Research Article

Boundedness and Compactness of Hankel Operators on Large Fock Space

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We introduce the BMO spaces and use them to characterize complex-valued functions f such that the big Hankel operators H_f and $H_{\bar{f}}$ are both bounded or compact from a weighted large Fock space $F^p(\phi)$ into a weighted Lebesgue space $L^p(\phi)$ when $1 \leq p < \infty$.

1. Introduction

Let $\mathbb{C}$ denote the set of complex numbers. For a fixed positive integer n , let $\mathbb{C}^n$ be the n -dimensional Euclidean space. Given $z \in \mathbb{C}^n$ and $r > 0$, let $B(z, r)$ denote the Euclidean ball defined as

$$B(z, r) = \{w \in \mathbb{C}^n : |w - z| < r\}. \quad (1)$$

Let Δ be the Laplace operator and dv denote the Lebesgue volume measure on $\mathbb{C}^n$. Suppose $\phi : \mathbb{C}^n \rightarrow \mathbb{R}$ is a C^2 -pluri-subharmonic function. We say that ϕ belongs to the weight class $\mathcal{W}$, if ϕ satisfies the following statements:

(A) There exists $c > 0$ such that for $z \in \mathbb{C}^n$

$$\inf_{z \in \mathbb{C}^n} \sup_{w \in B(z, c)} \Delta \phi(w) > 0. \quad (2)$$

(B) $\Delta \phi$ satisfies the reverse-Hölder's inequality

$$\|\Delta \phi\|_{L^\infty(B(z, r))} \leq Cr^{-2n} \int_{B(z, r)} \Delta \phi dv, \quad \forall z \in \mathbb{C}^n, r > 0, \quad (3)$$

for some $0 < C < \infty$

(C) The eigenvalues of H_ϕ are comparable; i.e., there exists $\delta_0 > 0$ such that

$$(H_\phi(z)u, u) \geq \delta_0 \Delta \phi(z) |u|^2, \quad \forall z, u \in \mathbb{C}^n, \quad (4)$$

where

$$H_\phi = \left(\frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} \right)_{j, k}. \quad (5)$$

Let $\phi \in \mathcal{W}$. For $0 < p < \infty$, the weighted Lebesgue space $L^p(\phi)$ is the set of those measurable functions f on $\mathbb{C}^n$ such that

$$\|f\|_{L^p(\phi)} := \left(\int_{\mathbb{C}^n} |f(z)|^p e^{-p\phi(z)} dv(z) \right)^{1/p} < \infty. \quad (6)$$

For $p = \infty$, the space $L^\infty(\phi)$ consists of all measurable functions f on $\mathbb{C}^n$ such that

$$\|f\|_{L^\infty(\phi)} := \sup_{z \in \mathbb{C}^n} |f(z)| e^{-\phi(z)} < \infty. \quad (7)$$

The weighted Fock space $F^p(\phi)$ consists of all analytic functions on $\mathbb{C}^n$ such that $\|f\|_{L^p(\phi)} < \infty$.

The weight class $\mathcal{W}$ is first introduced by Dall'Ara in [1] where the pointwise estimation of the Bergman kernel is given. Recently, the research on the weighted Fock space has been gradually improved. Lv [2] obtained characterizations on the boundedness of the Bergman projection. Arroussi and Tong [3] gave the $L^p(\phi)$ -norm estimation of the Bergman kernel and studied the boundedness, compactness, and Schatten class membership of the weighted composition operators. Liu and Wang [4] consider the duality of large Fock spaces and obtained equivalent conditions for compactness of weakly localized operators. The weight class $\mathcal{W}$ contains abundant functions. For example, when $\phi(z)$ is replaced by $|z|^2/2$, $F^2(\phi)$ is the classical Fock space (see [5]). The weight function ϕ satisfying $dd^c\phi \approx dd^c|z|^2$ in [6, 7] is the member of $\mathcal{W}$. It is interesting that the weight class $\mathcal{W}$ also contains some nonradial weights. An example of nonradial weight is given in [3].

For $1 \leq p < \infty$, $F^p(\phi)$ is a Banach space. In particular, $F^2(\phi)$ is a Hilbert space. Let $K_\phi(w, z)$ be the Bergman kernel of $F^2(\phi)$ and denote $K_z(w) = K_\phi(w, z)$ for simplicity. We know that $K_\phi(w, z) = K_\phi(z, w)$. The Bergman projection P can be represented as

$$Pg(z) = \int_{\mathbb{C}^n} K_\phi(z, w)g(w)e^{-2\phi(w)}d\nu(w), \quad z \in \mathbb{C}^n. \quad (8)$$

Moreover, P is bounded from $L^p(\phi)$ to $F^p(\phi)$ and $Pg = g$ for any $g \in F^p(\phi)$ when $1 \leq p < \infty$. See [2, 4] for more details. From [4], we know that

$$\Gamma = \left\{ \sum_{j=1}^N a_j K_\phi(\cdot, z_j) : N \in \mathbb{N}, a_j \in \mathbb{C}, z_j \in \mathbb{C}^n, \forall 1 \leq j \leq N \right\}, \quad (9)$$

is dense in $F^p(\phi)$. To let H_f make sense on Γ , we naturally consider those measurable functions f in the symbol class $\mathcal{S}$ defined as

$$\mathcal{S} = \{f : fg \in L^p(\phi), \forall g \in \Gamma\}. \quad (10)$$

It is obvious that the space $L^\infty(\mathbb{C}^n)$ of essentially bounded measurable functions on $\mathbb{C}^n$ is contained in $\mathcal{S}$. Given some symbol function $f \in \mathcal{S}$, one defines the big Hankel operator as

$$H_f g = fg - P(fg). \quad (11)$$

Using the property of reproducing kernel, it has the form of

$$H_f g(z) = \int_{\mathbb{C}^n} (f(z) - f(w))g(w)K_\phi(z, w)e^{-2\phi(w)}d\nu(w), \quad z \in \mathbb{C}^n. \quad (12)$$

Hankel operators on Fock spaces and Bergman spaces have been studied by many mathematicians during the past few decades; see [5, 8] for instance. The main problem we consider in this paper is to characterize the simultaneous boundedness and compactness of Hankel operators H_f and $H_{\bar{f}}$ from $F^p(\phi)$ to $L^p(\phi)$ with $1 \leq p < \infty$ in terms of suitable defined BMO spaces and VMO spaces. There are exclusive references for this issue both in the setting of Bergman spaces and Fock spaces. See [9–12] for Hankel operators that are simultaneously bounded or compact between the same exponents and [13–16] between different exponents.

This paper is organized as follows. In Section 2, we will give some basic useful results including the $L^p(\phi)$ -norm as well as pointwise estimation of the reproducing kernel. The space of BMO_r^p will be established in Section 3. The decomposition theorem of BMO_r^p is given. We will use the BMO_r^p space to characterize simultaneously bounded Hankel operators H_f and $H_{\bar{f}}$ in Section 4. The space VMO_r^p is introduced in Section 5, and it has similar results of the space BMO_r^p . Compact Hankel operators are characterized in Section 6.

Throughout this paper, we will use C to denote a positive constant whose value may change from line to line, but does not depend on functions being considered. We denote $A \leq B$ if there is a positive constant C such that $A \leq CB$. $A \approx B$ stands for $A \leq B$ and $B \leq A$.

2. Preliminaries

In this section, we are going to give some basic definitions and results, which will be frequently used. For $z \in \mathbb{C}^n$, set

$$\rho(z) := \sup \left\{ r > 0 : \sup_{w \in B(z, r)} \Delta\phi(w) \leq r^{-2} \right\}. \quad (13)$$

Given $r > 0$, write $B^r(z) = B(z, r\rho(z))$. We will call $\rho(z)$ a radius function.

Lemma 1. *Let $\phi \in \mathcal{W}$. Then, the function ρ satisfies the following properties:*

(A) ρ is bounded on $\mathbb{C}^n$, that is

$$\rho(z) \leq \sup_{z \in \mathbb{C}^n} \rho(z) =: M, \quad \forall z \in \mathbb{C}^n. \quad (14)$$

(B) For $0 < r < 1$, and $w \in B^r(z)$, there holds

$$(1-r)\rho(z) \leq \rho(w) \leq (1+r)\rho(z). \quad (15)$$

Proof. See [3]. $\square$

$\square$

Let $k_{p,z}$ be the normalized reproducing kernel in $F^p(\phi)$, that is

$$k_{p,z}(w) = \frac{K_z(w)}{\|K_z\|_{L^p(\phi)}}. \quad (16)$$

We are going to give some estimations of the reproducing kernel. One can find the proof in [3].

