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Research Article

Certain Concepts of $\mathscr{Q}$-Hesitant Fuzzy Ideals

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The hesitant fuzzy set model has attracted the interest of scholars in various fields. The striking framework of hesitant fuzzy sets is keen to provide a larger domain of preference for fuzzy information modeling of deployment membership. Starting from the hybrid properties of hesitant fuzzy ideals (HFI), this paper constructs a new generalized hybrid structure $\mathscr{Q}$-HFI. The concept of $\mathscr{Q}$-hesitant fuzzy exchange ideal in $\mathscr{BCK}$-algebra is considered. Lastly, Q-hesitant fuzzy exchange ideal features are described.

1. Introduction

When dealing with information on all aspects of uncertainty, nonclassical logic always makes use of classical logic. Nonclassical logic is a useful tool in computer science because it deals with fuzzy information and uncertainty. In the literature, the study of BCK/BCI-algebras was first proposed by Imai and Iséki [1] in 1966 and such algebras can be regarded as a generalization of propositional logic. The study BCK/BCI-algebras have been developed by many people and have been extended to the fuzzy setting. After the introduction of fuzzy sets introduced by Zadeh [2], there have been many generalizations of this fundamental concept. In 2010, Torra [3] considered hesitant fuzzy sets. The hesitant fuzzy set model is useful tool to deal with uncertainty, which can be accurately and perfectly described in terms of the opinions of decision-makers.

Algebraic structures provide sufficient motivation for researchers to examine various concepts and stem from the broader field of abstract algebra blur set frame. In 2011, Xia and Xu [4] described hesitant fuzzy information aggregation techniques, and this concept was applied to $\mathscr{BCK}/\mathscr{BCI}$-algebras, $\mathscr{EQ}$-algebras, residuated lattices, $\mathscr{MTL}$-algebras, and K-algebras [5–9]. Jun and Ahn [6] investigated the concept of hesitant fuzzy subalgebras and HFIs of $\mathscr{BCK}/\mathscr{BCI}$-algebras. In 2018, Alshehri et al. [10] put forward the concept of new types of HFIs in $\mathscr{BCK}$-algebras. As a continuation of this study, we describe certain concepts, including $\mathscr{Q}$-HFIs and $\mathscr{Q}$-hesitant fuzzy commutative ideals in $\mathscr{BCK}$-algebras.

2. Basic Notions

A set $\mathscr{U}$ with a constant element 0 and a binary operation $*$ is said to be a $\mathscr{BCK}$-algebra [1] if it satisfies the axioms:

For all $\sqcap, \sqsubseteq, \Updownarrow \in \mathscr{U}$,

$$\begin{gathered}(\mathscr{BCK}-1)((\sqsubseteq*\sqcap)*(\sqcap*m))*(\Updownarrow*\sqsubseteq)=0,\\(\mathscr{BCK}-2)(\sqcap*(\sqcap*\sqcap))*\sqsubseteq=0,\\(\mathscr{BCK}-3)\sqcap*\sqcap=0,\\(\mathscr{BCK}-4)0*\sqcap=0,\\(\mathscr{BCK}-5)\sqcap*\sqsubseteq=0,\sqsubseteq*\sqcap=0\text{ imply that }\sqcap=\sqsubseteq.\end{gathered}\tag{1}$$

In a $\mathscr{BCK}$-algebra $\mathscr{U}$, we can define the relation $\leq$ by $\sqcap\leq\sqsubseteq$ if and only if $\sqcap*\sqsubseteq=0$.

Then, $(\mathscr{U};\leq)$ is a partially ordered set with the least element 0. In any $\mathscr{BCK}$-algebra $\mathscr{U}$, the following properties hold:

$$\begin{gathered}(\sqcap*\sqsubseteq)*\Updownarrow=(\sqcap*\Updownarrow m)*\sqsubseteq,\\\sqcap*\sqsubseteq\leq\sqcap,\\\sqcap*0=\sqcap,\\(\sqcap*\Updownarrow)*(\sqsubseteq*\Updownarrow)\leq\sqcap*\sqsubseteq,\\\sqcap*(\sqcap*(\sqcap*\sqsubseteq))=\sqcap*\sqsubseteq.\end{gathered}\tag{2}$$

$\sqcap \leq \sqsubseteq$ implies $\sqcap * \Updownarrow \leq \sqsubseteq * \Updownarrow$ and $\Updownarrow * \sqsubseteq \leq \Updownarrow * \sqcap$ for all $\sqcap, \sqsubseteq, \Updownarrow \in \mathscr{U}$.

Let $\mathscr{U}$ be a $\mathscr{BCK}$-algebra and let $\mathscr{A}$ be a nonempty subset of $\mathscr{U}$. Then, $\mathscr{A}$ is called an ideal of $\mathscr{U}$ [11] if it satisfies the following:

(1) $(\mathscr{I}_{\infty}) 0 \in \mathscr{A}$

(2) $(\mathscr{I}_{\epsilon}) \sqsubseteq \in \mathscr{A}$ and $\sqcap * \sqsubseteq \in \mathscr{A}$ imply that $\sqcap \in \mathscr{A}$ for all $\sqcap, \sqsubseteq \in \mathscr{U}$

A subset $\mathscr{A}$ of a $\mathscr{BCK}$-algebra $\mathscr{U}$ is called a commutative ideal [12] of $\mathscr{U}$ if it satisfies the following:

(1) $(\mathscr{I}_{\infty}) 0 \in \mathscr{A}$

(2) $(\mathscr{C}) (\sqcap * \sqsubseteq) * \Updownarrow \in \mathscr{A}$ and $\Updownarrow \in \mathscr{A}$ imply that $\sqcap * (\sqsubseteq * (\sqsubseteq * \sqcap)) \in \mathscr{A}$ for all $\sqcap, \sqsubseteq, \Updownarrow \in \mathscr{U}$

A fuzzy set in $\mathscr{U}$ is said to be a fuzzy ideal of $\mathscr{U}$ if it satisfies the following:

(1) $(\mathscr{F}_{\infty}) \mu(0) \geq \mu(\sqcap)$ for all $\sqcap \in \mathscr{U}$

(2) $(\mathscr{F}_{\epsilon}) \mu(\sqcap) \geq \min \{\mu(\sqcap * \sqsubseteq), \mu(\sqsubseteq)\}$ for all $\sqcap, \sqsubseteq \in \mathscr{U}$

Let $\mathscr{U}$ be a reference set and $\mathscr{A}$ be a nonempty subset of $\mathscr{U}$, a hesitant fuzzy set.

$\mathscr{H}_{\mathscr{U}} = \{(\sqcap, \langle_{\mathscr{U}}(\sqcap)) : \sqcap \in \mathscr{U}\}$ on $\mathscr{U}$ [3] satisfying the following condition:

$$\langle_{\mathscr{U}}(\sqcap) = \phi \text{ for all } \sqcap \notin \mathscr{A}, \tag{3}$$

is called a hesitant fuzzy set related to $\mathscr{U}$ (briefly, $\mathscr{A}$-hesitant fuzzy set) on $\mathscr{U}$ and is represented by $\mathscr{H}_{\mathscr{A}} = \{(\sqcap, \langle_{\mathscr{A}}(\sqcap)) : \sqcap \in \mathscr{U})\}$, where $\langle_{\mathscr{A}}$ is a mapping from $\mathscr{U}$ to $\checkmark([0,1])$ with $\langle_{\mathscr{A}}(\sqcap) = \phi$ for all $\sqcap \notin \mathscr{A}$.

Let $\mathscr{U}$ be a reference set and $\mathscr{A}$ be a nonempty subset of $\mathscr{U}$, an $\mathscr{A}$-hesitant fuzzy set $\mathscr{H}_{\mathscr{A}} = \{(\sqcap, \langle_{\mathscr{A}}(\sqcap)) : \sqcap \in \mathscr{U}\}$ of $\mathscr{U}$ is called a HFI [6] of $\mathscr{U}$ related to $\mathscr{A}$ (briefly, $\mathscr{A}$-HFI of $\mathscr{U}$) if it satisfies the following:

(1) $(\mathscr{H}_{\infty}) \langle_{\mathscr{A}}(0) \geq \langle_{\mathscr{A}}(\sqcap)$ for all $\sqcap \in \mathscr{U}$

(2) $(\mathscr{H}_{\epsilon}) \langle_{\mathscr{A}}(\sqcap) \geq \min \{\langle_{\mathscr{A}}(\sqcap * \sqsubseteq), \langle_{\mathscr{A}}(\sqsubseteq)\}$ for all $u, v \in U$

Given a nonempty subset $\mathscr{A}$ of $\mathscr{U}$, an $\mathscr{A}$-hesitant fuzzy set $\mathscr{H}_{\mathscr{A}} = \{(\sqcap, \langle_{\mathscr{A}}) : \sqcap \in \mathscr{U}\}$ of $\mathscr{U}$ is called a hesitant fuzzy commutative ideal [10] of $\mathscr{U}$ related to $\mathscr{A}$ (briefly, $\mathscr{A}$-hesitant fuzzy commutative ideal of $\mathscr{U}$) if it satisfies

