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Research Article

Regularity for 3D Inhomogeneous Naiver–Stokes Equations in Vishik Spaces

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In this short paper, we consider the conditional regularity for the 3D inhomogeneous incompressible Navier–Stokes equations in Vishik spaces and give regularity criterion of strong solutions.

1. Introduction

We consider the regularity issue for solutions (ρ, u, Π): $Q_T \longrightarrow \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}$ to 3D inhomogeneous incompressible Navier–Stokes equations for $Q_T := \mathbb{R}^3 \times [0, T)$:

$$\partial_t \rho + u \cdot \nabla \rho = 0, \tag{1}$$

$$\begin{gathered} \rho u_t - \Delta u + \rho (u \cdot \nabla) u + \nabla \Pi = 0, \\ \operatorname{div} \ u = 0. \end{gathered} \tag{2}$$

Here, ρ is the density function of flow velocity, u is the flow velocity, and Π is the pressure. We consider the initial value problem of (1), which requires initial

$$\begin{gathered} \rho(x, 0) = \rho_0(x), \\ u(x, 0) = u_0(x), x \in \mathbb{R}^3. \end{gathered} \tag{3}$$

There is a very rich literature dedicated to the study of the above system. In the case of smooth data with no vacuum, Kazhikov [1] proved that the nonhomogeneous Navier–Stokes equations have at least one global weak solution in the energy space. When the initial data may contain vacuum states, Simon [2] proved the global existence of a weak solution to the equations of incompressible, viscous, nonhomogeneous fluid flow in a bounded domain of two or three spaces, under the no-slip boundary condition. Choe and Kim [3] proposed a compatibility condition and investigated the local existence of strong solutions. More precisely, under the compatibility condition,

$$\Delta u_0 - \Pi_0 = \rho_0^{1/2} g \text{ and } \operatorname{div} \ u_0 = 0, \quad \text{for } a.e. x \in \Omega, \tag{4}$$

For initial data,

$$0 \le \rho_0 \in \left(L^{3/2} \cap L^\infty \cap H^1\right)(\Omega) \, u_0 \in \left(H^1 \cap H^2\right)(\Omega), \tag{5}$$

they proved the local-in-time existence for solutions in the class

$$\begin{gathered} \rho \in L^\infty\left(0, T^* ; H^1(\Omega)\right), \\ \rho_t \in L^\infty\left(0, T^* ; L^2(\Omega)\right), \end{gathered} \tag{6}$$

$$\begin{gathered} \nabla^3 u \in L^2\left(0, T^* ; L^{3/2}\right), \\ \nabla \Pi \in L^\infty\left(0, T^*, L^2(\Omega)\right) \cap L^2\left(0, T^* ; L^6 \cap W^{1,3/2}(\Omega)\right). \end{gathered} \tag{7}$$

Here, $\Omega \subseteq \mathbb{R}^3$ is a bounded domain or whole space. After that, Craig et al. [4] improved the above result to global strong small solutions. Very recently, without compatibility conditions, for any initial data $(\rho_0, u_0) \in (W^{1,\gamma} \cap L^\infty) \times H^1_{0,\sigma}$ with $\gamma > 1$, Li showed the existence of local strong solution for the initial-boundary value problem to the nonhomogeneous incompressible Navier–Stokes equations in the class

$$\rho \in L^{\infty}\left(0, T ; W^{1,\gamma} \cap L^{\infty}\right) \cap C([0,T], L^{\gamma})((\Omega)),$$
$$u \in L^{\infty}\left(0, T, H^1_{0,\sigma}(\Omega)\right) \cap L^2\left(0, T, H^2(\Omega)\right), \rho u \in C\left(0, T, L^2(\Omega)\right),$$
$$\sqrt{t} u \in L^{\infty}\left(0, T, H^2(\Omega)\right) \cap L^2\left(0, T, W^{2,6}\right), \sqrt{t}\partial_t u \in L^2\left(0, T, H^1(\Omega)\right). \tag{8}$$

Moreover, if $\gamma \geq 2$, then, the strong solution is unique.