Lemma 2. *Let $\phi \in \mathcal{W}$, then one has*

(A) *For $0 < p \leq \infty$*

$$\|K_z\|_{L^p(\phi)} \simeq e^{\phi(z)} \rho(z)^{2n(1-p)/p}, \quad z \in \mathbb{C}^n. \quad (17)$$

(B) *There is some $\alpha \in (0, 1]$ such that*

$$|k_{p,z}(w)|^p e^{-p\phi(w)} \simeq \rho(z)^{-2n}, \quad w \in B^\alpha(z). \quad (18)$$

(C) *The set $\{k_{p,z} : z \in \mathbb{C}^n\}$ is bounded in $F^p(\phi)$ and $k_{p,z} \rightarrow 0$ uniformly on every compact subsets of $\mathbb{C}^n$ when $|z| \rightarrow \infty$.*

In the following argument, we will always assume α to be a constant as in (B) of Lemma 2. Notice that $\alpha \in (0, 1]$, so inequality (15) always holds when $0 < r < \alpha$. Before giving the pointwise estimation of the Bergman kernel, we need to introduce a distance function associated with the radius function ρ . In fact, we are interested with the Riemannian distance. To describe it explicitly, given $z, w \in \mathbb{C}^n$, we define

$$d(z, w) = \inf_{\gamma} \int_0^1 \frac{|\gamma'(t)|}{\rho(\gamma(t))} dt, \quad (19)$$

where the infimum is taken as $\gamma : [0, 1] \rightarrow \mathbb{C}^n$ which varies over the collection of piecewise C^1 -curves with $\gamma(0) = z$ and $\gamma(1) = w$. We define the ball of d -metric as

$$B_\rho(z, r) = \{w \in \mathbb{C}^n : d(z, w) < r\}. \quad (20)$$

The following lemma gives the pointwise estimation of $K_\phi(z, w)$.

Lemma 3. *There is a constant $\sigma > 0$ such that*

$$|K_\phi(z, w)| \leq e^{\phi(w) + \phi(z)} \frac{\exp\{-\sigma d(z, w)\}}{\rho(z)^n \rho(w)^n}, \quad \forall z \in \mathbb{C}^n. \quad (21)$$

Proof. See Theorem 20 of [1].

In what follows, we will always assume σ to be a constant as in Lemma 3. $\square$

3. The Space BMO

For $1 \leq p < \infty$, let $L_{loc}^p(\mathbb{C}^n)$ be the set of all p -th locally Lebesgue integrable functions on $\mathbb{C}^n$. Suppose $f \in L_{loc}^1(\mathbb{C}^n)$ and $r > 0$, the averaging function $\widehat{f}_r$ is defined as

$$\widehat{f}_r(z) = \frac{1}{|B^r(z)|} \int_{B^r(z)} f(w) dv(w), \quad (22)$$

where $|B^r(z)|$ is the volume of $B^r(z)$. For $f \in L_{loc}^p(\mathbb{C}^n)$, the p -th mean oscillation of f at z is given by

$$MO_r^p(f)(z) = \left(\frac{1}{|B^r(z)|} \int_{B^r(z)} |f(w) - \widehat{f}_r(z)|^p dv(w) \right)^{1/p}. \quad (23)$$

Suppose $r > 0$ and $1 \leq p < \infty$, the space BMO_r^p consists of those functions $f \in L_{loc}^p(\mathbb{C}^n)$ such that

$$\|f\|_{BMO_r^p} = \sup_{z \in \mathbb{C}^n} MO_r^p(f)(z) < \infty. \quad (24)$$

The following proposition reveals that the function $\widehat{f}_r(z)$ in (23) can be replaced by any complex-valued function in $\mathbb{C}^n$.

Proposition 4. *Let $1 \leq p < \infty$, $f \in L_{loc}^p(\mathbb{C}^n)$, and $r > 0$. Then, $f \in BMO_r^p$ if and only if there exists a complex-valued function $c(z)$ on $\mathbb{C}^n$ such that*

$$\frac{1}{|B^r(z)|} \int_{B^r(z)} |f(w) - c(z)|^p dv(w) \leq C, \quad z \in \mathbb{C}^n, \quad (25)$$

where C is a constant.

Proof. If $f \in BMO_r^p$, then (25) holds with $c(z) = \widehat{f}_r(z)$ and $C = \|f\|_{BMO_r^p}^p$.

Conversely, if (25) is satisfied, by the triangle inequality for the L^p -norm, we obtain

$$\begin{aligned} MO_r^p(f)(z) &= \left(\frac{1}{|B^r(z)|} \int_{B^r(z)} |f(w) - \widehat{f}_r(z)|^p dv(w) \right)^{1/p} \\ &\leq \left(\frac{1}{|B^r(z)|} \int_{B^r(z)} |f(w) - c(z)|^p dv(w) \right)^{1/p} \\ &\quad + \left(\frac{1}{|B^r(z)|} \int_{B^r(z)} |\widehat{f}_r(z) - c(z)|^p dv(w) \right)^{1/p} \\ &= \left(\frac{1}{|B^r(z)|} \int_{B^r(z)} |f(w) - c(z)|^p dv(w) \right)^{1/p} \\ &\quad + |\widehat{f}_r(z) - c(z)|. \end{aligned} \quad (26)$$

For $1 < p < \infty$, by the Hölder's inequality, one can get

$$\begin{aligned} |\hat{f}_r(z) - c(z)| &= \left| \frac{1}{|B^r(z)|} \int_{B^r(z)} f(w) - c(z) dv(w) \right| \\ &\leq \left(\frac{1}{|B^r(z)|} \int_{B^r(z)} |f(w) - c(z)|^p dv(w) \right)^{1/p}. \end{aligned} \quad (27)$$

For $p = 1$, it is obvious that

$$|\hat{f}_r(z) - c(z)| \leq \frac{1}{|B^r(z)|} \int_{B^r(z)} |f(w) - c(z)| dv(w). \quad (28)$$

So we have

$$MO_r^p(f)(z) \leq 2 \left(\frac{1}{|B^r(z)|} \int_{B^r(z)} |f(w) - c(z)|^p dv(w) \right)^{1/p}, \quad (29)$$

which implies

$$\|f\|_{BMO_r^p}^p \leq 2^p C. \quad (30)$$

This completes the proof. $\square$

Let $H(B^r(z))$ be the set of all analytic functions on $B^r(z)$.

Lemma 5. Let $1 \leq p < \infty$, $f \in L_{loc}^p(\mathbb{C}^n)$, and $r > 0$. If for any $z \in \mathbb{C}^n$, there exist functions $h_1, h_2 \in H(B^r(z))$ and constants $C_1, C_2 > 0$ such that

$$\begin{aligned} \frac{1}{|B^r(z)|} \int_{B^r(z)} |f(w) - h_1(w)|^p dv(w) &\leq C_1, \\ \frac{1}{|B^r(z)|} \int_{B^r(z)} |\bar{f}(w) - h_2(w)|^p dv(w) &\leq C_2, \end{aligned} \quad (31)$$

then $f \in BMO_r^p$.

Proof. The approach is similar to [13]. It is well known that, if $v : B(0, 1) \rightarrow \mathbb{R}$ is pluriharmonic, there exists some pluriharmonic function $u : B(0, 1) \rightarrow \mathbb{R}$ such that $u + iv \in H(B(0, 1))$. Theorem 1 from [17] tells us there is some constant $C > 0$ such that

$$\|u - u(0)\|_{L^p(B(0,1), dv)} \leq C \|v\|_{L^p(B(0,1), dv)}. \quad (32)$$

Hence, for any $r > 0$, by change of variables, we know that, if $v : B^r(z) \rightarrow \mathbb{R}$ is pluriharmonic, there exists a pluriharmonic function u such that $u + iv \in H(B^r(z))$ and

$$\|u - u(z)\|_{L^p(B^r(z), dv)} \leq C \|v\|_{L^p(B^r(z), dv)}. \quad (33)$$

For $f \in L_{loc}^p(\mathbb{C}^n)$, set $\|f\|_{p,z} = (1/|B^r(z)| \int_{B^r(z)} |f|^p dv)^{1/p}$.

