Research Article
Certain Concepts of $Q$-Hesitant Fuzzy Ideals

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The hesitant fuzzy set model has attracted the interest of scholars in various fields. The striking framework of hesitant fuzzy sets is keen to provide a larger domain of preference for fuzzy information modeling of deployment membership. Starting from the hybrid properties of hesitant fuzzy ideals (HFI), this paper constructs a new generalized hybrid structure $Q$-HFI. The concept of $Q$-hesitant fuzzy exchange ideal in $BCK$-algebra is considered. Lastly, $Q$-hesitant fuzzy exchange ideal features are described.

1. Introduction

When dealing with information on all aspects of uncertainty, nonclassical logic always makes use of classical logic. Nonclassical logic is a useful tool in computer science because it deals with fuzzy information and uncertainty. In the literature, the study of BCK/BCI-algebras was first proposed by Imai and Iséki [1] in 1966 and such algebras can be regarded as a generalization of propositional logic. The study BCK/BCI-algebras have been developed by many people and have been extended to the fuzzy setting. After the introduction of fuzzy sets introduced by Zadeh [2], there have been many generalizations of this fundamental concept. In 2010, Torra [3] considered hesitant fuzzy sets. The hesitant fuzzy set model is useful tool to deal with uncertainty, which can be accurately and perfectly described in terms of the opinions of decision-makers.

Algebraic structures provide sufficient motivation for researchers to examine various concepts and stem from the broader field of abstract algebra blur set frame. In 2011, Xia and Xu [4] described hesitant fuzzy information aggregation techniques, and this concept was applied to $BCK/BCI$-algebras, $EQ$-algebras, residuated lattices, $MTL$-algebras, and $K$-algebras [5–9]. Jun and Ahn [6] investigated the concept of hesitant fuzzy subalgebras and HFIs of $BCK/BCI$-algebras. In 2018, Alshehri et al. [10] put forward the concept of new types of HFIs in $BCK$-algebras. As a continuation of this study, we describe certain concepts, including $Q$-HFIs and $Q$-hesitant fuzzy commutative ideals in $BCK$-algebras.

2. Basic Notions

A set $\mathcal{U}$ with a constant element 0 and a binary operation $*$ is said to be a $BCK$-algebra [1] if it satisfies the axioms:

For all $\pi, \xi, \eta \in \mathcal{U}$,

\[
(BCK-1) (\xi \ast \pi) \ast (\eta \ast \pi) = 0,
\]

\[
(BCK-2) (\xi \ast (\eta \ast \pi)) \ast \xi = 0,
\]

\[
(BCK-3) \pi \ast \pi = 0,
\]

\[
(BCK-4) 0 \ast \pi = 0,
\]

\[
(BCK-5) \pi \ast \xi = 0, \xi \ast \pi = 0 \text{ imply that } \pi = \xi.
\]

In a $BCK$-algebra $\mathcal{U}$, we can define the relation $\leq$ by $\pi \leq \xi$ if and only if $\pi \ast \xi = 0$.

Then, $(\mathcal{U}; \leq)$ is a partially ordered set with the least element 0. In any $BCK$-algebra $\mathcal{U}$, the following properties hold:

\[
(\pi \ast \xi) \ast \eta = (\pi \ast m \ast \eta) \ast \xi,
\]

\[
\pi \ast \xi \leq \pi,
\]

\[
\pi \ast 0 = \pi,
\]

\[
(\pi \ast \eta) \ast (\xi \ast \psi) \leq \pi \ast \xi,
\]

\[
\pi \ast (\pi \ast (\pi \ast \xi)) = \pi \ast \xi.
\]
\[ \cap \subseteq \text{ implies } \cap \cap \subseteq \subseteq \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap \cap 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Table 1

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Proof.

(1) Suppose \( \cap \subseteq \exists \) implies \( \cap \subseteq = 0 \in \mathcal{A} \) (for all \( \cap, \exists \in \mathcal{A} \)) and so

\[
\langle x_0 (\cap, \exists) \rangle = \min \left\{ \langle x_0 (\cap), \langle x_0 (\exists, \exists) \rangle \right\}
\]

by \((Q, \mathcal{H})\).

(2) Suppose \( \cap \subseteq \exists \subseteq \delta \) implies \( (\cap \subseteq) \ast \delta = 0 \in \mathcal{A} \) (for all \( \cap, \exists, \delta \in \mathcal{U} \)) so

\[
\langle x_0 (\exists, \exists) \rangle = \min \left\{ \langle x_0 (\exists), \langle x_0 (\exists, \exists) \rangle \right\}
\]

by \((Q, \mathcal{H})\).

