

Research Article

A Self-Adaptive Technique for Solving Variational Inequalities: A New Approach to the Problem

Muhammad Bux ¹, Saleem Ullah ¹, Muhammad Shoaib Arif ^{1,2}
and Kamaleldin Abodayeh ²

¹Department of Mathematics, Air University, PAF Complex E-9, Islamabad 44000, Pakistan

²Department of Mathematics and Sciences, College of Humanities and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia

Correspondence should be addressed to Muhammad Shoaib Arif; maths_zas@hotmail.com

Received 25 January 2022; Accepted 24 March 2022; Published 26 April 2022

Academic Editor: Rale M. Nikoli

Copyright © 2022 Muhammad Bux et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Variational inequalities are considered the most significant field in applied mathematics and optimization because of their massive and vast applications. The current study proposed a novel iterative scheme developed through a fixed-point scheme and formulation for solving variational inequalities. Modification is done by using the self-adaptive technique that provides the basis for predicting a new predictor-corrector self-adaptive for solving nonlinear variational inequalities. The motivation of the presented study is to provide a meaningful extension to existing knowledge through convergence at mild conditions. The numerical interpretation provided a significant boost to the results.

1. Introduction

Earlier, most of the equilibrium-related queries were resolved by variational inequalities that are a mathematical theory. In this regard, Stampacchia [1] is considered a pioneer who initially introduced variational inequalities in 1964. At the end of 1964, Stampacchia extended his work by introducing partial differential equations. Since then, this field has become the most emerging and demanding with extensive applications in optimization and control, economics, movements, engineering sciences, and equilibrium problems. Massive utilization of variational inequalities in applied sciences made it branched and more generalized to interact with other fields [2–5], hence proved the novelty and productivity of variational inequalities. Most of the profound task for researchers is to work on extensions and generalized inequalities regarding their applications; consequently, it gives rise to pure and applied mathematics problems. Modifications in variational inequalities produced advances in numerical methods [6–10], sensitivity analysis, and the dynamical system that are efficient in solving mathematics-

related problems. Theory and algorithmic advancements meet in the theory of variational inequalities, opening up a brand-new field of application [7, 11, 12]. These issues necessitate a combination of convex, functional, and numerical analysis techniques. There are numerous exciting applications for this fascinating section of applied mathematics in the fields of business, finance, economics, and the social, as well as the pure and applied sciences (see [3, 9, 13, 14] and the references therein for applications and numerical approaches). Such extraordinary progress is based on the most basic and unidirectional linear and nonlinear approaches.

A fundamental problem associated with variational inequalities is the establishment of fast numerical methods. A projection-type method and its variant solve many optimization problems and are also related to variational inequalities. Variational inequalities and fixed-point issues with equivalent effects utilizing projection techniques have grown in popularity in recent years as a study focus. To prove the convergence of fixed-point iterative methods, quantitative knowledge of pseudocontractive and nonlinear

monotone (accretive) operators combined with Lipschitz type conditions is required (see [15–17]). The phenomena of variational inequalities have a significant contribution to solving the Wiener-Hopf equations. Salient features of Wiener-Hopf equations and optimization problems in the presence of variational inequalities are addressed by Shi [17]. Together with the Wiener-Hopf equation, the projection method is considered an important technique for approximating the solution of variational inequality problems. Constructing an equivalence between fixed-point problems and variational inequalities is made easier with the concept of the projection method. Utilizing variational problems, several conventional improved ways to establish solutions for open, moving boundary value problems, asymmetric obstacle, unilateral, even-order, and odd-order problems could be developed (see [4–7, 11, 15] and the references therein). An investigation into a new predictor-corrector self-adaptive strategy for solving nonlinear variational inequalities under known assumptions is suggested in the proposed study. It was possible to arrive at this fixed-point formulation using projection, variational inequalities, and Wiener-Hopf equations. Additionally, the convergence of the proposed method is discussed.

