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Research Article

Fuzzy Fixed Point Results in Convex C^*-Algebra-Valued Metric Spaces

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The purpose of this note is to come up with some new directions in fuzzy fixed point theory. To this effect, notions of a C^*-algebra-valued fuzzy λ-contraction and related concepts in a convex C^*-algebra-valued metric space (C^*-AVMS) are set-up. In line with the view of a Hausdorff distance function, an idea of a distance between two approximate quantities is proposed. Consequently, two fixed point results of a C^*-algebra-valued fuzzy mapping (C^*-AVFM) for the new type of contractions are established using Mann and Ishikawa iterative schemes. For some future investigations of our results, two open problems are noted concerning sufficient criteria guaranteeing the existence of fixed points of a C^*-algebra-valued fuzzy λ-contraction and whether or not the Picard iteration for a C^*-algebra-valued fuzzy λ-contraction converges.

1. Introduction and Preliminaries

We begin this section with specific notions of C^*-algebras as follows.

Definition 1 (see [1]). Let $\mathscr{A}$ be a unital algebra with the unit $I_{\mathscr{A}}$. An involution on $\mathscr{A}$ is a conjugate linear map $j \mapsto j^*$ such that $j^{**} = j$ and $(j\ell)^* = \ell^* j^*$, for all $j, \ell \in \mathscr{A}$. The pair $(\mathscr{A}, *)$ is called a $*$-algebra. A Banach $*$-algebra is a $*$-algebra $\mathscr{A}$ together with a submultiplicative norm such that $\|j^*\| = \|j\|$, for all $j \in \mathscr{A}$; where a norm $\|.\|$ on an algebra $\mathscr{A}$ is said to be submultiplicative if $\|j\ell\| \leq \|j\| \|\ell\|$, for all $j, \ell \in \mathscr{A}$. A C*-algebra is a Banach $*$-algebra such that $\|j^* j\| = \|j\|^2$, for all $j \in \mathscr{A}$.

Throughout this paper, $\mathscr{A}$ represents a unital C^*-algebra with a unit $I_{\mathscr{A}}$. Also, we take $\mathscr{A}_a = \{j \in \mathscr{A} : j = j^*\}$ and denote the zero element in $\mathscr{A}$ by $0_{\mathscr{A}}$. An element $j \in \mathscr{A}$ is called positive, written $j \pm 0_{\mathscr{A}}$, if $j \in \mathscr{A}_a$ and $\sigma(j) \subseteq \mathbb{R}_+ = [0, \infty)$, where $\sigma(j) = \{\lambda \in \mathbb{C} : \lambda I - j \text{ is not invertible}\}$ is the spectrum of j. Availing positive elements, we set up a partial ordering $^\circ$ on $\mathscr{A}_a$ as follows: $j^\circ \ell$ if and only if $\ell - j \pm 0_{\mathscr{A}}$. Hereafter, by $\mathscr{A}_+$, we mean the set $\{j \in \mathscr{A} : j \pm 0_{\mathscr{A}}\}$ and $|j| = (j^* j)^{1/2}$ (cf. [2]).

Remark 2. When $\mathscr{A}$ is a unital C^*-algebra, then for any $j \in \mathscr{A}_+$, we have $j^\circ I_{\mathscr{A}}$ if and only if $\|j\| \leq 1$ (cf. [1]).

With the aid of positive elements in $\mathscr{A}$, Ma et al. [2] launched the concept of C^*-AVMS in the following manner.

Definition 3 (see [2]). Let $\mho$ be a nonempty set. Suppose that the mapping $\sigma : \mho^2 \longrightarrow \mathscr{A}$ satisfies the following conditions:

(c1) $0_{\mathscr{A}}{}^{\circ}\sigma(j, \ell)$ and $\sigma(j, \ell) = 0_{\mathscr{A}}$ if and only if $j = \ell$

(c2) $\sigma(j, \ell) = \sigma(\ell, j)$, for all $j, \ell \in \mho$

(c3) $\sigma(j, \ell)^{\circ}\sigma(j, z) + \sigma(z, \ell)$, for all $j, \ell, z \in \mho$

Then, σ is called a C^*-algebra-valued metric, and $(\mho, \mathscr{A}, \sigma)$ is known as a C^*-AVMS.

It is clear that a C^*-AVMS generalizes the idea of a metric space, by replacing the set of real numbers with $\mathscr{A}_+$. For some recent fixed point results in C^*-AVMS, one can consult [3, 4] and the references therein.

Definition 4 (see [2]). Given a C^*-AVMS $(\mho, \mathscr{A}, \sigma)$. Suppose that the sequence $\{j_n\}_{n\in\mathbb{N}} \subset \mho$ and $u \in \mho$. If for any $\wp{>}0$, there exists $\zeta \in \mathbb{N}$ such that for all $n > \zeta$, $\|\sigma(j_n, u)\| {\le} \wp$, then $\{j_n\}_{n\in\mathbb{N}}$ is said to be convergent to u with respect to $\mathscr{A}$. In this case, we write $\lim_{n\longrightarrow\infty} j_n = u$.

If for any $\wp{>}0$, there is $\zeta \in \mathbb{N}$ such that $n, m > \zeta$, $\|\sigma(j_n, j_m)\| {\le} \wp$, then the sequence $\{j_n\}_{n\in\mathbb{N}}$ is said to be Cauchy with respect to $\mathscr{A}$. We say that $(\mho, \mathscr{A}, \sigma)$ is a complete C^*-AVMS, if every Cauchy sequence in $\mho$ is convergent with respect to $\mathscr{A}$.