$(\mathscr{H}_{\infty}) \langle_{\mathscr{A}}(0) \geq \langle_{\mathscr{A}}(\sqcap)$ for all $\sqcap \in \mathscr{U}$

$(\mathscr{H}_{\rfloor})$

$\langle_{\mathscr{A}}(\sqcap * (\sqsubseteq * (\sqsubseteq * \sqcap))) \geq \min \{\langle_{\mathscr{A}}((\sqcap * \sqsubseteq) * \Updownarrow), \langle_{\mathscr{A}}(\Updownarrow)\}$ for all $\sqcap, \sqsubseteq, \Updownarrow \in \mathscr{U}$

Let $\mathscr{U}$ be a nonempty finite universe and $\mathscr{Q}$ be a nonempty set. A $\mathscr{Q}$-hesitant fuzzy set $\mathscr{A}_{\mathscr{Q}}$ is a set given by

$$\mathscr{A}_{\mathscr{Q}} = \left\{ \left((\sqcap, \coprod), \langle_{\mathscr{A}_{\mathscr{Q}}}(\sqcap, \coprod) \right) : \sqcap \in \mathscr{U}, \coprod \in \mathscr{Q} \right\}, \tag{4}$$

where $\langle_{\mathscr{A}_{\mathscr{Q}}} : \mathscr{U} \times \mathscr{Q} \longrightarrow [0,1]$. The function $\langle_{\mathscr{A}_{\mathscr{Q}}}(\sqcap, \coprod)$ is called the membership function of $\mathscr{Q}$-hesitant fuzzy set, and the set of all $\mathscr{Q}$-hesitant fuzzy set over $\mathscr{U} \times \mathscr{Q}$ will be denoted by $\mathscr{QHF}(\mathscr{U} \times \mathscr{Q})$.

Let $\mathscr{H}_{\mathscr{U}}$ be a hesitant fuzzy set of a $\mathscr{BCK}$-algebra $\mathscr{U}$. The set

$$\mathscr{H}_{\mathscr{U}}\left(\int\right) = \left\{ \sqcap \in \mathscr{U} \mid \mathscr{H}_{\mathscr{U}}(\sqcap) \geq \int \right\}, \tag{5}$$

where $\int; \in_{\checkmark}([0,1])$ is called a hesitant fuzzy $\int$-level set of $\mathscr{H}_{\mathscr{U}}$.

Theorem 1 (see [6]). *For a subalgebra, $\mathscr{A}$ of a $\mathscr{BCK}$-algebra $\mathscr{U}$, every $\mathscr{A}$-HFI is an $\mathscr{A}$-hesitant fuzzy subalgebra.*

Proposition 2 (see [13]). *In $\mathscr{BCK}$-algebra $\mathscr{U}$ the following conditions hold, for all $\sqcap, \sqsubseteq, \Updownarrow \in \mathscr{U}$,*

$$\begin{gathered}((\sqcap * \Updownarrow) * \Updownarrow) * (\sqsubseteq * \Updownarrow) \leq (\sqcap * \sqsubseteq) * \Updownarrow, \\ (\sqcap * \Updownarrow) * (\sqcap * (\sqcap * \Updownarrow)) = (\sqcap * \Updownarrow) * \Updownarrow, \\ (\sqcap * (\sqsubseteq * (\sqsubseteq * \sqcap))) * (\sqsubseteq * (\sqcap * (\sqsubseteq * (\sqsubseteq * \sqcap)))) \leq \sqcap * \sqsubseteq.\end{gathered} \tag{6}$$

3. $\mathscr{Q}$-Hesitant Fuzzy Ideals

Definition 3. Let $\mathscr{U}$ be a nonempty finite universe, $\mathscr{Q}$ be a nonempty set and $\mathscr{A}$ be the subset of $\mathscr{U}$, a $\mathscr{Q}$-hesitant fuzzy ideal $\mathscr{H}_{\mathscr{A}_{\mathscr{Q}}}$ of $\mathscr{U} \times \mathscr{Q}$ (briefly: $\mathscr{QHF}$-ideal) if it satisfies the following assertion:

(1) $(\mathscr{Q}_{\infty}\mathscr{H}) \langle_{\mathscr{A}_{\mathscr{Q}}}(', \coprod) \geq \langle_{\mathscr{A}_{\mathscr{Q}}}(\sqcap, \coprod) \forall \sqcap \in \mathscr{U}, \coprod \in \mathscr{Q}$

(2) $(\mathscr{Q}_{\epsilon}\mathscr{H}) \langle_{\mathscr{A}_{\mathscr{Q}}}(\sqcap, \coprod) \geq \min \{\langle_{\mathscr{A}_{\mathscr{Q}}}(\sqcap * \sqsubseteq, \coprod), \langle_{\mathscr{A}_{\mathscr{Q}}}(\sqsubseteq, \coprod)\}$, $\forall \sqcap, \sqsubseteq \in \mathscr{U}, \coprod \in \mathscr{Q}$

Example 1 Denote $\mathscr{U} = \{0, \mathrm{a}, \mathrm{b}, \mathrm{c}\}$. The binary operation $*$ on $\mathscr{U}$ is given by Cayley (Table 1).

For a subset $\mathscr{A} = \{0, a, b\}$. Let $\mathscr{H}_{\mathscr{A}_{\mathscr{Q}}} = (((\sqcap, \coprod), \langle_{\mathscr{A}_{\mathscr{Q}}}(\sqcap, \coprod)) \mid \sqcap \in \mathscr{U}, \coprod \in \mathscr{Q})$ be a $\mathscr{QHFS}$ of $\mathscr{U} \times \mathscr{Q}$ defined by

$$\langle_{\mathscr{A}_{\mathscr{Q}}} : \mathscr{U} \times \mathscr{Q} \mapsto \begin{cases} \left[0, \dfrac{1}{2}\right], \text{if} & (\sqcap, \coprod) = \left(', \coprod\right), \\ \left[0, \dfrac{1}{3}\right], \text{if} & (\sqcap, \coprod) = (\{a, b, c\}, \coprod). \end{cases} \tag{7}$$

Then, $\mathscr{H}_{\mathscr{A}_{\mathscr{Q}}}$ is a $\mathscr{QHF}$-ideal of $\mathscr{U} \times \mathrm{Q}$.

Proposition 4. *Let $\mathscr{A}$ be a subset of $\mathscr{U}$ and $\mathscr{H}_{\mathscr{A}_{\mathscr{Q}}}$ be a $\mathscr{Q}$-HFI of $\mathscr{U} \times \mathscr{Q}$. Then, the following assertions are valid:*

(1) $\sqcap \leq \sqsubseteq \Longrightarrow \langle_{\mathscr{A}_{\mathscr{Q}}}(\sqcap, \coprod) \geq \langle_{\mathscr{A}_{\mathscr{Q}}}(\sqsubseteq, \coprod)$ *for all* $\sqcap, \sqsubseteq \in \mathscr{U}$, $\coprod \in \mathscr{Q}$

(2) $\sqcap * \sqsubseteq \leq \Updownarrow \Longrightarrow \langle_{\mathscr{A}_{\mathscr{Q}}}(\sqcap, \coprod) \geq \min \{\langle_{\mathscr{A}_{\mathscr{Q}}}(\sqsubseteq, \coprod), \langle_{\mathscr{A}_{\mathscr{Q}}}(\Updownarrow, \coprod)\}$ *for all* $\sqcap, \sqsubseteq, \Updownarrow \in \mathscr{U}, \coprod \in \mathscr{Q}$

Table 1

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Proof.

(1) Suppose $\sqcap \leq \sqsubseteq$ implies $\sqcap * \sqsubseteq = 0 \in \mathscr{A}$ (for all $\sqcap, \sqsubseteq \in \mathscr{A}$) and so

$$\left\langle_{\mathscr{A}_{\mathbb{Q}}}(\sqsubseteq,\coprod) = \min\left\{\left\langle_{\mathscr{A}_{\mathbb{Q}}}\left({}^{\prime},\coprod\right), \left\langle_{\mathscr{A}_{\mathbb{Q}}}(\sqsubseteq,\coprod)\right\}\right.\right. = \min\left\{\left\langle_{\mathscr{A}_{\mathbb{Q}}}(\sqcap*\sqsubseteq,\coprod), \left\langle_{\mathscr{A}_{\mathbb{Q}}}(\sqsubseteq,\coprod)\right\} \leq \left\langle_{\mathscr{A}_{\mathbb{Q}}}(\sqcap,\coprod),\right.\right. \tag{8}$$

by $(\mathbb{Q}_{\epsilon}\mathscr{H})$

(2) Suppose $\sqcap * \sqsubseteq \leq \Updownarrow$ implies $(\sqcap * \sqsubseteq) * \Updownarrow = 0 \in \mathscr{A}$ (for all $\sqcap, \sqsubseteq, \Updownarrow \in \mathscr{U}$) so

$$\left\langle_{\mathscr{A}_{\mathbb{Q}}}(\Updownarrow,\coprod) = \min\left\{\left\langle_{\mathscr{A}_{\mathbb{Q}}}\left({}^{\prime},\coprod\right), \left\langle_{\mathscr{A}_{\mathbb{Q}}}(\Updownarrow,\coprod)\right\}\right.\right. = \min\left\{\left\langle_{\mathscr{A}_{\mathbb{Q}}}((\sqcap*\sqsubseteq)*\Updownarrow,\coprod), \left\langle_{\mathscr{A}_{\mathbb{Q}}}(\Updownarrow,\coprod)\right\}\right.\right. \leq \left\langle_{\mathscr{A}_{\mathbb{Q}}}(\sqcap*\sqsubseteq,\coprod)\right. \tag{9}$$

