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Research Article

Invariant Tori for a Two-Dimensional Completely Resonant Beam Equation with a Quintic Nonlinear Term

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This paper focuses on a two-dimensional completely resonant beam equation with a quintic nonlinear term. This means studying $u_{tt} + \Delta^2 u + \varepsilon f(u) = 0, x \in \mathbb{T}^2, t \in \mathbb{R}$, under periodic boundary conditions, where ε is a small positive parameter and $f(u)$ is a real analytic odd function of the form $f(u) = f_5 u^5 + \sum_{\hat{i} \geq 3} f_{2\hat{i}+1} u^{2\hat{i}+1}, f_5 \neq 0$. It is proved that the equation admits small-amplitude, Whitney smooth, linearly stable quasiperiodic solutions on the phase-flow invariant subspace $\mathbb{Z}^2_\dagger = \{r = (r_1, r_2), r_1 \in 4\mathbb{Z} - 1, r_2 \in 4\mathbb{Z}\}$. Firstly, the corresponding Hamiltonian system of the equation is transformed into an angle-dependent block-diagonal normal form by using symplectic transformation, which can be achieved by selecting the appropriate tangential position. Finally, the existence of a class of invariant tori is proved, which implies the existence of quasiperiodic solutions for most values of frequency vector by an abstract KAM (Kolmogorov-Arnold-Moser) theorem for infinite dimensional Hamiltonian systems.

1. Introduction

In this paper, a two-dimensional completely resonant beam equation with a quintic nonlinear term

$$u_{tt} + \Delta^2 u + \varepsilon f(u) = 0, \quad x \in \mathbb{T}^2, t \in \mathbb{R}, \tag{1}$$

under periodic boundary conditions

$$u(t, x_1, x_2) = u(t, x_1 + 2\pi, x_2) = u(t, x_1, x_2 + 2\pi) \tag{2}$$

is considered, where ε is a small positive parameter and $f(u)$ is a real analytic odd function of the form

$$f(u) = f_5 u^5 + \sum_{\hat{i} \geq 3} f_{2\hat{i}+1} u^{2\hat{i}+1}, \quad f_5 \neq 0. \tag{3}$$

It is proved that the equation admits the existence of a class of invariant tori, which implies the existence of quasiperiodic solutions for most values of frequency vector by an abstract KAM (Kolmogorov-Arnold-Moser) theorem for infinite dimensional Hamiltonian systems.

In nature, periodic phenomenon is an ideal state, but in practical problems, such as data observation, extraction, and operation, it always has errors and even some interference. In fact, quasiperiodic functions are always needed to be introduced in a system when there are two disturbance factors with incommensurability of periods; thus, quasiperiodic phenomenon is more common than periodic phenomenon, such as the celestial mechanics, ecology system, and economic volatility in many practical problems which often can be classified as quasiperiodic problem of differential equations. Generally speaking, there is more than one variable that causes the change of a phenomenon, so it is of great practical value to study the quasiperiodic problem of partial differential equations (PDEs), and the quasiperiodic solution problem of nonlinear Hamiltonian system is an important branch of nonlinear scientific research. As a kind of important Hamiltonian system, beam equation has also received corresponding attention.

The classical KAM theory, proposed by Kolmogorov [1, 2], Arnold [3], and Moser [4], is a theory about the long-term state of the solution of the integrable Hamiltonian system after it is perturbed, which is a significant progress of Newtonian mechanics in the 20th century and enables people to study the Hamiltonian system in a new way. In the late 1980s, in order to construct quasiperiodic solutions of one-dimensional Hamiltonian PDEs, the classical KAM theory was developed into infinite dimensional space by Wayne [5], Kuksin [6], and Pöschel [7]. Since then, KAM theory of Hamiltonian PDEs with one-dimensional spatial variables has been well developed and produced a lot of results, which we will not repeat here.

When the dimension of the spatial variable exceeds 1, the multiplicity of the normal frequency tends to infinity, which makes the small divisor problem and its measure estimation in KAM iteration more difficult to solve, resulting in fewer corresponding results and larger research space. The conclusion of the existence of quasiperiodic solutions of high-dimensional Hamiltonian PDEs comes from Bourgain [8], but instead of using KAM theory, it uses multiscale analysis, so as to avoid a lot of tedious second Melnikov conditions. Since then, according to this idea, many important results have been obtained on high-dimensional Hamiltonian PDEs (refer to [9–13]). However, this method also has some disadvantages, such as it cannot give the normal form of the system, and thus, the linear stability and other related dynamic properties of small amplitude quasiperiodic solutions cannot be given. For these reasons, researchers have been trying to apply KAM theory to high-dimensional Hamiltonian PDEs. Yuan [14] and Geng and You [15, 16] first applied KAM theory to the existence of quasiperiodic solutions of high-dimensional Hamiltonian PDEs. In [17], Eliasson and Kuksin studied high-dimensional Schrödinger equations with convolutional-type potential and made a breakthrough in properly classifying normal frequencies by introducing the Toplitz-Lipschitz property, which perfectly solved the measure estimation problem brought by eigenvalue multiplicity. Eliasson et al. [18] considered a d-dimensional cubic beam equation that does not satisfy momentum conservation

$$u_{tt} + \Delta^2 u + mu + \partial_u G(x, u) = 0, \quad x \in \mathbb{T}^d, t \in \mathbb{R}, \tag{4}$$

whereG$(x, u) = u^4 + O(u^5)$ and $d \geq 2$. The existence of quasiperiodic solutions of (1) was proved by the KAM theory.

However, the above conclusions are dependent on external parameters and therefore cannot be applied to classical equations of complete resonance with physical background. Geng et al. [19] researched the KAM theory of two-dimensional completely resonant Schrodinger equation

$$\mathrm{i}u_t - \Delta u + |u|^2 u = 0, \quad x \in \mathbb{T}^2, t \in \mathbb{R}, \tag{5}$$

under periodic boundary conditions. This equation has no external parameters and can only be supplied by amplitude. They proved the existence of quasiperiodic solution with special tangential frequencies of the equation by combining the Toplitz-Lipschitz idea in [17] and the idea of solving the homology equation of dependent angular variables proposed by Xu and You in [20]. By using the similar idea in [19], Geng and Zhou [21] researched the existence of quasiperiodic solutions of the two-dimensional completely resonant cubic beam equation

$$u_{tt} + \Delta^2 u + u^3 = 0, \quad x \in \mathbb{T}^2, t \in \mathbb{R}, \tag{6}$$

on the phase flow invariant subspace $\mathbb{Z}^2_{\text{odd}} = \{r = (r_1, r_2), r_1 \in 2\mathbb{Z} - 1, r_2 \in 2\mathbb{Z}\}$ of $\mathbb{Z}^2$. The reason why the existence of solution is only discussed in the invariant subspace of phase flow is that the nonlinear term of the Hamiltonian system corresponding to beam equation is relatively complex, and this idea was first proposed by [22].

The KAM theory is the compound of Newton iterative method and Birkhoff normal type. Through the normal type, parameters are introduced to adjust the frequency; that is, the spoke frequency modulation is realized through the parameters, so as to overcome the problem of small divisor related to homology equation in KAM iterative, which is an important link of the KAM theory. The nonlinear term of the Hamiltonian system directly affects its normal form. Therefore, once the nonlinearity changes, the corresponding normal form should be adjusted accordingly, so the KAM theory needs to be reconstructed. In 2021, the authors of the present paper Zhang and Si [23] applied the idea in [19] to the existence of quasiperiodic solutions of the two-dimensional completely resonant quintic Schrödinger equation

$$\mathrm{i}u_t - \Delta u + |u|^4 u = 0, \quad x \in \mathbb{T}^2, t \in \mathbb{R}. \tag{7}$$

Although only the nonlinear term has changed from $|u|^2 u$ to $|u|^4 u$, its normal form is completely different, so its corresponding KAM theory has also undergone essential changes. In recent years, more attention has been paid to the existence of quasiperiodic solutions of quintic Hamiltonian PDEs in high-dimensional space. Relevant results can be referred to references [24–26]. However, using the KAM theory to prove the existence of quasiperiodic solutions for two-dimensional completed resonant beam equations with higher order nonlinear terms remains to be solved.

This paper is focused on the study of (1) + (2). The nonlinear term of the Hamiltonian system corresponding to (6) is $p_i^{\tau_1} p_j^{\tau_2} p_r^{\tau_3} p_s^{\tau_4} / \sqrt{\lambda_i \lambda_j \lambda_r \lambda_s}$ and that of (1) is $p_i^{\tau_1} p_j^{\tau_2} p_n^{\tau_3} p_m^{\tau_4} p_r^{\tau_5} p_s^{\tau_6} / \sqrt{\lambda_i \lambda_j \lambda_n \lambda_m \lambda_r \lambda_s}$, where $\tau_1 i + \tau_2 j + \tau_3 n + \tau_4 m + \tau_5 r + \tau_6 s = 0, \tau_1 = \pm, \tau_2 = \pm, \tau_3 = \pm, \tau_4 = \pm, \tau_5 = \pm, \tau_6 = \pm$, and $p^- = \bar{p}$. This difference leads to the essential difference of the normal form between the two, which leads to the fact that the KAM theory for (6) is not suitable for (1), and the phase flow invariant subspace where the quasiperiodic solution is located will also change. In this paper, we only discuss the existence of quasiperiodic solutions of (1) in the phase flow invariant subspace $\mathbb{Z}^2_\dagger = \{r = (r_1, r_2), r_1 \in 4\mathbb{Z} - 1, r_2 \in 4\mathbb{Z}\}$. By selecting phase flow invariant subspace $\mathbb{Z}^2_\dagger$, it can be ensured that the eigenvalue $\lambda_r \neq 0$, and the normal form is simple and beautiful enough. However, due to the change of the phase flow invariant subspace where the quasiperiodic solution is located, we will have to reselect the tangential sites, that is, reconstruct the admissible set. Although this process does not require advanced mathematical knowledge and only involves elementary operations, it is cumbersome enough and requires strong skills. For the selected admissible set, (1) is turned into normal form by symplectic coordinate transformation, whose integrable terms only come from $p_i\bar{p}_jp_n\bar{p}_mp_r\bar{p}_s/\sqrt{\lambda_i\lambda_j\lambda_n\lambda_m\lambda_r\lambda_s}$ where $i-j+n-m+r-s=0$; that is, the normal form of (1) is formally consistent with the normal form of (7). Of course, the two normal forms are only in the same form, and the coefficients of (1) are much more complex than (7). Therefore, we spend a lot of energy to prove that (1) meets the requirements of the KAM theory in [23] and then prove the existence of its quasiperiodic solution.

The following are the conditions that the tangential sites need to meet, which are proposed by Zhang and Si [23].

Definition 1. A set $\mathcal{K}=\{i_1^*=(b_1,c_1),\cdots,i_d^*=(b_d,c_d)\}\subset\mathbb{Z}_\dagger^2$ is admissible, if

(1) arbitrary three of them are not vertices of a rectangle

(2) for any $\{i,j,n,m,r,s\}\subset\mathbb{Z}_\dagger^2$, if $i-j+n-m+r-s=0$ and $|i|^2-|j|^2+|n|^2-|m|^2+|r|^2-|s|^2=0$, then the intersection of $\{i,j,n,m,r,s\}$ and $\mathcal{K}$ contains at most four elements, that is, $\#\{\{i,j,n,m,r,s\}\bigcap\mathcal{K}\}\le 4$

(3) for any $r\in\mathbb{Z}_\dagger^2\setminus\mathcal{K}$, there exists at most one triplet $\{i,j,s\}$, where $i,j\in\mathcal{K},s\in\mathbb{Z}_\dagger^2\setminus\mathcal{K}$ such that $i-j+r-s=0$ and $|i|^2-|j|^2+|r|^2-|s|^2=0$. If such triplet exists, let us say that r,s are resonant in the first type and denote all such r by $\mathcal{I}_1$

(4) for any $r\in\mathbb{Z}_\dagger^2\setminus\mathcal{K}$, there exists at most one triplet $\{i,j,s\}$, where $i,j\in\mathcal{K},s\in\mathbb{Z}_\dagger^2\setminus\mathcal{K}$ such that $i+j-r-s=0$ and $|i|^2+|j|^2-|r|^2-|s|^2=0$. If such triplet exists, let us say that r,s are resonant in the second type and denote all such r by $\mathcal{I}_2$

(5) for any $r\in\mathbb{Z}_\dagger^2\setminus\mathcal{K}$, there exists at most one quintuple $\{i,j,n,m,s\}$, where $i,j,n,m\in\mathcal{K},s\in\mathbb{Z}_\dagger^2\setminus\mathcal{K}$ such that $i-j+n-m+r-s=0$ and $|i|^2-|j|^2+|n|^2-|m|^2+|r|^2-|s|^2=0$. If such quintuple exists, let us say that r,s are resonant in the third type and denote all such r by $\mathcal{I}_3$

(6) for any $r\in\mathbb{Z}_\dagger^2\setminus\mathcal{K}$, there exists at most one quintuple $\{i,j,n,m,s\}$, where $i,j,n,m\in\mathcal{K},s\in\mathbb{Z}_\dagger^2\setminus\mathcal{K}$ such that $i-j+n+m-r-s=0$ and $|i|^2-|j|^2+|n|^2+|m|^2-|r|^2-|s|^2=0$. If such quintuple exists, let us say that r,s are resonant in the fourth type and denote all such r by $\mathcal{I}_4$

(7) Any $r\in\mathbb{Z}_\dagger^2\setminus\mathcal{K}$ is not resonant of any two of the above four classes. Geometrically, any two of the above defined graphs cannot share vertex in $\mathbb{Z}_\dagger^2\setminus\mathcal{K}$

Remark 2. There are sets satisfying the above definition. For example, for any integer $d\ge 4$, the first point $(b_1,c_1)\in\mathbb{Z}_\dagger^2$ is defined as $b_1>d^2$, $c_1=4b_1^5$, and the second one is defined as $b_2=4c_1^5-1$, $c_2=4b_2^5$; the others are defined inductively by

$$b_{\hat{j}+1}=4c_{\hat{j}}^5\prod_{2\le\hat{m}\le\hat{j},1\le\hat{l}<\hat{m}}\left(c_{\hat{m}}-c_{\hat{l}}\right)^2-1,\ c_{\hat{j}+1}=4b_{\hat{j}+1}^5,\ 2\le\hat{j}\le d-1. \tag{8}$$

The set $\mathcal{K}=\{i_1^*=(b_1,c_1),\cdots,i_d^*=(b_d,c_d)\}\subset\mathbb{Z}_\dagger^2$ given above is admissible, and the proof is shown in the appendix.

The main result of this paper is as follows.

Theorem 3 (main theorem). *Let $\mathcal{K}=\{i_1^*=(b_1,c_1),\cdots,i_d^*=(b_d,c_d)\}\subset\mathbb{Z}_\dagger^2$ be an admissible set with $d\ge 4$. There exist a Cantor set Ξ^* with positive measure, such that for arbitrary $(\zeta_1,\cdots,\zeta_d)\in\Xi^*$, the beam equation (1) + (2) has a solution*

$$u(t,x)=\sum_{l\in\mathcal{K}}\frac{1}{2\pi}\sqrt{\zeta_l}\left(e^{i\eta_lt}e^{i\langle l,x\rangle}+e^{-i\eta_lt}e^{-i\langle l,x\rangle}\right)+O\left(|\zeta|^{3/2}\right),\eta_l=|l|^2+O\left(|\zeta|^2\right). \tag{9}$$

2. The Hamiltonian Setting and Birkhoff Normal Form

Before turning equation (1) into a Hamiltonian system, we first introduce the following notations which will appear later.