Definition 5 (see [2]). Given a C^*-AVMS $(\mho, \mathscr{A}, \sigma)$, a mapping $Y : \mho \longrightarrow \mho$ is called a C^*-algebra-valued contractive mapping on $\mho$, if there exists $\lambda \in \mathscr{A}$ with $\|\lambda\|{<}1$ such that for all $j, \ell \in \mho$,

$$\sigma(Yj, Y\ell)^{\circ}\lambda^*\sigma(j, \ell)\lambda. \tag{1}$$

The following lemma is useful in discussing our main results.

Lemma 6 [1]. *Let $\mathscr{A}$ be a C^*-algebra. Then:*

(i) For any $\tau \in \mathscr{A}$, if $p, q \in \mathscr{A}_a$ with $p^{\circ}q$, then $r^*pr^{\circ}r^*qr$

(ii) For any $p, q \in \mathscr{A}_a$, if $0_{\mathscr{A}}{}^{\circ}p^{\circ}q$, then $0 \le \|p\| {\le} \|q\|$

In 1970, Takahashi [5] initiated the concept of convexity in metric spaces in the following fashion.

Definition 7 (see [5]). Let $(\mho, \sigma)$ be a metric space and $D = [0, 1]$. A mapping $\Psi : \mho\times \mho\times D \longrightarrow \mho$ is called a convex structure on $\mho$, if for all $j, \ell, a \in \mho$ and $t \in D$,

$$\sigma(a, \Psi(j, \ell, t)) \le t\sigma(a, j) + (1 - t)\sigma(a, \ell). \tag{2}$$

Using the notion of convexity, Takahashi [5] complemented some FP results originally obtained in Banach spaces. Following [5], several investigators have come up with FP notions in convex metric spaces; for such results, we can refer [6–9] and the references therein. Using the idea of a convex metric space and with the aid of positive elements in a C^*-algebra, Ghanifard et al. [10] brought up the next definition.

Definition 8 (see [10]). Let $(\mho, \mathscr{A}, \sigma)$ be a C^*-AVMS. A mapping $\Psi : \mho\times \mho\times D \longrightarrow \mho$ is called a convex structure on $\mho$ if for all $j, \ell, a \in \mho$, it satisfies the following:

$$\sigma(a, \Psi(j, \ell, t))^{\circ}t\sigma(a, j) + (1 - t)\sigma(a, \ell). \tag{3}$$

A C^*-AVMS equipped with a convex structure is said to be a convex C^*-AVMS, denoted by $(\mho, \mathscr{A}, \Psi, \sigma)$. A subset Θ of $\mho$ is called convex if for all $j, \ell \in \Theta$ and $t \in D$, $\Psi(j, \ell, t) \in \Theta$.

As an attempt at reducing uncertainties in dealing with practical problems for which conventional mathematics cannot cope effectively, the evolvement of fuzzy mathematics started with the introduction of the concepts of fuzzy sets by Zadeh [11] in 1965. Fuzzy set theory is now well-known as one of the mathematical tools for handling information with nonstatistical uncertainty. As a result, the theory of fuzzy sets has gained greater applications in diverse domains such as management sciences, engineering, environmental sciences, medical sciences, and in other emerging fields. In the meantime, the basic notions of fuzzy sets have been modified and improved in various settings; for example, see [12–15]. Along the lane, Heilpern [16] employed the concept of fuzzy sets to come up with the notion of fuzzy mappings and established a FP result for fuzzy contraction mappings which is a fuzzy version of FP theorems established by Nadler [17] and Banach [18].

Let $\mho$ be a universal set. A fuzzy set in $\mho$ is a map with domain $\mho$ and range set D. Let $I^{\mho}$ be the collection of all fuzzy sets in $\mho$. If ∇ is a fuzzy set in $\mho$, then the function value $\nabla(j)$ is called the grade of membership of j in ∇. The α-level set of a fuzzy set ∇ is denoted by $[\nabla]_\alpha$ and is defined as follows:

$$[\nabla]_\alpha = \begin{cases} \{j \in \mho : \overline{\nabla}(j) > 0\}, & \text{if } \alpha = 0, \\ \{j \in \mho : \nabla(j) \ge \alpha\}, & \text{if } \alpha \in D \setminus \{0\}, \end{cases} \tag{4}$$

where by $\bar{P}$, we mean the closure of the crisp set P.

Definition 9 (see [16]). Let $\mho$ be an arbitrary set and Y be a metric space. A mapping $Y : \mho \longrightarrow I^{\mho}$ is called a fuzzy mapping. A fuzzy mapping Y is a fuzzy subset of $\mho\times Y$ with membership function $Y(j)(\ell)$. The function value $Y(j)(\ell)$ is called the grade of membership of ℓ in $Y(j)$.

Definition 10 (see [16]). Let $\mho$ be a nonempty set and $Y : \mho \longrightarrow I^{\mho}$ be a fuzzy mapping. A point $u \in \mho$ is said to be a fuzzy FP of Y if there exists an $\alpha \in D \setminus \{0\}$ such that $u \in [Yu]_{\alpha(u)}$.

Hereafter, $\mathrm{Fix}(Y) = \{u \in \mho : u \in [Yu]_\alpha \text{ for some } \alpha \in D\}$.