It follows that

$$\min\left\{\left\langle_{\mathscr{A}_{\mathbb{Q}}}(\sqsubseteq,\coprod), \left\langle_{\mathscr{A}_{\mathbb{Q}}}(\Updownarrow,\coprod)\right\} \leq \min\left\{\left\langle_{\mathscr{A}_{\mathbb{Q}}}(\sqcap*\sqsubseteq,\coprod), \left\langle_{\mathscr{A}_{\mathbb{Q}}}(\sqsubseteq,\coprod)\right\}\right.\right.\right.\right. \leq \left\langle_{\mathscr{A}_{\mathbb{Q}}}(\sqcap,\coprod).\right. \tag{10}$$

□

Proposition 5. *Every A $\mathbb{Q}\mathscr{H}\mathscr{F}$-ideal of $\mathscr{U} \times \mathbb{Q}$ satisfies the following condition:*

(1) $\langle_{\mathscr{U}_{\mathbb{Q}}}(\sqsubseteq,\coprod) \leq \langle_{\mathscr{U}_{\mathbb{Q}}}(\sqcap,\coprod)$ with $\sqcap \leq \sqsubseteq$ for all $\sqcap, \sqsubseteq \in \mathscr{U}$, $\coprod \in \mathbb{Q}$

(2) $\min\{\langle_{\mathscr{U}_{\mathbb{Q}}}(\sqsubseteq,\coprod), \langle_{\mathscr{U}_{\mathbb{Q}}}(\Updownarrow,\coprod)\} \leq \langle_{\mathscr{U}_{\mathbb{Q}}}(\sqcap,\coprod)$ with $\sqcap * \sqsubseteq \leq \Updownarrow$ for all $\sqcap, \sqsubseteq, \Updownarrow \in \mathscr{U}, \coprod \in \mathbb{Q}$

Theorem 6. *If $\mathscr{H}_{\mathscr{A}_{\mathbb{Q}}}$ a $\mathbb{Q}$-HFI of $\mathscr{U} \times \mathbb{Q}$, then for any $\sqcap, \dashv_{\infty}, \dashv_{\epsilon}, \cdots, \dashv_{\backslash} \in \mathscr{U}$, and*

$$(\cdots((\sqcap*\dashv_{\infty})*\dashv_{\epsilon})*\cdots)*\dashv_{\backslash} = 0 \Longrightarrow \left\langle_{\mathscr{A}_{\mathbb{Q}}}(\sqcap,\coprod) \geq \min\left\{\left\langle_{\mathscr{A}_{\mathbb{Q}}}(\dashv_{\infty},\coprod), \left\langle_{\mathscr{A}_{\mathbb{Q}}}(\dashv_{\epsilon},\coprod), \cdots, \left\langle_{\mathscr{A}_{\mathbb{Q}}}(\dashv_{\backslash},\coprod)\right\}.\right.\right.\right.\right. \tag{11}$$

Theorem 7. *Let $\mathscr{H}_{\mathscr{A}_{\mathbb{Q}}}$ be a $\mathbb{Q}$-HFI of $\mathscr{U} \times \mathbb{Q}$. Then, the following are equivalent:*

(i) $\langle_{\mathscr{A}_{\mathbb{Q}}}((\sqcap*\sqsubseteq)*\sqsubseteq,\coprod) \leq \langle_{\mathscr{A}_{\mathbb{Q}}}(\sqcap*\sqsubseteq,\coprod)$ for all $\sqcap, \sqsubseteq \in \mathscr{U}$, $\coprod \in \mathbb{Q}$

(ii) $\langle_{\mathscr{A}_{\mathbb{Q}}}((\sqcap*\sqsubseteq)*\Updownarrow,\coprod) \leq \langle_{\mathscr{A}_{\mathbb{Q}}}((\sqcap*\Updownarrow)*(\sqsubseteq*\Updownarrow),\coprod)$ for all $\sqcap, \sqsubseteq, \Updownarrow \in \mathscr{U}, \coprod \in \mathbb{Q}$

Proof. $(i) \Longrightarrow (ii)$ Suppose condition $(\rangle)$ is valid. Since

$$((\sqcap*(\sqsubseteq*\Updownarrow))*\Updownarrow)*\Updownarrow = ((\sqcap*\Updownarrow)*(\sqsubseteq*\Updownarrow))*\Updownarrow \leq (\sqcap*\sqsubseteq)*\Updownarrow. \tag{12}$$

Applying, by Proposition 2 and $(\dashv)$, we have

$$\left\langle_{\mathscr{A}_{\mathbb{Q}}}((\sqcap*\sqsubseteq)*\Updownarrow,\coprod) \leq \left\langle_{\mathscr{A}_{\mathbb{Q}}}(((\sqcap*(\sqsubseteq*\Updownarrow))*\Updownarrow)*\Updownarrow,\coprod) \leq \left\langle_{\mathscr{A}_{\mathbb{Q}}}((\sqcap*(\sqsubseteq*\Updownarrow))*\Updownarrow,\coprod) = \left\langle_{\mathscr{A}_{\mathbb{Q}}}((\sqcap*\Updownarrow)*(\sqsubseteq*\Updownarrow),\coprod).\right.\right.\right.\right. \tag{13}$$

Hence, condition (ii) holds

$(ii) \Longrightarrow (i)$ Suppose condition (ii) is valid. If we put $\Updownarrow = \sqsubseteq$ in (ii) then

$$\left\langle_{\mathscr{A}_{\mathbb{Q}}}((\sqcap*\sqsubseteq)*\sqsubseteq,\coprod) \leq \left\langle_{\mathscr{A}_{\mathbb{Q}}}((\sqcap*\sqsubseteq)*(\sqsubseteq*\sqsubseteq),\coprod) = \left\langle_{\mathscr{A}_{\mathbb{Q}}}\left((\sqcap*\sqsubseteq)*\left({}^{\prime}\right),\coprod\right) = \left\langle_{\mathscr{A}_{\mathbb{Q}}}(\sqcap*\sqsubseteq,\coprod),\right.\right.\right.\right. \tag{14}$$

hence, the condition (i) holds.

The proof is complete. □

Theorem 8. *Let $\mathscr{H}_{\mathscr{U}_{\mathbb{Q}}}$ be a $\mathbb{Q}$-HFI of $\mathscr{U} \times \mathbb{Q}$, then the set*

$$\mathscr{H}_{\int} = \left\{\sqcap\in\mathscr{U},\coprod\in\mathbb{Q} \mid \left\langle_{\mathscr{U}_{\mathbb{Q}}}\left(\int,\coprod\right) \leq \left\langle_{\mathscr{U}_{\mathbb{Q}}}(\sqcap,\coprod)\right\},\right.\right. \tag{15}$$

is an ideal of $\mathscr{U} \times \mathbb{Q}$ for all $\int \in \mathscr{U}$.

Proof. Let $\sqcap, \sqsubseteq \in \mathscr{U}, \coprod \in \mathscr{Q}$ be such that $(\sqcap * \sqsubseteq, \coprod) \in \mathscr{H}_{\int}$ and $(\sqsubseteq, \coprod) \in \mathscr{H}_{\int}$. Then,

$$\left\langle_{\mathscr{U}_{\mathscr{Q}}}\left(\int, \coprod\right) \leq \left\langle_{\mathscr{U}_{\mathscr{Q}}}(\sqcap * \sqsubseteq, \coprod), \left\langle_{\mathscr{U}_{\mathscr{Q}}}\left(\int, \coprod\right) \leq \left\langle_{\mathscr{U}_{\mathscr{Q}}}(\sqsubseteq, \coprod). \right.\right.\right. \tag{16}$$

It follows from $(\mathscr{Q}_{\infty}\mathscr{H}), (\mathscr{Q}_{\epsilon}\mathscr{H})$ that

$$\begin{aligned} \left\langle_{\mathscr{U}_{\mathscr{Q}}}\left(\int, \coprod\right) &\leq \min\left\{\left\langle_{\mathscr{U}_{\mathscr{Q}}}(\sqcap * \sqsubseteq, \coprod), \left\langle_{\mathscr{U}_{\mathscr{Q}}}(\sqsubseteq, \coprod)\right.\right\}\right. \\ &\leq \left\langle_{\mathscr{U}_{\mathscr{Q}}}(\sqcap, \coprod) \leq \left\langle_{\mathscr{U}_{\mathscr{Q}}}\left(', \coprod\right).\right.\right. \end{aligned} \tag{17}$$

□

So that $(', \coprod) \in \mathscr{H}_{\int}$ and $(\sqcap, \coprod) \in \mathscr{H}_{\int}$, therefore, $\mathscr{H}_{\int}$ is an ideal of $\mathscr{U} \times \mathscr{Q}$ for all $\int \in \mathscr{U}$.