It follows that

\[
\min \left\{ \langle x_0 (\cap, \exists), \langle x_0 (\exists, \exists) \rangle \right\} \leq \min \left\{ \langle x_0 (\cap \subseteq, \exists), \langle x_0 (\exists, \exists) \rangle \right\}
\]

\[
\leq \langle x_0 (\cap, \exists) \rangle.
\]

Proposition 5. Every \( \mathcal{A} \)-ideal \( \mathcal{U} \times \mathcal{G} \) satisfies the following condition:

(1) \( \langle y_0 (\cap, \exists) \rangle \leq \langle y_0 (\cap, \exists) \rangle \) with \( \cap \subseteq \) for all \( \cap, \exists \in \mathcal{U} \), \( \exists \in \mathcal{G} \)

(2) \( \min \left\{ \langle x_0 (\cap, \exists), \langle y_0 (\exists, \exists) \rangle \right\} \leq \langle y_0 (\cap, \exists) \rangle \) with \( \cap \ast \subseteq \delta \) for all \( \cap, \exists, \delta \in \mathcal{U} \), \( \exists \in \mathcal{G} \)

Theorem 6. If \( \mathcal{H} \subseteq \mathcal{A} \)-HFI of \( \mathcal{U} \times \mathcal{G} \), then for any \( \cap, -\infty, -\infty, \cdots, -\infty \in \mathcal{U} \), and

\[
(\cdots((\cap \ast -\infty) * -\infty) * -\infty) * -\infty = 0 \implies \langle x_0 (\cap, \exists) \rangle
\]

\[
\geq \min \left\{ \langle x_0 (\cap \subseteq, \exists), \langle x_0 (-\infty, \exists) \rangle, \cdots, \langle x_0 (-\infty, \exists) \rangle \right\}.
\]

Theorem 7. Let \( \mathcal{H} \subseteq \mathcal{A} \)-HFI be a \( \mathcal{A} \)-HFI of \( \mathcal{U} \times \mathcal{G} \). Then, the following are equivalent:

(i) \( \langle x_0 (\cap \subseteq, \exists), \langle x_0 (\exists, \exists) \rangle \rangle \leq \langle x_0 (\cap \subseteq, \exists), \langle x_0 (\exists, \exists) \rangle \rangle \) for all \( \cap, \exists \in \mathcal{U} \), \( \exists \in \mathcal{G} \)

(ii) \( \langle x_0 (\cap \subseteq, \exists), \langle x_0 (\exists, \exists) \rangle \rangle \leq \langle x_0 (\cap \subseteq, \exists), \langle x_0 (\exists, \exists) \rangle \rangle \) for all \( \cap, \exists, \delta \in \mathcal{U} \), \( \delta \in \mathcal{G} \)

Proof. (i) \( \implies \) (ii) Suppose condition (i) is valid. Since

(\( (\cap \subseteq, \exists) \ast \delta = ((\cap \subseteq) \ast (\exists \ast \delta)) \ast \delta \leq (\cap \subseteq) \ast \delta \).

Applying, by Proposition 2 and (-i), we have

\[
\langle x_0 ((\cap \subseteq) \ast \delta, \exists) \rangle \leq \langle x_0 ((\cap \subseteq) \ast \delta, \exists) \rangle \leq \langle x_0 ((\cap \subseteq) \ast \delta, \exists) \rangle \leq \langle x_0 ((\cap \subseteq) \ast (\exists \ast \delta), \exists) \rangle.
\]

Hence, condition (ii) holds

(ii) \( \implies \) (i) Suppose condition (ii) is valid. If we put \( \delta = \exists \) in (ii) then

\[
\langle x_0 ((\cap \subseteq) \ast \exists, \exists) \rangle \leq \langle x_0 ((\cap \subseteq) \ast \exists, \exists) \rangle \leq \langle x_0 ((\cap \subseteq) \ast \exists, \exists) \rangle \leq \langle x_0 ((\cap \subseteq) \ast (\exists \ast \exists), \exists) \rangle.
\]

The proof is complete.