Motivated by the ideas of fuzzy mappings and C^*-AVMSs due to Heilpern [16] and Ma et al. [2], respectively, the aim of this research is to initiate the study of fuzzy FP results in convex C^*-AVMSs. To this effect, some new concepts of C^*-algebra-valued fuzzy contractions in convex

C^*-AVMSs are proposed, and related fuzzy FP theorems are established. The notions put forward herein are not only novel, but complement and unify a few corresponding results in the existing literature.

2. Main Results

In this section, we introduce notions of C^*-algebra-valued fuzzy contractions and some corresponding fixed point results. First, a few requisite auxiliary concepts are initiated as follows.

Definition 11. A fuzzy set Ω in a C*-AVMS $(\mho, \mathscr{A}, \sigma)$ is said to be convex if for all $j, \ell \in \mho$ and $t \in D \setminus \{0, 1\}$, min $\{\Omega(j), \Omega(\ell)\}^\circ \Omega(tj + (1-t)\ell)$. A fuzzy set Ω in $\mho$ is called an approximate quantity if its α-level set is a compact convex subset of $\mho$ for each $\alpha \in D$ and $\sup_{j\in\mho_{\mathscr{A}}} \Omega(j) = 1$.

Throughout, the collection of all approximate quantities in $(\mho, \mathscr{A}, \sigma)$ is denoted by $W_{\mathscr{A}}(\mho)$. We define a distance function between two approximate quantities in $W_{\mathscr{A}}(\mho)$ as follows.

Definition 12. Let $F, G \in W_{\mathscr{A}}(\mho)$ and $\alpha \in D$. Then, we define:

$$\begin{aligned} D_\alpha^*(F, G) &= \aleph([F]_\alpha, [G]_\alpha), \\ \sigma_\infty(F, G) &= \sup_\alpha D_\alpha^*(F, G), \end{aligned} \tag{5}$$

where the Hausdorff distance function $\aleph : W_{\mathscr{A}} \times W_{\mathscr{A}} \longrightarrow \mathscr{A}$ is set-up as follows:

$$\begin{aligned} \aleph([F]_\alpha, [G]_\alpha) &= \left(\max \left\{ \sup_{j\in[F]_\alpha} \left\| \sigma(j, [G]_\alpha) \right\|, \sup_{\ell\in[F]_\alpha} \left\| \sigma([F]_\alpha, \ell) \right\| \right\} \right) I_{\mathscr{A}}, \\ \sigma(a, \varpi) &= (\inf \{\|\sigma(a, \rho)\| : \rho \in \varpi\}) I_{\mathscr{A}}. \end{aligned} \tag{6}$$

Consistent with Heilpern [16], we call the function D_α^* an $(\alpha, *)$-distance and σ_∞ a distance between two approximate quantities in $W_{\mathscr{A}}(\mho)$.

We say that a subset Ψ of a C^*-AVMS $(\mho, \mathscr{A}, \sigma)$ is bounded if $\sup_{j,\ell\in\mathscr{A}} \{\|\sigma(j, \ell)\|\} < \infty$. The collection of all closed and bounded subsets of $(\mho, \mathscr{A}, \sigma)$ is represented by $B_{\mathscr{A}}(\mho)$.

Note that σ_∞ is a C^*-algebra-valued metric on $B_{\mathscr{A}}(\mho)$ (induced by the Hausdorff metric $\aleph$), and the completeness of $(\mho, \mathscr{A}, \sigma_\infty)$ implies the completeness of the corresponding C^*-AVMS $(K_\Omega(\mho), \mathscr{A}, \sigma_\infty)$. Moreover, $(\mho, \mathscr{A}, \sigma_\infty) \mapsto (B_{\mathscr{A}}(\mho), \mathscr{A}, \sigma_\infty) \mapsto (K_\Omega(\mho), \mathscr{A}, \sigma_\infty)$ are isometric embedding via the relation $j \mapsto \{j\}$ (crisp set) and $\sqsubseteq \mapsto \chi_\sqsubseteq$, respectively, where $\chi_\sqsubseteq$ is the characteristic function of $\sqsubseteq$, and

$$K_\Omega(\mho) = \left\{ \Omega \in I^\mho : [\Omega]_\alpha \in B_{\mathscr{A}}(\mho), \text{for each } \alpha \in D \right\}. \tag{7}$$

Similarly,

$$K_{(\Omega_1,\Omega_2)}(\mho) = \left\{ \Omega_1, \Omega_2 \in I^\mho : [\Omega_1]_\alpha, [\Omega_2]_\alpha \in B_{\mathscr{A}}(\mho), \text{for each } \alpha \in D \right\}. \tag{8}$$

We now define the idea of a C^*-AVFM in the following manner.

Definition 13. Let $\mho$ be an arbitrary set and $(\Upsilon, \mathscr{A}, \sigma)$ be a C^*-AVMS. A mapping $Y : \mho \longrightarrow W_{\mathscr{A}}(Y)$ is called a C^*-AVFM.

In line with the idea of fuzzy λ-contraction due to Heilpern [16], we introduce the next concept.