Let $\mathcal{K}=\{i_1^*,\cdots,i_d^*\}\subset\mathbb{Z}_\dagger^2$, $\mathbb{Z}^{2,\diamond}=\mathbb{Z}_\dagger^2\setminus\mathcal{K}$, and $l^{a,\varpi}$ be some Hilbert space of sequences $v=(\cdots,v_l,\cdots)_{l\in\mathbb{Z}^{2,\diamond}}$, where the norm be $\|v\|_{a,\varpi}=\sum_{l\in\mathbb{Z}^{2,\diamond}}|v_l|e^{a|l|}|l|^{\varpi}$, $(a>0,\varpi>1/2)$. Set $\theta=(\theta_l)_{l\in\mathcal{K}}$, $I=(I_l)_{l\in\mathcal{K}}$, $v=(v_l)_{l\in\mathbb{Z}^{2,\diamond}}$ and $\zeta=(\zeta_l)_{l\in\mathcal{K}}$, and introduce the phase space $\mathcal{P}^{a,\varpi}=\widehat{\mathbb{T}}^d\times\mathbb{C}^d\times l^{a,\varpi}\times l^{a,\varpi}\ni(\theta,I,v,\bar{v})$, where $\widehat{\mathbb{T}}^d$ is complex neighborhood of $\mathbb{T}^d$. Let

$$D_{a,\varpi}(r^*,s^*):=\left\{(\theta,I,v,\bar{v})\in\mathcal{P}^{a,\varpi}:|\mathrm{Im}\theta|<r^*,|I|<(s^*)^2,\|v\|_{a,\varpi}<s^*,\|\bar{v}\|_{a,\varpi}<s^*\right\}. \tag{10}$$

For any $V^*=(\theta,I,v,\bar{v})\in\mathcal{P}^{a,\varpi}$, define its norm be $|V^*|_{a,\varpi}=|\theta|+(1/(s^*)^2)|I|+(1/s^*)\|v\|_{a,\varpi}+(1/s^*)\|\bar{v}\|_{a,\varpi}$. Let $\mathcal{O}$ be the parameter set. Denote $\alpha\equiv(\cdots,\alpha_l,\cdots)_{l\in\mathbb{Z}^{2,\diamond}}$, $\beta\equiv(\cdots,\beta_l,\cdots)_{l\in\mathbb{Z}^{2,\diamond}}$, and α_l and $\beta_l\in\mathbb{N}$ have a finite number of positive integer components. Let $W(\theta,I,v,\bar{v})=\sum_{\alpha,\beta}W_{\alpha\beta}(\theta,I)v^\alpha\bar{v}^\beta$, where $v^\alpha\bar{v}^\beta$ is $\prod_l v_l^{\alpha_l}\bar{v}_l^{\beta_l}$ and $W_{\alpha\beta}=\sum_{k,l}W_{kl\alpha\beta}I^le^{i<k,\theta>}$ are C_W^8 functions of the parameter ζ. Set the norm of W as $\|W\|_{D_{a,\varpi}(r^*,s^*),\mathcal{O}}\equiv\sup_{\|v\|_{a,\varpi}<s^*,\|\bar{v}\|_{a,\varpi}<s^*}\sum_{\alpha,\beta}\|W_{\alpha\beta}\||v^\alpha||\bar{v}^\beta|$, where $\|W_{\alpha\beta}\|\equiv\sum_{k,l}|W_{kl\alpha\beta}|_{\mathcal{O}}(s^*)^{2|l|}e^{|k|r^*}$ and $|W_{kl\alpha\beta}|_{\mathcal{O}}\equiv\sup_{\zeta\in\mathcal{O}}\sum_{0\le\hat{i}\le 8}|\partial_\zeta^{\hat{i}}W_{kl\alpha\beta}|$. Denote X_W as the Hamiltonian field W corresponding to the symplectic structure $d\theta\wedge dI+\mathrm{i}dv\wedge d\bar{v}$, which is $X_W=(\partial_IW,-\partial_\theta W,\mathrm{i}\nabla_{\bar{v}}W,-\mathrm{i}\nabla_vW)$. Set its norm as

$$\|X_W\|_{D_{a,\varpi}(r^*,s^*),\mathcal{O}} \equiv \|W_I\|_{D_{a,\varpi}(r^*,s^*),\mathcal{O}} + \frac{1}{(s^*)^2}\|W_\theta\|_{D_{a,\varpi}(r^*,s^*),\mathcal{O}} + \frac{1}{s^*}\left(\sum_{l\in\mathbb{Z}^{2,\diamond}}\|W_{v_l}\|_{D_{a,\varpi}(r^*,s^*),\mathcal{O}} e^{|l|a}|l|^{\varpi} + \sum_{l\in\mathbb{Z}^{2,\diamond}}\|W_{\bar v_l}\|_{D_{a,\varpi}(r^*,s^*),\mathcal{O}} e^{|l|a}|l|^{\varpi}\right). \tag{11}$$

The vector function $\check{W} : D_{a,\varpi}(r^*,s^*)\times\mathcal{O}\longrightarrow\mathbb{C}^{\check m}(\check m<\infty)$ is C_W^8 function of the parameter ζ, and its norm is $\|\check W\|_{D_{a,\varpi}(r^*,s^*),\mathcal{O}} = \sum_{\hat i=1}^{\check m}\|\check W_{\hat i}\|_{D_{a,\varpi}(r^*,s^*),\mathcal{O}}$. The vector function $\widehat W : D_{a,\varpi}(r^*,s^*)\times\mathcal{O}\longrightarrow l^{\bar a,\varpi}$ is C_W^8 functions of the parameter ζ, and its norm is $\|\widehat W\|_{\bar a,D_{a,\varpi}(r^*,s^*),\mathcal{O}} = \|(\|\widehat W_{\hat i}\|_{D_{a,\varpi}(r^*,s^*),\mathcal{O}})_{\hat i}\|_{\bar a,\varpi}$.

2.1. The Hamiltonian Setting. Without losing generality, we suppose that $f(u)=u^5$. Rewrite the beam equation (1) as

$$u_{tt} + \Delta^2 u + \varepsilon u^5 = 0, \quad x\in\mathbb{T}^2, t\in\mathbb{R}. \tag{12}$$

Introducing a variable $u_t = v$, then (12) can be turned into

$$\begin{cases} u_t = v, \\ v_t = -\Delta^2 u - \varepsilon u^5. \end{cases} \tag{13}$$

Set $p = 1/\sqrt{2}((-\Delta)^{1/2}u - \mathrm{i}(-\Delta)^{-(1/2)}v)$; then, (13) is turned into $\dot p = i(\partial H^*/\partial\bar p)$ whose corresponding Hamiltonian is

$$H^*(p) = \int_{\mathbb{T}^2}\left[((-\Delta)p)\bar p + \frac{\varepsilon}{6}\left((-\Delta)^{-(1/2)}\left(\frac{p+\bar p}{\sqrt 2}\right)\right)^6\right]dx. \tag{14}$$

The eigenvalues and eigenvectors of operator $-\Delta$ with periodic boundary conditions are $\lambda_l = |l|^2$ and $\phi_l(x) = (1/2\pi)e^{\mathrm{i}<l,x>}$, respectively. p has coordinates $(p_l)_{l\in\mathbb{Z}_\dagger^2}\in l^{a,\varpi}$ with respect to the bases $\{\phi_l\}_{l\in\mathbb{Z}_\dagger^2}$. The corresponding symplectic structure is $\mathrm{i}\sum_{d\in\mathbb{Z}_\dagger^2} dp_l\wedge d\bar p_l$. In new coordinates, (13) becomes

$$\dot p_l = \mathrm{i}\frac{\partial H^*}{\partial\bar p_l}, \quad \forall l\in\mathbb{Z}_\dagger^2. \tag{15}$$

The corresponding Hamiltonian is

$$H^* = \Lambda + Q, \tag{16}$$

where

$$\begin{aligned}
\Lambda &= \sum_{l\in\mathbb{Z}_\dagger^2}\lambda_l|p_l|^2,\\
Q &= \frac{\varepsilon}{768\pi^4}\sum_{\substack{i+j+n+m+r+s=0\\ i,j,n,m,r,s\in\mathbb{Z}_\dagger^2}}\frac{1}{\sqrt{\lambda_i\lambda_j\lambda_n\lambda_m\lambda_r\lambda_s}}\cdot\left(p_ip_jp_np_mp_rp_s + \bar p_i\bar p_j\bar p_n\bar p_m\bar p_r\bar p_s\right)\\
&+ \frac{\varepsilon}{128\pi^4}\sum_{\substack{i+j+n+m+r-s=0\\ i,j,n,m,r,s\in\mathbb{Z}_\dagger^2}}\frac{1}{\sqrt{\lambda_i\lambda_j\lambda_n\lambda_m\lambda_r\lambda_s}}\cdot\left(p_ip_jp_np_mp_r\bar p_s + \bar p_i\bar p_j\bar p_n\bar p_m\bar p_r p_s\right)\\
&+ \frac{5\varepsilon}{256\pi^4}\sum_{\substack{i+j+n+m-r-s=0\\ i,j,n,m,r,s\in\mathbb{Z}_\dagger^2}}\frac{1}{\sqrt{\lambda_i\lambda_j\lambda_n\lambda_m\lambda_r\lambda_s}}\cdot\left(p_ip_jp_np_m\bar p_r\bar p_s + \bar p_i\bar p_j\bar p_n\bar p_m p_r p_s\right)\\
&+ \frac{5\varepsilon}{192\pi^4}\sum_{\substack{i+j+n-m-r-s=0\\ i,j,n,m,r,s\in\mathbb{Z}_\dagger^2}}\frac{1}{\sqrt{\lambda_i\lambda_j\lambda_n\lambda_m\lambda_r\lambda_s}}p_ip_jp_n\bar p_m\bar p_r\bar p_s.
\end{aligned} \tag{17}$$

The regularity of the Hamiltonian system (15) is shown below, and its proof is similar to [21], which is omitted here.

Lemma 4. *For a given $a\geq 0$ and $\varpi > 1/2$, the gradients Q_p^6, $Q_{\bar p}^6$ are real analytic as maps from some neighborhood of origin in $l^{a,\varpi}\times l^{a,\varpi}$ into $l^{a,\varpi}$ and $\|Q_p^6\|_{a,\varpi} = O(\|p\|_{a,\varpi}^5)$, $\|Q_{\bar p}^6\|_{a,\varpi} = O(\|p\|_{a,\varpi}^5)$.*

2.2. Partial Birkhoff Normal Form. $\mathcal{K}$ is an admissible set with d points. Set $\mathbb{Z}^{2,\diamond} = \mathbb{Z}_\dagger^2\setminus\mathcal{K}$, and define four sets as follows:

$$\begin{aligned}
\mathcal{K}_1 &= \left\{(i,j,n,m,r,s)\in(\mathbb{Z}_\dagger^2)^6 : \begin{array}{c} i-j+n-m+r-s=0 \\ |i|^2-|j|^2+|n|^2-|m|^2+|r|^2-|s|^2\neq 0 \\ \#(\mathcal{K}\cap\{i,j,n,m,r,s\})\geq 4\end{array}\right\},\\
\mathcal{K}_2 &= \left\{(i,j,n,m,r,s)\in(\mathbb{Z}_\dagger^2)^6 : \begin{array}{c} i+j+n+m+r+s=0 \\ |i|^2+|j|^2+|n|^2+|m|^2+|r|^2+|s|^2\neq 0 \\ \#(\mathcal{K}\cap\{i,j,n,m,r,s\})\geq 4\end{array}\right\},\\
\mathcal{K}_3 &= \left\{(i,j,n,m,r,s)\in(\mathbb{Z}_\dagger^2)^6 : \begin{array}{c} i+j+n+m+r-s=0 \\ |i|^2+|j|^2+|n|^2+|m|^2+|r|^2-|s|^2\neq 0 \\ \#(\mathcal{K}\cap\{i,j,n,m,r,s\})\geq 4\end{array}\right\},\\
\mathcal{K}_4 &= \left\{(i,j,n,m,r,s)\in(\mathbb{Z}_\dagger^2)^6 : \begin{array}{c} i+j+n+m-r-s=0 \\ |i|^2+|j|^2+|n|^2+|m|^2-|r|^2-|s|^2\neq 0 \\ \#(\mathcal{K}\cap\{i,j,n,m,r,s\})\geq 4\end{array}\right\}.
\end{aligned} \tag{18}$$

Obviously, the set

$$\left\{(i,j,n,m,r,s)\in\left(\mathbb{Z}_{\dagger}^{2}\right)^{6}:\begin{array}{c} i+j+n+m+r+s=0 \\ |i|^2+|j|^2+|n|^2+|m|^2+|r|^2+|s|^2=0 \end{array}\right\} \tag{19}$$

is an empty set. It is proved by the reduction to absurdity that any six points $i=(4\widehat{i}_1-1,4\widehat{i}_2)$, $j=(4\widehat{j}_1-1,4\widehat{j}_2)$, $n=(4\widehat{n}_1-1,4\widehat{n}_2)$, $m=(4\widehat{m}_1-1,4\widehat{m}_2)$, $r=(4\widehat{r}_1-1,4\widehat{r}_2)$, $s=(4\widehat{s}_1-1,4\widehat{s}_2)$ on $\mathbb{Z}_{\dagger}^{2}$ satisfy $|i|^2+|j|^2+|n|^2+|m|^2+|r|^2-|s|^2\neq 0$. Suppose that $|i|^2+|j|^2+|n|^2+|m|^2+|r|^2-|s|^2=0$, then

$$\begin{aligned}&\left(4\widehat{i}_1-1\right)^2+\left(4\widehat{j}_1-1\right)^2+(4\widehat{n}_1-1)^2+(4\widehat{m}_1-1)^2+(4\widehat{r}_1-1)^2\\&\quad-(4\widehat{s}_1-1)^2=-16\left(\widehat{i}_2^{\,2}+\widehat{j}_2^{\,2}+\widehat{n}_2^{\,2}+\widehat{m}_2^{\,2}+\widehat{r}_2^{\,2}-\widehat{s}_2^{\,2}\right).\end{aligned} \tag{20}$$

The first polynomial is divisible by 8, but the second polynomial is not, which is a contradiction. Therefore, the set

$$\left\{(i,j,n,m,r,s)\in\left(\mathbb{Z}_{\dagger}^{2}\right)^{6}:\begin{array}{c} i+j+n+m+r-s=0 \\ |i|^2+|j|^2+|n|^2+|m|^2+|r|^2-|s|^2=0 \end{array}\right\} \tag{21}$$

is an empty set. Similarly, any six points i,j,n,m,r,s on $\mathbb{Z}_{\dagger}^{2}$ satisfy $|i|^2+|j|^2+|n|^2+|m|^2-|r|^2-|s|^2\neq 0$. Therefore, the set

$$\left\{(i,j,n,m,r,s)\in\left(\mathbb{Z}_{\dagger}^{2}\right)^{6}:\begin{array}{c} i+j+n+m-r-s=0 \\ |i|^2+|j|^2+|n|^2+|m|^2-|r|^2-|s|^2=0 \end{array}\right\} \tag{22}$$

is an empty set.