Theorem 9. *Suppose that $\mathscr{H}_{\mathscr{A}_{\mathscr{Q}}}$ is a $\mathscr{Q}$-hesitant fuzzy set of $\mathscr{U} \times \mathscr{Q}$, where $\mathscr{A}$ is a nonempty subset of $\mathscr{U}$. Then, the following are equivalent:*

(i) $\mathscr{H}_{\mathscr{A}_{\mathscr{Q}}}$ is a $\mathscr{Q}$-HFI of $\mathscr{U} \times \mathscr{Q}$

(ii) For any $\int \in \checkmark([0,1])$, the set $\mathscr{H}_{\mathscr{A}_{\mathscr{Q}}}(\int) = \{\sqcap \in \mathscr{U} : \langle_{\mathscr{U}_{\mathscr{Q}}}(\sqcap, \coprod) \geq \int\}$ is an ideal of $\mathscr{U} \times \mathscr{Q}$

Proof. ⇒ Assume that $\mathscr{H}_{\mathscr{A}_{\mathscr{Q}}}$ is a $\mathscr{Q}\mathscr{H}\mathscr{F}$-ideal of $\mathscr{U} \times \mathscr{Q}$. Let $u, v \in \mathscr{U}$ and $\jmath \in \checkmark([0,1])$ be such that $(\sqcap * \sqsubseteq, \coprod) \in \mathscr{H}_{\mathscr{A}_{\mathscr{Q}}}(\int)$ and $(\sqsubseteq, \coprod) \in \mathscr{H}_{\mathscr{A}_{\mathscr{Q}}}(\int)$.Then,

$$\left\langle_{\mathscr{A}_{\mathscr{Q}}}(\sqcap * \sqsubseteq, \coprod) \geq \int, \left\langle_{\mathscr{A}_{\mathscr{Q}}}(\sqsubseteq, \coprod) \geq \int. \right.\right. \tag{18}$$

It follows that

$$\left\langle_{\mathscr{A}_{\mathscr{Q}}}\left(', \coprod\right) \geq \left\langle_{\mathscr{A}_{\mathscr{Q}}}(\sqcap, \coprod) \geq \min\left\{\left\langle_{\mathscr{A}_{\mathscr{Q}}}(\sqcap * \sqsubseteq, \coprod), \left\langle_{\mathscr{A}_{\mathscr{Q}}}(\sqsubseteq, \coprod)\right.\right\} \geq \int.\right.\right. \tag{19}$$

Hence, $(', \coprod) \in \mathscr{H}_{\mathscr{A}_{\mathscr{Q}}}(\int)$ and $(\sqcap, \coprod) \in \mathscr{H}_{\mathscr{A}_{\mathscr{Q}}}(\int)$

Therefore $\mathscr{H}_{\mathscr{A}_{\mathscr{Q}}}(\int)$ is an ideal of $\mathscr{U} \times \mathscr{Q}$

⟸ Suppose that $\mathscr{H}_{\mathscr{A}_{\mathscr{Q}}}(\int)$ is an ideal of $\mathscr{U} \times \mathscr{Q}$. For any $\sqcap \in \mathscr{U}, \coprod \in \mathscr{Q}$, let $\langle_{\mathscr{A}_{\mathscr{Q}}}(\sqcap, \coprod) = \int^{(\sqcap, \coprod)}$

Then $(\sqcap, \coprod) \in \mathscr{H}_{\mathscr{A}_{\mathscr{Q}}}(\int^{(\sqcap, \coprod)})$. Since $\mathscr{H}_{\mathscr{A}_{\mathscr{Q}}}(\int^{(\sqcap, \coprod)})$ is an ideal of $\mathscr{U} \times \mathscr{Q}$. we have

$$\left(', \coprod\right) \in \mathscr{H}_{\mathscr{A}_{\mathscr{Q}}}\left(\int^{(\sqcap, \coprod)}\right) \text{ and so } \left\langle_{\mathscr{A}_{\mathscr{Q}}}(\sqcap, \coprod) = \int^{(\sqcap, \coprod)} \leq \left\langle_{\mathscr{A}_{\mathscr{Q}}}\left(', \coprod\right)\right.\right. \tag{20}$$

Let $\langle_{\mathscr{A}_{\mathscr{Q}}}(\sqcap * \sqsubseteq) = \int^{(\sqcap * \sqsubseteq, \coprod)}$ and $\langle_{\mathscr{A}_{\mathscr{Q}}}(\sqsubseteq, \coprod)\int^{(\sqsubseteq, \coprod)}$. Then $(\sqcap * \sqsubseteq, \coprod) \in \mathscr{H}_{\mathscr{A}_{\mathscr{Q}}}(\int)$ and $(\sqsubseteq, \coprod) \in \mathscr{H}_{\mathscr{A}_{\mathscr{Q}}}(\int)$ such that $\int = \min\{\int^{(\sqcap * \sqsubseteq, \coprod)}, \int^{(\sqsubseteq, \coprod)}\}$, which imply that $(\sqcap, \coprod) \in \mathscr{H}_{\mathscr{A}_{\mathscr{Q}}}(\int)$. Thus,

$$\left\langle_{\mathscr{A}_{\mathscr{Q}}}(\sqcap, \coprod) \geq \int = \min\left\{\left\langle_{\mathscr{A}_{\mathscr{Q}}}(\sqcap * \sqsubseteq, \coprod), \left\langle_{\mathscr{A}_{\mathscr{Q}}}(\sqsubseteq, \coprod)\right.\right\}.\right. \tag{21}$$

Therefore, $\mathscr{H}_{\mathscr{A}_{\mathscr{Q}}}$ is a $\mathscr{Q}\mathscr{H}\mathscr{F}$-ideal of $\mathscr{U} \times \mathscr{Q}$. □

4. $\mathscr{Q}$-Hesitant Fuzzy Commutative Ideals

Definition 10. Let $\mathscr{U}$ be a universal set and $\mathscr{Q}$ be a nonempty set. A $\mathscr{Q}$-hesitant fuzzy commutative ideal ($\mathscr{Q}$-HFCI) of $\mathscr{U} \times \mathscr{Q}$ if it satisfies the following assertion:

(1) $(\mathscr{Q}_{\infty}\mathscr{H})\langle_{\mathscr{A}_{\mathscr{Q}}}(', \coprod) \geq \langle_{\mathscr{A}_{\mathscr{Q}}}(\sqcap, \coprod) \forall \sqcap \in \mathscr{U}, \coprod \in \mathscr{Q}$

(2) $(\mathscr{Q}_{\rfloor}\mathscr{H})\langle_{\mathscr{A}_{\mathscr{Q}}}(\sqcap * (\sqsubseteq * (\sqsubseteq * \sqcap)), \coprod) \geq \min\{\langle_{\mathscr{A}_{\mathscr{Q}}}(\sqcap * \sqsubseteq) * \Updownarrow, \coprod), \langle_{\mathscr{A}_{\mathscr{Q}}}(\Updownarrow, \coprod)\} \forall \sqcap, \sqsubseteq, \Updownarrow \in \mathscr{U}, \coprod \in \mathscr{Q}$

Example 1. Let $\mathscr{U} = \{0, a, b, c\}$ be a set with the binary operation $*$ which is defined in Cayley (Table 2).

Let $\sqcup', \sqcup_{\infty}, \sqcup_{\epsilon} \in_{\surd}([0,1])$ such that $\sqcup' \succ \sqcup_{\infty} \succ \sqcup_{\epsilon}$. We define a $\mathscr{Q}\mathscr{H}\mathscr{F}\mathscr{S}$ of $\mathscr{U} \times \mathscr{Q}$ as follows:

$$\langle_{\mathscr{U}_{\mathscr{Q}}} : \mathscr{U} \times \mathscr{Q} \longrightarrow_{\surd}([0,1]), (\sqcap, \coprod) \mapsto \begin{cases} t_0, \text{if} & (\sqcap, \coprod) = \left(', \coprod\right), \\ t_1, \text{if} & (\sqcap, \coprod) = (a, \coprod), \\ t_2, \text{if} & (\sqcap, \coprod) = (\{b, c\}, \coprod), \end{cases} \tag{22}$$

where $t_0 = [0,1] \succ t_1 = [0.2, 0.6] \succcurlyeq t_2 = (0.2, 0.3]$. By direct calculations, one can see that $\mathscr{H}_{\mathscr{Q}}$ is $\mathscr{Q}\mathscr{H}\mathscr{F}\mathscr{C}$-ideal of $\mathscr{U} \times \mathscr{Q}$.