By the triangle inequality, we have

$$\begin{aligned} \left\| \frac{f + \bar{f}}{2} - \frac{h_1 + h_2}{2} \right\|_{p,z} &\leq \left\| \frac{f - h_1}{2} \right\|_{p,z} + \left\| \frac{\bar{f} - h_2}{2} \right\|_{p,z} \\ &\leq \frac{1}{2} (C_1^{1/p} + C_2^{1/p}). \end{aligned} \quad (34)$$

Since $f + \bar{f}$ is real-valued, we get

$$\left\| \operatorname{Im} \frac{h_1 + h_2}{2} \right\|_{p,z} \leq \frac{1}{2} (C_1^{1/p} + C_2^{1/p}). \quad (35)$$

Notice that $h_1 + h_2 \in H(B^r(z))$, by (33), one has

$$\begin{aligned} \left\| \operatorname{Re} \frac{h_1 + h_2}{2} - \operatorname{Re} \frac{h_1 + h_2}{2}(z) \right\|_{p,z} \\ \leq C \left\| \operatorname{Im} \frac{h_1 + h_2}{2} \right\|_{p,z} \leq \frac{C}{2} (C_1^{1/p} + C_2^{1/p}). \end{aligned} \quad (36)$$

By the triangle inequality, there holds

$$\begin{aligned} \left\| \frac{f + \bar{f}}{2} - \operatorname{Re} \frac{h_1 + h_2}{2}(z) \right\|_{p,z} \\ \leq \left\| \frac{f + \bar{f}}{2} - \operatorname{Re} \frac{h_1 + h_2}{2} \right\|_{p,z} \\ + \left\| \operatorname{Re} \frac{h_1 + h_2}{2} - \operatorname{Re} \frac{h_1 + h_2}{2}(z) \right\|_{p,z} \\ \leq \left\| \frac{f + \bar{f}}{2} - \frac{h_1 + h_2}{2} \right\|_{p,z} + \left\| \operatorname{Re} \frac{h_1 + h_2}{2} - \operatorname{Re} \frac{h_1 + h_2}{2}(z) \right\|_{p,z} \\ \leq \frac{1+C}{2} (C_1^{1/p} + C_2^{1/p}). \end{aligned} \quad (37)$$

Similarly, there holds

$$\left\| \operatorname{Im} \frac{f - \bar{f}}{2} - \operatorname{Im} \frac{h_1 - h_2}{2}(z) \right\|_{p,z} \leq C_1^{1/p} + C_2^{1/p}. \quad (38)$$

Set $c(z) = \operatorname{Re}((h_1 + h_2)/2)(z) + i \cdot \operatorname{Im}((h_1 - h_2)/2)(z)$. By the triangle inequality again, we have

$$\|f - c(z)\|_{p,z} \leq C_1^{1/p} + C_2^{1/p}. \quad (39)$$

From this and Proposition 4, the desired result follows immediately. $\square$

To arrive at the decomposition theorem, we need to introduce two subspaces of BMO_r^p : one is BO_r , the other is BA_r^p .

For a continuous function f on $\mathbb{C}^n$ and radius $r > 0$, let

$$\omega_r(f)(z) = \sup \{ |f(z) - f(w)| : w \in B^r(z) \}. \quad (40)$$

The function $\omega_r(f)(z)$ is called the oscillation of f at z . For $r > 0$, let BO_r denote the space of all continuous functions f on $\mathbb{C}^n$ such that

$$\|f\|_{BO_r} = \sup_{z \in \mathbb{C}^n} \omega_r(f)(z) < \infty. \quad (41)$$

Before we give a characterization of functions $f \in BO_r$, we need the following lemma.

Lemma 6. *There holds*

$$B_\rho\left(z, \frac{r}{1+r}\right) \subseteq B^r(z) \subseteq B_\rho\left(z, \frac{r}{1-r}\right), \quad (42)$$

for $z \in \mathbb{C}^n$ and $r \in (0, 1)$.

Proof. If $w \in B^r(z)$, take the curve $\gamma(t) = z + t(z - w)$. For $t \in [0, 1]$, $\gamma(t) \in B^r(z)$. By (15), there holds

$$\rho(\gamma(t)) \geq (1-r)\rho(z), \quad \forall t \in [0, 1]. \quad (43)$$

It follows that

$$d(z, w) \leq \int_0^1 \frac{|z - w|}{\rho(\gamma(t))} dt \leq \frac{|z - w|}{(1-r)\rho(z)} \leq \frac{r\rho(z)}{(1-r)\rho(z)} = \frac{r}{1-r}, \quad (44)$$

which makes $B^r(z) \subseteq B_\rho(z, r/(1-r))$.

If $w \notin B^r(z)$, take any piecewise C^1 -curve $\gamma : [0, 1] \rightarrow \mathbb{C}^n$ connecting z and w , and let T_0 be the minimum time such that $|z - \gamma(T_0)| = r\rho(z)$. By (15), there holds

$$\rho(\gamma(t)) \leq (1+r)\rho(z), \quad \forall t \in [0, T_0]. \quad (45)$$

This implies that

$$\int_0^1 \frac{|\gamma'(t)|}{\rho(\gamma(t))} dt \geq \int_0^{T_0} \frac{|\gamma'(t)|}{\rho(\gamma(t))} dt \geq \int_0^{T_0} \frac{|\gamma'(t)|}{(1+r)\rho(z)} dt. \quad (46)$$

With the fact that $\int_0^{T_0} |\gamma'(t)| dt \geq r\rho(z)$, one has $d(z, w) \geq r/(1+r)$. It follows that $B_\rho(z, r/(1+r)) \subseteq B^r(z)$. The proof is completed. $\square$

Proposition 7. *Suppose $r \in (0, 1)$. Then, the space BO_r is independent of r . Moreover, a continuous function f on $\mathbb{C}^n$ belongs to BO_r if and only if for any $z, w \in \mathbb{C}^n$, there exists a constant $C_{f,r} > 0$ such that*

$$|f(z) - f(w)| \leq C_{f,r}(d(z, w) + 1), \quad (47)$$

where $C_{f,r} \approx \|f\|_{BO_r}$.

Proof. Suppose $f \in BO_r$. Set $M_r = r/(1+2r) < r/(1+r)$. According to Lemma 6, we get $w \in B^r(z)$ and $z \in B^r(w)$ whenever $d(z, w) \leq M_r$.

If $d(z, w) \leq M_r$, then for any $z \in \mathbb{C}^n$, there holds

$$|f(z) - f(w)| \leq \omega_r(f)(z) \leq \|f\|_{BO_r}(d(z, w) + 1). \quad (48)$$

Take $C_{f,r} = \|f\|_{BO_r}$, then the inequality (47) holds in the case of $d(z, w) \leq M_r$.

If $d(z, w) > M_r$, we replace points $z_0 = z, z_1, \dots, z_N = w$ on the geodesic in the d -metric from z to w in such a way: $d(z_i, z_{i-1}) = M_r$ for $i = 1, 2, \dots, N-1$ and $d(z_N, z_{N-1}) \leq M_r$. Note that it is possible to avoid using the completeness of the d -metric. More specifically, we do not have to use geodesics. Instead, we can simply use a curve from z and w whose length in the d -metric differs slightly from $d(z, w)$. Since $d(z, w) = (N-1)M_r + d(z_{N-1}, w)$, one can get

$$M_r + d(z, w) = NM_r + d(z_{N-1}, w) \geq NM_r. \quad (49)$$

It follows that

$$N \leq 1 + M_r^{-1}d(z, w) \leq \left(2 + \frac{1}{r}\right)(d(z, w) + 1). \quad (50)$$

By the triangle inequality, there holds

$$|f(z) - f(w)| \leq \sum_{i=1}^N |f(z_i) - f(z_{i-1})| \leq \sum_{i=1}^N \omega_r(f)(z_i) \leq N\|f\|_{BO_r}. \quad (51)$$

Take $C_{f,r} = (2 + 1/r)\|f\|_{BO_r}$, then the inequality (47) comes from combining (50) and (51).

Conversely, suppose (47) holds. Set $C_r = 2r/(1-r)$. According to Lemma 6, we have $d(z, w) \leq C_r$ whenever $w \in B^r(z)$. By (47), there holds

$$|f(z) - f(w)| \leq (C_r + 1)C_{f,r}, \quad \forall z \in \mathbb{C}^n, w \in B^r(z). \quad (52)$$

It follows that

$$\|f\|_{BO_r} \leq C_{f,r}. \quad (53)$$

This completes the proof. $\square$

From the above proposition, we have proved that BO_r is independent of the radius r for any $r \in (0, 1)$. So one can simply write BO for short.