On the other hand, for the regularity issue to system (1)–(3), Kim [5] proved the following regularity condition:

$$u \in L^s\left(0, T ; L^{p,\infty}\left(\mathbb{R}^3\right)\right), \frac{2}{s} + \frac{3}{p} = 1, 3 < p \leq \infty. \tag{9}$$

And Zhou and Fan [6] showed the following regularity condition:

$$u \in L^{2/1-r}\left(0, T, \dot{\mathscr{M}}_{2,3/r}\left(\mathbb{R}^3\right)\right), \quad \text{with } 0 < r < 1. \tag{10}$$

Here, $\mathscr{M}^{\cdot}_{2,3/r}(\mathbb{R}^3)$ stands for the homogeneous Morrey space (see Appendix).

Before stating our result, we now introduce a Banach space $\dot{V}^s_{p,\sigma,\theta}$ which is larger than the homogeneous Besov space; see [7, 8].

Definition 1. Let $s \in \mathbb{R}, p, \sigma \in [1,\infty], \theta \in [1, \sigma]$; the Vishik space $\dot{V}^s_{p,\sigma,\theta}$ is defined by

$$\dot{V}^s_{p,\sigma,\theta}\left(\mathbb{R}^3\right) \coloneqq \left\{f \in \mathscr{D}'\left(\mathbb{R}^3\right) : \|f\|_{\dot{V}^s_{p,\sigma,\theta}} < \infty\right\}, \tag{11}$$

with the norm

$$\|f\|_{\dot{V}^s_{p,\sigma,\theta}\left(\mathbb{R}^3\right)} \coloneqq \sup_{N=1,2,\cdots,} \frac{\left(\sum_{|j|<N} 2^{js\theta}\left\|\dot{\Delta}_j f\right\|^{\theta}_{L^p}\right)^{1/\theta}}{N^{1/\theta-1/\sigma}}, \quad \theta \neq \infty, \tag{12}$$

and if $\theta = \infty$, $\|f\|_{\dot{V}^s_{p,\sigma,\theta}(\mathbb{R}^3)} \coloneqq \|f\|_{\dot{B}^0_{p,\infty}(\mathbb{R}^3)}$.

Here, $\mathscr{D}'(\mathbb{R}^3)$ is the dual space of

$$\mathscr{D}\left(\mathbb{R}^3\right) = \left\{f \in \mathscr{S}\left(\mathbb{R}^3\right) ; D^{\alpha}\widehat{f}(0) = 0, \forall \alpha \in \mathbb{N}^3\right\}. \tag{13}$$

Motivated by [7, 9], now, we are ready to state our first main result.

Theorem 2. *Let $T > 0$. Assume that the initial data (ρ_0, u_0) satisfy the initial condition (5) and the compatibility condition (4). Let (ρ, u) be the corresponding unique local strong solution to system (1)–(3) with the properties stated in (6). If additionally for all $t \in [0, T)$*

$$\int_0^t \|u(\tau)\|^{2p/p-3}_{\dot{V}^0_{p,\sigma,\theta}} d\tau < \infty, \quad 3 < p \leq \infty, \sigma, \theta \in [1,\infty], \tag{14}$$

then, the solution (ρ, u) can be extended smoothly beyond time $t = T$.

Remark 3. As mentioned in [9], we remind that the following continuous embeddings hold:

$$\dot{B}^s_{p,\sigma}\left(\mathbb{R}^3\right) = \dot{V}^s_{p,\sigma,\sigma}\left(\mathbb{R}^3\right) \subset \dot{V}^s_{p,\sigma,\theta_1}\left(\mathbb{R}^3\right) \subset \dot{V}^s_{p,\sigma,\theta_2}\left(\mathbb{R}^3\right) \subset \dot{V}^s_{p,\sigma,1}\left(\mathbb{R}^3\right), \tag{15}$$

for $s \in \mathbb{R}$, $p, \sigma \in [1,\infty]$, and $\theta_1, \theta_2 \in [1, \sigma]$ with $\theta_1 \geq \theta_2$. For this reason, (13) is a stronger condition than

$$\int_0^t \|u(\tau)\|^{2p/p-3}_{\dot{B}^0_{p,\infty}} d\tau < \infty, \quad 3 < p \leq \infty, \sigma, \theta \in [1,\infty]. \tag{16}$$