Theorem 11. *Every $\mathscr{Q}$-hesitant fuzzy CI is a $\mathscr{Q}$-hesitant fuzzy ideal of $\mathscr{U} \times \mathscr{Q}$.*

Proof. Assume that $\mathscr{H}_{\mathscr{A}_{\mathscr{Q}}}$ is a $\mathscr{Q}\mathscr{H}\mathscr{F}\mathscr{C}$-ideal of a $\mathscr{B}\mathscr{C}\mathscr{K}$-algebra $\mathscr{U}$, for any $\sqcap, \Updownarrow \in \mathscr{U}, \coprod \in \mathscr{Q}$. We have

$$\begin{aligned} &\min\left\{\left\langle_{\mathscr{A}_{\mathscr{Q}}}(\sqcap * \Updownarrow, \coprod), \langle\mathscr{A}_{\mathscr{Q}}(\Updownarrow, \coprod)\right.\right\} \\ &\quad = \min\left\{\left\langle_{\mathscr{A}_{\mathscr{Q}}}\left(\left(\sqcap *'\right) * \Updownarrow, \coprod\right), \langle\mathscr{A}_{\mathscr{Q}}(\Updownarrow, \coprod)\right.\right\} \\ &\quad \leq \left\langle_{\mathscr{A}_{\mathscr{Q}}}\left(\left(\sqcap * \left(' * \left('* \sqcap\right)\right), \coprod\right) = \left\langle_{\mathscr{A}_{\mathscr{Q}}}(\sqcap, \coprod).\right.\right.\right. \end{aligned} \tag{23}$$

Therefore, $\mathscr{H}_{\mathscr{A}_{\mathscr{Q}}}$ is a $\mathscr{Q}\mathscr{H}\mathscr{F}$-ideal in $\mathscr{U} \times \mathscr{Q}$. □

The following example shows that the converse of theorem 6 is not true.

Example 2. Let $\mathscr{U} = \{0, a, b, c, d\}$ be a set with the Cayley (Table 3).

Table 2

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Table 3

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	d	d	c	0

Let $e_0, e_1, e_2 \in_{\checkmark}([0,1])$ such that $e_0 \succ e_1 \succ e_2$. We define a mapping

$$\Big\langle_{\mathscr{U}_{\mathscr{Q}}} \sim \mathscr{U} \times \mathscr{Q} \longrightarrow_{\checkmark}([0,1]), (\sqcap, \coprod) \mapsto \begin{cases} e_0, \text{if} & (\sqcap, \coprod) = (', \coprod), \\ e_1, \text{if} & (\sqcap, \coprod) = (\dashv, \coprod), \\ e_2, \text{if} & (\sqcap, \coprod) = (\{b, c, d\}, \coprod), \end{cases} \tag{24}$$

where $e_0 = [0,1) \succ e_1 = [0.4, 0.8] \pm e_2 = [0.5, 0.6]$. It is routine to verify that $\mathscr{H}_{\mathscr{U}_{\mathscr{Q}}}$ is $\mathscr{QHF}$-ideal of $\mathscr{U} \times \mathscr{Q}$. But it is not a $\mathscr{QHFC}$-ideal of $\mathscr{U} \times \mathscr{Q}$. Since

$$\Big\langle_{\mathscr{U}_{\mathscr{Q}}}(b * (c * (c * b), \coprod) \ngeq \min \left\{ \langle \mathscr{U}_{\mathscr{Q}}((b * c) * 0, \coprod), \Big\langle_{\mathscr{U}_{\mathscr{Q}}}(', \coprod) \right\}. \tag{25}$$

Theorem 12. *Let $\mathscr{H}_{\mathscr{U}_{\mathscr{Q}}}$ be a $\mathscr{Q}$-HFI of a $\mathscr{BCK}$-algebra $\mathscr{U}$. Then, $\mathscr{H}_{\mathscr{U}_{\mathscr{Q}}}$ is a $\mathscr{Q}$-hesitant fuzzy CI of $\mathscr{U} \times \mathscr{Q}$ if and only if it satisfies the following condition:*

$$\Big\langle_{\mathscr{U}_{\mathscr{Q}}}(\sqcap * (\sqsubseteq * (\sqsubseteq * \sqcap)), \coprod) \geq \Big\langle_{\mathscr{U}_{\mathscr{Q}}}(\sqcap * \sqsubseteq, \coprod) \forall \sqcap, \sqsubseteq \in \mathscr{U}, \coprod \in \mathscr{Q}. \tag{26}$$

Proof. Assume that $\mathscr{H}_{\mathscr{U}_{\mathscr{Q}}}$ is $\mathscr{QHFC}$-ideal. Taking $m = 0$ in $(\mathscr{Q}_{\mathscr{C}}\mathscr{H})$ and using $(\mathscr{Q}_{\infty}\mathscr{H})$. Also, we use $\sqcap * 0 = \sqcap$.

$$\begin{aligned} &\Big\langle_{\mathscr{U}_{\mathscr{Q}}}(\sqcap * (\sqsubseteq * (\sqsubseteq * \sqcap)), \coprod) \\ &\quad \geq \min \left\{ \Big\langle_{\mathscr{U}_{\mathscr{Q}}}(\sqcap * \sqsubseteq) * ' \Big), \coprod), \langle \mathscr{U}_{\mathscr{Q}}\left(', \coprod\right) \right\} \\ &\quad = \Big\langle_{\mathscr{U}_{\mathscr{Q}}}(\sqcap * \sqsubseteq). \end{aligned} \tag{27}$$

Conversely, Let $\mathscr{H}_{\mathscr{U}_{\mathscr{Q}}}$ As $\mathscr{H}_{\mathscr{U}_{\mathscr{Q}}}$ be a $\mathscr{QHF}$-ideal of $\mathscr{U} \times \mathscr{Q}$ satisfying condition (1).

Then,

$$\Big\langle_{\mathscr{U}_{\mathscr{Q}}}(\sqcap * \sqsubseteq, \coprod) \geq \min \left\{ \Big\langle_{\mathscr{U}_{\mathscr{Q}}}(\sqcap * \sqsubseteq) * \Updownarrow, \coprod), \Big\langle_{\mathscr{U}_{\mathscr{Q}}}(\Updownarrow, \coprod) \right\}, \forall \sqcap, \sqsubseteq, \Updownarrow \in \mathscr{U}, \coprod \in \mathscr{Q}, \tag{28}$$

combining (1) and (2), then we obtain $(\mathscr{Q}_c\mathscr{H})$.

The proof is complete. □

Lemma 13. *Any $\mathscr{Q}$-HFI of a $\mathscr{BCK}$-algebra $\mathscr{U}$ satisfies*

$$\sqcap * \sqsubseteq \leq \Updownarrow \Longrightarrow \Big\langle_{\mathscr{U}_{\mathscr{Q}}}(\sqcap, \coprod) \geq \min \left\{ \Big\langle_{\mathscr{U}_{\mathscr{Q}}}(\sqsubseteq, \coprod), \langle \mathscr{U}_{\mathscr{Q}}(\Updownarrow, \coprod) \right\}. \tag{29}$$

Proof. Assume that $\sqcap * \sqsubseteq \leq \Updownarrow$ holds. Then,

$$\begin{aligned} \Big\langle_{\mathscr{U}_{\mathscr{Q}}}(\sqcap * \sqsubseteq, \coprod) &\geq \min \left\{ \Big\langle_{\mathscr{U}_{\mathscr{Q}}}((\sqcap * \sqsubseteq) * \Updownarrow, \coprod), \Big\langle_{\mathscr{U}_{\mathscr{Q}}}(\Updownarrow, \coprod) \right\} \\ &= \min \left\{ \Big\langle_{\mathscr{U}_{\mathscr{Q}}}\left(', \coprod\right), \Big\langle_{\mathscr{U}_{\mathscr{Q}}}(\Updownarrow, \coprod) \right\} = \Big\langle_{\mathscr{U}_{\mathscr{Q}}}(\Updownarrow, \coprod). \end{aligned} \tag{30}$$

It follows that

$$\begin{aligned} \Big\langle_{\mathscr{U}_{\mathscr{Q}}}(\sqcap, \coprod) &\geq \min \left\{ \Big\langle_{\mathscr{U}_{\mathscr{Q}}}(\sqcap * \sqsubseteq, \coprod), \Big\langle_{\mathscr{U}_{\mathscr{Q}}}(\sqsubseteq, \coprod) \right\} \\ &\geq \min \left\{ \Big\langle_{\mathscr{U}_{\mathscr{Q}}}(\sqsubseteq, \coprod), \Big\langle_{\mathscr{U}_{\mathscr{Q}}}(\Updownarrow, \coprod) \right\}. \end{aligned} \tag{31}$$

The proof is complete. □

Theorem 14. *For any commutative in a $\mathscr{BCK}$-algebra $\mathscr{U}$. Every $\mathscr{Q}$-HFI is commutative.*

Proof. Let $\mathscr{H}_{\mathscr{U}_{\mathscr{Q}}}$ be a $\mathscr{QHF}$-ideal of a commutative $\mathscr{BCK}$-algebra $\mathscr{U}$. It is sufficient to show that $\mathscr{H}_{\mathscr{U}_{\mathscr{Q}}}$ satisfies condition $(\mathscr{Q}_{\rfloor}\mathscr{H})$ .Let $\sqcap, \sqsubseteq, \Updownarrow \in \mathscr{U}$. Then,

$$\begin{aligned} &((\sqcap * (\sqsubseteq * (\sqsubseteq * \sqcap))) * ((\sqcap * \sqsubseteq) * \Updownarrow)) * \Updownarrow \\ &\quad = ((\sqcap * (\sqsubseteq * (\sqsubseteq * \sqcap))) * \Updownarrow) * ((\sqcap * \sqsubseteq) * \Updownarrow) \\ &\quad \leq (\sqcap * (\sqsubseteq * (\sqsubseteq * \sqcap))) * (\sqcap * \sqsubseteq) \\ &\quad = (\sqcap * (\sqcap * \sqsubseteq)) * (\sqsubseteq * (\sqsubseteq * \sqcap)) = 0. \end{aligned} \tag{32}$$