Theorem 8. Let \( \mathcal{H} \subseteq \mathcal{A} \)-HFI be a \( \mathcal{A} \)-HFI of \( \mathcal{U} \times \mathcal{G} \), then the set

\[
\mathcal{H} = \{ \cap \in \mathcal{U} \mid \langle y_0 (\cap, \exists) \rangle \leq \langle y_0 (\cap, \exists) \rangle \}
\]

is an ideal of \( \mathcal{U} \times \mathcal{G} \) for all \( \exists \in \mathcal{U} \).
Proof. Let \( \cap, \supseteq \in \mathcal{H} \), \( \bigcup \in \mathcal{H} \) be such that \((\cap \star \supseteq, \bigcup) \in \mathcal{H}\) and \((\supseteq, \bigcup) \in \mathcal{H}\). Then,

\[
\langle w_0(\cap \star \supseteq, \bigcup) \rangle = \langle w_0(\cap \star \supseteq, \bigcup) \rangle 
\leq \langle w_0(\cap, \bigcup) \rangle 
\leq \langle w_0(\cap \star \supseteq, \bigcup) \rangle.
\] (16)

It follows from \((Q, \mathcal{H}), (Q, \mathcal{H})\) that

\[
\langle w_0(\bigcup) \rangle \leq \min \left\{ \langle w_0(\cap \star \supseteq, \bigcup) \rangle, \langle w_0(\cap, \bigcup) \rangle \right\} 
\leq \langle w_0(\bigcup) \rangle.
\] (17)

(2) \( \bigcup, \supseteq \in \mathcal{H} \), \( \cap \in \mathcal{H} \) such that \((\cap \star \supseteq, \bigcup) \in \mathcal{H}\) and \((\supseteq, \bigcup) \in \mathcal{H}\). Then, \( \langle w_0(\cap \star \supseteq, \bigcup) \rangle \geq \min \left\{ \langle w_0(\cap \star \supseteq, \bigcup) \rangle, \langle w_0(\cap, \bigcup) \rangle \right\} \geq \langle w_0(\bigcup) \rangle\).

\[
\langle w_0(\cap \star \supseteq, \bigcup) \rangle \geq \min \left\{ \langle w_0(\cap \star \supseteq, \bigcup) \rangle, \langle w_0(\cap, \bigcup) \rangle \right\} \geq \langle w_0(\bigcup) \rangle.
\] (18)

It follows that

\[
\langle w_0(\bigcup) \rangle \geq \min \left\{ \langle w_0(\cap \star \supseteq, \bigcup) \rangle, \langle w_0(\cap, \bigcup) \rangle \right\} \geq \langle w_0(\bigcup) \rangle.
\] (19)

Hence, \( \bigcup, \supseteq \in \mathcal{H} \), \( \cap \in \mathcal{H} \). Therefore \( \langle w_0(\cap \star \supseteq, \bigcup) \rangle \) is an ideal of \( \mathcal{H} \). Suppose that \( \langle w_0(\cap \star \supseteq, \bigcup) \rangle \) is an ideal of \( \mathcal{H} \). For any \( \cap \in \mathcal{H} \), let \( \langle w_0(\cap, \bigcup) \rangle = \langle w_0(\cap, \bigcup) \rangle \).

Then \( \langle w_0(\cap, \bigcup) \rangle \in \mathcal{H} \). Since \( \langle w_0(\cap, \bigcup) \rangle \) is an ideal of \( \mathcal{H} \), we have

\[
\langle w_0(\cap \star \supseteq, \bigcup) \rangle \in \mathcal{H} \) and so \( \langle w_0(\cap \star \supseteq, \bigcup) \rangle \) is an ideal of \( \mathcal{H} \). The following example shows that the converse of theorem 6 is not true.

**Example 1.** Let \( \mathcal{H} = \{0, a, b, c\} \) be a set with the Cayley (Table 2).

| \( \cdot \) | \( 0 \) | \( a \) | \( b \) | \( c \) |
|-----------------|---------|---------|---------|
| \( 0 \)         | 0       | a       | b       | c       |
| \( a \)         | a       | b \( = \) c | b \( = \) c | c \( = \) c |
| \( b \)         | b       | b \( = \) c | b \( = \) c | c \( = \) c |
| \( c \)         | c       | c \( = \) a | c \( = \) a | c \( = \) a |

where \( t_0 = [0, 1] \times t_0 = [0.2, 0.6] \times t_0 = [0.2, 0.3] \). By direct calculations, one can see that \( \mathcal{H} \) is an ideal of \( \mathcal{H} \) ideal of \( \mathcal{H} \).
Let $e_0, e_1, e_2 \in ([0, 1])$ such that $e_0 > e_1 > e_2$. We define a mapping

$$\langle w_e, (\pi \in \mathcal{Q}), [\mathcal{Q}] \rangle \mapsto \begin{cases} e_0, & (\pi, [\mathcal{Q}]) = ([\mathcal{Q}], [\mathcal{Q}]) \\ e_1, & (\pi, [\mathcal{Q}]) = ([\mathcal{Q}], [\mathcal{Q}]) \\ e_2, & (\pi, [\mathcal{Q}]) = ([\mathcal{Q}], [\mathcal{Q}]) \end{cases}$$