Definition 14. Let $(\mho, \mathscr{A}, \sigma)$ be a C^*-AVMS. A C^*-AVFM $Y : \mho \longrightarrow W_{\mathscr{A}}(\mho)$ is called a C^*-algebra-valued fuzzy λ-contraction, if there exists $\lambda \in \mathscr{A}$ with $\|\lambda\|<1$ such that for all $j, \ell \in \mho$,

$$\sigma_\infty(Yj, Y\ell)^\circ \lambda^* \sigma(j, \ell)\lambda. \tag{9}$$

Example 15. Let $\mho = \mathbb{R}$ and $\mathscr{A} = M_2(\mathbb{R})$ (the collection all 2×2 matrices with real entries) with the norm $\|\Phi\| = \max_{p',q'} |\varsigma_{p'q'}|$, where $\varsigma_{p'q'}$ are the entries of the matrix $\Phi \in M_2(\mathbb{R})$ and the involution given by $\Phi^* = \Phi^T$. Define $\sigma : \mho \times \mho \longrightarrow \mathscr{A}$ by $\sigma(j, \ell) = \text{diag}\,(|j - \ell|, |j - \ell|)$. Obviously, $(\mho, \mathscr{A}, \sigma)$ is a C^*-AVMS. We define a partial ordering on $\mathscr{A}$ as follows:

$$\begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 \\ q_3 & q_4 \end{bmatrix} \Leftrightarrow p_i \preccurlyeq \lambda^* q_i \lambda, \tag{10}$$

for $i = 1, 2, 3, 4$ and for some $\lambda \in \mathscr{A}$. Let $\beta \in (0, 1]$ and for each $j \in \mho$, define a fuzzy mapping $Y(j)$: $\mho \longrightarrow [0, 1]$ as follows:

$$Y(j)(t) = \begin{cases} \beta, & \text{if } \dfrac{j}{7} \le t \le \dfrac{j}{5} \\ \dfrac{\beta}{6}, & \text{otherwise.} \end{cases} \tag{11}$$

If we take the mapping $\alpha : \mho \longrightarrow (0, 1]$ as $\alpha(j) = \beta$ for all $j \in \mho$, then

$$[Yj]_{\alpha(j)} = \{t \in \mho : Y(j)(t) \ge \beta\} = \left[\frac{j}{7}, \frac{j}{5}\right]. \tag{12}$$

Obviously, $[Yj]_{\alpha(j)} \in B_{\mathscr{A}}(\mho)$, for each $j \in \mho$. We see that for all $j, \ell \in \mho$,

$$\begin{aligned}\sigma_\infty(Yj, Y\ell) &= \sup_\beta D_\beta(Yj, Y\ell) = \aleph\left([Yj]_\beta, [Y\ell]_\beta\right)\\ &= \left(\max\left\{\sup_{a\in[Yj]_\beta}\left\|\sigma\left(a, [Y\ell]_\beta\right)\right\|, \sup_{b\in[Y\ell]}\left\|\sigma\left([Yj]_\beta, b\right)\right\|\right\}\right)I_{\mathscr{A}}\\ &= \left(\max\left\{\sup \inf_{a\in[Yj]_\beta,\ell\in[Y\ell]_\beta}|a-b|, \sup \inf_{b\in[Y\ell]_\beta,a\in[Yj]_\beta}|a-b|\right\}\right)\\ &\cdot I_{\mathscr{A}}\left(\max\left\{\left|\frac{j}{7}-\frac{\ell}{7}\right|, \left|\frac{j}{5}-\frac{\ell}{5}\right|\right\}\right)I_{\mathscr{A}} = \left(\left|\frac{j}{5}-\frac{\ell}{5}\right|\right)I_{\mathscr{A}}\\ &= \operatorname{diag}\left(\left|\frac{j}{5}-\frac{\ell}{5}\right|\right).\end{aligned} \tag{13}$$

It follows that Y is a C^*-algebra-valued fuzzy λ-contraction with $\lambda = \operatorname{diag}(1/\sqrt{4}, 1/\sqrt{4})$. Clearly, $\|\lambda\| < 1$.

Definition 16. Let $\mho$ be a nonempty set. A point $u \in \mho$ is called a stationary point of a fuzzy mapping $Y : \mho \longrightarrow I^{\mho}$, if there exists an $\alpha \in D \setminus \{0\}$ such that $[Yu]_\alpha = \{u\}$. We say that u is a common stationary point of any two fuzzy mappings $Y, \Lambda : \mho \longrightarrow I^{\mho}$ if $[Yu]_\alpha = \{u\} = [\Lambda u]_\alpha$, for some $\alpha \in D \setminus \{0\}$.

Our main result is provided hereunder.

Theorem 17. *Let $(\mho, \mathscr{A}, \Psi, \sigma)$ be a complete convex C^*-AVMS. Suppose that the mapping $Y : \mho \longrightarrow K_Y(\mho)$ is a C^*-algebra-valued fuzzy λ-contraction such that $Fix(Y) \neq \varnothing$ and every $a \in Fix(Y)$ is a stationary point of Y. Let $\{j_n\}$ be the Mann iteration scheme given by*

$$j_{n+1} = \Psi(\ell_n, j_n, \eta_n), \tag{14}$$

where $\ell_n \in [Yj_n]_{\alpha(j_n)}$ and $\eta_n \in D$. Then, $\{j_n\}$ converges to a fuzzy FP of Y, provided $\lim_{n\longrightarrow\infty}\sigma(j_n, Fix(Y)) = 0_{\mathscr{A}}$.