That is,

$$(\sqcap * (\sqsubseteq * (\sqsubseteq * \sqcap))) * ((\sqcap * \sqsubseteq) * \Updownarrow) \leq \Updownarrow. \tag{33}$$

By Lemma 13, we have

$$\left\langle_{\mathscr{U}_{\mathcal{Q}}}(\sqcap*(\sqsubseteq*(\sqsubseteq*\sqcap)),\coprod) \geq \min\left\{\left\langle_{\mathscr{U}_{\mathcal{Q}}}((\sqcap*\sqsubseteq)*\Updownarrow,\coprod),\left\langle_{\mathscr{U}_{\mathcal{Q}}}(\Updownarrow,\coprod)\right.\right\}. \quad (34)$$

Thus, $(\mathcal{Q}_c\mathscr{H})$ holds. Therefore, $\mathscr{H}_{\mathscr{U}_{\mathcal{Q}}}$ is a $\mathcal{Q}$-HFCI. □

Definition 15. Let $\mathscr{H}_{\mathscr{U}_{\mathcal{Q}}}$ be a $\mathcal{Q}$-hesitant CI of a $\mathscr{BCK}$-algebra $\mathscr{U}$, for $\int \in_{\checkmark}([0,1])$, the set $\mathscr{H}_{\mathscr{U}_{\mathcal{Q}}}(\int) = \{\sqcap\in\mathscr{U},\coprod\in\mathcal{Q} \mid \mathscr{H}_{\mathscr{U}_{\mathcal{Q}}}(',\coprod) \geq \int\}$ of a CI is called $\mathcal{Q}$-hesitant $\int$-level CI of $\mathscr{H}_{\mathscr{U}_{\mathcal{Q}}}$.

Theorem 16. *In $\mathscr{BCK}$-algebra $\mathscr{U}$, any CI of can be realized as $\mathcal{Q}$-hesitant $\int$-level CI of some $\mathcal{Q}$-HFCI of $\mathscr{U}\times\mathcal{Q}$.*

Proof. Let $\mathscr{C}$ be a CI of $\mathscr{BCK}$-algebra $\mathscr{U}$ and let $\mathscr{H}_{\mathscr{A}_{\mathcal{Q}}}$ be a $\mathcal{Q}$-hesitant fuzzy set of $\mathscr{U}\times\mathcal{Q}$ defined by

$$\left\langle_{\mathscr{A}_{\mathcal{Q}}}(\sqcap,\coprod) = \begin{cases} \int, \text{if} & \sqcap\in\mathscr{C}, \\ 0, \text{if} & \text{otherwise}, \end{cases}\right. \quad (35)$$

where $\int \in_{\checkmark}([0,1])$. Let $\sqcap,\sqsubseteq,\Updownarrow\in\mathscr{U}$.

If $(\sqcap*\sqsubseteq)*\Updownarrow\in\mathscr{C}$ and $\Updownarrow\in\mathscr{C}$, then $(\sqcap*(\sqsubseteq*(\sqsubseteq*\sqcap)))\in\mathscr{C}$ Thus,

$$\left\langle_{\mathscr{A}_{\mathcal{Q}}} = \left\langle_{\mathscr{A}_{\mathcal{Q}}}(\Updownarrow,\coprod) = \left\langle_{\mathscr{A}_{\mathcal{Q}}}(\sqcap*(\sqsubseteq*(\sqsubseteq*\sqcap)),\coprod) = \int,\right.\right.\right. \quad (36)$$

and so

$$\left\langle_{\mathscr{A}_{\mathcal{Q}}}(\sqcap*(\sqsubseteq*(\sqsubseteq*\sqcap)),\coprod) \geq \min\left\{\left\langle_{\mathscr{A}_{\mathcal{Q}}}((\sqcap*\sqsubseteq)*\Updownarrow,\coprod),\left\langle_{\mathscr{A}_{\mathcal{Q}}}(\Updownarrow,\coprod)\right.\right\},\right. \quad (37)$$

(i) If $(\sqcap*\sqsubseteq)*\Updownarrow\notin\mathscr{C}$ and $\Updownarrow\notin\mathscr{C}$, then $\langle_{\mathscr{A}_{\mathcal{Q}}}((\sqcap*\sqsubseteq)*\Updownarrow,\coprod) = \langle_{\mathscr{A}_{\mathcal{Q}}}(\Updownarrow,\coprod) = 0$

Hence,

$$\left\langle_{\mathscr{A}_{\mathcal{Q}}}(\sqcap*(\sqsubseteq*(\sqsubseteq*\sqcap)),\coprod) \geq \min\left\{\left\langle_{\mathscr{A}_{\mathcal{Q}}}((\sqcap*\sqsubseteq)*\Updownarrow,\coprod),\left\langle_{\mathscr{A}_{\mathcal{Q}}}(\Updownarrow,\coprod)\right.\right\}.\right. \quad (38)$$

(ii) If exactly one of $((\sqcap*\sqsubseteq)*\coprod)$ and $\Updownarrow$ belongs to $\mathscr{C}$, then exactly one of $\langle_{\mathscr{A}_{\mathcal{Q}}}((\sqcap*\sqsubseteq)*\Updownarrow,\coprod)$ and $\langle_{\mathscr{A}_{\mathcal{Q}}}(\Updownarrow,\coprod)$ is equal to zero. So,

$$\left\langle_{\mathscr{A}_{\mathcal{Q}}}(\sqcap*(\sqsubseteq*(\sqsubseteq*\sqcap)),\coprod) \geq \min\left\{\left\langle_{\mathscr{A}_{\mathcal{Q}}}((\sqcap*\sqsubseteq)*\Updownarrow,\coprod),\left\langle_{\mathscr{A}_{\mathcal{Q}}}(\Updownarrow,\coprod)\right.\right\}\right. \quad (39)$$

The results above show

$$\left\langle_{\mathscr{A}_{\mathcal{Q}}}(\sqcap*(\sqsubseteq*(\sqsubseteq*\sqcap)),\coprod) \geq \min\left\{\left\langle_{\mathscr{A}_{\mathcal{Q}}}(\sqcap*\sqsubseteq)*\Updownarrow,\coprod),\left\langle_{\mathscr{A}_{\mathcal{Q}}}(\Updownarrow,\coprod)\right.\right\} \text{ for all } \sqcap,\sqsubseteq,\Updownarrow\in\mathscr{U},\coprod\in\mathcal{Q}.\right. \quad (40)$$

It is clear that $\langle_{\mathscr{A}_{\mathcal{Q}}}(',\coprod) \geq \langle_{\mathscr{A}_{\mathcal{Q}}}(\sqcap,\coprod)$ for all $\sqcap\in\mathscr{U}$. Therefore, $\mathscr{H}_{\mathscr{A}_{\mathcal{Q}}}$ is a $\mathcal{QHFC}$-ideal of $\mathscr{U}\times\mathcal{Q}$. Obviously, $\mathscr{H}_{\mathscr{A}_{\mathcal{Q}}}(\int) = \mathscr{C}$. □

Theorem 17. *If $\mathscr{H}_{\mathscr{A}_{\mathcal{Q}}}$ a $\mathcal{QHFC}$-ideal of a $\mathscr{BCK}$-algebra $\mathscr{U}$. Then, two-level CI $\mathscr{H}_{\mathscr{A}_{\mathcal{Q}}}(\int_\infty)$ and $\mathscr{H}_{\mathscr{A}_{\mathcal{Q}}}(\int_\epsilon)$ where $\int_\infty < \int_\epsilon$ of $\mathscr{H}_{\mathscr{A}_{\mathcal{Q}}}$ are equal if and only if there is no $u\in\mathscr{U}$ such that $\int_\infty \leq \langle_{\mathscr{A}_{\mathcal{Q}}}(\sqcap,\coprod) < \int_\epsilon$.*

Proof. Let $\mathscr{H}_{\mathscr{A}_{\mathcal{Q}}}(\int_\infty) = \mathscr{H}_{\mathscr{A}_{\mathcal{Q}}}(\int_\epsilon)$. If there exists $\sqcap\in\mathscr{U}$ such that $\int_\infty \leq \langle_{\mathscr{A}_{\mathcal{Q}}}(\sqcap,\coprod) < \int_\epsilon$, then $\mathscr{H}_{\mathscr{A}_{\mathcal{Q}}}(\int_\epsilon) \subseteq \mathscr{H}_{\mathscr{A}_{\mathcal{Q}}}(\int_\infty)$. This is impossible. Conversely, assume that there is no $\sqcap\notin\mathscr{U}$ such that $\int_\infty \leq \langle_{\mathscr{A}_{\mathcal{Q}}}(\sqcap,\coprod) < \int_\epsilon$.