Let us introduce some partial Birkhoff form of order six.

Proposition 5. *$\mathcal{K}$ is an admissible set with d points; there exists a symplectic transformation X_R^1 that converts the Hamiltonian (16) into*

$$H=\Lambda+\mathcal{A}_1+\mathcal{B}_1+\bar{\mathcal{B}}_1+\mathcal{A}_2+\mathcal{B}_2+\bar{\mathcal{B}}_2+W, \tag{23}$$

with

$$\Lambda=\sum_{i\in\mathcal{K}}\eta_i(\zeta)I_i+\sum_{r\in\mathbb{Z}^{2,\diamond}}\Omega_r(\zeta)|v_r|^2, \tag{24}$$

$$\begin{aligned}\eta_i(\zeta)&=\varepsilon^{-4}\lambda_i+\frac{5\zeta_i^2}{64\pi^4\lambda_i^3}+\sum_{j\in\mathcal{K},j\neq i}\\&\quad\cdot\left[\frac{15\zeta_j^2}{64\pi^4\lambda_i\lambda_j^2}+\frac{15\zeta_i\zeta_j}{32\pi^4\lambda_i^2\lambda_j}+\sum_{n\in\mathcal{K},n\neq j,n\neq i}\frac{15\zeta_j\zeta_n}{32\pi^4\lambda_i\lambda_j\lambda_n}\right],\end{aligned} \tag{25}$$

$$\Omega_r(\zeta)=\varepsilon^{-4}\lambda_r+\sum_{i\in\mathcal{K}}\left[\frac{15\zeta_i^2}{64\pi^4\lambda_i^2\lambda_r}+\sum_{j\in\mathcal{K},j\neq i}\frac{15\zeta_i\zeta_j}{32\pi^4\lambda_i\lambda_j\lambda_r}\right], \tag{26}$$

$$\mathcal{A}_1=\sum_{r\in\mathcal{J}_1}\sum_{n\in\mathcal{K}}\frac{15}{16\pi^4\sqrt{\lambda_i\lambda_j\lambda_n^2\lambda_r\lambda_s}}\sqrt{\zeta_i\zeta_j}\zeta_n e^{i\left(\theta_i-\theta_j\right)}v_r\bar{v}_s, \tag{27}$$

$$\mathcal{B}_1=\sum_{r\in\mathcal{J}_2}\sum_{n\in\mathcal{K}}\frac{15}{16\pi^4\sqrt{\lambda_i\lambda_j\lambda_n^2\lambda_r\lambda_s}}\sqrt{\zeta_i\zeta_j}\zeta_n e^{i\left(-\theta_i-\theta_j\right)}v_r v_s, \tag{28}$$

$$\bar{\mathcal{B}}_1=\sum_{r\in\mathcal{J}_2}\sum_{n\in\mathcal{K}}\frac{15}{16\pi^4\sqrt{\lambda_i\lambda_j\lambda_n^2\lambda_r\lambda_s}}\sqrt{\zeta_i\zeta_j}\zeta_n e^{i\left(\theta_i+\theta_j\right)}\bar{v}_r\bar{v}_s, \tag{29}$$

$$\mathcal{A}_2=\sum_{r\in\mathcal{J}_3}\frac{15}{16\pi^4\sqrt{\lambda_i\lambda_j\lambda_n\lambda_m\lambda_r\lambda_s}}\sqrt{\zeta_i\zeta_j\zeta_n\zeta_m}e^{i\left(\theta_i-\theta_j+\theta_n-\theta_m\right)}v_r\bar{v}_s, \tag{30}$$

$$\mathcal{B}_2=\sum_{r\in\mathcal{J}_4}\frac{15}{16\pi^4\sqrt{\lambda_i\lambda_j\lambda_n\lambda_m\lambda_r\lambda_s}}\sqrt{\zeta_i\zeta_j\zeta_n\zeta_m}e^{i\left(-\theta_i+\theta_j-\theta_n-\theta_m\right)}v_r v_s, \tag{31}$$

$$\bar{\mathcal{B}}_2=\sum_{r\in\mathcal{J}_4}\frac{15}{16\pi^4\sqrt{\lambda_i\lambda_j\lambda_n\lambda_m\lambda_r\lambda_s}}\sqrt{\zeta_i\zeta_j\zeta_n\zeta_m}e^{i\left(\theta_i-\theta_j+\theta_n+\theta_m\right)}\bar{v}_r\bar{v}_s, \tag{32}$$

$$\begin{aligned}W&=O\Big(\varepsilon^2|I|^2+\varepsilon^2|I|\|v\|_{a,\varpi}^2+\varepsilon^{7/4}|\zeta|^{3/2}|I|^{1/2}\|v\|_{a,\varpi}^2\\&\quad+\varepsilon^{7/4}|\zeta|^{3/2}\|v\|_{a,\varpi}^3+\varepsilon^{7/2}|\zeta|\|v\|_{a,\varpi}^4+\varepsilon^{9/4}|\zeta|^{9/2}\|v\|_{a,\varpi}\\&\quad+\varepsilon^4|\zeta|^4\|v\|_{a,\varpi}^2+\varepsilon^{23/4}|\zeta|^{7/2}\|v\|_{a,\varpi}^3\Big).\end{aligned} \tag{33}$$

Proof. Let

$$\begin{aligned}
R = {} & \frac{5\varepsilon}{192\pi^4} \sum_{(i,j,n,m,r,s)\in\mathscr{K}_1} \frac{\mathrm{i}\cdot p_i\bar{p}_j p_n \bar{p}_m p_r \bar{p}_s}{\sqrt{\lambda_i\lambda_j\lambda_n\lambda_m\lambda_r\lambda_s}\cdot\left(\lambda_i-\lambda_j+\lambda_n-\lambda_m+\lambda_r-\lambda_s\right)} \\
& + \frac{\varepsilon}{768\pi^4} \sum_{(i,j,n,m,r,s)\in\mathscr{K}_2} \frac{\mathrm{i}\cdot\left(p_i p_j p_d p_l p_r p_s - \bar{p}_i\bar{p}_j\bar{p}_n\bar{p}_m\bar{p}_r\bar{p}_s\right)}{\sqrt{\lambda_i\lambda_j\lambda_n\lambda_m\lambda_r\lambda_s}\cdot\left(\lambda_i+\lambda_j+\lambda_n+\lambda_m+\lambda_r+\lambda_s\right)} \\
& + \frac{\varepsilon}{128\pi^4} \sum_{(i,j,n,m,r,s)\in\mathscr{K}_3} \frac{\mathrm{i}\cdot\left(p_i p_j p_n p_m p_r \bar{p}_s - \bar{p}_i\bar{p}_j\bar{p}_n\bar{p}_m\bar{p}_r p_s\right)}{\sqrt{\lambda_i\lambda_j\lambda_n\lambda_m\lambda_r\lambda_s}\cdot\left(\lambda_i+\lambda_j+\lambda_n+\lambda_m+\lambda_r-\lambda_s\right)} \\
& + \frac{5\varepsilon}{256\pi^4} \sum_{(i,j,n,m,r,s)\in\mathscr{K}_4} \frac{\mathrm{i}\cdot\left(p_i p_j p_n p_m \bar{p}_r\bar{p}_s - \bar{p}_i\bar{p}_j\bar{p}_d\bar{p}_l p_r p_s\right)}{\sqrt{\lambda_i\lambda_j\lambda_n\lambda_m\lambda_r\lambda_s}\cdot\left(\lambda_i+\lambda_j+\lambda_n+\lambda_m-\lambda_r-\lambda_s\right)}
\end{aligned} \tag{34}$$

and X_R^1 be the time − 1 mapping of the Hamiltonian vector field of R. Set

$$p_l = \begin{cases} p_l, & l\in\mathscr{K}, \\ v_l, & l\in\mathbb{Z}^{2,\diamond}. \end{cases} \tag{35}$$

Then, X_R^1 converts H^* into

$$\begin{aligned}
\tilde{H} = {} & H^*\circ X_R^1 = \Lambda + Q + \{\Lambda, R\} + \{Q, R\} \\
& + \int_0^1 (1-t)\{\{H^*, R\}, R\}\circ X_R^t dt \\
= {} & \sum_{i\in\mathscr{K}} \lambda_i |p_i|^2 + \sum_{r\in\mathbb{Z}^{2,\diamond}} \lambda_r |v_r|^2 + \sum_{i\in\mathscr{K}} \frac{5\varepsilon}{192\pi^4\lambda_i^3}|p_i|^6 \\
& + \sum_{i,j\in\mathscr{K}, i\neq j} \frac{15\varepsilon}{64\pi^4\lambda_i^2\lambda_j}|p_i|^4\left|p_j\right|^2 \\
& + \sum_{i,j,n\in\mathscr{K}, i\neq j, i\neq n, j\neq n} \frac{15\varepsilon}{16\pi^4\lambda_i\lambda_j\lambda_n}|p_i|^2\left|p_j\right|^2|p_n|^2 \\
& + \sum_{i\in\mathscr{K}, r\in\mathbb{Z}^{2,\diamond}} \frac{15\varepsilon}{64\pi^4\lambda_i^2\lambda_r}|p_i|^4|v_r|^2 \\
& + \sum_{i,j\in\mathscr{K}, r\in\mathbb{Z}^{2,\diamond}, i\neq j} \frac{15\varepsilon}{16\pi^4\lambda_i\lambda_j\lambda_r}|p_i|^2\left|p_j\right|^2|v_r|^2 \\
& + \sum_{r\in\mathscr{I}_1}\sum_{n\in\mathscr{K}} \frac{15\varepsilon}{16\pi^4\sqrt{\lambda_i\lambda_j\lambda_n^2\lambda_r\lambda_s}} p_i\bar{p}_j|p_n|^2 v_r\bar{v}_s \\
& + \sum_{r\in\mathscr{I}_2}\sum_{n\in\mathscr{K}} \frac{15\varepsilon}{16\pi^4\sqrt{\lambda_i\lambda_j\lambda_n^2\lambda_r\lambda_s}} \bar{p}_i\bar{p}_j|p_n|^2 v_r v_s \\
& + \sum_{r\in\mathscr{I}_2}\sum_{n\in\mathscr{K}} \frac{15\varepsilon}{16\pi^4\sqrt{\lambda_i\lambda_j\lambda_n^2\lambda_r\lambda_s}} p_i p_j|p_n|^2 \bar{v}_r\bar{v}_s \\
& + \sum_{r\in\mathscr{I}_3} \frac{15\varepsilon}{16\pi^4\sqrt{\lambda_i\lambda_j\lambda_n\lambda_m\lambda_r\lambda_s}} p_i\bar{p}_j p_n\bar{p}_m v_r\bar{v}_s \\
& + \sum_{r\in\mathscr{I}_4} \frac{15\varepsilon}{16\pi^4\sqrt{\lambda_i\lambda_j\lambda_n\lambda_m\lambda_r\lambda_s}} \bar{p}_i p_j\bar{p}_n\bar{p}_m v_r v_s \\
& + \sum_{r\in\mathscr{I}_4} \frac{15\varepsilon}{16\pi^4\sqrt{\lambda_i\lambda_j\lambda_n\lambda_m\lambda_r\lambda_s}} p_i\bar{p}_j p_n p_m \bar{v}_r\bar{v}_s \\
& + O\left(\varepsilon|p|^3\|v\|_{a,\varpi}^3 + \varepsilon|p|^2\|v\|_{a,\varpi}^4 + \varepsilon|p|\|v\|_{a,\varpi}^5 + \varepsilon\|v\|_{a,\varpi}^6\right. \\
& \left. + \varepsilon^2|p|^{10} + \varepsilon^2|p|^9\|v\|_{a,\varpi} + \varepsilon^2|p|^8\|v\|_{a,\varpi}^2 + \varepsilon^2|p|^7\|v\|_{a,\varpi}^3\right),
\end{aligned} \tag{36}$$

where $\sum_{l\in\mathscr{K}} \mathrm{i} dp_l\wedge d\bar{p}_l + \sum_{r\in\mathbb{Z}^{2,\diamond}} \mathrm{i} dv_r\wedge d\bar{v}_r$ is the corresponding symplectic structure of Poisson bracket $\{\cdot,\cdot\}$, (r, s) is a resonant pair, and (i, j) and (i, j, n, m) are uniquely determined by (r, s).

Introduce the action-angle variable

$$p_l = \sqrt{I_l + \zeta_l}e^{\mathrm{i}\theta_l}, \bar{p}_l = \sqrt{I_l + \zeta_l}e^{-\mathrm{i}\theta_l}, l\in\mathscr{K}. \tag{37}$$

Equation (37) converts the Hamiltonian $\tilde{H}$ into

$$\begin{aligned}
\tilde{H} = {} & \sum_{i\in\mathscr{K}} \left\{ \lambda_i + \frac{5\varepsilon\zeta_i^2}{64\pi^4\lambda_i^3} + \sum_{\substack{j\in\mathscr{K}\\ j\neq i}} \left[\frac{15\varepsilon\zeta_j^2}{64\pi^4\lambda_i\lambda_j^2} + \frac{15\varepsilon\zeta_i\zeta_j}{32\pi^4\lambda_i^2\lambda_j} + \sum_{\substack{n\in\mathscr{K}\\ n\neq j\\ n\neq i}} \frac{15\varepsilon\zeta_j\zeta_n}{32\pi^4\lambda_i\lambda_j\lambda_n} \right] \right\} I_i \\
& + \sum_{r\in\mathbb{Z}^{2,\diamond}} \left\{ \lambda_r + \sum_{i\in\mathscr{K}} \left[\frac{15\varepsilon}{64\pi^4\lambda_i^2\lambda_r}\zeta_i^2 + \sum_{j\in\mathscr{K}, j\neq i} \frac{15\varepsilon}{32\pi^4\lambda_i\lambda_j\lambda_r}\zeta_i\zeta_j \right] \right\} |v_r|^2 \\
& + \sum_{r\in\mathscr{I}_1}\sum_{n\in\mathscr{K}} \frac{15\varepsilon}{16\pi^4\sqrt{\lambda_i\lambda_j\lambda_n^2\lambda_r\lambda_s}} \sqrt{\zeta_i\zeta_j}\zeta_n e^{\mathrm{i}\left(\theta_i-\theta_j\right)} v_r\bar{v}_s \\
& + \sum_{r\in\mathscr{I}_2}\sum_{n\in\mathscr{K}} \frac{15\varepsilon}{16\pi^4\sqrt{\lambda_i\lambda_j\lambda_n^2\lambda_r\lambda_s}} \sqrt{\zeta_i\zeta_j}\zeta_n e^{\mathrm{i}\left(-\theta_i-\theta_j\right)} v_r v_s \\
& + \sum_{r\in\mathscr{I}_2}\sum_{n\in\mathscr{K}} \frac{15\varepsilon}{16\pi^4\sqrt{\lambda_i\lambda_j\lambda_n^2\lambda_r\lambda_s}} \sqrt{\zeta_i\zeta_j}\zeta_n e^{\mathrm{i}\left(\theta_i+\theta_j\right)} \bar{v}_r\bar{v}_s \\
& + \sum_{r\in\mathscr{I}_3} \frac{15\varepsilon}{16\pi^4\sqrt{\lambda_i\lambda_j\lambda_n\lambda_m\lambda_r\lambda_s}} \sqrt{\zeta_i\zeta_j\zeta_n\zeta_m} e^{\mathrm{i}\left(\theta_i-\theta_j+\theta_d-\theta_l\right)} v_r\bar{v}_s \\
& + \sum_{r\in\mathscr{I}_4} \frac{15\varepsilon}{16\pi^4\sqrt{\lambda_i\lambda_j\lambda_n\lambda_m\lambda_r\lambda_s}} \sqrt{\zeta_i\zeta_j\zeta_n\zeta_m} e^{\mathrm{i}\left(-\theta_i+\theta_j-\theta_n-\theta_m\right)} v_r v_s \\
& + \sum_{r\in\mathscr{I}_4} \frac{15\varepsilon}{16\pi^4\sqrt{\lambda_i\lambda_j\lambda_n\lambda_m\lambda_r\lambda_s}} \sqrt{\zeta_i\zeta_j\zeta_n\zeta_m} e^{\mathrm{i}\left(\theta_i-\theta_j+\theta_n+\theta_m\right)} \bar{v}_r\bar{v}_s \\
& + O\left(\varepsilon|I|^2 + \varepsilon|I|\|v\|_{a,\varpi}^2 + \varepsilon|\zeta|^{3/2}|I|^{1/2}\|v\|_{a,\varpi}^2 + \varepsilon|\zeta|^{3/2}\|v\|_{a,\varpi}^3\right. \\
& \left. + \varepsilon|\zeta|\|v\|_{a,\varpi}^4 + \varepsilon^2|\zeta|^{9/2}\|v\|_{a,\varpi} + \varepsilon^2|\zeta|^4\|v\|_{a,\varpi}^2 + \varepsilon^2|\zeta|^{7/2}\|v\|_{a,\varpi}^3\right).
\end{aligned} \tag{38}$$