(24)

where $e_0 = [0, 1] > e_1 = [0.4, 0.8] \geq e_2 = [0.5, 0.6]$. It is routine to verify that $\mathcal{H}_{\mathcal{Q}}$ is $\mathcal{Q}$-HFI of $\mathcal{U} \times \mathcal{Q}$. But it is not a $\mathcal{Q}$-HFIC-ideal of $\mathcal{U} \times \mathcal{Q}$. Since

$$\langle w_e, ((b + c) * b), [\mathcal{Q}] \rangle \geq \min \left\{ \langle w_e, (b * c), [\mathcal{Q}] \rangle, \langle w_e, (\mathcal{Q}), [\mathcal{Q}] \rangle \right\}$$

(25)

Theorem 12. Let $\mathcal{H}_{\mathcal{Q}}$ be a $\mathcal{Q}$-HFI of a $\mathcal{BCK}$-algebra $\mathcal{U}$. Then, $\mathcal{H}_{\mathcal{Q}}$ is a $\mathcal{Q}$-resistant fuzzy CI of $\mathcal{U} \times \mathcal{Q}$ if and only if it satisfies the following condition:

$$\langle w_e, ([\pi \in \mathcal{Q}], [\mathcal{Q}]) \rangle \geq \langle w_e, ([\pi \in \mathcal{Q}], \mathcal{U} \times \mathcal{Q}) \rangle \forall \pi \in \mathcal{E}, \mathcal{Q} \in \mathcal{Q} \in \mathcal{E}. \in \mathcal{Q}.$$  

(26)

Proof. Assume that $\mathcal{H}_{\mathcal{Q}}$ is $\mathcal{Q}$-HFIC-ideal. Taking $m = 0$ in $(\mathcal{Q}, \mathcal{H})$ and using $(\mathcal{Q}, \mathcal{H})$. Also, we use $\pi \cdot 0 = \pi$.

$$\langle w_e, ([\pi \in \mathcal{Q}], [\mathcal{Q}]) \rangle \geq \min \left\{ \langle w_e, ([\pi \in \mathcal{Q}], [\mathcal{Q}]) \rangle, (\mathcal{Q}, [\mathcal{Q}]) \right\}$$

(27)

Conversely, let $\mathcal{H}_{\mathcal{Q}}$. As $\mathcal{H}_{\mathcal{Q}}$ be a $\mathcal{Q}$-HFIC-ideal of $\mathcal{U} \times \mathcal{Q}$ satisfying condition (1). Then,

$$\langle \langle w_e, ([\pi \in \mathcal{Q}], [\mathcal{Q}]) \rangle \geq \min \left\{ \langle \langle w_e, ([\pi \in \mathcal{Q}], [\mathcal{Q}]) \rangle, \langle 0, \mathcal{Q} \rangle \right\} \rangle \in \mathcal{Q}. \in \mathcal{Q}.$$  

(28)

combining (1) and (2), then we obtain $(\mathcal{Q}, \mathcal{H})$. The proof is complete.

Lemma 13. Any $\mathcal{Q}$-HFI of a $\mathcal{BCK}$-algebra $\mathcal{U}$ satisfies

$$\langle \langle \pi \in \mathcal{Q}, \mathcal{Q} \rangle \rangle \geq \min \left\{ \langle \langle \pi \in \mathcal{Q}, \mathcal{Q} \rangle, \langle \mathcal{Q}, \mathcal{Q} \rangle \rangle \right\}.$$  

(29)

Proof. Assume that $\pi \in \mathcal{Q}$ holds. Then,

$$\langle \langle w_e, ([\pi \in \mathcal{Q}], [\mathcal{Q}]) \rangle \geq \min \left\{ \langle \langle w_e, ([\pi \in \mathcal{Q}], [\mathcal{Q}]) \rangle, \langle \mathcal{Q}, [\mathcal{Q}]) \rangle \rangle \right\}$$

(30)

It follows that

$$\langle \langle w_e, ([\pi \in \mathcal{Q}], [\mathcal{Q}]) \rangle \geq \min \left\{ \langle \langle w_e, ([\pi \in \mathcal{Q}], [\mathcal{Q}]) \rangle, \langle \mathcal{Q}, [\mathcal{Q}]) \rangle \rangle \right\} \rangle \in \mathcal{Q}.$$  

(31)

The proof is complete.

Theorem 14. For any commutative in a $\mathcal{BCK}$-algebra $\mathcal{U}$. Every $\mathcal{Q}$-HFI is commutative.