$\int_\infty < \int_\epsilon$ implies $\mathscr{H}_{\mathscr{A}_{\mathcal{Q}}}(\int_\epsilon) \subseteq \mathscr{H}_{\mathscr{A}_{\mathcal{Q}}}(\int_\infty)$ If $(\sqcap,\coprod)\in\mathscr{H}_{\mathscr{A}_{\mathcal{Q}}}(\int_\infty)$, then $\langle_{\mathscr{A}_{\mathcal{Q}}}(\sqcap,\coprod) \geq \int_\infty$ and so $\langle_{\mathscr{A}_{\mathcal{Q}}}(\sqcap,\coprod) \geq \int_\epsilon$, because $\langle_{\mathscr{A}_{\mathcal{Q}}}(\sqcap,\coprod) \nless \int_\epsilon$.

Hence, $(\sqcap,\coprod)\in\mathscr{H}_{\mathscr{A}_{\mathcal{Q}}}(\int_\epsilon)$ which says that $\mathscr{H}_{\mathscr{A}_{\mathcal{Q}}}(\int_\infty) \subseteq \mathscr{H}_{\mathscr{A}_{\mathcal{Q}}}(\int_\epsilon)$.

Thus, $\mathscr{H}_{\mathscr{A}_{\mathcal{Q}}}(\int_\infty) = \mathscr{H}_{\mathscr{A}_{\mathcal{Q}}}(\int_\epsilon)$.

This completes the proof. □

Let $\langle\mathscr{A}_{\mathcal{Q}}$ be a $\mathcal{Q}$-hesitant fuzzy set in $\mathscr{U}$ and let Im $(\langle_{\mathscr{A}_{\mathcal{Q}}})$ denote the image of $\langle_{\mathscr{A}_{\mathcal{Q}}}$.

Theorem 18. *Let $\mathscr{U}$ be a $\mathscr{BCK}$-algebra and $\mathscr{H}_{\mathscr{A}_{\mathcal{Q}}}$ a $\mathcal{Q}$-HFCI of $\mathscr{U}\times\mathcal{Q}$. If* Im $(\langle\mathscr{A}_{\mathcal{Q}}) = \{\int_\infty,\int_\epsilon,\cdots,\int_\backslash\}$, *where $\int_\infty \lhd \int_\epsilon \lhd \cdots \int_\backslash$, then the family of CIs $\mathscr{H}_{\mathscr{A}_{\mathcal{Q}}}(\int_\succ)(\succ=\infty,\ \epsilon\ \cdots,\ \backslash)$ constitutes all the level CIs of $\langle_{\mathscr{A}_{\mathcal{Q}}}$.*

Proof. Let $\int\in_{\checkmark}([0,1])$ and $\int\notin$ Im $(\hbar_{\mathscr{A}_{\mathcal{Q}}})$. If $\int < \int_\infty$, then $\langle_{\mathscr{A}_{\mathcal{Q}}}(\int_\infty) \subseteq \langle_{\mathscr{A}_{\mathcal{Q}}}(\int)$. Since $\langle_{\mathscr{A}_{\mathcal{Q}}}(\int_\infty) = \mathscr{U}$, we have $\langle\mathscr{A}_{\mathcal{Q}}(\int) = \mathscr{U}$ and $\langle_{\mathscr{A}_{\mathcal{Q}}}(\int) = \langle_{\mathscr{A}_{\mathcal{Q}}}(\int_\infty)$.

If $\int_\succ < \int < \int_{\succ+\infty}(1\leq\succ\leq\backslash-\infty)$, then there is no $\sqcap\in\mathscr{U}$ such that $\int \leq \langle_{\mathscr{A}_{\mathcal{Q}}}(\sqcap,\coprod) < \int_{\succ+\infty}$. From above theorem 10, it follows that $\langle_{\mathscr{A}_{\mathcal{Q}}}(\int) = \langle_{\mathscr{A}_{\mathcal{Q}}}(\int_{\succ+\infty})$. This shows that for any $\int\in_{\checkmark}([0,1])$ with $\int \leq \langle_{\mathscr{A}_{\mathcal{Q}}}(',\coprod)$, the level CI $\langle_{\mathscr{A}_{\mathcal{Q}}}(\int)$ is in $\{\langle_{\mathscr{A}_{\mathcal{Q}}}(\int_\succ): 1\leq\succ\leq\backslash\}$. □

Lemma 19. *Given a $\mathscr{BCK}$-algebra $\mathscr{U}$ and $\mathscr{H}_{\mathscr{A}_{\mathcal{Q}}}$ a $\mathcal{QHFC}$-ideal over $\mathscr{U}\times\mathcal{Q}$. If $\int_\infty$ and $\int_\epsilon$ belong to* Im $(\langle_{\mathscr{A}_{\mathcal{Q}}})$ *such that $\langle_{\mathscr{A}_{\mathcal{Q}}}(\int_\infty) = \langle_{\mathscr{A}_{\mathcal{Q}}}(\int_\epsilon)$, then $\int_\infty = \int_\epsilon$.*

Proof. Assume that $\int_\infty \neq \int_\epsilon$, say $\int_\infty < \int_\epsilon$. Then, there is $\sqcap\in\mathscr{U}$, $\coprod\in\mathcal{Q}$ such that $\langle_{\mathscr{A}_{\mathcal{Q}}}(\sqcap,\coprod) = \int_\infty < \int_\epsilon$, and so $(\sqcap,\coprod)\in$

$\langle_{\mathscr{A}_{\mathcal{Q}}}(\int_\infty)$ and $(\sqcap, \coprod) \notin \langle_{\mathscr{A}_{\mathcal{Q}}}(\int_\epsilon)$. Thus, $\langle_{\mathscr{A}_{\mathcal{Q}}}(\int_\infty) \neq \langle_{\mathscr{A}_{\mathcal{Q}}}(\int_\epsilon)$, which is a contradiction to our fact. This completes the proof. □

5. $\mathcal{Q}$-Hesitant Fuzzy Characteristic CIs

A mapping $\{: \mathscr{U} \longrightarrow \mathscr{V}$ of a $\mathscr{BCK}$-algebra is called a homomorphism if satisfying the identity $\{(\sqcap * \sqsubseteq) = \{(\sqcap) * \{(\sqsubseteq)$ for all $\sqcap, \sqsubseteq \in \mathscr{U}$. Throughout, $\mathrm{Aut}(\mathscr{U})$ will denote the $\mathscr{BCK}$-algebra of automorphisms of $\mathscr{U}$.

Definition 20. Let $\{: \mathscr{U} \longrightarrow \mathscr{V}$ be a homomorphism of $\mathscr{BCK}$-algebras. For any $\mathcal{Q}\mathscr{HFC}$-ideal $\langle_{\mathscr{A}_{\mathcal{Q}}}$ of $\mathscr{V}$, we define a new $\mathcal{Q}\mathscr{HFC}$-ideal $\langle^{\{}_{\mathscr{A}_{\mathcal{Q}}}$ in $\mathscr{U}$ by

$$\left\langle^{\{}_{\mathscr{A}_{\mathcal{Q}}}(\sqcap, \coprod) = \left\langle_{\mathscr{A}_{\mathcal{Q}}}(\{(\sqcap), \coprod) \text{ for all } \sqcap \in \mathscr{U}, \coprod \in \mathcal{Q}. \right.\right. \quad (41)$$

Theorem 21. *Let $\{: \mathscr{U} \longrightarrow \mathscr{V}$ be a homomorphism of $\mathscr{BCK}$-algebra $\mathscr{U}$. If $\langle_{\mathscr{A}_{\mathcal{Q}}}$ is a $\mathcal{Q}\mathscr{HFC}$-ideal of $\mathscr{V}$, then $\langle^{\{}_{\mathscr{A}_{\mathcal{Q}}}$ is a $\mathcal{Q}\mathscr{HFC}$-ideal of $\mathscr{U}$.*

Proof. Let $\sqcap \in \mathscr{U}, \coprod \in \mathcal{Q}$

$$\begin{aligned} \left\langle^{\{}_{\mathscr{A}_{\mathcal{Q}}}(\sqcap, \coprod) &= \left\langle_{\mathscr{A}_{\mathcal{Q}}}(\{(\sqcap), \coprod) \leq \left\langle_{\mathscr{A}_{\mathcal{Q}}}\left(', \coprod\right)\right.\right. \\ &= \left\langle_{\mathscr{A}_{\mathcal{Q}}}\left(\left\{\left(\,'\,\right), \coprod\right) = \left\langle^{\{}_{\mathscr{A}_{\mathcal{Q}}}\left(', \coprod\right).\right.\right.\right. \end{aligned} \quad (42)$$

Let $\sqcap, \sqsubseteq, \Updownarrow \in \mathscr{U}, \coprod \in \mathcal{Q}$

$$\begin{aligned} &\min\left\{\left\langle^{\{}_{\mathscr{A}_{\mathcal{Q}}}((\sqcap * \sqsubseteq) * \Updownarrow, \coprod), \left\langle^{\{}_{\mathscr{A}_{\mathcal{Q}}}(\Updownarrow, \coprod)\right.\right.\right\} \\ &= \min\left\{\left\langle_{\mathscr{A}_{\mathcal{Q}}}(\{((\sqcap * \sqsubseteq) * \Updownarrow), \coprod), \left\langle_{\mathscr{A}_{\mathcal{Q}}}(\{(\Updownarrow), \coprod)\right.\right.\right\} \\ &= \min\left\{\langle_{\mathscr{A}_{\mathcal{Q}}}(\{(\sqcap * \sqsubseteq) * \{(\Updownarrow), \coprod), \left\langle_{\mathscr{A}_{\mathcal{Q}}}(\{(\Updownarrow), \coprod)\right.\right\} \\ &= \min\left\{\left\langle_{\mathscr{A}_{\mathcal{Q}}}((\{(\sqcap) * \{(\sqsubseteq) * \{(\Updownarrow), \coprod), \left\langle_{\mathscr{A}_{\mathcal{Q}}}(\{(\Updownarrow), \coprod)\right.\right.\right\} \\ &\leq \left\langle_{\mathscr{A}_{\mathcal{Q}}}(\{(\sqcap * (\sqsubseteq * (\sqsubseteq * \sqcap))), \coprod) = \left\langle^{\{}_{\mathscr{A}_{\mathcal{Q}}}(\sqcap * (\sqsubseteq * (\sqsubseteq * \sqcap)), \coprod).\right.\right. \end{aligned} \quad (43)$$

□

Definition 22. A CI $\mathscr{C}$ of a $\mathscr{BCK}$-algebra $\mathscr{U}$ is called a characteristic CI (CCI) of $\mathscr{U}$ if $\{(\mathscr{C}) = \mathscr{C}$ for all $\{ \in \mathrm{Aut}(\mathscr{U})$.