Scaling through time

$$\begin{aligned}
\zeta & \longrightarrow \varepsilon^{3/2}\zeta, \\
I & \longrightarrow \varepsilon^5 I, \\
\theta & \longrightarrow \theta, \\
v & \longrightarrow \varepsilon^{5/2} v, \\
\bar{v} & \longrightarrow \varepsilon^{5/2}\bar{v},
\end{aligned} \tag{39}$$

the scaled Hamiltonian is $H = \varepsilon^{-9}\tilde{H}(\varepsilon^{3/2}\zeta, \varepsilon^5 I, \theta, \varepsilon^{5/2} v, \varepsilon^{5/2}\bar{v})$. The Hamiltonian H satisfies (23) and (24) where $\zeta\in[\varepsilon^{3/2}, 2\varepsilon^{3/2}]^d$. □

3. An Infinite-Dimensional KAM Theorem for PDEs

We will use the KAM theorem in [23] to prove the main result (Theorem 3). For easy understanding, the KAM theorem in [23] is introduced below. Denote $H^\diamond = \Lambda^\diamond + \mathscr{A}_1^\diamond + \mathscr{B}_1^\diamond + \bar{\mathscr{B}}_1^\diamond + \mathscr{A}_2^\diamond + \mathscr{B}_2^\diamond + \bar{\mathscr{B}}_2^\diamond$, where

$$\begin{aligned}
\Lambda^{\diamond} &= \sum_{j\in\mathcal{K}} \eta_j(\zeta) I_j + \sum_{r\in\mathbb{Z}^{2,\diamond}} \Omega_r(\zeta) v_r \bar{v}_r,\\
\mathcal{A}_1^{\diamond} &= \sum_{r\in\mathcal{K}_1} a_r(\zeta) e^{\mathrm{i}\left(\theta_i-\theta_j\right)} v_r \bar{v}_s,\\
\mathcal{B}_1^{\diamond} &= \sum_{r\in\mathcal{K}_2} a_r(\zeta) e^{-\mathrm{i}\left(\theta_i+\theta_j\right)} v_r v_s,\\
\bar{\mathcal{B}}_1^{\diamond} &= \sum_{r\in\mathcal{K}_2} \bar{a}_r(\zeta) e^{\mathrm{i}\left(\theta_i+\theta_j\right)} \bar{v}_r \bar{v}_s,\\
\mathcal{A}_2^{\diamond} &= \sum_{r\in\mathcal{K}_3} a_r(\zeta) e^{\mathrm{i}\left(\theta_i-\theta_j+\theta_d-\theta_l\right)} v_r \bar{v}_s,\\
\mathcal{B}_2^{\diamond} &= \sum_{r\in\mathcal{K}_4} a_r(\zeta) e^{\mathrm{i}\left(-\theta_i+\theta_j-\theta_d-\theta_l\right)} v_r v_s,\\
\bar{\mathcal{B}}_2^{\diamond} &= \sum_{r\in\mathcal{K}_4} \bar{a}_r(\zeta) e^{\mathrm{i}\left(\theta_i-\theta_j+\theta_d+\theta_l\right)} \bar{v}_r \bar{v}_s.
\end{aligned} \tag{40}$$

(θ, I) are d-dimensional angle-action coordinates, $(v, \bar{v})$ are infinite-dimensional coordinates, and the corresponding symplectic structure is $\sum_{i\in\mathcal{K}} d\theta_i \wedge dI_i + \mathrm{i}\sum_{r\in\mathbb{Z}^{2,\diamond}} dv_r \wedge d\bar{v}_r$. Frequencies $\eta = (\eta_i)_{i\in\mathcal{K}}$ and $\Omega = (\Omega_r)_{r\in\mathbb{Z}^{2,\diamond}}$ depend on the parameter $\zeta \in \Xi \subset \mathbb{R}^d$, where Ξ is a closed bounded set with positive Lebesgue measure. In order to prove the existence of the invariant torus of small perturbation $H = H^{\diamond} + W^{\diamond}$ of $H^{\diamond}$, we need the following assumptions.

Assumption 6 (nondegeneracy). Suppose that $\forall \zeta \in \Xi$,

$$\begin{cases} \operatorname{rank}\left\{\dfrac{\partial \eta_{i_1^*}}{\partial \zeta}, \cdots, \dfrac{\partial \eta_{i_d^*}}{\partial \zeta}\right\} = \sigma, \\ \operatorname{rank}\left\{\dfrac{\partial^{\iota} \eta}{\partial \zeta^{\iota}} \middle| \forall \iota, 1 \le |\iota| \le \min\{d-\sigma+1, 5\}\right\} = d, \end{cases} \tag{41}$$

where σ is a given integer with $1 \le \sigma \le d$, $\partial \eta_{i_1^*}/\partial\zeta, \cdots, \partial \eta_{i_d^*}/\partial\zeta$ are vectors of all $1-$order partial derivatives in ζ, and for a fixed ι, $\partial^{\iota}\eta/\partial\zeta^{\iota} = (\partial^{\iota}\eta_{i_1^*}/\partial\zeta^{\iota}, \cdots, \partial^{\iota}\eta_{i_d^*}/\partial\zeta^{\iota})$. Moreover, $\eta(\zeta)$ belongs to $C_W^8(\Xi)$.

Assumption 7 (asymptotics of normal frequencies).

$$\Omega_r = \varepsilon^{-\varsigma}|r|^2 + \tilde{\Omega}_r, \quad \varsigma \ge 0, \tag{42}$$

where $\tilde{\Omega}_r$'s are C_W^8 functions of ζ with C_W^8-norm bounded by some small positive constant L.

Assumption 8 (Melnikov's nondegeneracy). Let $\mathcal{G}_r = \Omega_r$ for $r \in \mathbb{Z}^{2,\diamond} \setminus (\mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_3 \cup \mathcal{I}_4)$,

$$\begin{aligned}
\mathcal{G}_r &= \begin{pmatrix} \Omega_r + \eta_i & a_r \\ a_s & \Omega_s + \eta_j \end{pmatrix}, \quad r \in \mathcal{I}_1,\\
\mathcal{G}_r &= \begin{pmatrix} \Omega_r - \eta_i & -a_r \\ \bar{a}_s & -\Omega_s + \eta_j \end{pmatrix}, \quad r \in \mathcal{I}_2,\\
\mathcal{G}_r &= \begin{pmatrix} \Omega_r + \eta_i + \eta_n & a_r \\ a_s & \Omega_s + \eta_j + \eta_m \end{pmatrix}, \quad r \in \mathcal{I}_3,\\
\mathcal{G}_r &= \begin{pmatrix} \Omega_r + \eta_j & -a_r \\ \bar{a}_s & -\Omega_s + \eta_i + \eta_n + \eta_m \end{pmatrix}, \quad r \in \mathcal{I}_4,
\end{aligned} \tag{43}$$

where (r, s) are resonant pairs and (i, j) and (i, j, n, m) are uniquely determined by (r, s). There is $\gamma', \tau > 0$ (here, I is identity matrix) such that

$$\begin{aligned}
|<k, \eta>| &\ge \frac{\gamma'}{|k|^{\tau}}, \quad k \ne 0,\\
|\det(<k, \eta> I \pm \mathcal{G}_r)| &\ge \frac{\gamma'}{|k|^{\tau}},\\
|\det(<k, \eta> I \pm \mathcal{G}_r \otimes I \pm I \otimes \mathcal{G}_{r'})| &\ge \frac{\gamma'}{|k|^{\tau}}.
\end{aligned} \tag{44}$$

Assumption 9 (regularity). $\mathcal{A}_1^{\diamond} + \mathcal{B}_1^{\diamond} + \bar{\mathcal{B}}_1^{\diamond} + \mathcal{A}_2^{\diamond} + \mathcal{B}_2^{\diamond} + \bar{\mathcal{B}}_2^{\diamond} + W^{\diamond}$ is real analytic in $\theta, I, v, \bar{v}$ and C_W^8 in ζ and

$$\begin{aligned}
&\left\|X_{\mathcal{A}_1^{\diamond}}\right\|_{D_{a,\varpi}(r^*,s^*),\Xi} + \left\|X_{\mathcal{B}_1^{\diamond}}\right\|_{D_{a,\varpi}(r^*,s^*),\Xi} + \left\|X_{\mathcal{A}_2^{\diamond}}\right\|_{D_{a,\varpi}(r^*,s^*),\Xi} + \left\|X_{\mathcal{B}_2^{\diamond}}\right\|_{D_{a,\varpi}(r^*,s^*),\Xi} < 1,\\
&\left\|X_{W^{\diamond}}\right\|_{D_{a,\varpi}(r^*,s^*),\Xi} < \varepsilon.
\end{aligned} \tag{45}$$

Assumption 10 (special form). $\mathcal{A}_1^{\diamond} + \mathcal{B}_1^{\diamond} + \bar{\mathcal{B}}_1^{\diamond} + \mathcal{A}_2^{\diamond} + \mathcal{B}_2^{\diamond} + \bar{\mathcal{B}}_2^{\diamond} + W^{\diamond}$ has the following special form:

$$\mathcal{Y}^{\diamond} = \left\{ \mathcal{A}_1^{\diamond} + \mathcal{B}_1^{\diamond} + \bar{\mathcal{B}}_1^{\diamond} + \mathcal{A}_2^{\diamond} + \mathcal{B}_2^{\diamond} + \bar{\mathcal{B}}_2^{\diamond} + W^{\diamond} : \mathcal{A}_1^{\diamond} + \mathcal{B}_1^{\diamond} + \bar{\mathcal{B}}_1^{\diamond} + \mathcal{A}_2^{\diamond} + \mathcal{B}_2^{\diamond} + \bar{\mathcal{B}}_2^{\diamond} + W^{\diamond} = \sum_{k\in\mathbb{Z}^d, l\in\mathbb{N}^d, \alpha, \beta} \left(\mathcal{A}_1^{\diamond} + \mathcal{B}_1^{\diamond} + \bar{\mathcal{B}}_1^{\diamond} + \mathcal{A}_2^{\diamond} + \mathcal{B}_2^{\diamond} + \bar{\mathcal{B}}_2^{\diamond} + W^{\diamond}\right)_{kl\alpha\beta}(\zeta) I^l e^{\mathrm{i}<k,\theta>} v^{\alpha} \bar{v}^{\beta} \right\}, \tag{46}$$

where k, α, β satisfy $\sum_{\hat{s}=1}^{d} k_{\hat{s}} i_{\hat{s}}^{*} + \sum_{r\in\mathbb{Z}^{2,\diamond}} (\alpha_r - \beta_r) r = 0$.

Assumption 11 (Töplitz-Lipschitz property). For given $r, s \in \mathbb{Z}_{\dagger}^{2}, \tilde{c} \in \mathbb{Z}_{\dagger}^{2}$, the limits

$$\lim_{\tilde{t}\longrightarrow\infty} \frac{\partial^2 (\mathscr{B}_1^{\diamond} + \mathscr{B}_2^{\diamond} + W_2^{\diamond})}{\partial v_{r+\tilde{c}\tilde{t}} \partial v_{s-\tilde{c}\tilde{t}}}, \quad \lim_{\tilde{t}\longrightarrow\infty} \frac{\partial^2 (\sum_{l\in\mathbb{Z}^{2,\diamond}} \Omega_l v_l \bar{v}_l + \mathscr{A}_1^{\diamond} + \mathscr{A}_2^{\diamond} + W^{\diamond})}{\partial v_{r+\tilde{c}\tilde{t}} \partial \bar{v}_{s+\tilde{c}\tilde{t}}}, \quad \lim_{\tilde{t}\longrightarrow\infty} \frac{\partial^2 \left(\bar{\mathscr{B}}_1^{\diamond} + \bar{\mathscr{B}}_2^{\diamond} + W^{\diamond}\right)}{\partial \bar{v}_{r+\tilde{c}\tilde{t}} \partial \bar{v}_{s-\tilde{c}\tilde{t}}} \tag{47}$$

exist. Moreover, there is $K^* > 0$, so that if $\tilde{t} > K^*$, then $\Lambda^{\diamond} + \mathscr{A}_1^{\diamond} + \mathscr{B}_1^{\diamond} + \bar{\mathscr{B}}_1^{\diamond} + \mathscr{A}_2^{\diamond} + \mathscr{B}_2^{\diamond} + \bar{\mathscr{B}}_2^{\diamond} + W^{\diamond}$ satisfies

$$\left\| \frac{\partial^2 (\mathscr{B}_1^{\diamond} + \mathscr{B}_2^{\diamond} + W^{\diamond})}{\partial v_{r+\tilde{c}\tilde{t}} \partial v_{s-\tilde{c}\tilde{t}}} - \lim_{\tilde{t}\longrightarrow\infty} \frac{\partial^2 (\mathscr{B}_1^{\diamond} + \mathscr{B}_2^{\diamond} + W^{\diamond})}{\partial v_{r+\tilde{c}\tilde{t}} \partial v_{s-\tilde{c}\tilde{t}}} \right\|_{D_{a,\varpi}(r^*,s^*),\Xi} \le \frac{\varepsilon}{\tilde{t}} e^{-|r+s|a},$$

$$\left\| \frac{\partial^2 (\sum_{l\in\mathbb{Z}^{2,\diamond}} \Omega_l v_l \bar{v}_l + \mathscr{A}_1^{\diamond} + \mathscr{A}_2^{\diamond} + W^{\diamond})}{\partial v_{r+\tilde{c}\tilde{t}} \partial \bar{v}_{s+\tilde{c}\tilde{t}}} - \lim_{\tilde{t}\longrightarrow\infty} \frac{\partial^2 (\sum_{l\in\mathbb{Z}^{2,\diamond}} \Omega_l v_l \bar{v}_l + \mathscr{A}_1^{\diamond} + \mathscr{A}_2^{\diamond} + W^{\diamond})}{\partial v_{r+\tilde{c}\tilde{t}} \partial \bar{v}_{s+\tilde{c}\tilde{t}}} \right\|_{D_{a,\varpi}(r^*,s^*),\Xi} \le \frac{\varepsilon}{\tilde{t}} e^{-|r-s|a},$$

$$\left\| \frac{\partial^2 \left(\bar{\mathscr{B}}_1^{\diamond} + \bar{\mathscr{B}}_2^{\diamond} + W^{\diamond}\right)}{\partial \bar{v}_{r+\tilde{c}\tilde{t}} \partial \bar{v}_{s-\tilde{c}\tilde{t}}} - \lim_{\tilde{t}\longrightarrow\infty} \frac{\partial^2 \left(\bar{\mathscr{B}}_1^{\diamond} + \bar{\mathscr{B}}_2^{\diamond} + W^{\diamond}\right)}{\partial \bar{v}_{r+\tilde{c}\tilde{t}} \partial \bar{v}_{s-\tilde{c}\tilde{t}}} \right\|_{D_{a,\varpi}(r^*,s^*),\Xi} \le \frac{\varepsilon}{\tilde{t}} e^{-|r+s|a}. \tag{48}$$

The following KAM theorem comes from Zhang and Si [23].