Proof. Let $\mathcal{H}_{\mathcal{Q}}$ be a $\mathcal{Q}$-HFIC-ideal of a commutative $\mathcal{BCK}$-algebra $\mathcal{U}$. It is sufficient to show that $\mathcal{H}_{\mathcal{Q}}$ satisfies condition $(Q, H)$. Let $\pi, \mathcal{Q}, \mathcal{Q} \in \mathcal{Q}$. Then,

$$\langle (\langle \pi \in \mathcal{Q}, [\mathcal{Q}] \rangle), (\langle \mathcal{Q}, [\mathcal{Q}] \rangle), \mathcal{Q} \rangle \rangle \in \mathcal{Q}.$$  

(32)

That is,

$$\langle (\langle \pi \in \mathcal{Q}, [\mathcal{Q}] \rangle), (\langle \mathcal{Q}, [\mathcal{Q}] \rangle), \mathcal{Q} \rangle \rangle \in \mathcal{Q}.$$  

(33)
By Lemma 13, we have
\[
\langle u_\alpha (\pi * (\Xi * (\Xi * \Pi))), [I] \rangle \geq \ min \ \{ \langle u_\alpha (\pi * [I]), [I] \rangle, \langle u_\alpha ([I]), [I] \rangle \}. \tag{34}
\]

Thus, \( \langle \alpha, H \rangle \) holds. Therefore, \( H_{\alpha} \) is a \( Q \)-HFCI.

**Definition 15.** Let \( H_{\alpha} \) be a \( Q \)-hesitant CI of a \( BCK \)-algebra \( U \), for \( \in \epsilon \{[0, 1]\} \), the set \( H_{\alpha} (\epsilon) = \{ \pi \in U | [I] \in \epsilon \} \) \( \geq \) \{ \} of a CI is called \( Q \)-hesitant \( \dashv \)-level CI of \( H_{\alpha} \).

**Theorem 16.** In \( BCK \)-algebra \( U \), any CI of can be realized as \( Q \)-hesitant \( \dashv \)-level CI of some \( Q \)-HFCI of \( U \times Q \).

**Proof.** Let \( C \) be a CI of \( BCK \)-algebra \( U \) and let \( H_{\alpha} \) be a \( Q \)-hesitant fuzzy set of \( U \times Q \) defined by
\[
\langle H_{\alpha} (\pi, [I]) \rangle = \begin{cases} \text{if } \pi \in C, & 1 \\ \text{else,} & 0 \end{cases}, \tag{35}
\]

where \( \in \epsilon \{[0, 1]\} \). Let \( \pi \in C \) and \( [I] \in C \), then \( \pi * (\Xi * (\Xi * \Pi)) \in C \). Thus,
\[
\langle H_{\alpha} (\pi, [I]) \rangle = \langle H_{\alpha} ([I]), [I] \rangle = 1. \tag{36}
\]
and so
\[
\langle H_{\alpha} (\pi * (\Xi * (\Xi * \Pi))), [I] \rangle \geq \ min \ \{ \langle H_{\alpha} (\pi * [I]), [I] \rangle, \langle H_{\alpha} ([I]), [I] \rangle \}. \tag{37}
\]
(i) If \( (\pi * [I]) \neq \emptyset \) and \( [I] \neq \emptyset \), then \( \langle H_{\alpha} (\pi * [I]), [I] \rangle = \langle H_{\alpha} ([I]), [I] \rangle = 0 \).

Hence,
\[
\langle H_{\alpha} (\pi * (\Xi * (\Xi * \Pi))), [I] \rangle \geq \ min \ \{ \langle H_{\alpha} (\pi * [I]), [I] \rangle, \langle H_{\alpha} ([I]), [I] \rangle \}. \tag{38}
\]

(ii) If exactly one of \( (\pi * [I]) \) and \( [I] \) belongs to \( C \), then exactly one of \( \langle H_{\alpha} (\pi * [I]), [I] \rangle \) and \( \langle H_{\alpha} ([I]), [I] \rangle \) is equal to zero. So,
\[
\langle H_{\alpha} (\pi * (\Xi * (\Xi * \Pi))), [I] \rangle \geq \ min \ \{ \langle H_{\alpha} (\pi * [I]), [I] \rangle, \langle H_{\alpha} ([I]), [I] \rangle \}. \tag{39}
\]

The results above show
\[
\langle H_{\alpha} (\pi * (\Xi * (\Xi * \Pi))), [I] \rangle \geq \ min \ \{ \langle H_{\alpha} (\pi * [I]), [I] \rangle, \langle H_{\alpha} ([I]), [I] \rangle \} \text{ for all } \pi, [I], [I] \in C. \tag{40}
\]

It is clear that \( \langle H_{\alpha} (\pi), [I] \rangle \geq \langle H_{\alpha} (\pi), [I] \rangle \) for all \( \pi \in U \).