Remark 4. By the same calculations as those in [10, 11], for the initial data (ρ_0, u_0) satisfying that $\rho \in L^{6/5}(\mathbb{R}^3)$, $\rho_0|u_0|^2 \in L^1(\mathbb{R}^3)$, $\rho_0 u_0 \in L^1(\mathbb{R}^3)$, and $\int_{\mathbb{R}^3} \rho_0 u_0 \, dx \neq 0$, there exists no global-in-time smooth solution to Cauchy problem (1)–(3).

2. Proof of Theorem 2

We first introduce some notations. Let $(X, \|\cdot\|)$ be a normed space. By $L^q(0, T ; X)$, we denote the space of all Bochner measurable functions $\varphi : (0, T) \longrightarrow X$ such that

$$\|\varphi\|_{L^q(0,T;X)} \coloneqq \left(\int_0^T \|\varphi(t)\|^q dt\right)^{1/q} < \infty, \quad 1 \leq q < \infty,$$
$$\|\varphi\|_{L^{\infty}(0,T;X)} \coloneqq \sup_{t \in (0,T)} \|\varphi(t)\| < \infty, \quad q = \infty. \tag{17}$$

Unless specifically mentioned, letter C is used to represent a generic constant, which may change from line to line.

Proof. By the maximum principle, we note that

$$\sup_{0 \leq t \leq T} \|\rho(\cdot, t)\|_{L^{\infty}} \leq \|\rho_0\|_{L^{\infty}} < \infty. \tag{18}$$

And also, by L^2-energy estimate, we know that

$$\sup_{0 \leq t \leq T}\left(\left\|\rho^{1/2} u(\cdot, t)\right\|^2_{L^2\left(\mathbb{R}^3\right)} + 2\int_0^T \|\nabla u(\cdot, t)\|^2_{L^2\left(\mathbb{R}^3\right)} dt \leq C\left\|\rho_0^{1/2} u_0\right\|^2_{L^2\left(\mathbb{R}^3\right)}. \tag{19}$$

To exclude the pressure term, multiplying $(1.1)_2$ by u_t and using Hölder's inequality, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2+\int_{\mathbb{R}^3}\rho|u_t|^2\,dx &\le \int_{\mathbb{R}^3}\left|\rho^{1/2}u\cdot\nabla u\cdot\rho^{1/2}u_t\right|dx\\ &\le \int_{\mathbb{R}^3}\left|\rho^{1/2}\sum_{j<-N}\Delta_j u\|\nabla u\|\rho^{1/2}u_t\right|dx\\ &\quad+\int_{\mathbb{R}^3}\left|\rho^{1/2}\sum_{j=-N}^{j=N}\Delta_j u\|\nabla u\|\rho^{1/2}u_t\right|dx\\ &\quad+\int_{\mathbb{R}^3}\left|\rho^{1/2}\sum_{j>-N}\Delta_j u\|\nabla u\|\rho^{1/2}u_t\right|dx\\ &\coloneqq I+II+III,\end{aligned} \tag{20}$$

where we use the decomposition of u. Let us control each term sequentially: the term (I):

$$\begin{aligned} I &\le \left\|\rho^{1/2}\right\|_{L^\infty}\left\|\sum_{j<-N}\Delta_j u\right\|_{L^\infty}\|\nabla u\|_{L^2}\left\|\rho^{1/2}u_t\right\|_{L^2}\\ &\le C\sum_{j<-N}2^{3/2j}\|u\|_{L^2}\|\nabla u\|_{L^2}\left\|\rho^{1/2}u_t\right\|_{L^2}\\ &\le C2^{-3/2N}x\left\|\rho^{1/2}u_t\right\|_{L^2}\|\nabla u\|_{L^2}\left\|\rho^{1/2}u_t\right\|_{L^2}\\ &\le C2^{-3N^2}\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2+\frac{1}{32}\left\|\rho^{1/2}u_t\right\|_{L^2}^2,\end{aligned} \tag{21}$$