Theorem 12 ([23] Theorem 2.1). *If $H^* = H^{\diamond} + W^{\diamond} = \Lambda^{\diamond} + \mathscr{A}_1^{\diamond} + \mathscr{B}_1^{\diamond} + \bar{\mathscr{B}}_1^{\diamond} + \mathscr{A}_2^{\diamond} + \mathscr{B}_2^{\diamond} + \bar{\mathscr{B}}_2^{\diamond} + W^{\diamond}$ satisfies the Assumptions (6)–(11), and $\gamma' > 0$ is small enough. Then, there exists a positive constant $\varepsilon = \varepsilon(d, L, K^*, \tau, \gamma', r^*, s^*, a, \varpi)$, so that if $\|X_{W^{\diamond}}\|_{D_{a,\varpi}(r^*,s^*),\Xi} < \varepsilon$, the following results true: there exists a Cantor subset $\Xi_{\gamma'} \subset \Xi$ with $\mathrm{meas}(\Xi \setminus \Xi_{\gamma'}) = O(\sqrt[8]{\gamma'})$ and two transformations (analytic in θ and C_W^8 in ζ) $\Psi : \mathbb{T}^d \times \Xi_{\gamma'} \longrightarrow D_{a,\varpi}(r^*, s^*), \eta : \Xi_{\gamma'} \longrightarrow \mathbb{R}^d$, where Ψ is $\varepsilon/(\gamma')^8$-close to the trivial embedding $\Psi_0 : \mathbb{T}^d \times \Xi \longrightarrow \mathbb{T}^d \times \{0, 0, 0\}$ and $\tilde{\eta}$ is ε-close to the unperturbed frequency η, so that for $\zeta \in \Xi_{\gamma'}$ and $\theta \in \mathbb{T}^d$, the curve $t \longrightarrow \Psi(\theta + \tilde{\eta}(\zeta)t, \zeta)$ is a quasiperiodic solution of the Hamiltonian equations corresponding to H^*.*

4. Proof of the Main Theorem

Let us show that the Hamiltonian (23) satisfies the Assumptions (6)–(11).

Verifying Assumption (6): from (24), for $\hat{n} = 1, \cdots, d$,

$$\frac{\partial^2 \eta_{i_{\hat{n}}^*}(\zeta)}{\partial \zeta_{i_{\hat{n}}^*}^2} = \frac{5}{32\pi^4 \lambda_{i_{\hat{n}}^*}^3}, \tag{49}$$

$$\frac{\partial^2 \eta_{i_{\hat{n}}^*}(\zeta)}{\partial \zeta_{i_{\hat{j}}^*} \partial \zeta_{i_{\hat{n}}^*}} = \frac{\partial^2 \eta_{i_{\hat{n}}^*}(\zeta)}{\partial \zeta_{i_{\hat{n}}^*} \partial \zeta_{i_{\hat{j}}^*}} = \frac{15}{32\pi^4 \lambda_{i_{\hat{n}}^*}^2 \lambda_{i_{\hat{j}}^*}}, \quad 1 \le \hat{j} \le d, \hat{j} \ne \hat{n}. \tag{50}$$

Let

$$\mathscr{W} = \begin{pmatrix} \dfrac{\partial^2 \eta_{i_1^*}(\zeta)}{\partial \zeta_{i_1^*}^2} & \cdots & \dfrac{\partial^2 \eta_{i_d^*}(\zeta)}{\partial \zeta_{i_d^*} \partial \zeta_{i_1^*}} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial^2 \eta_{i_1^*}(\zeta)}{\partial \zeta_{i_1^*} \partial \zeta_{i_d^*}} & \cdots & \dfrac{\partial^2 \eta_{i_d^*}(\zeta)}{\partial \zeta_{i_d^*}^2} \end{pmatrix}, \tag{51}$$

where $\zeta \in \Xi$; then, $\mathscr{W}$ is the submatrix of matrix $\{\partial^2 \eta / \partial \zeta^2\}$. According to (49) and (50), then

$$\lim_{\varepsilon\longrightarrow 0} \mathscr{W} = \frac{5}{32\pi^4} \begin{pmatrix} \dfrac{1}{\lambda_{i_1^*}^3} & \dfrac{3}{\lambda_{i_2^*}^2 \lambda_{i_1^*}} & \cdots & \dfrac{3}{\lambda_{i_d^*}^2 \lambda_{i_1^*}} \\ \dfrac{3}{\lambda_{i_1^*}^2 \lambda_{i_2^*}} & \dfrac{1}{\lambda_{i_2^*}^3} & \cdots & \dfrac{3}{\lambda_{i_d^*}^2 \lambda_{i_2^*}} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{3}{\lambda_{i_1^*}^2 \lambda_{i_d^*}} & \dfrac{3}{\lambda_{i_2^*}^2 \lambda_{i_d^*}} & \cdots & \dfrac{1}{\lambda_{i_d^*}^3} \end{pmatrix}_{d\times d} := \frac{5}{32\pi^4} \widehat{\mathscr{W}}. \tag{52}$$

From $\det(\widehat{\mathscr{W}}) = ([1 + 3(d-1)](-2)^{d-1}) / \prod_{\hat{j}=1}^{d} \lambda_{i_{\hat{j}}^*}^3 \ne 0$, we have $\det(\mathscr{W}) \ne 0$ when $0 < \varepsilon \ll 1$, that is, $\mathrm{rank}(\mathscr{W}) = d$. Hence, Assumptions (6) is verified.

Verifying Assumption (7): take $\varsigma = 4$; the proof is obvious.

Verifying Assumption (8): from (23), $\mathscr{G}_r$ is represented as follows:

$$\mathscr{G}_r = \Omega_r, r \in \mathbb{Z}^{2,\diamond} \setminus (\mathscr{I}_1 \cup \mathscr{I}_2 \cup \mathscr{I}_3 \cup \mathscr{I}_4),$$

$$\mathscr{G}_r = \begin{pmatrix} \Omega_r + \eta_i & \sum_{n^*\in\mathscr{K}} \dfrac{15\sqrt{\zeta_i \zeta_j} \zeta_{n^*}}{16\pi^4 \sqrt{\lambda_i \lambda_j \lambda_{n^*}^2 \lambda_r \lambda_s}} \\ \sum_{n^*\in\mathscr{K}} \dfrac{15\sqrt{\zeta_i \zeta_j} \zeta_{n^*}}{16\pi^4 \sqrt{\lambda_i \lambda_j \lambda_{n^*}^2 \lambda_r \lambda_s}} & \Omega_s + \eta_j \end{pmatrix}, \quad r \in \mathscr{I}_1,$$

$$\mathscr{G}_r = \begin{pmatrix} \Omega_r - \eta_i & -\sum_{n^*\in\mathscr{K}} \dfrac{15\sqrt{\zeta_i \zeta_j} \zeta_{n^*}}{16\pi^4 \sqrt{\lambda_i \lambda_j \lambda_{n^*}^2 \lambda_r \lambda_s}} \\ \sum_{n^*\in\mathscr{K}} \dfrac{15\sqrt{\zeta_i \zeta_j} \zeta_{n^*}}{16\pi^4 \sqrt{\lambda_i \lambda_j \lambda_{n^*}^2 \lambda_r \lambda_s}} & -\Omega_s + \eta_j \end{pmatrix}, \quad r \in \mathscr{I}_2,$$

$$\mathscr{G}_r = \begin{pmatrix} \Omega_r + \eta_i + \eta_n & \dfrac{15\sqrt{\zeta_i\zeta_j\zeta_n\zeta_m}}{16\pi^4\sqrt{\lambda_i\lambda_n\lambda_m\lambda_r\lambda_s}} \\ \dfrac{15\sqrt{\zeta_i\zeta_j\zeta_n\zeta_m}}{16\pi^4\sqrt{\lambda_i\lambda_j\lambda_n\lambda_m\lambda_r\lambda_s}} & \Omega_s + \eta_j + \eta_m \end{pmatrix}, \quad r \in \mathscr{I}_3,$$

$$\mathscr{G}_r = \begin{pmatrix} \Omega_r + \eta_j & -\dfrac{15\sqrt{\zeta_i\zeta_j\zeta_n\zeta_m}}{16\pi^4\sqrt{\lambda_i\lambda_j\lambda_n\lambda_m\lambda_r\lambda_s}} \\ \dfrac{15\sqrt{\zeta_i\zeta_j\zeta_n\zeta_m}}{16\pi^4\sqrt{\lambda_i\lambda_j\lambda_n\lambda_m\lambda_r\lambda_s}} & -\Omega_s + \eta_i + \eta_n + \eta_m \end{pmatrix}, \quad r \in \mathscr{I}_4, \tag{53}$$

where (i, j) and (i, j, n, m) are uniquely determined by (r, s). We only prove (A3) for det $[<k, \eta(\eta) > I \pm \mathscr{G}_r \otimes I \pm I \otimes \mathscr{G}_{r'}]$ which is the most complicated. Let

$$\mathscr{F}(\zeta) = <k, \eta(\zeta) > I \pm \mathscr{G}_r \otimes I \pm I \otimes \mathscr{G}_{r'}. \tag{54}$$

We will prove that $|\mathscr{F}(\zeta)| \geq (\gamma'/|k|^\tau)$, $(k \neq 0)$. Let us only prove the case $r, r' \in \mathscr{I}_4$, and everything else is similar. Let

$$\mathscr{G}_r = \varepsilon^{-4}\mathscr{G}_{r,1} + \mathscr{G}_{r,2}, \quad \forall r \in \mathscr{I}_4, \tag{55}$$

where

$$\mathscr{G}_{r,1} = \begin{pmatrix} \lambda_r + \lambda_j & 0 \\ 0 & -\lambda_s + \lambda_i + \lambda_n + \lambda_m \end{pmatrix}, \quad \mathscr{G}_{r,2} = \begin{pmatrix} \mathscr{G}_{r,2}^{11} & \mathscr{G}_{r,2}^{12} \\ \mathscr{G}_{r,2}^{21} & \mathscr{G}_{r,2}^{22} \end{pmatrix}, \tag{56}$$

with

$$\mathscr{G}_{r,2}^{11} = \frac{5}{16\pi^4\lambda_j^3}\zeta_j^2 - \sum_{j^*\in\mathscr{K}} \frac{15}{32\pi^4\lambda_j^2\lambda_{j^*}}\zeta_j\zeta_{j^*} + \frac{15}{64\pi^4}\left(\frac{1}{\lambda_r} + \frac{1}{\lambda_j}\right) < \zeta, A\zeta >,$$

$$\mathscr{G}_{r,2}^{12} = -\frac{15\sqrt{\zeta_i\zeta_j\zeta_n\zeta_m}}{16\pi^4\sqrt{\lambda_i\lambda_j\lambda_n\lambda_m\lambda_r\lambda_s}}, \mathscr{G}_{r,2}^{21} = \frac{15\sqrt{\zeta_i\zeta_j\zeta_n\zeta_m}}{16\pi^4\sqrt{\lambda_i\lambda_j\lambda_n\lambda_m\lambda_r\lambda_s}},$$

$$\begin{aligned} \mathscr{G}_{r,2}^{22} = {} & \frac{5}{16\pi^4}\left(\frac{\zeta_i^2}{\lambda_i^3} + \frac{\zeta_n^2}{\lambda_n^3} + \frac{\zeta_m^2}{\lambda_m^3}\right) - \sum_{j^*\in\mathscr{K}} \frac{15\zeta_{j^*}}{32\pi^4\lambda_{j^*}}\left(\frac{\zeta_i}{\lambda_i^2} + \frac{\zeta_n}{\lambda_n^2} + \frac{\zeta_m}{\lambda_m^2}\right) \\ & + \frac{15}{64\pi^4}\left(\frac{1}{\lambda_i} + \frac{1}{\lambda_n} + \frac{1}{\lambda_m} - \frac{1}{\lambda_s}\right) < \zeta, A\zeta >, \end{aligned} \tag{57}$$

with $\zeta = (\zeta_{i_1^*}, \zeta_{i_2^*}, \cdots, \zeta_{i_d^*})$ and

$$A = \begin{pmatrix} \dfrac{1}{\lambda_{i_1^*}^2} & \dfrac{2}{\lambda_{i_2^*}\lambda_{i_1^*}} & \cdots & \dfrac{2}{\lambda_{i_d^*}\lambda_{i_1^*}} \\ \dfrac{2}{\lambda_{i_1^*}\lambda_{i_2^*}} & \dfrac{1}{\lambda_{i_2^*}^2} & \cdots & \dfrac{2}{\lambda_{i_d^*}\lambda_{i_2^*}} \\ \cdots & \cdots & \cdots\cdots & \\ \dfrac{2}{\lambda_{i_1^*}\lambda_{i_d^*}} & \dfrac{2}{\lambda_{i_2^*}\lambda_{i_d^*}} & \cdots & \dfrac{1}{\lambda_{i_d^*}^2} \end{pmatrix}_{d\times d}. \tag{58}$$

