). \quad (108)$$

Also, the nonlinearity $\mathcal{F}_j(\mathcal{Q})$, $j=1,2,3$, can be decomposed by an infinite series of polynomials represented by $\mathcal{F}_1(\mathcal{Q})=\mathcal{Q}^2_{\mathbf{w}_1\mathbf{w}_1}=\sum_{\ell=0}^{\infty}\mathcal{A}_\ell$ and $\mathcal{F}_2(\mathcal{Q})=\mathcal{Q}^2_{\mathbf{w}_2\mathbf{w}_2}=\sum_{\ell=0}^{\infty}\mathcal{B}_\ell$ along with $\mathcal{F}_3(\mathcal{Q})=(1/96\mathcal{Q})-(1/2)=\sum_{\ell=0}^{\infty}\mathcal{C}_\ell$ defined in (44) and (101), respectively.

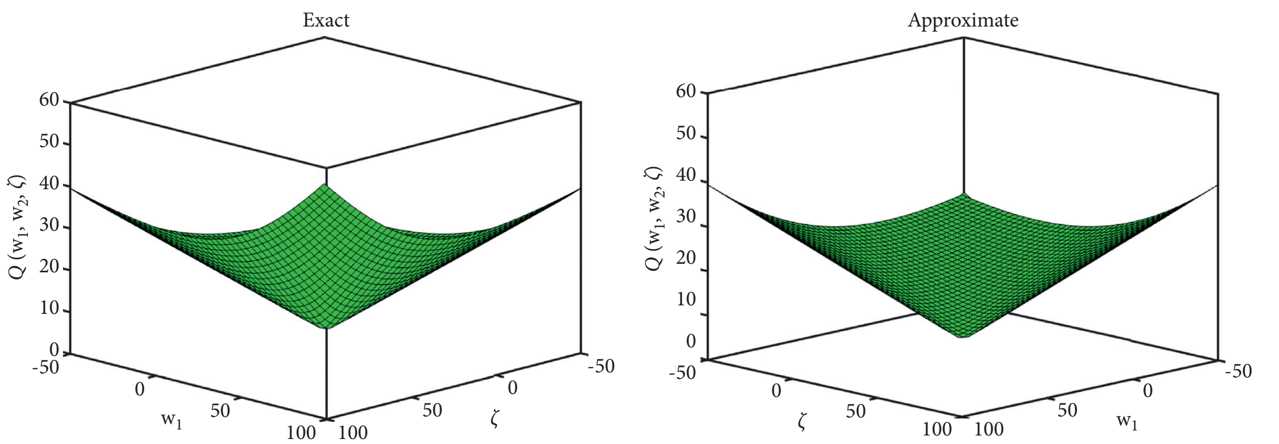
For $\ell=0,1,2,3,\ldots$,

$$\mathcal{Q}_1(\mathbf{w}_1,\mathbf{w}_2,\varsigma)=\mathbb{J}^{-1}\left[\frac{\chi+(1-\chi)\psi^{\chi}(\mathfrak{s})}{\psi^{\chi}(\mathfrak{s})\mathbb{A}(\chi)}\mathbb{J}\left[\mathcal{A}_0+\mathcal{B}_0+\mathcal{D}_0\right]\right]$$
$$=\frac{1}{24\mathbb{A}(\chi)}\left(2\left(\mathbf{w}_1^2+\mathbf{w}_2^2+\mathbf{w}_2+5\right)\right)^{-1/2}\left[\frac{\chi\varsigma^{\chi}}{\Gamma(\chi+1)}+(1-\chi)\right],$$
$$\mathcal{Q}_2(\mathbf{w}_1,\mathbf{w}_2,\varsigma)=\mathbb{J}^{-1}\left[\frac{\chi+(1-\chi)\psi^{\chi}(\mathfrak{s})}{\psi^{\chi}(\mathfrak{s})\mathbb{A}(\chi)}\mathbb{J}\left[\mathcal{A}_1+\mathcal{B}_1+\mathcal{D}_1\right]\right]$$
$$=-\frac{1}{144\mathbb{A}^2(\chi)}\left(2\left(\mathbf{w}_1^2+\mathbf{w}_2^2+\mathbf{w}_2+5\right)\right)^{-3/2}\left[\frac{\chi^2\varsigma^{2\chi}}{\Gamma(2\chi+1)}+2\chi(1-\chi)\frac{\varsigma^{\chi}}{\Gamma(\chi+1)}+(1-\chi)^2\right],$$
$$\mathcal{Q}_3(\mathbf{w}_1,\mathbf{w}_2,\varsigma)=\mathbb{J}^{-1}\left[\frac{\chi+(1-\chi)\psi^{\chi}(\mathfrak{s})}{\psi^{\chi}(\mathfrak{s})\mathbb{A}(\chi)}\mathbb{J}\left[\mathcal{A}_2+\mathcal{B}_2+\mathcal{D}_2\right]\right]$$
$$=\frac{1}{288\mathbb{A}^3(\chi)}\left(2\left(\mathbf{w}_1^2+\mathbf{w}_2^2+\mathbf{w}_2+5\right)\right)^{-5/2}\left[\frac{\chi^3\varsigma^{3\chi}}{\Gamma(3\chi+1)}+3\chi^2(1-\chi)\frac{\varsigma^{2\chi}}{\Gamma(2\chi+1)}+3\chi(1-\chi)^2\frac{\varsigma^{\chi}}{\Gamma(\chi+1)}+(1-\chi)^3\right],$$
$$\vdots. \quad (109)$$