the term (II):

$$\begin{aligned} II &\le \sum_{j=-N}^{j=N}\left\|\rho^{\frac{1}{2}}\Delta_j u\right\|_{L^p}\|\nabla u\|_{L^{p-2/2p}}\left\|\rho^{1/2}u_t\right\|_{L^2}\\ &\le C\sum_{j=-N}^{j=N}\left\|\Delta_j u\right\|_{L^p}\|\nabla u\|_{L^2}^{1-3/p}\|\nabla^2 u\|_{L^2}^{3/p}\left\|\rho^{1/2}u_t\right\|_{L^2}\\ &\le CN^{1-1/\sigma}\sup_{N=1,2,\cdots}\frac{\sum_{j=-N}^{j=N}\left\|\dot{\Delta}_j u\right\|_{L^p}}{N^{1-1/\sigma}}\|\nabla u\|_{L^2}^{1-3/p}\|\nabla^2 u\|_{L^2}^{3/p}\left\|\rho^{1/2}u_t\right\|_{L^2}\\ &\le C\left(N^{1-1/\sigma}\sup_{N=1,2,\cdots}\frac{\sum_{j=-N}^{j=N}\left\|\dot{\Delta}_j u\right\|_{L^p}}{N^{1-1/\sigma}}\right)^2\|\nabla u\|_{L^2}^{2-6/p}\|\nabla^2 u\|_{L^2}^{6/p}+\frac{1}{32}\left\|\rho^{1/2}u_t\right\|_{L^2}\\ &\le CN\|u\|_{\dot{V}_{p,\sigma,1}^0}^{2p/p-3}\|\nabla u\|_{L^2}^2+\frac{1}{32}\left(\left\|\rho^{1/2}u_t\right\|_{L^2}+\left\|\nabla^2 u\right\|_{L^2}^2\right),\end{aligned} \tag{22}$$

and the term (III):

$$\begin{aligned} III &\le \sum_{j>N}\left\|\rho^{1/2}\Delta_j u\right\|_{L^3}\|\nabla u\|_{L^6}\left\|\rho^{1/2}u_t\right\|_{L^2}\\ &\le C\|\nabla u\|_{L^6}\sum_{j>N}2^{1/2j}\|u\|_{L^2}\left\|\rho^{1/2}u_t\right\|_{L^2}\\ &\le C2^{-N/2}\|u\|_{L^2}\left\|\nabla^2 u\right\|_{L^2}\left\|\rho^{1/2}u_t\right\|_{L^2}\\ &\le 2^{-N^2}\|u\|_{L^2}^2\left\|\nabla^2 u\right\|_{L^2}^2+\frac{1}{32}\left\|\rho^{1/2}u_t\right\|_{L^2}^2.\end{aligned} \tag{23}$$

Summing up the estimate above with the energy estimate, we get

$$\begin{aligned}&\frac{d}{dt}\left(\left\|\rho^{1/2}u\right\|_{L^2}^2+\|\nabla u\|_{L^2}^2\right)+\int_{\mathbb{R}^3}\left(|\nabla u|^2+\rho|u_t|^2\right)dx\\ &\quad\le C2^{-3N^2}\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2+CN\|u\|_{\dot{V}_{p,\sigma,1}^0}^{2p/p-3}\|\nabla u\|_{L^2}^2\\ &\qquad+C2^{-N^2}\|u\|_{L^2}^2\left\|\nabla^2 u\right\|_{L^2}^2+\frac{1}{16}\left\|\nabla^2 u\right\|_{L^2}^2.\end{aligned} \tag{24}$$