2. Formulation and Basic Results

A convex set is denoted by K in H (Hilbert space). We denote norm and inner by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively. We consider a variational inequality: for general operator T , find $y \in K$ such that

$$\langle Ty, x - y \rangle \geq 0, \forall x \in K. \quad (1)$$

The inequality (1) is called the variational inequality (VI) introduced by Stampacchia [1]. A large number of problems related to equilibrium, nonsymmetric, physical sciences, engineering, moving boundary value problem, unified, obstacle, unilateral contact, and applied sciences can be discussed via the inequalities (1) [1, 6, 7, 12, 13].

Lemma 1. [13].

For $z \in H, y \in K$ holds for the inequality

$$\langle y - z, x - y \rangle \geq 0, \forall x \in K, \quad (2)$$

if and only if

$$y = P_K z, \quad (3)$$

where P_K is the projection of H onto K (convex set).

It is also known that the P_K is called projection operator, which is also nonexpansive and holds for the inequality.

$$\|P_K z - y\| \leq \|z - y\| - \|z - P_K z\|. \quad (4)$$

Lemma 2. If y is a solution of VI (1), then $y \in K$ satisfies the relation

$$y = P_K [y - \rho Ty], \quad (5)$$

where $\rho \geq 0$ is taken as constant and P_K is considered the projection operator H onto K .

From Lemma 2, it is obvious that y is a solution of VI (1), if and only if y satisfies the residue vector $r(y, \rho)$ defined by

$$r(y, \rho) = y - P_K [y - \rho Ty]. \quad (6)$$

Related to the original inequality (1), we see the Wiener-Hopf equations (WHE) problem. To be more precise, let $Q_K = I - P_K$, where P_K is the projection operator and I is the identity operator. For the operator $T : H \rightarrow H$, then for finding $z \in H$, we have

$$\rho TP_K z + Q_K z = 0. \quad (7)$$

Here, Equation (7) is the Wiener-Hopf equation (WHE), investigated by Shi [17]. This WHE (7) is considered more general and gives a unified framework to establish the various powerful and efficient iterative methods and numerical techniques (for the application of the WHE (7), see [17, 18]).

Lemma 3. The inequality (1) has a unique solution $y \in K$, if and only if $z \in H$ satisfies the WHE (7), provided

$$y = P_K z, \quad (8)$$

$$z = y - \rho Ty. \quad (9)$$

Lemma 3 implies that the VI (1) is equivalent to WHE (7). Noor et al. [8, 18] considers this fixed-point formulation to establish various iterative schemes for solving the VI and other optimization and related problems.

3. Main Results and Algorithm

To solve the variational inequality (1), we will use an iterative approach that we are developing in this study. The relevant results, algorithm, and theory will be established to make an iterative process for solving the inequality. The convergence of the new technique will also be provided.

We use the fixed-point formulation and suggest a predictor-corrector technique for upgrading the solution for VI.

$$w = P_K [y - \gamma Ty], \text{ for } \gamma > 0, \quad (10)$$

$$y = P_K [w - \rho Tw] = P_K [P_K [y - \gamma Ty] - \rho TP_K [y - \gamma Ty]]. \quad (11)$$

Using (6), (8), and (10), the WHE (7) can be written in the form

$$\begin{aligned} 0 &= y - P_K [y - \rho Ty] - \rho Ty + \rho TP_K [y - \rho Ty] \\ &= r(y, \rho) - \rho Ty + \rho TP_K [y - \rho Ty]. \end{aligned} \quad (12)$$

We define the relation

$$D(y, \rho) = r(y, \rho) - \rho Ty + \rho TP_K[y - \rho Ty]. \quad (13)$$

It is obvious that $y \in K$ is a solution of the VI if and only if $y \in K$ is satisfied with Equation (13).

$$D(y, \rho) = 0. \quad (14)$$

Using (10) and (13), we can rewrite as

$$w = P_K[y - \gamma D(y, \rho) - \gamma Ty], \quad (15)$$

This fact has motivated us to establish the new predictor-corrector self-adaptive iterative method for solving the VI (1).