Proof. Let $a \in \operatorname{Fix}(Y)$ and $\alpha(j) \in D \setminus \{0\}$ for each $j \in \mho$. Then, we have

$$\sigma(j_{n+1}, a) = \sigma(\Psi(\ell_n, j_n, \eta_n), a) \preccurlyeq \eta_n\sigma(\ell_n, a) + (1-\eta_n)\sigma(j_n, a). \tag{15}$$

From (15),

$$\begin{aligned}\|\sigma(j_{n+1}, a)\| &\le \eta_n\|\sigma(\ell_n, a)\| + (1-\eta_n)\|\sigma(j_n, a)\|\\ &\le \eta_n\left(\sup_{\ell_n\in[Yj_n]_{\alpha(j_n)},a\in[Ya]_{\alpha(a)}}\|\sigma(\ell_n, a)\|\right)\\ &\quad + (1-\eta_n)\|\sigma(j_n, a)\|\\ &= \eta_n\left\|\sigma\left([Yj_n]_{\alpha(j_n)}, [Ya]_{\alpha(a)}\right)\right\| + (1-\eta_n)\|\sigma(j_n, a)\|\\ &\le \eta_n\left\|\aleph\left([Yj_n]_{\alpha(j_n)}, [Ya]_{\alpha(a)}\right)\right\| + (1-\eta_n)\|\sigma(j_n, a)\|\\ &\le \eta_n\|\sigma_\infty(Yj_n, Ya)\| + (1-\eta_n)\|\sigma(j_n, a)\|\\ &\le \eta_n\|\lambda^*\|\|\sigma(j_n, a)\|\|\lambda\| + (1-\eta_n)\|\sigma(j_n, a)\|\\ &= \eta_n\|\lambda\|^2\|\sigma(j_n, a)\| + (1-\eta_n)\|\sigma(j_n, a)\|\\ &< \eta_n\|\sigma(j_n, a)\| + (1-\eta_n)\|\sigma(j_n, a)\| = \|\sigma(j_n, a)\|.\end{aligned} \tag{16}$$

We observe that the strict inequality in (16) is valid whenever $j_n \neq a$, for each $n \in \mathbb{N}$. Indeed, if we take $j_\kappa = a$ for a finite $\kappa \in \mathbb{N}$, then $j_n = a$ for each $n \ge \kappa$, from which it yields that $\{j_n\}_{n\in\mathbb{N}}$ converges to a for finite number of iterations, and hence, we obtain the conclusion of our result.

Now, we prove that the sequence $\{j_n\}_{n\in\mathbb{N}}$ is Cauchy with respect to $\mathscr{A}$. Since $\lim_{n\longrightarrow\infty}\sigma(j_n, \operatorname{Fix}(Y)) = 0_{\mathscr{A}}$, for each $\wp>0$, there exists $m(\wp) \in \mathbb{N}$ such that for all $n \ge m(\wp)$,

$$\|\sigma(j_n, \operatorname{Fix}(Y))\| \le \frac{\wp}{7}. \tag{17}$$

By (17), there exists $r_1 \in \operatorname{Fix}(Y)$ such that for all $n \ge m(\wp)$,

$$\|\sigma(j_n, r_1)\| \le \frac{\wp}{2}. \tag{18}$$

Using the triangle inequality in $(\mho, \mathscr{A}, \sigma)$,

$$\sigma(j_{n+k}, j_n) \preccurlyeq \sigma(j_{n+k}, r_1) + \sigma(r_1, j_n). \tag{19}$$

Therefore, taking (18) into consideration, we get

$$\begin{aligned}\|\sigma(j_{n+k}, j_n)\| &\le \|\sigma(j_{n+k}, r_1)\| + \|\sigma(r_1, j_n)\|\\ &< \|\sigma(j_n, r_1)\| + \|\sigma(r_1, j_n)\| \le \frac{\wp}{2} + \frac{\wp}{2} = \wp,\end{aligned} \tag{20}$$

for $j_n \neq r_1$. This proves that $\{j_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $\mathscr{A}$. The completeness of $\mho_{\mathscr{A}}$ implies that there exists $a^* \in \mho_{\mathscr{A}}$ such that $\lim_{n\longrightarrow\infty} j_n = a^*$. Next, we establish that a^* is a fuzzy FP of Y. For this, take $\wp' > 0$. Since $j_n \longrightarrow a^*$ as $n \longrightarrow \infty$, there exists $m(\wp') \in \mathbb{N}$ such that for all $n \ge m(\wp')$,

$$\|\sigma(j_n, a^*)\| \le \frac{\wp'}{4}. \tag{21}$$

Moreover, $\lim_{n\longrightarrow\infty}\sigma(j_n, \operatorname{Fix}(Y)) = 0_{\mathscr{A}}$ yields that there exists $m'(\wp) \ge m(\wp')$ such that for all $m(\wp) \ge m'(\wp)$,

$$\|\sigma(j_n, \operatorname{Fix}(Y))\| \le \frac{\wp'}{10}. \tag{22}$$

Hence, there exists $r_2 \in \operatorname{Fix}(Y)$ such that for all $m(\wp) \ge m'(\wp)$,

$$\|\sigma(j_n, r_2)\| \le \frac{\wp'}{12}. \tag{23}$$

By triangle inequality in $\mho_{\mathscr{A}}$, there exists write

$$\begin{aligned}\sigma\left([Ya^*]_{\alpha(a^*)}, a^*\right) &\preccurlyeq \sigma\left([Ya^*]_{\alpha(a^*)}, r_2\right) + \sigma\left(r_2, [Yj_3]_{\alpha(j_3)}\right)\\ &\quad + \sigma\left([Yj_3]_{\alpha(j_3)}, r_2\right) + \sigma\left(r_2, j_{n_3}\right)\\ &\quad + \sigma\left(j_{n_3}, a^*\right).\end{aligned} \tag{24}$$