Therefore, \( H_{\alpha} \) is a \( Q \)-HFCI of \( U \times Q \). Obviously, \( H_{\alpha} \) is a \( Q \)-HFCI of \( U \times Q \).

**Theorem 17.** If \( H_{\alpha} \) a \( Q \)-HFCI of a \( BCK \)-algebra \( U \). Then, two-level CI \( H_{\alpha} (\pi, [I]) \) and \( H_{\alpha} (\pi, [I]) \) where \( \pi \in U \) and \( [I] \in Q \) are equal if and only if there is no \( \pi \in U \) such that \( \pi \in U \times Q \).

**Proof.** Let \( H_{\alpha} (\pi, [I]) = H_{\alpha} (\pi, [I]) \). If there exists \( \pi \in U \) such that \( \pi \in U \times Q \) then \( H_{\alpha} (\pi, [I]) \leq H_{\alpha} (\pi, [I]) \). Conversely, assume that there is no \( \pi \in U \) such that \( \pi \in U \times Q \). If \( \pi \in U \) and \( [I] \in Q \) then \( \pi \in U \times Q \). Hence, \( \pi \in U \times Q \).

Thus, \( H_{\alpha} (\pi, [I]) = H_{\alpha} (\pi, [I]) \).

This completes the proof.

Let \( H_{\alpha} \) be a \( Q \)-hesitant fuzzy set in \( U \) and let \( \text{Im}(\langle H_{\alpha} \rangle) \) denote the image of \( \langle H_{\alpha} \rangle \).

**Theorem 18.** Let \( U \) be a \( BCK \)-algebra and \( H_{\alpha} \) a \( Q \)-HFCI of \( U \times Q \). If \( \text{Im}(\langle H_{\alpha} \rangle) \) \( = \{ \pi, \pi, \ldots, \pi \} \) then the family of CI s \( H_{\alpha} (\pi, [I]) \) \( = \{ \pi, \pi, \ldots, \pi \} \) constitutes all the level CI s of \( \langle H_{\alpha} \rangle \).

**Proof.** Let \( \pi \in \{[0, 1]\} \) and \( \pi \in \text{Im}(\langle H_{\alpha} \rangle) \). If \( \pi \in \pi \), then \( \langle H_{\alpha} (\pi, [I]) \rangle \leq \langle H_{\alpha} (\pi, [I]) \rangle \). Since \( \langle H_{\alpha} (\pi, [I]) \rangle = U \), we have \( \langle H_{\alpha} (\pi, [I]) \rangle = U \). Hence, \( \langle H_{\alpha} (\pi, [I]) \rangle = \langle H_{\alpha} (\pi, [I]) \rangle \).

If \( \pi < \pi < \pi \), then there is no \( \pi \in U \) such that \( \pi < \pi \). From above theorem 10, it follows that \( \langle H_{\alpha} (\pi, [I]) \rangle = \langle H_{\alpha} (\pi, [I]) \rangle \). This shows that for any \( \pi \in \{[0, 1]\} \)

\[ \text{with } \langle H_{\alpha} (\pi, [I]) \rangle \leq \langle H_{\alpha} (\pi, [I]) \rangle \], the level CI \( \langle H_{\alpha} (\pi, [I]) \rangle \) is in \( \{ \pi \in U \times Q \} \).

**Lemma 19.** Given a \( BCK \)-algebra \( U \) and \( H_{\alpha} \) a \( Q \)-HFCI over \( U \times Q \). If \( \pi \in U \) and \( [I] \in Q \) belong to \( \text{Im}(\langle H_{\alpha} \rangle) \) such that \( \langle H_{\alpha} (\pi, [I]) \rangle = \langle H_{\alpha} (\pi, [I]) \rangle \), then \( \pi = \pi \).