Algorithm 1. Step 1: Give $\epsilon > 0, \gamma > 0, \delta \in (0, 1), \delta_0 \in (0, 1), \mu \in (0, 1), \rho > 0$, and $y^* \in H$ set $n = 0$

Step 2: Set $\rho_n = \rho$; if $\|r(y^n, \rho)\| < \epsilon$, then computation stops; otherwise, the iteration will continue to find the m_n nonnegative integer, and take $\rho_n = \rho \mu^{m_n}$ which satisfies the inequality

$$\|\rho_n(T(y^n) - T(w^n))\| \leq \delta \|r(y^n, \rho_n)\|, \quad (16)$$

where

$$w^n = P_K[y^n - \gamma_n D(y^n, \rho_n) - \gamma_n Ty^n] \quad (17)$$

Step 3: Compute

$$d(y^n, \rho_n) = r(y^n, \rho_n) - \rho_n T(y^n) + \rho_n T(P_K[y^n - \rho_n Ty^n]), \quad (18)$$

where

$$r(y^n, \rho_n) = y^n - P_K[y^n - \rho Ty^n] \quad (19)$$

Step 4: Get the next iterate

$$w^n = P_K[y^n - \gamma D(y^n, \rho_n) - \gamma T(y^n)], \quad (20)$$

$$y^{n+1} = P_K[w^n - \rho Tw^n], \quad (21)$$

and then set $\rho = \rho_n / \mu$, else set $\rho = \rho_n, n = n + 1$, and go to Step 2

We observe that Algorithm 1 is refinement and addition of the standard procedure. Here, we consider $-\gamma D(y^n, \rho_n) - \gamma T(y^n)$, the self-adaptive technique, or we can say the step-size. This technique and procedure are closely related to the projection residue.

The convergence of the newly established result of Algorithm 1 is the important part to consider under some suitable and mild conditions, which is the paper's main target and motivation.

Theorem 4. Let real Hilbert space be denoted by H and $T : K \rightarrow H$; we take α as strongly monotone, where β is Lipschitz continuous mapping on a convex subset K of H . Let $y^* \in K$ be a solution of VI (1) and let the sequences $\{y^n\}$ be generated by Algorithm 1. If $\theta = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}(1 + \gamma\beta) < 1$, then the sequences $\{y^n\}$ converges to y^* , for

$$0 < \rho < \frac{2\alpha}{\beta^2}. \quad (22)$$

Proof. Since y^* is a solution of NVI (1), from Lemma 1, we have

$$w^* = P_K[y^* - \gamma^* Ty^*], \text{ for } \gamma > 0, \quad (23)$$

$$y^* = P_K[w^* - \rho Tw^*], \text{ for } \rho > 0. \quad (24)$$

□

Applying Algorithm 1, from (19) and (24), we know that P_K is nonexpansive:

$$\begin{aligned} \|y^{n+1} - y^*\| &= \|P_K[w^n - \rho Tw^n] - P_K[w^* - \rho Tw^*]\| \\ &\leq \|w^n - w^* - \rho Tw^n + \rho Tw^*\|. \end{aligned} \quad (25)$$

Since T is considered as strongly monotone and Lipschitz continuous with constant α and β . From (25), we have

$$\begin{aligned} &\|w^n - w^* - \rho(Tw^n - Tw^*)\|^2 \\ &= \|w^n - w^*\|^2 - 2\rho \langle Tw^n - Tw^*, w^n - w^* \rangle \\ &\quad + \rho^2 \|Tw^n - Tw^*\|^2 \leq \|w^n - w^*\|^2 \\ &\quad - 2\rho\alpha \|w^n - w^*\|^2 + \rho^2\beta^2 \|w^n - w^*\|^2 \\ &= (1 - 2\rho\alpha + \rho^2\beta^2) \|w^n - w^*\|^2. \end{aligned} \quad (26)$$

From (25) and (22), we get

$$\|y^{n+1} - y^*\| \leq \sqrt{1 - 2\rho\alpha + \rho^2\beta^2} \|w^n - w^*\|. \quad (27)$$