Thus,

$$\begin{aligned} \mathscr{F}(\zeta) = {} & \varepsilon^{-4}\left(<k, \beta^*>+<k, B>\pm\left(|r|^2 + |j|^2\right) \pm \left(\left|r'\right|^2 + \left|j'\right|^2\right)\right)I \\ & \pm \mathscr{G}_{r,2} \otimes I \pm I \otimes \mathscr{G}_{r',2}, \end{aligned} \tag{59}$$

where $\beta^* = (|i_1^*|^2, |i_2^*|^2, \cdots, |i_d^*|^2)$ and $B = (\cdots, B_{i^*}, \cdots)_{i^*\in\mathscr{K}}^T$ with

$$B_{i^*} = \frac{5\zeta_{i^*}^2}{64\pi^4\lambda_{i^*}^3} + \sum_{\substack{j^*\in\mathscr{K} \\ j^*\neq i^*}} \cdot \left[\frac{15\zeta_{j^*}^2}{64\pi^4\lambda_{i^*}\lambda_{j^*}^2} + \frac{15\zeta_{i^*}\zeta_{j^*}}{32\pi^4\lambda_{i^*}^2\lambda_{j^*}} + \sum_{\substack{n^*\in\mathscr{K} \\ n^*\neq j^* \\ n^*\neq i^*}} \frac{15\zeta_{j^*}\zeta_{n^*}}{32\pi^4\lambda_{i^*}\lambda_{j^*}\lambda_{n^*}}\right]. \tag{60}$$

The eigenvalues of $\mathscr{F}(\zeta)$ are

$$\begin{aligned} & \varepsilon^{-4}\left(<k, \beta^*>\pm\left(|r|^2 + |j|^2\right) \pm \left(\left|r'\right|^2 + \left|j'\right|^2\right)+<k, B >\right) \\ & \pm \frac{5}{128\pi^4}\left\{4\left(\frac{\zeta_i^2}{\lambda_i^3} + \frac{\zeta_j^2}{\lambda_j^3} + \frac{\zeta_n^2}{\lambda_n^3} + \frac{\zeta_m^2}{\lambda_m^3}\right)\right. \\ & - 6\sum_{j^*\in\mathscr{K}} \frac{\zeta_{j^*}}{\lambda_{j^*}}\left(\frac{\zeta_i}{\lambda_i^2} + \frac{\zeta_j}{\lambda_j^2} + \frac{\zeta_n}{\lambda_n^2} + \frac{\zeta_m}{\lambda_m^2}\right) \\ & + 3\left(\frac{1}{\lambda_i} + \frac{1}{\lambda_j} + \frac{1}{\lambda_n} + \frac{1}{\lambda_m} + \frac{1}{\lambda_r} - \frac{1}{\lambda_s}\right) \\ & \left. < \zeta, A\zeta>\pm\sqrt{h_{\mathscr{I}_4,\mathscr{I}_4}(\zeta_i, \zeta_j, \zeta_n, \zeta_m)}\right\} \\ & \pm \frac{5}{128\pi^4}\left\{4\left(\frac{\zeta_{i'}^2}{\lambda_{i'}^3} + \frac{\zeta_{j'}^2}{\lambda_{j'}^3} + \frac{\zeta_{n'}^2}{\lambda_{n'}^3} + \frac{\zeta_{m'}^2}{\lambda_{m'}^3}\right)\right. \\ & - 6\sum_{j^*\in\mathscr{K}} \frac{\zeta_{j^*}}{\lambda_{j^*}}\left(\frac{\zeta_{i'}}{\lambda_{i'}^2} + \frac{\zeta_{j'}}{\lambda_{j'}^2} + \frac{\zeta_{n'}}{\lambda_{n'}^2} + \frac{\zeta_{m'}}{\lambda_{m'}^2}\right) \\ & + 3\left(\frac{1}{\lambda_{i'}} + \frac{1}{\lambda_{j'}} + \frac{1}{\lambda_{n'}} + \frac{1}{\lambda_{m'}} + \frac{1}{\lambda_{r'}} - \frac{1}{\lambda_{s'}}\right) \\ & \left. < \zeta, A\zeta>\pm\sqrt{h_{\mathscr{I}_4,\mathscr{I}_4}\left(\zeta_{i'}, \zeta_{j'}, \zeta_{n'}, \zeta_{m'}\right)}\right\}, \end{aligned} \tag{61}$$

where

$$
\begin{aligned}
&h_{\mathscr{I}_4,\mathscr{I}_4}(\zeta_i,\zeta_j,\zeta_n,\zeta_m)\\
&\quad=\left[4\left(\frac{\zeta_i^2}{\lambda_i^3}-\frac{\zeta_j^2}{\lambda_j^3}+\frac{\zeta_n^2}{\lambda_n^3}+\frac{\zeta_m^2}{\lambda_m^3}\right)-6\sum_{j^*\in\mathscr{K}}\frac{\zeta_{j^*}}{\lambda_{j^*}}\left(\frac{\zeta_i}{\lambda_i^2}-\frac{\zeta_j}{\lambda_j^2}+\frac{\zeta_n}{\lambda_n^2}+\frac{\zeta_m}{\lambda_m^2}\right)\right.\\
&\qquad\left.+3\left(\frac{1}{\lambda_i}-\frac{1}{\lambda_j}+\frac{1}{\lambda_n}+\frac{1}{\lambda_m}-\frac{1}{\lambda_r}-\frac{1}{\lambda_s}\right)\langle\zeta,A\zeta\rangle\right]^2-576\frac{\zeta_i\zeta_j\zeta_n\zeta_m}{\lambda_i\lambda_j\lambda_n\lambda_m\lambda_r\lambda_s}.
\end{aligned}\tag{62}
$$

It has been proved that none of the eigenvalues of $\mathscr{F}(\zeta)$ are zero in [23]. Moreover, when $r\in\mathscr{I}_1, r'\in\mathscr{I}_1$ or $r\in\mathscr{I}_1, r'\in\mathscr{I}_2$ or $r\in\mathscr{I}_1, r'\in\mathscr{I}_3$ and so on, the situations are similar, so omit the proofs. That is, none of the eigenvalues of $\mathscr{F}(\zeta)$ are zero for $k\neq 0$. From Lemma 3.1 in [19], then det $(\mathscr{F}(\zeta))$ is a polynomial function in the components of ζ with order at most eight and $|\partial_\zeta^8(det(\mathscr{W}(\zeta)))|\geq(1/2)|k|\neq 0$. By excluding some parameter set with measure $O(\sqrt[8]{\gamma'})$, then $|\det(\mathscr{W}(\zeta))|\geq(\gamma'/|k|^\tau), k\neq 0$.(A3) is verified.

Verifying Assumption (9): similar to [19], from Lemma 4, it is obvious that (A4) holds, so omit its proof.

Verifying Assumption (10): similar to [19], the result is obvious.

Verifying Assumption (11): similar to [23], the result is obvious.

By Theorem 12 ([23] Theorem 2.1), we have Theorem 3.

Appendix

A. Proof of Non Empty Admissible Set

Lemma 13. *Set $\mathscr{K}$ as the given set with d points in Remark 2. For $n, m, r, s, \tilde n, \tilde m, \tilde r, \tilde s\in\mathscr{K}$, the following conclusion is true:*

(i) *$n-m=0$, $n=m$, and $|n|=|m|$ are equivalent*

(ii) *$n-m+r-s=0$, $\{n,r\}=\{m,s\}$, and $\{|n|,|r|\}=\{|m|,|s|\}$ are equivalent*

(iii) *$n-m+r-s+\tilde n-\tilde m=0$, $\{n,r,\tilde n\}=\{m,s,\tilde m\}$, and $\{|n|,|r|,|\tilde n|\}=\{|m|,|s|,|\tilde m|\}$ are equivalent*

(iv) *$n-m+r-s+\tilde n-\tilde m+\tilde r-\tilde s=0$, $\{n,r,\tilde n,\tilde r\}=\{m,s,\tilde m,\tilde s\}$, and $\{|n|,|r|,|\tilde n|,|\tilde r|\}=\{|m|,|s|,|\tilde m|,|\tilde s|\}$ are equivalent*

(v) *$\langle n,r\rangle=\langle m,s\rangle$, $\{n,r\}=\{m,s\}$, and $\{|n|,|r|\}=\{|m|,|s|\}$ are equivalent*

(vi) *$\langle n-m,r-s\rangle=0$ is equivalent to $n=m$ or $r=s$*

The above results are obvious, and we omit their proof.

(I) Let us prove the property (2) by reduction to absurdity. The proof for the property (1) is similar and simpler. Let us say $i,j,n,m,r\in\mathscr{K}$ satisfies $\langle r-m,i-j+n-m\rangle=\langle i-j,j-n\rangle$

Lemma 14. *The set $\mathscr{K}$ given in Remark 2 is admissible.*

Proof. According to $i-j+r-s=0$ and $|i|^2-|j|^2+|r|^2-|s|^2=0$, $\langle r-j,i-j\rangle=0$. By $i-j+n-m+r-s=0$ and $|i|^2-|j|^2+|n|^2-|m|^2+|r|^2-|s|^2=0$, then $\langle r-m,i-j+n-m\rangle=\langle i-j,j-n\rangle$. In order to prove that $\mathscr{K}$ has the properties (3)–(12) in Definition 1, we need to prove that

$$\begin{cases}\langle r-j,i-j\rangle=0,\\ \langle\tilde r-\tilde j,\tilde i-\tilde j\rangle=0,\end{cases}\tag{A.1}$$

$$\begin{cases}\langle r-i,r-j\rangle=0,\\ \langle\tilde r-\tilde i,\tilde r-\tilde j\rangle=0,\end{cases}\tag{A.2}$$

$$\begin{cases}\langle r-m,i-j+n-m\rangle=\langle i-j,j-n\rangle,\\ \langle r-\tilde m,\tilde i-\tilde j+\tilde n-\tilde m\rangle=\langle\tilde i-\tilde j,\tilde j-\tilde n\rangle,\end{cases}\tag{A.3}$$

$$\begin{cases}\langle m-r,i-j+n-r\rangle=\langle i-j,j-n\rangle,\\ \langle\tilde m-r,\tilde i-\tilde j+\tilde n-r\rangle=\langle\tilde i-\tilde j,\tilde j-\tilde n\rangle,\end{cases}\tag{A.4}$$

$$\begin{cases}\langle r-j,i-j\rangle=0,\\ \langle\tilde r-\tilde i,\tilde r-\tilde j\rangle=0,\end{cases}\tag{A.5}$$

$$\begin{cases}\langle r-j,i-j\rangle=0,\\ \langle r-\tilde m,\tilde i-\tilde j+\tilde n-\tilde m\rangle=\langle\tilde i-\tilde j,\tilde j-\tilde n\rangle,\end{cases}\tag{A.6}$$

$$\begin{cases}\langle r-j,i-j\rangle=0,\\ \langle\tilde m-r,\tilde i-\tilde j+\tilde n-r\rangle=\langle\tilde i-\tilde j,\tilde j-\tilde n\rangle,\end{cases}\tag{A.7}$$

$$\begin{cases}\langle r-i,r-j\rangle=0,\\ \langle r-\tilde m,\tilde i-\tilde j+\tilde n-\tilde m\rangle=\langle\tilde i-\tilde j,\tilde j-\tilde n\rangle,\end{cases}\tag{A.8}$$

$$\begin{cases}\langle r-i,r-j\rangle=0,\\ \langle\tilde m-r,\tilde i-\tilde j+\tilde n-r\rangle=\langle\tilde i-\tilde j,\tilde j-\tilde n\rangle\end{cases}\tag{A.9}$$

$$\begin{cases}\langle r-m,i-j+n-m\rangle=\langle i-j,j-n\rangle,\\ \langle\tilde m-r,\tilde i-\tilde j+\tilde n-r\rangle=\langle\tilde i-\tilde j,\tilde j-\tilde n\rangle\end{cases}\tag{A.10}$$

have no solution in $r=(r_1,r_2)\in\mathbb{Z}^{2,\diamond}$ for $\tilde i,\tilde j,\tilde n,\tilde m,i,j,n,m\in\mathscr{K}$ and $\{\tilde i,\tilde j,\tilde n,\tilde m\}\neq\{i,j,n,m\}$. □

Case 1.1. Only one element of $\{|i|,|j|,|n|,|m|,|r|\}$ gets the maximum value.

Case 1.1.1. Suppose that $|r|=\max\{|i|,|j|,|n|,|m|,|r|\}$. We write $\langle r-m,i-j+n-m\rangle=\langle i-j,j-n\rangle$ in terms of the following components:

$$
\begin{aligned}
&(r_1-m_1)(i_1-j_1+n_1-m_1)+(r_2-m_2)(i_2-j_2+n_2-m_2)\\
&\quad=(i_1-j_1)(j_1-n_1)+(i_2-j_2)(j_2-n_2).
\end{aligned}\tag{A.11}
$$

By the calculation, we have

$$\begin{aligned} r_2 &= m_2 + \frac{(m_1-r_1)(i_1-j_1+n_1-m_1)+(i_1-j_1)(j_1-n_1)+(i_2-j_2)(j_2-n_2)}{i_2-j_2+n_2-m_2} \\ &\le m_2 + |(m_1-r_1)(i_1-j_1+n_1-m_1)| + |(i_1-j_1)(j_1-n_1)| + |(i_2-j_2)(j_2-n_2)| \\ &\le m_2 + 2r_1^2 + r_1^2 + r_1^2 \le m_2 + 4r_1^2 < 5r_1^2. \end{aligned} \tag{A.12}$$

This is contradictory to $r_2 = r_1^5$.

Case 1.1.2. Suppose that $|j| = \max\{|i|, |j|, |n|, |m|, |r|\}$; then, we have

$$|<i-j, j-n>| > \left|<\frac{j}{2}, \frac{j}{2}>\right| = \frac{1}{4}|j|^2 > |<r-m, i-j+n-m>|. \tag{A.13}$$

This is contradictory to the hypothesis $<i-j, j-n> = <r-m, i-j+n-m>$.

Case 1.2. Two elements of $\{|i|, |j|, |n|, |m|, |r|\}$ get the maximum value.

Case 1.2.1. Suppose that $|m| = |j| = \max\{|i|, |j|, |n|, |m|, |r|\}$, then

$$\begin{aligned} &<r-m, i-j+n-m> \\ &\quad = <i-j, j-n> \Longrightarrow <r-j, i+n-2j> \\ &\quad = <i-j, j-n> \Longrightarrow 3<j,j> -2<r,j> -2<i,j> -2<n,j> \\ &\qquad + <i,r> + <n,r> + <i,n> = 0. \end{aligned} \tag{A.14}$$

But we have

$$\begin{aligned} &3<j,j> -2<r,j> -2<i,j> -2<n,j> + <i,r> + <n,r> + <i,n> \\ &\quad = 3<j,j> -2<r+i+n, j> + <i,r> + <n,r> + <i,n> > 0. \end{aligned} \tag{A.15}$$

It is a contradiction.

Other situations are similar to the three above. Thus, the hypothesis $<r-m, i-j+n-m> = <i-j, j-n>$ is wrong. That is, $\mathscr{K}$ satisfies the property (2) in Definition 1. Similarly, $\mathscr{K}$ satisfies the property (1) in Definition 1.