For $1 \leq p < \infty$ and $r > 0$, let BA_r^p denote the space of all locally p -th integrable functions f on $\mathbb{C}^n$ such that

$$\|f\|_{BA_r^p} = \sup_{z \in \mathbb{C}^n} \left[\left(\widehat{|f|^p} \right)_r(z) \right]^{1/p} < \infty. \quad (54)$$

Given a Borel measure μ , a radius $r > 0$, and $1 \leq t < \infty$, we define the averaging function with respect to μ as follows:

$$\widehat{\mu}_r(z) = \frac{1}{|B^r(z)|} \int_{B^r(z)} d\mu(w), \quad \forall z \in \mathbb{C}^n. \quad (55)$$

Suppose μ is a Borel measure. Let $r > 0$ and $1 \leq t < \infty$; we define the t -Berezin transform with respect to μ as follows:

$$\tilde{\mu}_t(z) = \int_{\mathbb{C}^n} |k_{t,z}(w)|^t e^{-t\phi(w)} d\mu(w). \quad (56)$$

In particular, for a function f on $\mathbb{C}^n$, set $d\mu = f dv$, then

$$\tilde{f}_t(z) = \int_{\mathbb{C}^n} f(w) |k_{t,z}(w)|^t e^{-t\phi(w)} dv(w). \quad (57)$$

For $1 \leq p < \infty$, we call a positive Borel measure μ on $\mathbb{C}^n$ a Carleson measure if there is a positive constant C such that

$$\left(\int_{\mathbb{C}^n} |g(w) e^{-\phi(w)}|^p d\mu(w) \right)^{1/p} \leq C \|g\|_{L^p(\phi)}, \quad (58)$$

for every function $g \in F^p(\phi)$. This definition means that the identity mapping

$$I_\mu : F^p(\phi) \longrightarrow L^p(\mathbb{C}^n, e^{-p\phi} d\mu), \quad (59)$$

is bounded. The following lemma characterizes the Carleson measure, and one can find it from Theorem 3.2 in [3] and its proof.

Lemma 8. *Let μ be a finite positive Borel measure on $\mathbb{C}^n$. Assume $0 < p < \infty$. Then, the following conditions are all equivalent:*

- (A) *The measure μ is a Carleson measure for $F^p(\phi)$*
- (B) *The function $\tilde{\mu}_t \in L^\infty(\mathbb{C}^n, dv)$ for $t > 0$*
- (C) *The function $\hat{\mu}_r \in L^\infty(\mathbb{C}^n, dv)$ for $0 < r < \alpha$. Moreover*

$$\|I_\mu\|^p \lesssim \|\tilde{\mu}_t\|_{L^\infty(\mathbb{C}^n, dv)} \approx \|\hat{\mu}_r\|_{L^\infty(\mathbb{C}^n, dv)}. \quad (60)$$

Set $d\mu = |f|^p dv$. Then, we have $\hat{\mu}_r(z) = (\widehat{|f|^p})_r(z)$ and $\tilde{\mu}_t(z) = (\widehat{|f|^p})_t(z)$. The following lemma comes immediately.

Lemma 9. *Let $1 \leq p < \infty$ and $f \in L^p_{loc}(\mathbb{C}^n)$. Then, the following conditions are all equivalent:*

- (A) *The measure $d\mu = |f|^p dv$ is a Carleson measure for $F^p(\phi)$*
- (B) *The function $(\widehat{|f|^p})_t \in L^\infty(\mathbb{C}^n, dv)$ for $t > 0$*
- (C) *The function $(\widehat{|f|^p})_r \in L^\infty(\mathbb{C}^n, dv)$ for $0 < r < \alpha$, i.e., $f \in BA^p_r$. Moreover*

$$\|I_\mu\|^p \lesssim \|(\widehat{|f|^p})_t\|_{L^\infty(\mathbb{C}^n, dv)} \approx \|f\|_{BA^p_r}^p. \quad (61)$$

From the previous lemma, one can see that the space BA^p_r is independent of the choice of $r \in (0, \alpha)$. So we will write BA^p for short.

Lemma 10. *Let $p > 0, k \in \mathbb{R}$, then there holds*

$$\int_{\mathbb{C}^n} \rho(w)^k e^{-pd(z,w)} dv(w) \lesssim \rho(z)^{k+2n}. \quad (62)$$

Proof. See [2]. $\square$

Theorem 11. *Let $0 < r < \alpha$, $1 \leq p < \infty$, and $0 < t, l < \infty$. Then, the following conditions are all equivalent:*

- (A) $f \in BMO^p_r$
- (B) f admits a decomposition $f = f_1 + f_2$ where $f_1 \in BO_r$ and $f_2 \in BA^p_r$
- (C) *The function $\int_{\mathbb{C}^n} |f(w) - \tilde{f}_t(z)|^p \cdot |k_{l,z}(w)|^l e^{-l\phi(w)} dv(w)$ is bounded. Moreover*

$$\|f_1\|_{BO_r} + \|f_2\|_{BA^p_r} \approx \|f\|_{BMO^p_r}. \quad (63)$$

Proof. (A) $\Rightarrow$ (B): suppose $f \in BMO^p_r$. For $w \in B^{r/4}(z)$, we have $B^{r/4}(w) \subseteq B^r(z)$. In fact, let $\xi \in B^{r/4}(w)$. Since $0 < r < \alpha \leq 1$, using inequality (15), we know that

$$\left(1 - \frac{r}{4}\right) \rho(z) \leq \rho(w) \leq \left(1 + \frac{r}{4}\right) \rho(z). \quad (64)$$

Applying the triangle inequality, we get

$$\begin{aligned} |\xi - z| &\leq |\xi - w| + |w - z| < \frac{r}{4} \rho(w) + \frac{r}{4} \rho(z) \\ &\leq \left(1 + \frac{r}{4}\right) \frac{r}{4} \rho(z) + \frac{r}{4} \rho(z) < r \rho(z). \end{aligned} \quad (65)$$

Set $f_1 = \hat{f}_{r/4}$ and $f_2 = f - f_1$. For $1 < p < \infty$ and $w \in B^{r/4}(z)$, by the triangle inequality and Hölder's inequality, one can get

$$\begin{aligned} |f_1(z) - f_1(w)| &\leq |f_1(z) - \hat{f}_r(z)| + |\hat{f}_r(z) - f_1(w)| \\ &\leq \frac{1}{|B^{r/4}(z)|} \int_{B^{r/4}(z)} |f(u) - \hat{f}_r(z)| dv(u) \\ &\quad + \frac{1}{|B^{r/4}(w)|} \int_{B^{r/4}(w)} |f(u) - \hat{f}_r(z)| dv(u) \\ &\leq \left(\frac{1}{|B^r(z)|} \int_{B^r(z)} |f(u) - \hat{f}_r(z)|^p dv(u) \right)^{1/p}. \end{aligned} \quad (66)$$

For $p = 1$ and $w \in B^{r/4}(z)$, there also holds

$$|f_1(z) - f_1(w)| \leq \frac{1}{|B^r(z)|} \int_{B^r(z)} |f(u) - \hat{f}_r(z)| dv(u). \quad (67)$$

Thus $f_1 \in BO_{r/4}$. By Proposition 7, we have $f_1 \in BO_r$. Moreover,

$$\|f_1\|_{BO_r} \leq \|f\|_{BMO_r^p}. \quad (68)$$

Next, we prove the part with regard to f_2 . By the triangle inequality, one can get

$$\begin{aligned} MO_{r/4}^p(f)(z) &\leq \left(\frac{1}{|B^{r/4}(z)|} \int_{B^{r/4}(z)} |f(u) - \widehat{f}_r(z)|^p dv(u) \right)^{1/p} \\ &\quad + |\widehat{f}_r(z) - \widehat{f}_{r/4}(z)|. \end{aligned} \quad (69)$$