From (18) and (22), we get

$$\begin{aligned} \|w^n - w^*\| &= \|P_K[y^n - \gamma D(y^n, \rho_n) - \gamma Ty^n] - P_K[y^* - \gamma Ty^*]\| \\ &\leq \|y^n - \gamma D(y^n, \rho_n) - \gamma Ty^n - y^* + \gamma Ty^*\| \\ &\leq \|y^n - y^* - \gamma D(y^n, \rho_n)\| + \gamma \|Ty^n - Ty^*\| \\ &\leq \|y^n - y^* - \gamma D(y^n, \rho_n)\| + \gamma\beta \|y^n - y^*\|. \end{aligned} \quad (28)$$

Consider

$$\begin{aligned} &\|y^n - y^* - \gamma D(y^n, \rho_n)\|^2 \\ &= \|y^n - y^*\|^2 - 2\gamma \langle y^n - y^*, D(y^n, \rho_n) \rangle + \gamma^2 \|D(y^n, \rho_n)\|^2. \end{aligned} \quad (29)$$

TABLE 1: For Algorithm 1 (numerical results).

Order of matrix n	Numerical results	Numerical results
	Algorithm 1 No. It.	[8] No. It.
100	44	44
200	55	55
300	48	48
500	31	31
700	43	43

We use the definition of $D(y^n, \rho_n)$, and we obtain

$$\|y^n - y^* - \gamma D(y^n, \rho_n)\| \leq \|y^n - y^*\|. \quad (30)$$

From (28) and (30), we have

$$\|w^n - w^*\| \leq (1 + \gamma\beta)\|y^n - y^*\|. \quad (31)$$

From (27) and (31), we get

$$\|y^{n+1} - y^*\| \leq \sqrt{1 - 2\rho\alpha + \rho^2\beta^2(1 + \gamma\beta)}\|y^n - y^*\| = \theta\|y^n - y^*\|, \quad (32)$$

where $\theta = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2(1 + \gamma\beta)}$, since $0 < \theta < 1$, $\sum_{n=0}^{\infty} \theta^n = \infty$, thus from (32) and Algorithm 1 for an arbitrarily chosen and consider initial points y_0 and y^n obtained from Algorithm 1, which converge strongly to y^* .

4. Numerical Example

Example 1. We take the nonlinear complementarity problems: for finding $y \in R^n$, we have

$$y \geq 0, T(y) \geq 0, \langle y, T(y) \rangle = 0. \quad (33)$$

Here, $T(y) = D_1(y) + D_2(y) + q$, we consider $D_1(y)$ as nonlinear part, and $D_2(y) + q$ is taken as a linear part, and in ((33)), we take a special case of the VI (1). The matrix $D_2 = B^t B + C$, where B is $n \times n$ matrix whose entries we generate randomly in the interval $(-5, +5)$, and skew-symmetric matrix C is considered in the same way. The vector is denoted by q and is obtained in the interval $(-500, +500)$. This is distributed uniformly. For easy problems, we take $(-500, +500)$ and $(-500, 0)$ considered for the hard problem. In $D_1(y)$, the nonlinear part of $T(y)$, the components are $D_j(y) = d_j * \arctan(y_j)$, and d_j is a random variable generated in $(0, 1)$.

For the output of the result, we consider, $\mu = 2/3$, $\delta = 0.95$, $\delta_0 = 0.95$, $\rho > 0$ and $\gamma = 1.95$; the initial guess $y_0 = (0, 0, 0, \dots, 0)^T$. The computation starts with $\rho_0 = 1$ and stops as soon as $\|r(y_n, \rho_n)\| \leq 10^{-7}$. MATLAB is used for all codes. Table 1 represents the outcomes of Algorithm 1.

5. Conclusion

We have considered the new technique for solving inequality (1). We have applied the self-adaptive technique to control the step size under some mild conditions. Results have been compared with the published paper. It has been observed that the number of iterations is reduced by applying the new suggested method. This is an extension of the previously known results. This work can be enhanced further when the operator is pseudomonotone which is considered a weaker condition when the operator is strongly monotonicity. The numerical results reflect the output of our newly established algorithms well for the considered problems.

Data Availability

The manuscript included all required data and information for its implementation.