**Proof.** Assume that \( \pi \neq \pi \). Then, there is \( \pi \in \{[0, 1]\} \) such that \( \pi < \pi \). Therefore, \( \pi \in \pi \). If \( \pi < \pi \), then there is no \( \pi \in U \) such that \( \pi < \pi \). From above theorem 10, it follows that \( \langle H_{\alpha} (\pi, [I]) \rangle = \langle H_{\alpha} (\pi, [I]) \rangle \). This shows that for any \( \pi \in \{[0, 1]\} \) such that \( \pi < \pi \), then \( \pi = \pi \).
\[ \left\langle \langle x_{\alpha}(\langle \rangle) \right\rangle \right\rangle \text{ and } (\pi, \emptyset) \notin \left\langle \langle x_{\alpha}(\langle \rangle) \right\rangle \right\rangle. \] Thus, \( \langle x_{\alpha}(\langle \rangle) \rangle \neq \langle x_{\alpha}(\langle \rangle) \rangle \), which is a contradiction to our fact. This completes the proof.

\[ \square \]

### 5. \(G\)-Hesitant Fuzzy Characteristic CIs

A mapping \( \{ \cdot \} : \mathcal{U} \rightarrow \mathcal{V} \) of a \(BCK\)-algebra is called a homomorphism if satisfying the identity \( \{(e \ast e) \ast e \} = \{(e) \ast (e) \ast (e)\} \) for all \(e \in \mathcal{V} \). Throughout, \( \text{Aut}(\mathcal{U}) \) will denote the \(BCK\)-algebra of automorphisms of \( \mathcal{U} \).

**Definition 20.** Let \( \{ \cdot \} : \mathcal{U} \rightarrow \mathcal{V} \) be a homomorphism of \(BCK\)-algebras. For any \(Q\)-\(HFC\)-ideal \( \langle x_{\alpha} \rangle \) of \( \mathcal{V} \), we define a new \(Q\)-\(HFC\)-ideal \( \langle x_{\alpha} \rangle \) in \( \mathcal{U} \) by

\[ \langle x_{\alpha}(\langle \rangle) \rangle = \langle x_{\alpha}(\langle \rangle) \rangle \text{ for all } \pi \in \mathcal{U}, \emptyset \in \mathcal{Q}. \] (41)

**Theorem 21.** Let \( \{ \cdot \} : \mathcal{U} \rightarrow \mathcal{V} \) be a homomorphism of \(BCK\)-algebras. If \( \langle x_{\alpha} \rangle \) is a \(Q\)-\(HFC\)-ideal of \( \mathcal{V} \), then \( \langle x_{\alpha} \rangle \) is a \(Q\)-\(HFC\)-ideal of \( \mathcal{U} \).

**Proof.** Let \( \pi \in \mathcal{U}, \emptyset \in \mathcal{Q} \in \mathcal{U}, \emptyset \in \mathcal{Q} \). Then

\[ \langle x_{\alpha}(\langle \rangle) \rangle = \langle x_{\alpha}(\langle \rangle) \rangle \leq \langle x_{\alpha}(\langle \rangle) \rangle = \langle x_{\alpha}(\langle \rangle) \rangle. \] (42)

Let \( \pi, \emptyset, \emptyset \in \mathcal{U}, \emptyset \in \mathcal{Q} \in \mathcal{U}, \emptyset \in \mathcal{Q} \). Then

\[ \langle x_{\alpha}(\langle \rangle) \rangle = \langle x_{\alpha}(\langle \rangle) \rangle \leq \langle x_{\alpha}(\langle \rangle) \rangle = \langle x_{\alpha}(\langle \rangle) \rangle. \] (43)

**Definition 22.** A CI \( \mathcal{C} \) of a \(BCK\)-algebra \( \mathcal{U} \) is called a characteristic CI (CCI) of \( \mathcal{U} \) if \( \langle (\emptyset) \rangle = \mathcal{C} \) for all \( \alpha \in \text{Aut}(\mathcal{U}) \).

**Definition 23.** A \(G\)-HFCI of a \(BCK\)-algebra \( \mathcal{U} \) is called a \(G\) -hesitant fuzzy CCI of \( \mathcal{U} \) if

\[ \langle x_{\alpha}(\langle \rangle) \rangle = \langle x_{\alpha}(\langle \rangle) \rangle \text{ for all } \pi \in \mathcal{U}, \emptyset \in \mathcal{Q} \text{ and } \forall \alpha \in \text{Aut}(\mathcal{U}). \] (44)

**Theorem 24.** Let \( \mathcal{H}_{\alpha} \) be a \(G\)-hesitant fuzzy characteristic CI of \( \mathcal{U} \times \mathcal{Q} \). Then, each \( I \)-level CI of \( \langle x_{\alpha} \rangle \) is a characteristic commutative ideal of \( \mathcal{U} \times \mathcal{Q} \).

**Proof.** Assume \( \langle (\langle \rangle) \rangle \in \text{Im}(\langle x_{\alpha} \rangle) \), \( \alpha \in \text{Aut}(\mathcal{U}) \) and \( \langle x_{\alpha}(\langle \rangle) \rangle \in \langle x_{\alpha}(\langle \rangle) \rangle \). Since \( \mathcal{H}_{\alpha} \) is a \(G\)-hesitant fuzzy characteristic commutative ideal of \( \mathcal{U} \times \mathcal{Q} \), we have \( \langle x_{\alpha}(\langle \rangle) \rangle \geq \langle x_{\alpha}(\langle \rangle) \rangle \).