Consequently,

$$\begin{aligned}\|\sigma\left([Ya^*]_{\alpha(a^*)}, a^*\right)\| &\leq \|\sigma\left([Ya^*]_{\alpha(a^*)}, r_2\right)\| + \|\sigma\left(r_2, \left[Yj_{n_3}\right]_{\alpha\left(j_{n_3}\right)}\right)\| \\ &+ \|\sigma\left(\left[Yj_{n_3}\right]_{\alpha\left(j_{n_3}\right)}, r_2\right)\| + \|\sigma\left(r_2, j_{n_3}\right)\| \\ &+ \|\sigma\left(j_{n_3}, a^*\right)\| \leq \|\aleph\left([Ya^*]_{\alpha(a^*)}, [Yr_2]_{\alpha(r_2)}\right)\| \\ &+ 2\|\aleph\left([Yr_2]_{\alpha(r_2)}, \left[Yj_{n_3}\right]_{\alpha\left(j_{n_3}\right)}\right)\| + \|\sigma\left(r_2, j_{n_3}\right)\| \\ &+ \|\sigma\left(j_{n_3}, a^*\right)\| \leq \|\sigma_\infty(Ya^*, Yr_2)\| \\ &+ 2\|\sigma_\infty\left(Yr_2, Yj_{n_3}\right)\| + \|\sigma\left(r_2, j_{n_3}\right)\| \\ &+ \|\sigma\left(j_{n_3}, a^*\right)\| \leq \|\lambda^*\|\|\sigma(a^*, r_2)\|\|\lambda\| \\ &+ 2\|\lambda^*\|\|\sigma\left(r_2, j_{n_3}\right)\|\|\lambda\| + \|\sigma(j_n, a^*)\| \\ &+ \|\sigma\left(r_2, j_{n_3}\right)\| + \|\sigma\left(j_{n_3}, a^*\right)\| \\ &= \|\lambda\|^2\|\sigma(a^*, r_2)\| + 2\|\lambda\|^2\|\sigma\left(r_2, j_{n_3}\right)\| + \\ &\|\sigma\left(j_{n_3}, a^*\right)\| < \|\sigma(a^*, r_2)\| + 2\|\sigma\left(r_2, j_{n_3}\right)\| \\ &+ \|\sigma\left(j_{n_3}, a^*\right)\| \leq \|\sigma\left(a^*, j_{n_3}\right)\| + \|\sigma\left(j_{n_3}, r_2\right)\| \\ &+ \|\sigma\left(j_{n_3}, r_2\right)\| + 2\|\sigma\left(j_{n_3}, r_2\right)\| + 2\|\sigma\left(j_{n_3}, r_2\right)\| \\ &+ \|\sigma\left(j_{n_3}, r_2\right)\| + \|\sigma\left(j_{n_3}, a^*\right)\| = 2\|\sigma\left(j_{n_3}, a^*\right)\| \\ &+ 6\|\sigma\left(j_{n_3}, r_2\right)\| \leq 2\left(\frac{\wp'}{4}\right) + 6\left(\frac{\wp'}{12}\right) = \wp',\end{aligned} \tag{25}$$

whenever $j_{n_3} \neq r_2$. It follows that $\sigma([Ya^*]_{\alpha(a^*)}, a^*) = 0_{\mathscr{A}}$, and thus, $a^* \in [Ya^*]_{\alpha(a^*)}$, for some $\alpha(a^*) \in D \setminus \{0\}$. □

In what follows, we present a fuzzy coincidence theorem for two C^*-algebra fuzzy mappings using Ishikawa iterative scheme.

Theorem 18. *Let $(\mho, \mathscr{A}, \sigma)$ be a complete convex C^*-AVMS and $\Lambda, Y : \mho \longrightarrow K_{(Y,\Lambda)}(\mho)$ be any two C^*-AVFMs satisfying:*

$$\sigma_\infty(Yj, \Lambda\ell) \leq \lambda^*\sigma(j, \ell)\lambda, \tag{26}$$

for all $j, \ell \in \mho$ with $\lambda \in \mathscr{A}$ such that $\|\lambda\|<1$. Assume further that $C_{(Y,\Lambda)} \coloneqq Fix(Y) \cap Fix(\Lambda) \neq \varnothing$, and every $a \in C_{(Y,\Lambda)}$ is a common stationary point of Y and Λ. Then, the sequence of Ishikawa iterative scheme set-up by

$$j_{n+1} = \Psi(z_n, j_n, \eta_n), \ell_n = \Psi\left(z', j_n, \xi_n\right), \tag{27}$$

where $z_n \in [\Lambda j_n]_{\alpha(j_n)}, z'_n \in [Yj_n]_{\alpha(j_n)}$, and $\eta_n, \xi_n \in D$, converges to an element of $C_{(Y,\Lambda)}$, provided $\lim_{n\longrightarrow\infty}\sigma(j_n, C_{(Y,\Lambda)}) = 0_{\mathscr{A}}$.