The approximate solution for Example 4 is expressed as follows:

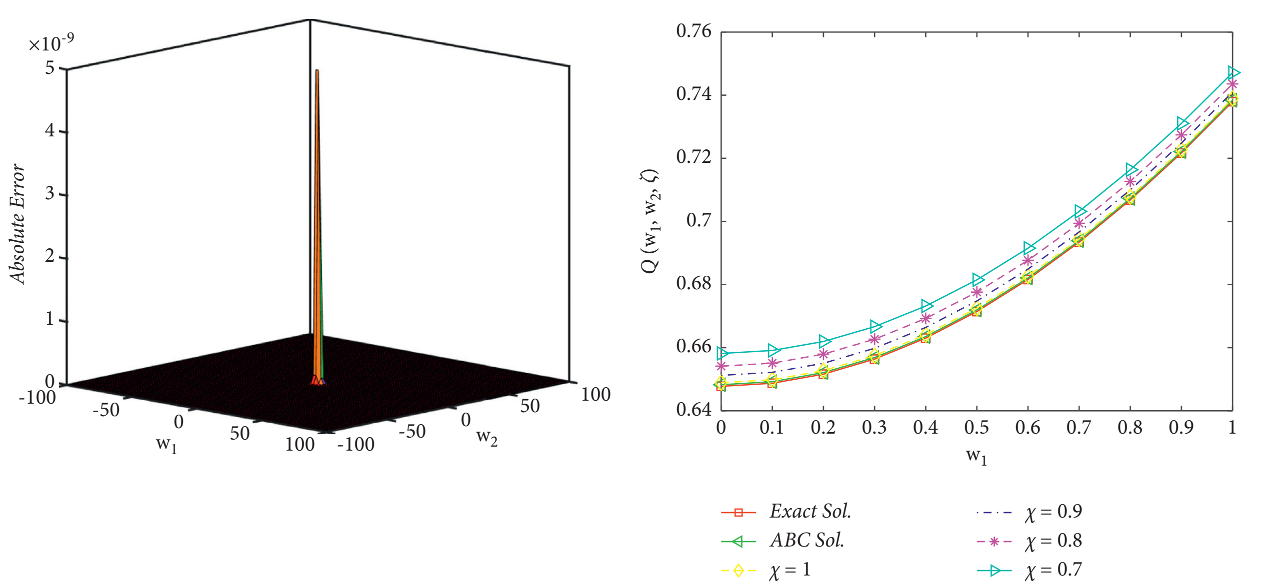
Table 4: Exact $\mathcal{Q}_E$ and approximate solution $\mathcal{Q}_A$ of $\mathcal{Q}(\mathbf{w}_1, \mathbf{w}_2, \varsigma)$ of Example 3 having absolute error when $\chi = 1, \varsigma = 10, r_1 = -8/9$, and $\hbar = -1$ for different values of $\mathbf{w}_1$ and $\mathbf{w}_2$.

$(\mathbf{w}_1, \mathbf{w}_2)$	$\mathcal{Q}_E$ sol.	$\mathcal{Q}_{\mathrm{CFD}}$ sol.	$\mathcal{Q}_{\mathrm{ABC}}$ sol.	VIMHP sol. [23]	Error = E_1	Error = E_2
(−450, 450)	224.939112	224.939000	224.938855	224.939112	$9.8139e-16$	$9.8127e-16$
(−400, 400)	199.939313	199.939304	199.939295	199.939316	$1.7688e-15$	$1.7679e-15$
(−300, 300)	149.939918	149.939822	149.939800	149.939925	$7.4574e-15$	$7.4570e-15$
(−250, 250)	124.940402	124.940400	124.940398	124.940445	$7.4465e-15$	$7.4456e-15$
(0, 0)	0.8531256	0.8531119	0.8531108	0.8531260	$7.3883e-3$	$7.3880e-3$
(50, 50)	25.07696486	25.07696467	25.07696402	25.07696508	$5.7070e-11$	$5.7004e-11$
(100, 100)	50.06964861	50.06964789	50.06964702	50.06964940	$1.7965e-12$	$1.7953e-12$
(200, 200)	100.0661239	100.0661130	100.0661009	100.0661400	$5.6333e-14$	$5.6300e-14$
(350, 350)	175.06457142	175.06457103	175.06457100	175.06457300	$3.4370e-15$	$3.4355e-15$
(500, 500)	250.0639500	250.0635573	250.0634435	250.0639946	$5.7707e-16$	$5.7700e-16$



(a) (b)

Figure 10: Three-dimensional illustration of exact and approximate solution of Example 4 when $\alpha = -1, \beta = 1, r_1 = 48, \varsigma = 0.01, \hbar = (1/96)$, and $\chi = 1$.