(II) Let us show that equation (A.3) has no solution in $\mathbb{Z}^{2,\diamond}$. The proof for (A.1) and (A.6) is similar and simpler than the proof for (A.3)

Case 2.1. Only one of $\{|\tilde{i}|, |\tilde{j}|, |\tilde{d}|, |\tilde{l}|, |i|, |j|, |d|, |l|\}$ gets the maximum value.

Case 2.1.1. Suppose that $|n| = \max\{|\tilde{i}|, |\tilde{j}|, |\tilde{n}|, |\tilde{m}|, |i|, |j|, |n|, |m|\}$. According to the calculation,

$$r_2 = m_2 + j_2 - i_2 + \frac{\beta_{211}}{\alpha_{211}}, \tag{A.16}$$

where

$$\begin{aligned} \beta_{211} &= n_1\left[<\tilde{j}-\tilde{i}, \tilde{j}-\tilde{n}> + <j-i+m-\tilde{m}, \tilde{i}-\tilde{j}+\tilde{n}-\tilde{m}>\right] \\ &\quad + (i_1-j_1-m_1)\left[<\tilde{j}-\tilde{i}, \tilde{j}-\tilde{n}> + <j-i+m-\tilde{m}, \tilde{i}-\tilde{j}+\tilde{n}-\tilde{m}>\right] \\ &\quad + \left(\tilde{i}_1-\tilde{j}_1+\tilde{n}_1-\tilde{m}_1\right) <i-j, i-m>, \\ \alpha_{211} &= \left(\tilde{i}_1-\tilde{j}_1+\tilde{n}_1-\tilde{n}_1\right)(i_2-j_2+n_2-m_2) \\ &\quad + (j_1-i_1+m_1-n_1)\left(\tilde{i}_2-\tilde{j}_2+\tilde{n}_2-\tilde{m}_2\right). \end{aligned} \tag{A.17}$$

To prove $\beta_{211} \neq 0$ by contradiction, suppose that $\beta_{211} = 0$, then

$$\begin{cases} <\tilde{j}-\tilde{i}, \tilde{j}-\tilde{n}> + <j-i+m-\tilde{m}, \tilde{i}-\tilde{j}+\tilde{n}-\tilde{m}> = 0, \\ (i_1-j_1-m_1)\left[<\tilde{j}-\tilde{i}, \tilde{j}-\tilde{n}> + <j-i+m-\tilde{m}, \tilde{i}-\tilde{j}+\tilde{n}-\tilde{m}>\right] + \left(\tilde{i}_1-\tilde{j}_1+\tilde{n}_1-\tilde{m}_1\right) <i-j, i-m> = 0. \end{cases} \tag{A.18}$$

From the system above, we get $\left(\tilde{i}_1-\tilde{j}_1+\tilde{n}_1-\tilde{m}_1\right) <i-j, i-m> = 0$. And by $\tilde{i}_1-\tilde{j}_1+\tilde{n}_1-\tilde{m}_1 \neq 0$, then $<i-j, i-m> = 0$ is obtained. In view of Lemma 13, then $i=j$ or $i=m$. It is contradictory to $r \in \mathscr{I}_3$. That is, $\beta_{211} \neq 0$. Because the order of the numerator β_{211} is no more than n_1 and the order of the divisor α_{211} is n_2, we have $r_2 \in \mathbb{Z}$.

Case 2.1.2. Suppose that $|m| = \max\{|i|, |j|, |n|, |m|, |\tilde{i}|, |\tilde{j}|, |\tilde{n}|, |\tilde{m}|\}$. By the calculation, then

$$r_2 = m_2 + \frac{\beta_{212}m_2 + \alpha_{212}}{\delta_{212}m_2 + \gamma_{212}} = m_2 + \frac{\beta_{212}}{\delta_{212}} + \frac{\alpha_{212}\delta_{212} - \beta_{212}\gamma_{212}}{\delta_{212}(\delta_{212}m_2 + \gamma_{212})}, \tag{A.19}$$

where

$$\beta_{212} = (i_1-j_1+n_1-m_1)\left(\tilde{i}_2-\tilde{j}_2+\tilde{n}_2-\tilde{m}_2\right),$$

$$\alpha_{212} = m_1(\tilde{i}_1 - \tilde{j}_1 + \tilde{n}_1 - \tilde{m}_1)(i_1 - j_1 + n_1 - m_1) - (\tilde{i}_1 - \tilde{j}_1 + \tilde{n}_1 - \tilde{m}_1) < j - i, j - n > + (i_1 - j_1 + n_1 - m_1) \cdot (<\tilde{j} - \tilde{i}, \tilde{j} - \tilde{n}> - <\tilde{m}, \tilde{i} - \tilde{j} + \tilde{n} - \tilde{m}>),$$
$$\delta_{212} = -(\tilde{i}_1 - \tilde{j}_1 + \tilde{n}_1 - \tilde{m}_1),$$
$$\gamma_{212} = (\tilde{i}_1 - \tilde{j}_1 + \tilde{n}_1 - \tilde{m}_1)(i_2 - j_2 + n_2) - (i_1 - j_1 + n_1 - m_1)(\tilde{i}_2 - \tilde{j}_2 + \tilde{n}_2 - \tilde{m}_2). \tag{A.20}$$

Let

$$\mu_{212} := \alpha_{212}\delta_{212} - \beta_{212}\gamma_{212}, \sigma_{212} := \delta_{212}(\delta_{212}m_2 + \gamma_{212}), \tag{A.21}$$

then by the calculation,

$$\mu_{212} = |\tilde{i} - \tilde{j} + \tilde{n} - \tilde{m}|^2 m_1^2 + \tilde{\mu}_{212}, \sigma_{212} = (\tilde{i}_1 - \tilde{j}_1 + \tilde{n}_1 - \tilde{m}_1)^2 m_2 + \tilde{\sigma}_{212}, \tag{A.22}$$

where the order of $\tilde{\mu}_{212}$ and $\tilde{\sigma}_{212}$ does not exceed m_1. By Lemma 13, then $\tilde{i} - \tilde{j} + \tilde{n} - \tilde{m} \neq 0$. Thus, we have that β_{212}/δ_{212} is an integer or $|(\beta_{212}/\delta_{212}) - [\beta_{212}/\delta_{212}]| > 1/\sqrt[25]{m_1}$, and $0 < |(\mu_{212}/\sigma_{212}) - [\mu_{212}/\sigma_{212}]| < 1/\sqrt[5]{m_1^2}$, where $[\bullet]$ is the integer operation, and then

$$\left|\frac{\beta_{212}}{\delta_{212}} + \frac{\mu_{212}}{\sigma_{212}} - \left[\frac{\beta_{212}}{\delta_{212}} + \frac{\mu_{212}}{\sigma_{212}}\right]\right| \in (0, 1). \tag{A.23}$$

Thus, $r_2 \in \mathbb{Z}$.

Other situations are similar to the above cases.

Case 2.2. Two elements of $\{|i|, |j|, |n|, |m|, |\tilde{i}|, |\tilde{j}|, |\tilde{n}|, |\tilde{m}|\}$ get the maximum value.

Case 2.2.1. Suppose that $|i| = |n| = \max\{|i|, |j|, |n|, |m|, |\tilde{i}|, |\tilde{j}|, |\tilde{n}|, |\tilde{m}|\}$, then $i = n$. By the calculation, then

$$r_2 = m_2 + \frac{\beta_{221} i_2^2 + \alpha_{221} i_2 + \delta_{221}}{\gamma_{221} i_2 + \mu_{221}} = m_2 + \frac{\beta_{221}}{\gamma_{221}} i_2 + \frac{\alpha_{221}\gamma_{221} - \beta_{221}\mu_{221}}{\gamma_{221}^2} + \frac{\delta_{221}\gamma_{221}^2 - \alpha_{221}\gamma_{221}\mu_{221} + \beta_{221}\mu_{221}^2}{\gamma_{221}^2(\gamma_{221} i_2 + \mu_{221})}, \tag{A.24}$$

where

$$\beta_{221} = -(\tilde{i}_1 - \tilde{j}_1 + \tilde{n}_1 - \tilde{m}_1), \alpha_{221} = 2(\tilde{i}_1 - \tilde{j}_1 + \tilde{n}_1 - \tilde{m}_1) j_2,$$

$$\delta_{221} = -(\tilde{i}_1 - \tilde{j}_1 + \tilde{n}_1 - \tilde{m}_1)(i_1^2 - 2i_1 j_1 + |j|^2) - (j_1 - 2i_1 + m_1) \cdot (<m - \tilde{m}, \tilde{i} - \tilde{j} + \tilde{n} - \tilde{m}> + <\tilde{j} - \tilde{i}, \tilde{j} - \tilde{n}>),$$

$$\gamma_{221} = 2(\tilde{i}_1 - \tilde{j}_1 + \tilde{n}_1 - \tilde{m}_1),$$

$$\mu_{221} = -(\tilde{i}_1 - \tilde{j}_1 + \tilde{n}_1 - \tilde{m}_1)(j_2 + m_2) + (j_1 - 2i_1 + m_1)(\tilde{i}_2 - \tilde{j}_2 + \tilde{n}_2 - \tilde{m}_2). \tag{A.25}$$

Denote

$$\sigma_{221} := \frac{\alpha_{221}\gamma_{221} - \beta_{221}\mu_{221}}{\gamma_{221}^2},$$
$$\nu_{221} := \delta_{221}\gamma_{221}^2 - \alpha_{221}\gamma_{221}\mu_{221} + \beta_{221}\mu_{221}^2, \omega_{221} := \gamma_{221}^2(\gamma_{221} i_2 + \mu_{221}), \tag{A.26}$$

then by the calculation,

$$\nu_{221} = 4|\tilde{i} - \tilde{j} + \tilde{n} - \tilde{m}|^2 i_1^2 + \tilde{\nu}_{221}, \omega_{221} = -8(\tilde{i}_1 - \tilde{j}_1 + \tilde{n}_1 - \tilde{m}_1)^2 i_2 + \tilde{\omega}_{221}, \tag{A.27}$$

where the order of $\tilde{\nu}_{221}$ and $\tilde{\omega}_{221}$ is no more than i_1. By Lemma 13, then $-\tilde{i} + \tilde{j} - \tilde{n} + \tilde{m} \neq 0$. So we know that $\beta_{221}/\gamma_{221} = -(1/2)$, σ_{221} is an integer or $|\sigma_{221} - [\sigma_{221}]| > 1/\sqrt[25]{i_1}$, and $0 < |(\nu_{221}/\omega_{221}) - [\nu_{221}/\omega_{221}]| < 1/i_1^2$, and then,

$$\left|\frac{\beta_{221}}{\gamma_{221}} i_2 + \sigma_{221} + \frac{\nu_{221}}{\omega_{221}} - \left[\frac{\beta_{221}}{\gamma_{221}} i_2 + \sigma_{221} + \frac{\nu_{221}}{\omega_{221}}\right]\right| \in (0, 1). \tag{A.28}$$

Hence, $r_2 \in \mathbb{Z}$.

Case 2.2.2. Suppose that $|m| = |\tilde{m}| = \max\{|i|, |j|, |n|, |m|, |\tilde{i}|, |\tilde{j}|, |\tilde{n}|, |\tilde{m}|\}$, then $m = \tilde{m}$. By the calculation, then

$$r_2 = m_2 + \frac{\beta_{222}}{\alpha_{222}}, \tag{A.29}$$

where

$$\beta_{222} = (<j - i, j - n> - <\tilde{j} - \tilde{i}, \tilde{j} - \tilde{n}>) m_1 - (\tilde{i}_1 - \tilde{j}_1 + \tilde{n}_1) < j - i, j - n > + (i_1 - j_1 + n_1) < \tilde{j} - \tilde{i}, \tilde{j} - \tilde{n} >,$$

$$\alpha_{222} = (i_1 - j_1 + n_1 - \tilde{i}_1 + \tilde{j}_1 - \tilde{n}_1) m_2 + (\tilde{i}_1 - \tilde{j}_1 + \tilde{n}_1 - m_1)(i_2 - j_2 + n_2) - (i_1 - j_1 + n_1 - m_1)(\tilde{i}_2 - \tilde{j}_2 + \tilde{n}_2). \tag{A.30}$$

We prove $\beta_{222} \neq 0$ by contradiction. Suppose that $\beta_{222} = 0$, then

$$\begin{cases} <j - i, j - n> - <\tilde{j} - \tilde{i}, \tilde{j} - \tilde{n}> = 0, \\ -(\tilde{i}_1 - \tilde{j}_1 + \tilde{n}_1) < j - i, j - n> + (i_1 - j_1 + n_1) < \tilde{j} - \tilde{i}, \tilde{j} - \tilde{n}> = 0. \end{cases} \tag{A.31}$$

From the system above, we have $(\tilde{i}_1 - \tilde{j}_1 + \tilde{n}_1 - i_1 + j_1 - n_1) < j - i, j - n > = 0$. And from $\tilde{i}_1 - \tilde{j}_1 + \tilde{n}_1 - i_1 + j_1 - n_1 \neq$

0, then we have $<j-i, j-n> = 0$. From Lemma 13, then $j = i$ or $j = n$. It is contradictory to $r \in \mathscr{I}_3$. That is, $\beta_{222} \neq 0$. Due to the order of the numerator β_{222} which is no more than m_1 and the order of the divisor α_{222} which is m_2, we have $r_2 \in \mathbb{Z}$.

Other situations are similar to the above cases.

Case 2.3. Three elements of $\{|i|, |j|, |n|, |m|, |\tilde{i}|, |\tilde{j}|, |\tilde{n}|, |\tilde{m}|\}$ get the maximum value. It can be seen by [23] that such situation is similar to those mentioned above; thus, omit the proof.

Case 2.4. Four elements of $\{|i|, |j|, |n|, |m|, |\tilde{i}|, |\tilde{j}|, |\tilde{n}|, |\tilde{m}|\}$ get the maximum value. It can be seen by [23] that such situation is similar to those mentioned above; thus, omit the proof. It is shown that equation (A.3) has no solution in $\mathbb{Z}^{2,\diamond}$. That is, $\mathscr{K}$ satisfies the property (6) in Definition 1. Similarly, $\mathscr{K}$ satisfies the property (3) in Definition 1 and $\mathscr{I}_2 \bigcap \mathscr{I}_4 = \varnothing$.

(III) Let us show that equation (A.4) has no solution in $\mathbb{Z}^{2,\diamond}$. The proof for (A.2), (A.5), and (A.7)–(A.10), is similar and simpler than the proof for (A.4)

By the calculation, equation (A.4) is equivalent to

$$\begin{cases} <m-r, i-j+n-r> = <i-j, j-n>, \\ <r, \tilde{i}+\tilde{n}+\tilde{m}-\tilde{j}-i-n-m+j> = <i-j, j-n> - <\tilde{i}-\tilde{j}, \tilde{j}-\tilde{n}> + <\tilde{i}-\tilde{j}+\tilde{n}, \tilde{m}> - <i-j+n, m>. \end{cases} \tag{A.32}$$

We assert $|j|$ and $|\tilde{j}|$ will not be the maximum. Suppose that $|j| = \max\{|\tilde{i}|, |\tilde{j}|, |\tilde{n}|, |\tilde{m}|, |i|, |j|, |n|, |m|\}$. According to $|m|^2 + |n|^2 + |i|^2 = |r|^2 + |s|^2 + |j|^2 \geq |j|^2$ and the definition of $\mathscr{K}$, then there is one element of the set $\{m, n, i\}$ that is identical to j. If $i = j$, then $r + s - n - m = 0$ and $|r|^2 + |s|^2 - |n|^2 - |m|^2 = 0$. That is, $r \in \mathscr{I}_2$. This is contradictory to $r \in \mathscr{I}_4$. That is, $|j| \neq \max\{|\tilde{i}|, |\tilde{j}|, |\tilde{n}|, |\tilde{m}|, |i|, |j|, |n|, |m|\}$. Similarly, we have $|\tilde{j}| \neq \max\{|\tilde{i}|, |\tilde{j}|, |\tilde{n}|, |\tilde{m}|, |i|, |j|, |n|, |m|\}$.