For $1 < p < \infty$, using the Hölder's inequality, there holds

$$\begin{aligned} |\widehat{f}_r(z) - \widehat{f}_{r/4}(z)| &= \left| \frac{1}{|B^{r/4}(z)|} \int_{B^{r/4}(z)} f(u) - \widehat{f}_r(z) dv(u) \right| \\ &\leq \left(\frac{1}{|B^{r/4}(z)|} \int_{B^{r/4}(z)} |f(u) - \widehat{f}_r(z)|^p dv(u) \right)^{1/p}. \end{aligned} \quad (70)$$

For $p = 1$, the above inequality is obvious. So we conclude that $MO_{r/4}^p(f)(z) \leq MO_r^p(f)(z)$ for all $z \in \mathbb{C}^n$.

$$\begin{aligned} \left((\widehat{|f_2|^p})_{r/4}(z) \right)^{1/p} &= \left(\frac{1}{|B^{r/4}(z)|} \int_{B^{r/4}(z)} |f(u) - f_1(u)|^p dv(u) \right)^{1/p} \\ &\leq \left(\frac{1}{|B^{r/4}(z)|} \int_{B^{r/4}(z)} |f(u) - \widehat{f}_{r/4}(z)|^p dv(u) \right)^{1/p} \\ &\quad + \left(\frac{1}{|B^{r/4}(z)|} \int_{B^{r/4}(z)} |f_1(u) - f_1(z)|^p dv(u) \right)^{1/p} \\ &\leq MO_{r/4}^p(f)(z) + \omega_{r/4}(f_1)(z) \\ &\leq MO_r^p(f)(z) + \omega_r(f_1)(z). \end{aligned} \quad (71)$$

Thus, $f_2 \in BA_{r/4}^p$. By Lemma 9, we have $f_2 \in BA_r^p$. Moreover,

$$\|f_2\|_{BA_r^p} \leq \|f\|_{BMO_r^p}. \quad (72)$$

(B) $\Rightarrow$ (C): set $f = f_1 + f_2$ where $f_1 \in BO_r$ and $f_2 \in BA_r^p$.

By the triangle inequality, there holds

$$\begin{aligned} &\left(\int_{\mathbb{C}^n} |f_1(w) - \widetilde{(f_1)_t}(z)|^p \cdot |k_{t,z}(w)|^l e^{-l\phi(w)} dv(w) \right)^{1/p} \\ &\leq \left(\int_{\mathbb{C}^n} |f_1(w) - f_1(z)|^p \cdot |k_{t,z}(w)|^l e^{-l\phi(w)} dv(w) \right)^{1/p} \\ &\quad + \left(\int_{\mathbb{C}^n} |f_1(z) - \widetilde{(f_1)_t}(z)|^p \cdot |k_{t,z}(w)|^l e^{-l\phi(w)} dv(w) \right)^{1/p} \\ &= \left(\int_{\mathbb{C}^n} |f_1(w) - f_1(z)|^p \cdot |k_{t,z}(w)|^l e^{-l\phi(w)} dv(w) \right)^{1/p} \\ &\quad + |f_1(z) - \widetilde{(f_1)_t}(z)|. \end{aligned} \quad (73)$$

By (17) and Lemma 3, we have

$$|k_{t,z}(w)|^l e^{-l\phi(w)} \simeq \rho(z)^{nl-2n} \rho(w)^{-nl} e^{-\sigma l d(z,w)}. \quad (74)$$

Since $(x+1)^a e^{-bx} \rightarrow 0$ as $x \rightarrow +\infty$ for any $a, b > 0$, there exists a constant C such that

$$\sup_{z,w \in \mathbb{C}^n} (d(z,w) + 1)^p e^{-(1/2)\sigma l d(z,w)} \leq C. \quad (75)$$

By Proposition 7, Lemma 10, (74), and (75), one can get

$$\begin{aligned} &\int_{\mathbb{C}^n} |f_1(w) - f_1(z)|^p \cdot |k_{t,z}(w)|^l e^{-l\phi(w)} dv(w) \\ &\leq \|f_1\|_{BO_r}^p \int_{\mathbb{C}^n} (d(z,w) + 1)^p \rho(z)^{nl-2n} \rho(w)^{-nl} e^{-\sigma l d(z,w)} dv(w) \\ &\leq \|f_1\|_{BO_r}^p \int_{\mathbb{C}^n} \rho(z)^{nl-2n} \rho(w)^{-nl} e^{-(1/2)\sigma l d(z,w)} dv(w) \leq \|f_1\|_{BO_r}^p. \end{aligned} \quad (76)$$

For $1 < p < \infty$, by Hölder's inequality and (76), we obtain

$$\begin{aligned} |f_1(z) - \widetilde{(f_1)_t}(z)| &= \left| \int_{\mathbb{C}^n} (f_1(z) - f_1(w)) \cdot |k_{t,z}(w)|^t e^{-t\phi(w)} dv(w) \right| \\ &\leq \left(\int_{\mathbb{C}^n} |f_1(w) - f_1(z)|^p \cdot |k_{t,z}(w)|^t e^{-t\phi(w)} dv(w) \right)^{1/p} \\ &\leq \|f_1\|_{BO_r}. \end{aligned} \quad (77)$$

The same is true for (77) when $p = 1$. By (73), (76), and (77), there holds

$$\left(\int_{\mathbb{C}^n} |f_1(w) - \widetilde{(f_1)_t}(z)|^p \cdot |k_{t,z}(w)|^l e^{-l\phi(w)} dv(w) \right)^{1/p} \leq \|f_1\|_{BO_r}. \quad (78)$$

Next, we prove the part with regard to f_2 . For $1 < p < \infty$,

applying Hölder's inequality, we obtain

$$|\widetilde{(f_2)_t}(z)| \lesssim \left[(\widetilde{|f_2|^p})_t(z) \right]^{1/p}. \quad (79)$$

For $p = 1$, inequality (79) is obvious. By the triangle inequality, (79), and Lemma 9, there holds

$$\begin{aligned} & \left(\int_{\mathbb{C}^n} |f_2(w) - \widetilde{(f_2)_t}(z)|^p \cdot |k_{t,z}(w)|^l e^{-l\phi(w)} dv(w) \right)^{1/p} \\ & \leq \left(\int_{\mathbb{C}^n} |f_2(w)|^p \cdot |k_{t,z}(w)|^l e^{-l\phi(w)} dv(w) \right)^{1/p} \\ & \quad + \left(\int_{\mathbb{C}^n} |\widetilde{(f_2)_t}(z)|^p \cdot |k_{t,z}(w)|^l e^{-l\phi(w)} dv(w) \right)^{1/p} \quad (80) \\ & = \left[(\widetilde{|f_2|^p})_t(z) \right]^{1/p} + |\widetilde{(f_2)_t}(z)| \\ & \leq \left[(\widetilde{|f_2|^p})_t(z) \right]^{1/p} + \left[(\widetilde{|f_2|^p})_t(z) \right]^{1/p} \lesssim \|f_2\|_{BA_r^p}. \end{aligned}$$

By (78), (80), and the triangle inequality, one can get

$$\begin{aligned} & \left(\int_{\mathbb{C}^n} |f(w) - \widetilde{f}_t(z)|^p \cdot |k_{t,z}(w)|^l e^{-l\phi(w)} dv(w) \right)^{1/p} \quad (81) \\ & \leq \|f_1\|_{BO_r} + \|f_2\|_{BA_r^p}. \end{aligned}$$

(C) $\Rightarrow$ (A): by (18) and $|B^r(z)| \approx \rho(z)^{2n}$, we get

$$\frac{1}{|B^r(z)|} \approx |k_{t,z}(w)|^l e^{-l\phi(w)}, \quad w \in B^r(z). \quad (82)$$

Hence, there holds

$$\begin{aligned} & \frac{1}{|B^r(z)|} \int_{B^r(z)} |f(w) - \widetilde{f}_t(z)|^p dv(w) \\ & \approx \int_{B^r(z)} |f(w) - \widetilde{f}_t(z)|^p \cdot |k_{t,z}(w)|^l e^{-l\phi(w)} dv(w) \quad (83) \\ & \leq \int_{\mathbb{C}^n} |f(w) - \widetilde{f}_t(z)|^p \cdot |k_{t,z}(w)|^l e^{-l\phi(w)} dv(w). \end{aligned}$$

By Proposition 4 with $c(z) = \widetilde{f}_t(z)$, we have $f \in \text{BMO}_r^p$.