(a) (b)

Figure 11: Three-dimensional illustration of the absolute error and two-dimensional view of multiple fractional orders for Example 4 when $\alpha = -1, \beta = 1, r_1 = 48, \hbar = (1/96)$, and $\varsigma = 0.01$.

$$\begin{aligned}\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) &= \Phi_0(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\Phi_1(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\Phi_2(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\Phi_3(\mathbf{w}_1,\mathbf{w}_2,\varsigma)+\cdots\\
&= \frac{1}{4}\sqrt{2\left(\mathbf{w}_1^2+\mathbf{w}_2^2\right)+\mathbf{w}_2+5}+\frac{1}{24\mathbb{A}(\chi)}\left(2\left(\mathbf{w}_1^2+\mathbf{w}_2^2+\mathbf{w}_2+5\right)\right)^{-1/2}\left(\frac{\varsigma^{\chi}}{\Gamma(\chi+1)}+(1-\chi)\right)\\
&\quad-\frac{1}{144\mathbb{A}^2(\chi)}\left(2\left(\mathbf{w}_1^2+\mathbf{w}_2^2+\mathbf{w}_2+5\right)\right)^{-3/2}\left(\frac{\chi^2\varsigma^{2\chi}}{\Gamma(2\chi+1)}+2\chi(1-\chi)\frac{\varsigma^{\chi}}{\Gamma(\chi+1)}+(1-\chi)^2\right)\\
&\quad+\frac{1}{288\mathbb{A}^3(\chi)}\left(2\left(\mathbf{w}_1^2+\mathbf{w}_2^2+\mathbf{w}_2+5\right)\right)^{-5/2}\left(\frac{\chi^3\varsigma^{3\chi}}{\Gamma(3\chi+1)}+3\chi^2(1-\chi)\frac{\varsigma^{2\chi}}{\Gamma(2\chi+1)}+3\chi(1-\chi)^2\frac{\varsigma^{\chi}}{\Gamma(\chi+1)}+(1-\chi)^3\right)+\cdots.\end{aligned} \tag{110}$$

For $\chi = 1$, we obtained the exact solution of Example 4 as

$$\mathcal{Q}(\mathbf{w}_1,\mathbf{w}_2,\varsigma) = \frac{1}{4}\sqrt{2\left(\mathbf{w}_1^2+\mathbf{w}_2^2\right)+\mathbf{w}_2+\frac{\varsigma}{3}+5}. \tag{111}$$

Table 4 shows the analytical approximate solutions with some free parameters that are provided by the proposed technique. The comparison analysis is conducted with the VIMHP that predicts the preciseness of the suggested scheme due to their lower error. The analytical findings are extremely useful in deciphering the internal components of acts of nature. The exact and numerical solution for $\alpha = -1, \beta = 1, r_1 = 48, \varsigma = 0.01, \hbar = (1/96)$, and $\chi = 1$ is shown in Figure 10.

Furthermore, the absolute error for the aforementioned assumptions is depicted in Figure 11(a).

Finally, Figure 11(b) represents the two-dimensional behavior of exact, approximate by Caputo, and approximate solutions derived by ABC fractional operators with varying fractional orders. These solutions have a distinctive characteristic that allows them to interact with other solutions derived by [23]. The proposed findings have particle-like geometries in their solutions. The synthesized trajectory is either a success or a descent from one asymptotic state to the next. The accuracy of the proposed method can be enhanced by increasing the recursive terms. □

6. Conclusion

This article investigated the more general integral transform with the Adomian decomposition method. The Caputo and ABC fractional derivative operators have been implemented to deal with the biological population model. Several distinct solutions have been proposed with the assumptions of Malthusian law, Verhulst law, and porous media. Various representations were used to elucidate these solutions, which clarified the significant properties of the fractional models in consideration. Without any restrictive assumptions, discretization, or linearization, the proposed methodology locates the solutions. Elegance and originality have been invoked to describe our trajectory. Contrasting proposed findings to those acquired in earlier scholarly articles demonstrates the peculiarity of our solutions. The strategy's powerful and successful implementation is explored and validated in order to demonstrate its applicability to additional nonlinear evolution equations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

S. Rashid provided the main ideas of the article, constructed the main algorithm, proved the convergence, and also submitted the article. R. Ashraf drafted the manuscript and provided two test examples with their illustration and uniqueness analysis. E. Bonyah provided the solution of Examples 3 and 4 and completed the final revision. All authors read and approved the final manuscript.

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