Proof. Let $a \in C_{(Y,\Lambda)}$. Assume that $j_n \neq a$ for all $n \in \mathbb{N}$. Then

$$\sigma(\ell_n, a) = \sigma\left(\Psi\left(z'_n\right), j_n, \xi_n\right)a \preccurlyeq \xi_n\sigma\left(z'_n, a\right) + (1-\xi_n)\sigma(j_n, a), \tag{28}$$

from which it follows that

$$\begin{aligned}\|\sigma(\ell_n, a)\| &\leq \xi_n\|\sigma\left(z'_n, a\right)\| + (1-\xi_n)\|\sigma(j_n, a)\| \\ &\leq \xi_n\|\aleph\left([Yj_n]_{\alpha(j_n)}, [\Lambda a]_{\alpha(a)}\right)\| + (1-\xi_n)\|\sigma(j_n, a)\| \\ &\leq \xi_n\|\sigma_\infty(Yj_n, \Lambda a)\| + (1-\xi_n)\|\sigma(j_n, a)\| \\ &\leq \|\lambda^*\sigma(\ell_n, a)\lambda\| + (1-\xi_n)\|\sigma(j_n, a)\| \\ &\leq \xi_n\|\lambda\|^2\|\sigma(j_n, a)\| + (1-\xi_n)\|\sigma(j_n, a)\| \\ &< \xi_n\|\sigma(j_n, a)\| + (1-\xi_n)\|\sigma(j_n, a)\| = \|\sigma(j_n, a)\|.\end{aligned} \tag{29}$$

Similarly,

$$\sigma(j_{n+1}, a) = \sigma(\Psi(z_n, j_n, \eta_n), a) \preccurlyeq \eta_n\sigma(z_n, a) + (1-\eta_n)\sigma(j_n, a), \tag{30}$$

which gives

$$\begin{aligned}\|\sigma(j_{n+1}, a)\| &\leq \eta_n\|\sigma(z_n, a)\| + (1-\eta_n)\|\sigma(j_n, a)\| \\ &\leq \eta_n\|\aleph\left([\Lambda j_n]_{\alpha(j_n)}, [Ya]_{\alpha(a)}\right)\| + (1-\eta_n)\|\sigma(j_n, a)\| \\ &\leq \eta_n\|\lambda\|^2\|\sigma(j_n, a)\| + (1-\eta_n)\|\sigma(j_n, a)\| \\ &< \eta_n\|\sigma(j_n, a)\| + (1-\eta_n)\|\sigma(j_n, a)\| = \|\sigma(j_n, a)\|.\end{aligned} \tag{31}$$

Hence, in line with the proof of Theorem 17, we can prove that $\{j_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $\mathscr{A}$, and the completeness of $(\mho, \mathscr{A}, \Psi, \sigma)$ implies that $\{j_n\}_{n\in\mathbb{N}}$ converges to some $u^* \in \mho$. Thus, consistent with Theorem 17, we obtain that $u^* \in \mathrm{Fix}(Y) \cap \mathrm{Fix}(\Lambda)$. □

As some consequences of Theorem 17 and Theorem 18, the following two results, using C^*-algebra-valued Hausdorff distance function, can be deduced easily.

Corollary 19. *Let $(\mho, \mathscr{A}, \Psi, \sigma)$ be a complete convex C^*-AVMS and $Y : \mho \longrightarrow K_Y(\mho)$ be a C^*-AVFM. Suppose that $Fix(Y) \neq \varnothing$ and every $a \in Fix(Y)$ is a stationary point of Y. Let $\{j_n\}_{n\in\mathbb{N}}$ be the Mann iterative scheme set-up by (14). If there exist $\alpha \in D \setminus \{0\}$ and $\lambda \in \mathscr{A}$ with $\|\lambda\|<1$ such that for all $j, \ell \in \mho$,*

$$D^*_\alpha(Yj, Y\ell) \leq \lambda^*\sigma(j, \ell)\lambda, \tag{32}$$

then $\{j_n\}_{n\in\mathbb{N}}$ converges to a fuzzy FP of Y, provided $\lim_{n\longrightarrow\infty}\sigma(j_n, Fix(Y)) = 0_{\mathscr{A}}$.

Proof. Since for all $j, \ell \in \mho$ and $\alpha \in D \setminus \{0\}$,

$$\|D_\alpha^*(Yj, Y\ell)\| \le \|\sigma_\infty(Yj, Y\ell)\|. \quad (33)$$

Theorem 17 can be followed to complete the proof. □

On the same steps in deriving Corollary 19, we can also deduce the following result.

Corollary 20. *Let $(\mho, \mathscr{A}, \Psi, \sigma)$ be a complete convex C^*-AVMS and $Y, \Lambda : \mho \longrightarrow K_{(Y,\Lambda)}(\mho)$ be any two C^*-AVFMs. Assume further that $C_{(Y,\Lambda)} = Fix(Y) \cap Fix(\Lambda) \neq \emptyset$ and every $a \in Fix(Y)$ is a common stationary point of Y and Λ. Let $\{j_n\}_{n\in\mathbb{N}}$ be the Ishikawa iterative scheme set-up by (23). If there exist $\alpha \in D \setminus \{0\}$ and $\lambda \in \mathscr{A}$ with $\|\lambda\|<1$ such that for all $j, \ell \in \mho$,*

$$D_\alpha^*(Yj, \Lambda\ell) \le \lambda^* \sigma(j, \ell)\lambda, \quad (34)$$

then $\{j_n\}_{n\in\mathbb{N}}$ converges to a common fuzzy FP of Y and Λ, provided

$$\lim_{n\longrightarrow\infty} \sigma\left(j_n, C_{(Y,\Lambda)}\right) = 0_{\mathscr{A}}. \quad (35)$$

3. Open Problems

For some future examinations of our main results, the following two problems are highlighted:

(P1) It is well-known that the importance of contractive mapping is to guarantee the existence and uniqueness of a fixed point of certain self-mappings in complete spaces. On this note, following Theorem 17 and Theorem 18, sufficient criteria guaranteeing the existence of fixed points of C^*-algebra-valued fuzzy λ-contractions is still a gap that needed to be filled.