On the other hand, we note that

$$\begin{aligned}\left\|\nabla^2 u\right\|_{L^2(\mathbb{R}^3)}^2 &\le C\left(\|\sqrt{\rho}u_t\|_{L^2(\mathbb{R}^3)}^2+\|\rho u\cdot\nabla u\|_{L^2(\mathbb{R}^3)}^2\right)\\ &\le C\|\sqrt{\rho}u_t\|_{L^2(\mathbb{R}^3)}^2+C2^{-3N^2}\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2\\ &\quad+C2^{-N^2}\|u\|_{L^2}^2\left\|\nabla^2 u\right\|_{L^2}^2+CN\|u\|_{\dot{V}_{p,\sigma,1}^0}^{2p/p-3}\|\nabla u\|_{L^2}^2.\end{aligned} \tag{25}$$

Collecting (23) and (24), we have

$$\begin{aligned}&\frac{d}{dt}\left(\left\|\rho^{1/2}u\right\|_{L^2}^2+\|\nabla u\|_{L^2}^2\right)+\int_{\mathbb{R}^3}\left(|\nabla u|^2+\left|\nabla^2 u\right|^2+\rho|u_t|^2\right)dx\\ &\quad\le C\|\sqrt{\rho}u_t\|_{L^2(\mathbb{R}^3)}^2+C2^{-3N^2}\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2\\ &\qquad+C2^{-N^2}\|u\|_{L^2}^2\left\|\nabla^2 u\right\|_{L^2}^2+CN\|u\|_{\dot{V}_{p,\sigma,1}^0}^{2p/p-3}\|\nabla u\|_{L^2}^2.\end{aligned} \tag{26}$$

Now, choosing $N>0$ sufficiently large such that $C2^{-N^2}\|u\|_{L^2}^2\le 1/128$, (indeed, the constant $C>0$ is also depending on $\|\rho_0^{1/2}u_0\|_{L^2}^2$), the estimate (25) becomes

$$\begin{aligned}&\frac{d}{dt}\left(\left\|\rho^{1/2}u\right\|_{L^2}^2+\|\nabla u\|_{L^2}^2\right)+\int_{\mathbb{R}^3}\left(|\nabla u|^2+\left|\nabla^2 u\right|^2+\rho|u_t|^2\right)dx\\ &\quad\le CN\|u\|_{\dot{V}_{p,\sigma,1}^0}^{2p/p-3}\|\nabla u\|_{L^2}^2.\end{aligned} \tag{27}$$

By Grönwall's inequality under assumption (13), we obtain

$$\rho^{1/2}u,\quad \nabla u\in L^\infty\left(0,T;L^2\left(\mathbb{R}^3\right)\right),\nabla u,\nabla^2 u,\rho^{1/2}u_t\in L^2\left(0,T;L^2\left(\mathbb{R}^3\right)\right). \tag{28}$$

Lastly, according to the arguments in [6], Lemma 2.3, differentiating $(1)_2$ with respect to time t and multiplying the equations by u_t, we can obtain

$$\begin{aligned}\rho^{1/2}u_t &\in L^\infty\left(0,T;L^2\left(\mathbb{R}^3\right)\right),\\ \nabla u_t &\in L^2\left(0,T;H^1\left(\mathbb{R}^3\right)\right).\end{aligned} \tag{29}$$

This is the desired result, and thus, the proof in Theorem 2 is completed. □

Appendix

Let $1 < p < r < +\infty$; the homogeneous Morrey space $\dot{M}^{p,r}(\mathbb{R}^3)$ is the set of functions $f \in L^p_{loc}(\mathbb{R}^3)$ such that

$$\|f\|_{\dot{M}^{p,r}} = \sup_{R>0, x_0 \in \mathbb{R}^3} R^{3/r} \left(\frac{1}{R^3} \int_{B(x_0,R)} |f(x)|^p dx \right)^{1/p} < +\infty, \tag{A.1}$$

where $B(x_0, R)$ denotes the ball centered at x_0 and with radio R. It is well known that $L^r(\mathbb{R}^3) \subset L^{r,q}(\mathbb{R}^3) \subset \dot{M}^{p,r}(\mathbb{R}^3)$, where for $r \le q \le +\infty$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflict of interest.

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