The estimation (63) comes from (68), (72), (81), and (83). The proof is finished. $\square$

4. Bounded Hankel Operators

In this section, we are devoted to characterizing complex-valued functions f in $\mathcal{S}$ such that the big Hankel operators H_f and $H_{\bar{f}}$ are simultaneously bounded from $F^p(\phi)$ into $L^p(\phi)$ with $1 \leq p < \infty$.

Theorem 12. *Let $1 \leq p < \infty$, $0 < r < \alpha$, $\phi \in \mathcal{W}$, and $f \in \mathcal{S}$. If $f \in \text{BO}_r$, then $H_f : F^p(\phi) \longrightarrow L^p(\phi)$ is bounded. Moreover,*

$$\|H_f\| \lesssim \|f\|_{\text{BO}_r}. \quad (84)$$

Proof. By Lemma 3 and Proposition 7, we have

$$\begin{aligned} |H_f g(z)| e^{-\phi(z)} &= |(I - P)(f g)(z)| e^{-\phi(z)} \\ &\leq e^{-\phi(z)} \int_{\mathbb{C}^n} |f(z) - f(w)| \cdot |g(w)| \cdot |K_\phi(z, w)| e^{-2\phi(w)} dv(w) \\ &\leq \|f\|_{\text{BO}_r} \int_{\mathbb{C}^n} (d(z, w) + 1) \rho(z)^{-n} \rho(w)^{-n} e^{-\sigma d(z, w)} |g(w)| e^{-\phi(w)} dv(w). \end{aligned} \quad (85)$$

By Lemma 10, one can get

$$\int_{\mathbb{C}^n} \rho(z)^{-n} \rho(w)^{-n} e^{-(1/2)\sigma d(z, w)} \leq C. \quad (86)$$

By (75) and (86), there holds

$$|H_f g(z)| e^{-\phi(z)} \lesssim \|f\|_{\text{BO}_r} \|g\|_{L^\infty(\phi)}, \quad (87)$$

for every $z \in \mathbb{C}^n$. It follows that H_f is bounded from $F^\infty(\phi)$ to $L^\infty(\phi)$ and $\|H_f\| \lesssim \|f\|_{\text{BO}_r}$.

For $p = 1$, applying Fubini's theorem, (75), and (86), we have

$$\begin{aligned} \int_{\mathbb{C}^n} |H_f g(z)| e^{-\phi(z)} dv(z) &\leq \|f\|_{\text{BO}_r} \int_{\mathbb{C}^n} |g(w)| e^{-\phi(w)} dv(w) \int_{\mathbb{C}^n} (d(z, w) \\ &\quad + 1) \rho(z)^{-n} \rho(w)^{-n} e^{-\sigma d(z, w)} dv(z) \\ &\lesssim \|f\|_{\text{BO}_r} \|g\|_{L^1(\phi)}. \end{aligned} \quad (88)$$

Thus, we see that $H_f : F^1(\phi) \longrightarrow L^1(\phi)$ is bounded and $\|H_f\| \lesssim \|f\|_{\text{BO}_r}$.

By the Riesz-Thorin interpolation theorem, we conclude that $H_f : F^p(\phi) \longrightarrow L^p(\phi)$ is bounded for all $1 \leq p < \infty$ and $\|H_f\| \lesssim \|f\|_{\text{BO}_r}$. $\square$

Lemma 13. *Let $1 \leq p < \infty$, $0 < r < \alpha$, $\phi \in \mathcal{W}$, and $f \in \mathcal{S}$. If $f \in \text{BA}_r^p$, then H_f is bounded from $F^p(\phi)$ to $L^p(\phi)$. Moreover,*

$$\|H_f\| \lesssim \|f\|_{\text{BA}_r^p}. \quad (89)$$

Proof. By Lemma 9, $f \in \text{BA}_r^p$ if and only if $|f|^p dv$ is a Carleson measure for $F^p(\phi)$. Moreover,

$$\left(\int_{\mathbb{C}^n} |f(w) g(w) e^{-\phi(w)}|^p dv(w) \right)^{1/p} \lesssim \|f\|_{\text{BA}_r^p} \|g\|_{L^p(\phi)}. \quad (90)$$

for any $g \in F^p(\phi)$. Since P is bounded on $L^p(\phi)$, there holds

$$\|H_f g\|_{L^p(\phi)} \leq \|I - P\| \cdot \|f g\|_{L^p(\phi)} \lesssim \|f\|_{\text{BA}_r^p} \|g\|_{L^p(\phi)}. \quad (91)$$

It follows that H_f is bounded and $\|H_f\| \lesssim \|f\|_{\text{BA}_r^p}$. $\square$

Theorem 14. *Let $1 \leq p < \infty$, $0 < r < \alpha$, $\phi \in \mathcal{W}$, and $f \in \mathcal{S}$. Then, H_f and $H_{\bar{f}}$ are both bounded from $F^p(\phi)$ to $L^p(\phi)$ if*

and only if $f \in \text{BMO}_r^p$. Moreover,

$$\|H_{\bar{f}}\| + \|H_f\| \approx \|f\|_{\text{BMO}_r^p}. \quad (92)$$

Proof. By (18), for any $z \in \mathbb{C}^n$, we have

$$\begin{aligned} \|H_f\|^p &\geq \|H_f k_{p,z}\|_{L^p(\phi)}^p = \int_{\mathbb{C}^n} |f(w)k_{p,z}(w) - P(fk_{p,z})(w)|^p e^{-p\phi(w)} dv(w) \\ &\geq \int_{B^r(z)} |f(w) - k_{p,z}^{-1}(w)P(fk_{p,z})(w)|^p |k_{p,z}(w)|^p e^{-p\phi(w)} dv(w) \\ &\approx \frac{1}{|B^r(z)|} \int_{B^r(z)} |f(w) - k_{p,z}^{-1}(w)P(fk_{p,z})(w)|^p dv(w). \end{aligned} \quad (93)$$

Similarly, there holds

$$\|H_{\bar{f}}\|^p \geq \frac{1}{|B^r(z)|} \int_{B^r(z)} |\bar{f}(w) - k_{p,z}^{-1}(w)P(\bar{f}k_{p,z})(w)|^p dv(w). \quad (94)$$

By (18), for $0 < r < \alpha$ and $w \in B^r(z)$, we have $k_{p,z}(w) \neq 0$. Thus,

$$k_{p,z}(w)^{-1}P(fk_{p,z})(w) \in H(B^r(z)), \quad (95)$$

$$k_{p,z}(w)^{-1}P(\bar{f}k_{p,z})(w) \in H(B^r(z)). \quad (96)$$

Therefore, by Lemma 5, (93), and (94), we have $f \in \text{BMO}_r^p$. Furthermore, following the proof of Lemma 5, we will get $\|f - c(z)\|_{p,z} \leq \|H_f\| + \|H_{\bar{f}}\|$. Then, checking the proof of Proposition 4, we obtain

$$\|f\|_{\text{BMO}_r^p} \leq \|H_f\| + \|H_{\bar{f}}\|. \quad (97)$$

Conversely, suppose $f \in \text{BMO}_r^p$. By Theorem 11, f has a decomposition $f = f_1 + f_2$ where $f_1 \in \text{BO}_r$ and $f_2 \in \text{BA}_r^p$. Then, by Lemma 12 and Lemma 13, H_f is bounded from $F^p(\phi)$ to $L^p(\phi)$. Moreover,

$$\|H_f\| \leq \|f_1\|_{\text{BO}_r} + \|f_2\|_{\text{BA}_r^p} \approx \|f\|_{\text{BMO}_r^p}. \quad (98)$$

Since $f \in \text{BMO}_r^p$ if and only if $\bar{f} \in \text{BMO}_r^p$, it follows that $H_{\bar{f}}$ is bounded from $F^p(\phi)$ to $L^p(\phi)$. Moreover,

$$\|H_{\bar{f}}\| \leq \|f_1\|_{\text{BO}_r} + \|f_2\|_{\text{BA}_r^p} \approx \|f\|_{\text{BMO}_r^p}. \quad (99)$$

Then, the norm estimation follows from these two inequalities above and (97). This completes the proof. $\square$

We immediately have the following corollaries.

Corollary 15. Let $1 \leq p < \infty$, $0 < r < \alpha$, $\phi \in \mathcal{W}$, and $f \in \mathcal{S}$. If f is real-valued on $\mathbb{C}^n$, then H_f is bounded from $F^p(\phi)$ to $L^p(\phi)$ if and only if $f \in \text{BMO}_r^p$.