It follows that \( \langle (\langle \rangle) \rangle \in \langle x_{\alpha}(\langle \rangle) \rangle \) and hence \( \langle (\langle \rangle) \rangle \leq \langle x_{\alpha}(\langle \rangle) \rangle \).

To show the reverse inclusion, let \( \langle x_{\alpha}(\langle \rangle) \rangle \) and let \( \alpha \in \text{Aut}(\mathcal{U}) \) be such that \( \langle (\langle \rangle) \rangle = \langle x_{\alpha}(\langle \rangle) \rangle \). Then, \( \langle x_{\alpha}(\langle \rangle) \rangle \leq \langle x_{\alpha}(\langle \rangle) \rangle \). Since \( \langle x_{\alpha}(\langle \rangle) \rangle \geq \langle x_{\alpha}(\langle \rangle) \rangle \), it follows that \( \langle (\langle \rangle) \rangle \in \langle x_{\alpha}(\langle \rangle) \rangle \). Thus, \( \langle x_{\alpha}(\langle \rangle) \rangle \). The proof of the following lemma is obvious, and we omit the proof.

**Lemma 25.** Let \( \mathcal{H}_{\alpha} \) be a \(G\)-HFCI of \( \mathcal{U} \times \mathcal{Q} \) and let \( \pi \in \mathcal{U} \).

Then, \( \langle x_{\alpha}(\langle \rangle) \rangle \in \langle x_{\alpha}(\langle \rangle) \rangle \) if and only if \( \langle (\langle \rangle) \rangle \leq \langle x_{\alpha}(\langle \rangle) \rangle \) and \( \langle (\langle \rangle) \rangle \notin \langle x_{\alpha}(\langle \rangle) \rangle \), for all \( \emptyset \geq \langle (\langle \rangle) \rangle \).

Now, we consider the inverse of Theorem 24.

**Theorem 26.** Let \( \mathcal{H}_{\alpha} \) be a \(G\)-HFCI of \( \mathcal{U} \times \mathcal{Q} \). If each level CI of \( \langle x_{\alpha} \rangle \) is a CCI of \( \mathcal{U} \), then \( \mathcal{H}_{\alpha} \) is a \(G\)-hesitant fuzzy characteristic commutative ideal of \( \mathcal{U} \times \mathcal{Q} \).

**Proof.** Let \( \pi \in \mathcal{U}, \emptyset \in \mathcal{Q}, \alpha \in \text{Aut}(\mathcal{U}) \) and \( \langle x_{\alpha}(\langle \rangle) \rangle \). Then, \( \langle (\langle \rangle) \rangle \in \langle x_{\alpha}(\langle \rangle) \rangle \) and \( \langle (\langle \rangle) \rangle \notin \langle x_{\alpha}(\langle \rangle) \rangle \) for all \( \emptyset \geq \langle (\langle \rangle) \rangle \), by Lemma 25 Since \( \langle (\langle \rangle) \rangle \leq \langle x_{\alpha}(\langle \rangle) \rangle \) by hypothesis, we have \( \langle (\langle \rangle) \rangle \in \langle x_{\alpha}(\langle \rangle) \rangle \) and hence \( \langle x_{\alpha}(\langle \rangle) \rangle \). Since \( \langle (\langle \rangle) \rangle \) is one to one, it follows that \( \langle (\langle \rangle) \rangle \in \langle x_{\alpha}(\langle \rangle) \rangle \), which is a contradiction. Hence, \( \langle x_{\alpha}(\langle \rangle) \rangle \). It follows that \( \mathcal{H}_{\alpha} \) is a \(G\)-hesitant fuzzy CCI of \( \mathcal{U} \) in \( \mathcal{Q} \). This completes the proof.

**6. Conclusions**

A new concept of HFI is considered by applying a two-dimensional membership function, namely, \(G\)-HFI. Several properties and theorems of \(G\)-HFI are proved. In this regard, we propose the concept of \(G\)-HFCI in \(BCK\)-algebra and prove some related properties. We have considered the features of \(G\)-HFCI. We study some feature properties related to \(G\)-HFCI. Our future research is to find ways to apply \(G\)-HFI to a wide range of logical algebraic systems, such as pseudo-\(BCK\)-algebras [14, 15]. For other notions, the readers are suggested to see [16–28].

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.
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