Definition 23. A $\mathcal{Q}$-HFCI of a $\mathscr{BCK}$-algebra $\mathscr{U}$ is called a $\mathcal{Q}$-hesitant fuzzy CCI of $\mathscr{U} \times \mathcal{Q}$ if

$$\left\langle_{\mathscr{A}_{\mathcal{Q}}}(\{(\sqcap), \coprod) = \left\langle_{\mathscr{A}_{\mathcal{Q}}}(\sqcap, \coprod) \text{ for all } \sqcap \in \mathscr{U}, \coprod \in \mathcal{Q} \text{ and } \forall \{ \in \mathrm{Aut}(\mathscr{U}).\right.\right. \quad (44)$$

Theorem 24. *Let $\mathscr{H}_{\mathscr{A}_{\mathcal{Q}}}$ be a $\mathcal{Q}$-hesitant fuzzy characteristic CI of $\mathscr{U} \times \mathcal{Q}$. Then, each $\int$-level CI of $\langle_{\mathscr{A}_{\mathcal{Q}}}$ is a characteristic commutative ideal of $\mathscr{U} \times \mathcal{Q}$.*

Proof. Assume $\int \in \mathrm{Im}\,(\langle_{\mathscr{A}_{\mathcal{Q}}})$, $\{ \in \mathrm{Aut}(\mathscr{U})$ and $(\sqcap, \coprod) \in \langle_{\mathscr{A}_{\mathcal{Q}}}(\int)$. Since $\mathscr{H}_{\mathscr{A}_{\mathcal{Q}}}$ is a $\mathcal{Q}$-hesitant fuzzy characteristic commutative ideal of $\mathscr{U} \times \mathcal{Q}$, we have $\langle_{\mathscr{A}_{\mathcal{Q}}}(\{(\sqcap), \coprod) = \langle_{\mathscr{A}_{\mathcal{Q}}}(\sqcap, \coprod) \geq \int$.

It follows that $(\{(\sqcap), \coprod) \in \langle_{\mathscr{A}_{\mathcal{Q}}}(\int)$ and hence $\{((\langle_{\mathscr{A}_{\mathcal{Q}}}(\int)) \subseteq \langle_{\mathscr{A}_{\mathcal{Q}}}(\int)$.

To show the reverse inclusion, let $(\sqcap, \coprod) \in \langle_{\mathscr{A}_{\mathcal{Q}}}(\int)$ and let $\sqsubseteq \in \mathscr{U}$ be such that $\{(\sqsubseteq) = \sqcap$. Then, $\langle_{\mathscr{A}_{\mathcal{Q}}}(\sqsubseteq, \coprod) = \langle_{\mathscr{A}_{\mathcal{Q}}}(\{(\sqsubseteq), \coprod) = \langle_{\mathscr{A}_{\mathcal{Q}}}(\sqcap, \coprod) \geq \int$ whence $(\sqsubseteq, \coprod) \in \langle_{\mathscr{A}_{\mathcal{Q}}}(\int)$. It follows that $\sqcap = \{(\sqsubseteq) \in \{((\langle_{\mathscr{A}_{\mathcal{Q}}}(\int))$ so that $\langle_{\mathscr{A}_{\mathcal{Q}}}(\int) \subseteq \{((\langle_{\mathscr{A}_{\mathcal{Q}}}(\int))$. Thus, $\langle_{\mathscr{A}_{\mathcal{Q}}}(\int)$, $\int \in \mathrm{Im}\,((\langle_{\mathscr{A}_{\mathcal{Q}}})$ is a CCI of $\mathscr{U} \times \mathcal{Q}$.

The proof of the following lemma is obvious, and we omit the proof. □

Lemma 25. *Let $\mathscr{H}_{\mathscr{A}_{\mathcal{Q}}}$ be a $\mathcal{Q}$-HFCI of $\mathscr{U} \times \mathcal{Q}$ and let $\sqcap \in \mathscr{U}$. Then, $\langle_{\mathscr{A}_{\mathcal{Q}}}(\sqcap, \coprod) = \int$ if and only if $(\sqcap, \coprod) \in \langle_{\mathscr{A}_{\mathcal{Q}}}(\int_\infty)$ and $(\sqcap, \coprod) \notin \langle_{\mathscr{A}_{\mathcal{Q}}}(\int_\epsilon)$, for all $\int_\epsilon \geq \int_\infty$.*

Now, we consider the inverse of Theorem 24

Theorem 26. *Let $\mathscr{H}_{\mathscr{A}_{\mathcal{Q}}}$ be a $\mathcal{Q}$-HFCI of $\mathscr{U} \times \mathcal{Q}$. If each level CI of $\langle_{\mathscr{A}_{\mathcal{Q}}}$ is a CCI of $\mathscr{U}$, then $\mathscr{H}_{\mathscr{A}_{\mathcal{Q}}}$ is a $\mathcal{Q}$-hesitant fuzzy characteristic commutative ideal of $\mathscr{U} \times \mathcal{Q}$.*

Proof. Let $\sqcap \in \mathscr{U}, \coprod \in \mathcal{Q}, \{ \in \mathrm{Aut}(\mathscr{U})$ and $\langle_{\mathscr{A}_{\mathcal{Q}}}(\sqcap, \coprod) = \int_\infty$. Then, $(\sqcap, \coprod) \in \langle_{\mathscr{A}_{\mathcal{Q}}}(\int_\infty)$ and $(\sqcap, \coprod) \notin \langle_{\mathscr{A}_{\mathcal{Q}}}(\int_\epsilon)$ for all $\int_\epsilon \geq \int_\infty$, by Lemma 25 Since $\{((\langle_{\mathscr{A}_{\mathcal{Q}}})(\int_\infty) = \langle_{\mathscr{A}_{\mathcal{Q}}}(\int_\infty)$ by hypothesis, we have $(\{(\sqcap), \coprod) \in \langle_{\mathscr{A}_{\mathcal{Q}}}(\int_\infty)$ and hence $\langle_{\mathscr{A}_{\mathcal{Q}}}(\{(\sqcap), \coprod) \geq \int_\infty$. Let $\int_\epsilon = \langle_{\mathscr{A}_{\mathcal{Q}}}(\{(\sqcap), \coprod)$. If possible, let $\int_\epsilon \geq \int_\infty$.

Then, $(\{(\sqcap), \coprod) \in \langle_{\mathscr{A}_{\mathcal{Q}}}(\int_\epsilon) = \{((\langle_{\mathscr{A}_{\mathcal{Q}}}(\int_\epsilon))$. Since $\{$ is one to one, it follows that $(\sqcap, \coprod) \in \langle_{\mathscr{A}_{\mathcal{Q}}}(\int_\epsilon)$, which is a contradiction. Hence, $\langle_{\mathscr{A}_{\mathcal{Q}}}(\{(\sqcap), \coprod)) = \int_\infty = \langle_{\mathscr{A}_{\mathcal{Q}}}(\sqcap, \coprod)$. It follows that $\mathscr{H}_{\mathscr{A}_{\mathcal{Q}}}$ is a $\mathcal{Q}$-hesitant fuzzy CCI of $\mathscr{U} \times \mathcal{Q}$. This completes the proof. □

6. Conclusions

A new concept of HFI is considered by applying a two-dimensional membership function, namely, $\mathcal{Q}$-HFI. Several properties and theorems of $\mathcal{Q}$-HFI are proved. In this regard, we propose the concept of $\mathcal{Q}$-HFCI in $\mathscr{BCK}$-algebra and prove some related properties. We have considered the features of $\mathcal{Q}$-HFCI. We study some feature properties related to $\mathcal{Q}$-HFCI. Our future research is to find ways to apply $\mathcal{Q}$-HFI to a wide range of logical algebraic systems, such as pseudo-$\mathscr{BCI}$-algebras [14, 15]. For other notions, the readers are suggested to see [16–28].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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