Case 3.1. Only one element of $\{|\tilde{i}|, |\tilde{j}|, |\tilde{n}|, |\tilde{m}|, |i|, |j|, |n|, |m|\}$ gets the maximum value. Suppose that $|n| = \max\{|\tilde{i}|, |\tilde{j}|, |\tilde{n}|, |\tilde{m}|, |i|, |j|, |n|, |m|\}$, then

$$|r|^2 = |\tilde{i}|^2 - |\tilde{j}|^2 + |\tilde{n}|^2 + |\tilde{m}|^2 - |\tilde{s}|^2 \ll n_1. \tag{A.33}$$

By the calculation to (A.32),

$$r_2 = i_2 - j_2 + m_2 + \frac{\beta_{31}}{\alpha_{31}}, \tag{A.34}$$

where

$$\begin{aligned} \beta_{31} = {} & (i_1 - j_1 + m_1 - r_1)n_1 + <j-i, j-m> - <\tilde{j}-\tilde{n}, \tilde{j}-\tilde{m}> \\ & + <\tilde{i}, \tilde{j}-\tilde{n}-\tilde{m}> - r_1\left(i_1 - j_1 + m_1 - \tilde{i}_1 + \tilde{j}_1 - \tilde{n}_1 - \tilde{m}_1\right) \\ & + (i_2 + m_2 - j_2)\left(-i_2 + j_2 - m_2 + \tilde{i}_2 + \tilde{n}_2 + \tilde{m}_2 - \tilde{j}_2\right), \end{aligned}$$

$$\alpha_{31} = n_2 + \left(i_2 + m_2 - j_2 + \tilde{j}_2 - \tilde{i}_2 - \tilde{n}_2 - \tilde{m}_2\right). \tag{A.35}$$

We prove $\beta_{31} \neq 0$ by contradiction. Suppose that $\beta_{31} = 0$, then

$$i_1 - j_1 + m_1 - r_1 = 0, r_2 = i_2 - j_2 + m_2. \tag{A.36}$$

That is, $i - j + m - r = 0$. Thus, we have $n = s$ by $i - j + n + m - r - s = 0$. This is contradictory to $n \in \mathscr{K}$ and $s \in \mathbb{Z}^{2,\diamond}$. That is, $\beta_{31} \neq 0$. Due to the order of the numerator β_{31} which is no more than n_1 and the order of the divisor α_{31} which is n_2, we have $r_2 \in \mathbb{Z}$.

Other situations are similar to the above cases.

Case 3.2. Two of $\{|\tilde{i}|, |\tilde{j}|, |\tilde{n}|, |\tilde{m}|, |i|, |j|, |n|, |m|\}$ get the maximum value.

Case 3.2.1. Suppose that $|m| = |\tilde{m}| = \max\{|\tilde{i}|, |\tilde{j}|, |\tilde{n}|, |\tilde{m}|, |i|, |j|, |n|, |m|\}$, then $m = \tilde{m}$. By equation (A.32), then

$$\beta_{321} r_1^2 + \alpha_{321} r_1 + \delta_{321} = 0, \tag{A.37}$$

where

$$\begin{aligned} \beta_{321} = {} & \left|i + n - j + \tilde{j} - \tilde{i} - \tilde{n}\right|^2, \\ \alpha_{321} = {} & 2\left(i_1 + n_1 - m_1 + \tilde{j}_1 - \tilde{i}_1 - \tilde{n}_1\right)\left(-<j-i, j-n> + <\tilde{j}-\tilde{i}, \tilde{j}-\tilde{n}>\right) \\ & - 2\left(i_1 - j_1 + n_1 + \tilde{j}_1 - \tilde{i}_1 - \tilde{n}_1\right) <m, i-j+n+\tilde{j}-\tilde{i}-\tilde{n}> \\ & + (i_1 - j_1 + n_1 + m_1)\left(i_2 - j_2 + n_2 + \tilde{j}_2 - \tilde{i}_2 - \tilde{n}_2\right)\left(\tilde{i}_2 - \tilde{j}_2 + \tilde{n}_2 + l_2\right) \\ & - \left(\tilde{i}_1 - \tilde{j}_1 + \tilde{n}_1 + m_1\right)\left(i_2 - j_2 + n_2 + \tilde{j}_2 - \tilde{i}_2 - \tilde{n}_2\right)(i_2 - j_2 + n_2 + m_2), \\ \delta_{321} = {} & \left(-<j-i, j-n> + <\tilde{j}-\tilde{i}, \tilde{j}-\tilde{n}> - <m, i-j+n+\tilde{j}-\tilde{i}-\tilde{n}>\right)^2 \\ & + \left(i_2 - j_2 + n_2 + \tilde{j}_2 - \tilde{i}_2 - \tilde{n}_2\right)(i_2 - j_2 + n_2 + m_2) \\ & \cdot \left(<\tilde{j}-\tilde{i}, \tilde{j}-\tilde{n}> + <m, \tilde{i}-\tilde{j}+\tilde{n}>\right) - \left(i_2 - j_2 + n_2 + \tilde{j}_2 - \tilde{i}_2 - \tilde{n}_2\right) \\ & \cdot \left(\tilde{i}_2 - \tilde{j}_2 + \tilde{n}_2 + m_2\right)\left(<j-i, j-n> + <m, i-j+n>\right). \end{aligned} \tag{A.38}$$

Therefore,

$$\begin{aligned}\Delta &= \alpha_{321}^2 - 4\beta_{321}\delta_{321} \\ &= \gamma_{321}^2 \cdot \left[m_2 + \frac{\mu_{321}}{\left(i_1 - j_1 + n_1 + \tilde{j}_1 - \tilde{i}_1 - \tilde{n}_1\right)^2}\right]^2 \\ &\quad + 4\left(i_2 - j_2 + n_2 + \tilde{j}_2 - \tilde{i}_2 - \tilde{n}_2\right)^2 \\ &\quad \cdot \frac{\left|i - j + n + \tilde{j} - \tilde{i} - \tilde{n}\right|^2}{\left(i_1 - j_1 + n_1 + \tilde{j}_1 - \tilde{i}_1 - \tilde{n}_1\right)} \cdot \sigma_{321},\end{aligned} \tag{A.39}$$

where

$$\gamma_{321} = \left(i_1 - j_1 + n_1 + \tilde{j}_1 - \tilde{i}_1 - \tilde{n}_1\right)\left(i_2 - j_2 + n_2 + \tilde{j}_2 - \tilde{i}_2 - \tilde{n}_2\right),$$

$$\begin{aligned}\mu_{321} &= -\left(i_1 - j_1 + n_1 + \tilde{j}_1 - \tilde{i}_1 - \tilde{n}_1\right)\left(i_2 - j_2 + n_2 + \tilde{j}_2 - \tilde{i}_2 - \tilde{n}_2\right)m_1 \\ &\quad + 2\left(i_2 - j_2 + n_2 + \tilde{j}_2 - \tilde{i}_2 - \tilde{n}_2\right)\left(<\tilde{j} - \tilde{i}, \tilde{j} - \tilde{n}> - <j - i, j - n>\right) \\ &\quad + \left(i_1 - j_1 + n_1 + \tilde{j}_1 - \tilde{i}_1 - \tilde{n}_1\right)\left(\tilde{i}_1 - \tilde{j}_1 + \tilde{n}_1\right)\left(i_2 - j_2 + n_2\right) \\ &\quad - \left(i_1 - j_1 + n_1 + \tilde{j}_1 - \tilde{i}_1 - \tilde{n}_1\right)\left(i_1 - j_1 + n_1\right)\left(\tilde{i}_2 - \tilde{j}_2 + \tilde{n}_2\right),\end{aligned}$$

$$\begin{aligned}\sigma_{321} &= \left(<\tilde{j} - \tilde{i}, \tilde{j} - \tilde{n}> - <j - i, j - n>\right)m_1 + <j - i, j - n>\left(\tilde{i}_1 - \tilde{j}_1 + \tilde{n}_1\right) \\ &\quad - <\tilde{j} - \tilde{i}, \tilde{j} - \tilde{n}>\left(i_1 - j_1 + n_1\right) \\ &\quad - \frac{\left(<\tilde{j} - \tilde{i}, \tilde{j} - \tilde{n}> - <j - i, j - n>\right)^2}{\left(i_1 - j_1 + n_1 + \tilde{j}_1 - \tilde{i}_1 - \tilde{n}_1\right)}.\end{aligned} \tag{A.40}$$

Due to σ_{321} which is of order m_1 which is far less than m_2, we have

$$\Delta = \gamma_{321}^2 \cdot \left[m_2 + \frac{\mu_{321}}{\left(i_1 - j_1 + n_1 + \tilde{j}_1 - \tilde{i}_1 - \tilde{n}_1\right)^2} - \nu_{321}\right]^2, \tag{A.41}$$

where

$$\nu_{321} \sim \frac{m_1}{m_2} \ll \frac{1}{\left(i_1 - j_1 + n_1 + \tilde{j}_1 - \tilde{i}_1 - \tilde{n}_1\right)^2}. \tag{A.42}$$

Therefore,

$$\begin{aligned}r_1 &= \frac{-\alpha_{321} \pm \sqrt{\Delta}}{2\beta_{321}} = \frac{-\alpha_{321}}{2\beta_{321}} \\ &\quad \pm \frac{|\gamma_{321}|m_2 + \left(|\gamma_{321}|\mu_{321}/\left(i_1 - j_1 + n_1 + \tilde{j}_1 - \tilde{i}_1 - \tilde{n}_1\right)^2\right) - |\gamma_{321}|\nu_{321}}{2\beta_{321}}.\end{aligned} \tag{A.43}$$

Due to $0 < \left||\gamma_{321}|\nu_{321}/2\beta_{321}\right| \ll 1/\left(2\beta_{321}\left(i_1 - j_1 + n_1 + \tilde{j}_1 - \tilde{i}_1 - \tilde{n}_1\right)^2\right)$, we have $r_1 \in \mathbb{Z}$.

Other situations are similar to the above cases.

Case 3.3. Three elements of $\{|\tilde{i}|, |\tilde{j}|, |\tilde{n}|, |\tilde{m}|, |i|, |j|, |n|, |m|\}$ get the maximum value. It can be seen by [23] that such situation is similar to that mentioned above; thus, omit the proof.

Case 3.4. Four elements of $\{|\tilde{i}|, |\tilde{j}|, |\tilde{n}|, |\tilde{m}|, |i|, |j|, |n|, |m|\}$ get the maximum value.

Case 3.4.1. Suppose that $|i| = |n| = |m| = |\tilde{m}| = \max\{|\tilde{i}|, |\tilde{j}|, |\tilde{n}|, |\tilde{m}|, |i|, |j|, |n|, |m|\}$, then $i = n = m = \tilde{m}$. By the calculation to (A.32),

$$\beta_{341} r_1^2 + \alpha_{341} r_1 + \delta_{341} = 0, \tag{A.44}$$

where

$$\beta_{341} = \left|2m - j + \tilde{j} - \tilde{i} - \tilde{n}\right|^2,$$

$$\begin{aligned}\alpha_{341} &= 2\left(2m_1 - j_1 + \tilde{j}_1 - \tilde{i}_1 - \tilde{n}_1\right)\left(-|j - m|^2 + <\tilde{j} - \tilde{i}, \tilde{j} - \tilde{n}>\right) \\ &\quad + 2\left(2m_1 - j_1 + \tilde{j}_1 - \tilde{i}_1 - \tilde{n}_1\right) < m, -2m + j + \tilde{i} - \tilde{j} + \tilde{n}> \\ &\quad + (3m_1 - j_1)\left(2m_2 - j_2 + \tilde{j}_2 - \tilde{i}_2 - \tilde{n}_2\right)\left(m_2 + \tilde{i}_2 - \tilde{j}_2 + \tilde{n}_2\right) \\ &\quad - \left(m_1 + \tilde{i}_1 - \tilde{j}_1 + \tilde{n}_1\right)\left(2m_2 - j_2 + \tilde{j}_2 - \tilde{i}_2 - \tilde{n}_2\right)(3m_2 - j_2),\end{aligned}$$

$$\begin{aligned}\delta_{341} &= -|j - m|^2 + \left(<\tilde{j} - \tilde{i}, \tilde{j} - \tilde{n}> + <m, -2m + j + \tilde{i} - \tilde{j} + \tilde{n}>\right)^2 \\ &\quad + \left(2m_2 - j_2 + \tilde{j}_2 - \tilde{i}_2 - \tilde{n}_2\right)(3m_2 - j_2) \\ &\quad \cdot \left(<\tilde{j} - \tilde{i}, \tilde{j} - \tilde{n}> + <m, \tilde{i} - \tilde{j} + \tilde{n}>\right) \\ &\quad - \left(2m_2 - j_2 + \tilde{j}_2 - \tilde{i}_2 - \tilde{n}_2\right)\left(m_2 + \tilde{i}_2 - \tilde{j}_2 + \tilde{n}_2\right) \\ &\quad \cdot \left(|j - m|^2 + <m, 2m - j>\right).\end{aligned} \tag{A.45}$$

It is shown that equation (A.32) has no solution in $\mathbb{Z}^{2,\diamond}$ because

$$\begin{aligned}\Delta &= \alpha_{341}^2 - 4\beta_{341}\delta_{341} \\ &= \left(2m_2 - j_2 + \tilde{j}_2 - \tilde{i}_2 - \tilde{n}_2\right)^2 \cdot m_2^2 \cdot \left(-8m_2^2 + \gamma_{341}\right) < 0,\end{aligned} \tag{A.46}$$

where the order of γ_{341} does not exceed m_2.

Other situations are similar to the above cases.

Case 3.5. Five elements of $\{|\tilde{i}|, |\tilde{j}|, |\tilde{n}|, |\tilde{m}|, |i|, |j|, |n|, |m|\}$ get the maximum value. It can be seen by [23] that such situation is similar to that mentioned above; thus, omit the proof.

Case 3.6. Six elements of $\{|\tilde{i}|, |\tilde{j}|, |\tilde{n}|, |\tilde{m}|, |i|, |j|, |n|, |m|\}$ get the maximum value. It can be seen by [23] that such situation is similar to that mentioned above; thus, omit the proof.

It is shown that equation (A.4) has no solution in $\mathbb{Z}^{2,\diamond}$. That is, $\mathscr{K}$ satisfies the property (7) in Definition 1. Similarly, $\mathscr{K}$ satisfies the properties (5) and (12) in Definition 1.

Data Availability

No external data has been used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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