Corollary 16. Let $1 \leq p < \infty$, $0 < r < \alpha$, $\phi \in \mathcal{W}$, and $f \in \mathcal{S}$. If f is analytic on $\mathbb{C}^n$, then $H_{\bar{f}}$ is bounded from $F^p(\phi)$ to $L^p(\phi)$ if and only if $f \in \text{BMO}_r^p$.

5. The Space VMO

Suppose $r > 0$. We say that f belongs to VMO_r^p if $f \in \text{BMO}_r^p$ and

$$\lim_{|z| \rightarrow \infty} \text{MO}_r^p(f)(z) = 0. \quad (100)$$

As in the space BMO, we have the following two lemmas. The proof is similar to the corresponding case of BMO. One can prove them with easy modifications, so we omit details here.

Lemma 17. Let $1 \leq p < \infty$, $f \in L_{\text{loc}}^p(\mathbb{C}^n)$, and $r > 0$. Then, $f \in \text{VMO}_r^p$ if and only if there exists a complex-valued function $c(z)$ on $\mathbb{C}^n$ such that

$$\lim_{|z| \rightarrow \infty} \frac{1}{|B^r(z)|} \int_{B^r(z)} |f(w) - c(z)|^p dv(w) = 0. \quad (101)$$

Lemma 18. Let $1 \leq p < \infty$, $f \in L_{\text{loc}}^p(\mathbb{C}^n)$, and $r > 0$. If for any $z \in \mathbb{C}^n$, there exist functions $h_1, h_2 \in H(B^r(z))$ such that

$$\begin{aligned} \lim_{|z| \rightarrow \infty} \frac{1}{|B^r(z)|} \int_{B^r(z)} |f(w) - h_1(w)|^p dv(w) &= 0, \\ \lim_{|z| \rightarrow \infty} \frac{1}{|B^r(z)|} \int_{B^r(z)} |\bar{f}(w) - h_2(w)|^p dv(w) &= 0, \end{aligned} \quad (102)$$

then $f \in \text{VMO}_r^p$.

Let $r > 0$. We define VO_r as the space of complex-valued functions f in BO_r satisfying

$$\lim_{|z| \rightarrow \infty} \omega_r(f)(z) = 0. \quad (103)$$

Define VA_r^p as the space of complex-valued functions f in BA_r^p satisfying

$$\lim_{|z| \rightarrow \infty} \left[\left(\widehat{|f|^p} \right)_r(z) \right]^{1/p} = 0. \quad (104)$$

The next result shows that VO_r is independent of the radius r .

Lemma 19. Let $0 < r_1, r_2 < \alpha$. Then, $f \in \text{VO}_{r_1}$ if and only if $f \in \text{VO}_{r_2}$.

Proof. Without loss of generality, we may assume $r_1 < r_2$. Since $\omega_{r_1}(f)(z) \leq \omega_{r_2}(f)(z)$, we have $f \in \text{VO}_{r_1}$ if $f \in \text{VO}_{r_2}$.

Conversely, suppose $f \in VO_{r_1}$. For any $z \in \mathbb{C}^n$, since f is continuous on $\mathbb{C}^n$, there holds

$$\omega_{r_2}(f)(z) = \sup \{ |f(z) - f(u)| : u \in B^{r_2}(z) \}. \quad (105)$$

Hence, there exists some $w \in B^{r_2}(z)$ such that

$$\omega_{r_2}(f)(z) = |f(z) - f(w)|. \quad (106)$$

For a sufficiently small $\varepsilon > 0$, $(1 + \varepsilon)r_2 < 1$, we can apply Lemma 6 with $r = (1 + \varepsilon)r_2$ and obtain

$$B^{r_2}(z) \subseteq B^{(1+\varepsilon)r_2}(z) \subseteq B_\rho \left(z, \frac{(1+\varepsilon)r_2}{1 - (1+\varepsilon)r_2} \right). \quad (107)$$

Therefore,

$$d(z, w) < \frac{(1+\varepsilon)r_2}{1 - (1+\varepsilon)r_2}. \quad (108)$$

Let $\varepsilon \rightarrow 0$, we get $d(z, w) \leq r_2/(1 - r_2)$. We then follow a similar argument as Proposition 7. We replace points

$$z_0 = z, z_1, \dots, z_N = w, \quad (109)$$

on the geodesic in the d -metric from z to w in such a way: $d(z_i, z_{i-1}) = M_{r_1}$ for $i = 1, 2, \dots, N-1$ and $d(z_N, z_{N-1}) \leq M_{r_1}$ where $M_{r_1} = r_1/(1 - 2r_1)$ is an appropriate constant such that $z_i \in B^{r_1}(z_{i-1})$. We will get a similar inequality as (50):

$$N \leq \left(2 + \frac{1}{r_1} \right) (d(z, w) + 1) \leq \left(2 + \frac{1}{r_1} \right) \left(\frac{r_2}{1 - r_2} + 1 \right) =: N_0. \quad (110)$$

Since $f \in VO_{r_1}$, for any $\varepsilon > 0$, there exists some positive number M_1 such that

$$\omega_{r_1}(f)(z) < \frac{\varepsilon}{N_0}, \quad |z| > M_1. \quad (111)$$

By the triangle inequality, we get

$$\begin{aligned} |z - z_i| &\leq |z - z_1| + \dots + |z_i - z_{i-1}| \leq r_1 \rho(z) + \dots + r_1 \rho(z_{i-1}) \\ &\leq N_0 r_1 \sup_{z \in \mathbb{C}^n} \rho(z) =: M_2. \end{aligned} \quad (112)$$

For $|z| \geq M_1 + M_2$, we clearly have $|z_i| \geq |z| - |z_i - z| \geq M_1$. Thus, if we set $M_0 = M_1 + M_2$, then for $|z| > M_0$, there

holds

$$\begin{aligned} \omega_{r_2}(f)(z) &= |f(z) - f(w)| \leq \sum_{i=1}^N |f(z_i) - f(z_{i-1})| \\ &\leq \sum_{i=1}^N \omega_{r_1}(f)(z_i) \leq \frac{N\varepsilon}{N_0} \leq \varepsilon. \end{aligned} \quad (113)$$

The proof is finished. $\square$

A positive Borel measure μ on $\mathbb{C}^n$ is called a vanishing Carleson measure if

$$\lim_{j \rightarrow \infty} \int_{\mathbb{C}^n} |g_j(w) e^{-\phi(w)}|^p d\mu(w) = 0, \quad (114)$$

whenever $\{g_j\}_{j=1}^\infty$ is a bounded sequence in $F^p(\phi)$ and converges to 0 uniformly on any compact subset of $\mathbb{C}^n$ as $j \rightarrow \infty$.

The following lemma characterizes the vanishing Carleson measure, and one can find it from Theorem 3.9 in [3].

Lemma 20. *Let μ be a finite positive Borel measure on $\mathbb{C}^n$. Assume $0 < p < \infty$. Then, the following conditions are all equivalent:*

- (A) *The measure μ is a vanishing Carleson measure*
- (B) *The function $\tilde{\mu}_t \rightarrow 0$ as $|z| \rightarrow \infty$ for $t > 0$*
- (C) *The function $\hat{\mu}_r \rightarrow 0$ as $|z| \rightarrow \infty$ for $0 < r < \alpha$*

Set $d\mu = |f|^p dv$. Then, we have $\hat{\mu}_r(z) = (\widehat{|f|^p})_r(z)$ and $\tilde{\mu}_t(z) = (\widetilde{|f|^p})_t(z)$. The following lemma comes immediately.

Lemma 21. *Let $1 \leq p < \infty$ and $f \in L_{loc}^p(\mathbb{C}^n)$. Then, the following conditions are all equivalent:*

- (A) *The measure $d\mu = |f|^p dv$ is a vanishing Carleson measure*
- (B) *The function $(\widetilde{|f|^p})_t \rightarrow 0$ as $|z| \rightarrow \infty$ for $t > 0$*
- (C) *The function $(\widehat{|f|^p})_r \rightarrow 0$ as $|z| \rightarrow \infty$ for $0 < r < \alpha$, i.e., $f \in VA_r^p$*

From the previous lemma, one can see that the space VA_r^p is independent of the choice of $r \in (0, \alpha)$. So we will write VA^p for short.

By the following decomposition theorem, we know that the space VMO_r^p is independent of the choice of $r \in (0, \alpha)$.