Conflicts of Interest

All authors declare no conflicts of interest in this paper.

Acknowledgments

We are thankful to the anonymous referees for their valuable comments which helped us to improve the quality of the paper. The authors wish to express their gratitude to Prince Sultan University for facilitating the publication of this article through the research lab Theoretical and Applied Sciences Lab. The authors would like to acknowledge the support of Prince Sultan University for paying the article processing charges (APC) of this publication.

References

- [1] G. Stampacchia, "Formes bilineaires coercivites sur les ensembles convexes," *Comptes Rendus de l'Academie des Sciences de Paris*, vol. 258, pp. 4413–4416, 1964.
- [2] S. S. Chang, H. W. J. Lee, and C. K. Chan, "Generalized system for relaxed cocoercive variational inequalities in Hilbert spaces," *Applied Mathematics Letters*, vol. 20, no. 3, pp. 329–334, 2007.
- [3] R. U. Verma, "A class of quasivariational inequalities involving cocoercive mappings," *Advances in Nonlinear Variational Inequalities*, vol. 2, 1999.
- [4] S. Ullah and M. A. Noor, "An efficient method for solving new general mixed variational inequalities," *Journal of Inequalities and Special Functions*, vol. 11, no. 3, pp. 1–9, 2020.
- [5] S. Wu, M. U. Awan, M. V. Mihai, M. A. Noor, and S. Talib, "Estimates of upper bound for a kth order differentiable functions involving Riemann-Liouville integrals via higher order strongly h-preinvex functions," *Journal of Inequalities and Applications*, vol. 2019, no. 1, 2019.
- [6] M. A. Noor and S. Ullah, "Predictor-corrector self-adaptive methods for variational inequalities," *Transylvanian Review*, vol. 16, pp. 4147–4152, 2017.
- [7] B.-s. He, Z.-h. Yang, and X.-m. Yuan, "An approximate proximal-extragradient type method for monotone variational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 300, no. 2, pp. 362–374, 2004.

- [8] M. A. Noor, A. Bnouhachem, and S. Ullah, "Self-adaptive methods for general variational inequalities," *Analysis*, vol. 71, no. 9, pp. 3728–3738, 2009.
- [9] A. Bnouhachem and M. A. Noor, "Numerical method for general mixed quasi-variational inequalities," *Applied Mathematics and Computation*, vol. 204, no. 1, pp. 27–36, 2008.
- [10] T. Guo-j, Z. Ming, and L. Huan-wen, "A new extragradient-type method for mixed variational inequalities," *Operations Research Letters*, vol. 43, no. 6, pp. 567–572, 2015.
- [11] J. Y. B. Cruz and A. N. Iusem, "Full convergence of an approximate projection method for nonsmooth variational inequalities," *Mathematics and Computers in Simulation*, vol. 114, pp. 2–13, 2015.
- [12] F. Alvarez and H. Attouch, "An inertial proximal method for maximalmonotone operators via discretization of a nonlinear oscillator with damping," *Set-Valued Analysis*, vol. 9, no. 1/2, pp. 3–11, 2001.
- [13] M. A. Noor, "General variational inequalities," *Applied Mathematics Letters*, vol. 1, no. 2, pp. 119–122, 1988.
- [14] B. S. He and L. Z. Lio, "Inexact implicit methods for monotone general variational inequalities," *Mathematical Programming*, vol. 86, no. 1, pp. 199–217, 1999.
- [15] A. Bnouhachem, "A self-adaptive method for solving general mixed variational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 309, no. 1, pp. 136–150, 2005.
- [16] M. A. Noor, "Differentiable non-convex functions and general variational inequalities," *Applied Mathematics and Computation*, vol. 199, no. 2, pp. 623–630, 2008.
- [17] P. Shi, "Equivalence of variational inequalities with Wiener-Hopf equations," *Proceedings of the American Mathematical Society*, vol. 111, no. 2, pp. 339–346, 1991.
- [18] M. A. Noor, *Principles of Variational Inequalities*, Lap-Lambest Academic Publishing AG and Co, Saarbrucken, Germany, 2009.