(P2) In this article, Mann and Ishikawa iterations are used to develop the ideas of C^*-algebra-valued fuzzy λ-contractions and associated fixed point theorems. Hence, it is natural to ask whether Picard iteration for C^*-algebra-valued fuzzy λ-contraction mapping converges or not.

4. Conclusions

Based on the ideas of fuzzy mappings and C^*-AVMSs in the sense of Heilpern [16] and Ma et al. [2], respectively, analogue notions of C^*-algebra-valued fuzzy contractions in convex C^*-AVMSs and associated FP theorems are established. The obtained fuzzy FP results are analysed using Mann and Ishikawa iterative schemes. It is pertinent to note that the ideas of this paper being discussed in fuzzy setting are very fundamental. Hence, it can be improved upon when presented in the framework of some generalized fuzzy mappings such as L-fuzzy, intuitionistic fuzzy, and soft set-valued mappings. The underlying space can also be fine-tuned in some other pseudo or quasi metric spaces. For some future considerations of our results, two open problems are posed regarding sufficient conditions under which C^*-algebra-valued fuzzy λ-contraction has a fixed point and whether or not the Picard iteration for C^*-algebra-valued fuzzy λ-contraction converges.

Data Availability

No data were used to support this study.

Disclosure

The statements made and views expressed are solely the responsibility of the author.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors contributed equally to this work.

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References

[1] G. J. Murphy, *C ∗-Algebras and Operator Theory*, Academic Press, Inc., Boston, MA, 1990.

[2] Z. H. Ma, L. N. Jiang, and H. K. Sun, "C ∗-Algebras-valued metric spaces and related fixed point theorems," *Fixed Point Theory and Applications*, vol. 2014, no. 1, 2014.

[3] M. Asim and M. Imdad, "C ∗-Algebra valued extended *b*-metric spaces and fixed point results with an application," *U.P.B. Science Bulletin, Series A*, vol. 82, no. 1, pp. 207–218, 2020.

[4] M. Mlaiki, M. Asim, and M. Imdad, "C ∗-Algebra valued partial *b*-metric spaces and fixed point results with an application," *Mathematics*, vol. 8, no. 1381, 2020.

[5] W. Takahashi, "A convexity in metric spaces and nonexpansive mappings," *Kodai Mathematical Seminar Reports*, vol. 22, pp. 142–149, 1970.

[6] I. Beg, M. Abbas, and J. K. Kim, "Convergence theorems for the iterative schemes in convex metric spaces," *Nonlinear Functional Analysis and Application*, vol. 11, pp. 421–436, 2006.

[7] J. K. Kim, K. H. Kim, and K. S. Kim, "Three-step iterative sequences with errors for asymptotically quasi-nonexpansive mappings in convex metric spaces," *Nonlinear Analysis and Convex Analysis*, pp. 156–165, 2004.

[8] R. Shukla, R. Pant, Z. Kadelburg, and H. K. Nashine, "Existence and convergence results for monotone nonexpansive type mappings in partially ordered hyperbolic metric spaces," *Bulleting of Iranian Mathematical Society*, vol. 43, no. 7, pp. 2547–2565, 2017.

[9] R. Shukla, R. Pant, and P. Kumam, "On the α-nonexpansive mapping in partially ordered hyperbolic metric spaces," *Journal of Mathematical Analysis*, vol. 8, no. 1, pp. 1–15, 2017.

[10] A. Ghanifard, H. Parvaneh Masiha, and M. De La Sen, "Approximation of fixed points of C^*-algebra-multi-valued contractive mappings by the Mann and Ishikawa processes in convex C^*-algebra-valued metric spaces," *Mathematics*, vol. 8, no. 3, p. 392, 2020.

[11] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[12] A. Azam, M. Arshad, and P. Vetro, "On a pair of fuzzy φ -contractive mappings," *Mathematical and Computer Modelling*, vol. 52, no. 1-2, pp. 207–214, 2010.

[13] S. S. Mohammed and A. Azam, "Fixed points of soft-set valued and fuzzy set-valued maps with applications," *Journal of Intelligent and Fuzzy Systems*, vol. 37, no. 3, pp. 3865–3877, 2019.

[14] S. S. Mohammed, I. A. Fulatan, and Y. Sirajo, "Fixed points of p-hybrid L-fuzzy contractions," *Sahand Communications in Mathematical Analysis*, vol. 18, no. 3, pp. 1–25, 2021.

[15] M. S. Shagari, S. Rashid, K. M. Abualnaja, and M. Alansari, "On nonlinear fuzzy set-valued $\ominus$-contractions with applications," *AIMS Mathematics*, vol. 6, no. 10, pp. 10431–10448, 2021.

[16] S. Heilpern, "Fuzzy mappings and fixed point theorem," *Mathematical Analysis and Applications*, vol. 83, no. 2, pp. 566–569, 1981.

[17] S. B. Nadler Jr., "Multi-valued contraction mappings," *Pacific Journal of Mathematics*, vol. 30, no. 2, pp. 475–488, 1969.

[18] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, no. 1, pp. 133–181, 1922.