Theorem 22. *Let $0 < r < \alpha$, $1 \leq p < \infty$, and $0 < t, l < \infty$. Then, the following conditions are all equivalent:*

- (A) *$f \in VMO_r^p$*

(B) f admits a decomposition $f = f_1 + f_2$ where $f_1 \in VO_r$ and $f_2 \in V\mathcal{A}_r^p$

(C) The function $\int_{\mathbb{C}^n} |f(w) - \tilde{f}_t(z)|^p \cdot |k_{t,z}(w)|^l e^{-l\phi} dv(w) \rightarrow 0$ as $|z| \rightarrow \infty$

Proof. The proof is similar to Theorem 11. Notice that $\{k_{t,z}\}$ converges uniformly to 0 as $|z| \rightarrow \infty$ on any compact subset of $\mathbb{C}^n$. One can conduct the same argument to prove that (B) implies (C). $\square$

6. Compact Hankel Operators

Theorem 23. Let $1 \leq p < \infty$, $0 < r < \alpha$, $\phi \in \mathcal{W}$, and $f \in \mathcal{S}$. Then, H_f and $H_{\bar{f}}$ are both compact from $F^p(\phi)$ to $L^p(\phi)$ if and only if $f \in \text{VMO}_r^p$.

Proof. Notice that $\{k_{p,z}\}$ converges to 0 on any compact subset of $\mathbb{C}^n$. Similar to the proof of Theorem 14, we have

$$\begin{aligned} \lim_{|z| \rightarrow \infty} \frac{1}{|B^r(z)|} \int_{B^r(z)} |f(w) - k_{p,z}^{-1}(w) P(fk_{p,z})(w)|^p dv(w) &= 0, \\ \lim_{|z| \rightarrow \infty} \frac{1}{|B^r(z)|} \int_{B^r(z)} |\bar{f}(w) - k_{p,z}^{-1}(w) P(\bar{f}k_{p,z})(w)|^p dv(w) &= 0. \end{aligned} \quad (115)$$

By (95), (96), and Lemma 18, we get $f \in \text{VMO}_r^p$.

Conversely, suppose $f \in \text{VMO}_r^p$. Let $f_R := f \chi_{B(0,R)}$ where $\chi_{B(0,R)}$ is the characteristic function of $B(0,R)$. Since $f \in \text{VMO}_r^p$, for any $\varepsilon > 0$, there is some $R > 0$ such that $MO_r^p(f)(z) < \varepsilon$ when $|z| > R/2$. Without loss of generality, we may suppose $r < R/4M$ where $M = \sup_{z \in \mathbb{C}^n} \rho(z)$. Then, we have $MO_r^p(f - f_R)(z) = MO_r^p(f)(z) < \varepsilon$ for $|z| > 2R$. Therefore, $\|f - f_R\|_{\text{BMO}_r^p} \rightarrow 0$ as $R \rightarrow \infty$. Applying Theorem 14 to $f - f_R$, there holds

$$\|H_{f-f_R}\| \lesssim \|f - f_R\|_{\text{BMO}_r^p}. \quad (116)$$

By the fact that the space of compact operators is norm closed, we only need to prove that $H_{f_R} : F^p(\phi) \rightarrow L^p(\phi)$ is compact.

Fix $R > 0$. Let $\{g_j\}_{j=1}^\infty$ be a bounded sequence in $F^p(\phi)$ that converges to 0 uniformly on any compact subset of $\mathbb{C}^n$. From Theorem 11, we know that $(\widehat{f_R})_r$ belongs to BO_r , since $f_R \in \text{BMO}_r^p$. It is clear that $(\widehat{f_R})_r$ has compact support. For z in some compact subset of $\mathbb{C}^n$, using Proposition 7, we obtain

$$|(\widehat{f_R})_r(z) - (\widehat{f_R})_r(0)| \lesssim d(z, 0) + 1 < \infty. \quad (117)$$

Hence, $(\widehat{f_R})_r$ is bounded. From Lemma 3.3 of [3], we

know that

$$\begin{aligned} \int_{\mathbb{C}^n} |g_j(z)|^p e^{-p\phi(z)} |f_R|^p(z) dv(z) \\ \leq C \int_{\mathbb{C}^n} |g_j(z)|^p e^{-p\phi(z)} (\widehat{|f_R|^p})_r(z) dv(z). \end{aligned} \quad (118)$$

Therefore,

$$\begin{aligned} \|f_R g_j\|_{L^p(\phi)} &= \left(\int_{\mathbb{C}^n} |f_R(z) g_j(z)|^p e^{-p\phi(z)} dv(z) \right)^{1/p} \\ &\leq \left(\int_{\mathbb{C}^n} |g_j(z)|^p e^{-p\phi(z)} (\widehat{|f_R|^p})_r(z) dv(z) \right)^{1/p} \\ &= \left(\int_E |g_j(z)|^p e^{-p\phi(z)} \frac{1}{|B^r(z)|} \int_{B^r(z)} |f_R(w)|^p dv(w) dv(z) \right)^{1/p}, \end{aligned} \quad (119)$$

where E is the support of $(\widehat{|f_R|^p})_r$. We also have

$$\begin{aligned} |f_R(w) - (\widehat{f_R})_r(z) + (\widehat{f_R})_r(z)|^p \\ \lesssim |f_R(w) - (\widehat{f_R})_r(z)|^p + |(\widehat{f_R})_r(z)|^p. \end{aligned} \quad (120)$$

For every $\varepsilon > 0$, there is some j_0 such that

$$\sup_{z \in E} |g_j(z)| < \varepsilon, \quad (121)$$

for $j \geq j_0$. Consequently,

$$\begin{aligned} \|f_R g_j\|_{L^p(\phi)} &\leq \left(\int_E |g_j(z)|^p e^{-p\phi(z)} \frac{1}{|B^r(z)|} \int_{B^r(z)} [|f_R(w) - (\widehat{f_R})_r(z)|^p \right. \\ &\quad \left. + |(\widehat{f_R})_r(z)|^p] dv(w) dv(z) \right)^{1/p} \\ &= \left(\int_E |g_j(z)|^p e^{-p\phi(z)} [MO_r^p(f_R)(z)^p \right. \\ &\quad \left. + |(\widehat{f_R})_r(z)|^p] dv(z) \right)^{1/p} \\ &\leq \varepsilon \|f_R\|_{\text{BMO}_r^p} + \varepsilon \|(\widehat{f_R})_r\|_{L^\infty(\mathbb{C}^n, dv)}, \end{aligned} \quad (122)$$

for $j \geq j_0$. This means that $\lim_{j \rightarrow \infty} \|f_R g_j\|_{L^p(\phi)} = 0$. Since $f_R g_j \in L^p(\phi)$ and P is bounded on $L^p(\phi)$, there holds

$$\|H_{f_R} g_j\|_{L^p(\phi)} \leq \|I - P\| \cdot \|f_R g_j\|_{L^p(\phi)}. \quad (123)$$

Thus, H_{f_R} is compact. Since $f \in \text{VMO}_r^p$ if and only if $\bar{f} \in \text{VMO}_r^p$, $H_{\bar{f}}$ is also compact. $\square$

Corollary 24. Let $1 \leq p < \infty$, $0 < r < \alpha$, $\phi \in \mathcal{W}$, and $f \in \mathcal{S}$. If $f \in VO_r$, then $H_f : F^p(\phi) \rightarrow L^p(\phi)$ is compact.

Corollary 25. Let $1 \leq p < \infty$, $0 < r < \alpha$, $\phi \in \mathcal{W}$, and $f \in \mathcal{S}$. If $f \in VA_r^p$, then $H_f : F^p(\phi) \longrightarrow L^p(\phi)$ is compact.

Corollary 26. Let $1 \leq p < \infty$, $0 < r < \alpha$, $\phi \in \mathcal{W}$, and $f \in \mathcal{S}$. If f is real-valued on $\mathbb{C}^n$, then H_f is compact from $F^p(\phi)$ to $L^p(\phi)$ if and only if $f \in VMO_r^p$.

Corollary 27. Let $1 \leq p < \infty$, $0 < r < \alpha$, $\phi \in \mathcal{W}$, and $f \in \mathcal{S}$. If f is analytic on $\mathbb{C}^n$, then $H_{\bar{f}}$ is compact from $F^p(\phi)$ to $L^p(\phi)$ if and only if $f \in VMO_r^p$.

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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