Hindawi
Journal of Function Spaces
Volume 2022, Article ID 7117244, 8 pages
https://doi.org/10.1155/2022/7117244

Research Article

A Forward-Backward-Forward Algorithm for Solving Quasimonotone Variational Inequalities

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Received 24 August 2021; Accepted 27 October 2021; Published 4 January 2022

Academic Editor: Calogero Vetro

In this paper, we continue to investigate the convergence analysis of Tseng-type forward-backward-forward algorithms for solving quasimonotone variational inequalities in Hilbert spaces. We use a self-adaptive technique to update the step sizes without prior knowledge of the Lipschitz constant of quasimonotone operators. Furthermore, we weaken the sequential weak continuity of quasimonotone operators to a weaker condition. Under some mild assumptions, we prove that Tseng-type forward-backward-forward algorithm converges weakly to a solution of quasimonotone variational inequalities.

1. Introduction

Let H be a real Hilbert space endowed with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\|\cdot\|$. Let C be a nonempty closed and convex subset of H. Let $f : H \longrightarrow H$ be an operator. Our purpose of this paper is to investigate the following Stampacchia-type variational inequality (shortly, $\mathrm{VI}(C, f)$).

Find $u \in C$ such that

$$\langle f(u), x - u \rangle \geq 0, \quad \forall x \in C. \tag{1}$$

Denote the solution set of (1) by $\mathrm{Sol}(C, f)$.

Variational inequality problem (1) was introduced by Stampacchia [1] in 1964. Now it is well-known that variational inequality problem (1) provides a natural, convenient, and unified framework for the study of a large number of problems in economics, operation research, and engineering (see [2–5]). Variational inequality (1) contains, as special cases, such well-known problems in mathematical programming as systems of nonlinear equations, optimization problems ([3, 6]), complementarity problems ([7–9], and fixed-point problems ([10–20]). Many iterative algorithms for solving variational inequalities and related problems have been proposed and investigated (see, for example, [1, 6, 9, 16, 21–40]). Among them, one of the influential algorithms for solving $\mathrm{VI}(C, f)$) is the projection-gradient algorithm ([28, 39, 40]) which defines a sequence $\{u^k\}$ by

$$u^{k+1} = P_C\left(u^k - \lambda f\left(u^k\right)\right), \quad \forall k \geq 0, \tag{2}$$

where P_C is the orthogonal projection operator onto C and $\lambda > 0$ is the step size.

The projection-gradient algorithm guarantees the convergence of the sequence $\{u^k\}$ defined by (2) if f is strongly (pseudo-)monotone (see [8, 41]) or f is inverse strongly monotone (see [3, 42]). However, if f is plain monotone, then the sequence $\{u^k\}$ generated by (2) does not necessarily converge. Consequently, Korpelevich [43] proposed an extragradient algorithm which generates a sequence $\{u^k\}$ by

$$\begin{cases} u^0 \in H, \\ v^k = P_C\left(u^k - \lambda f\left(u^k\right)\right), \\ u^{k+1} = P_C\left(u^k - \lambda f\left(v^k\right)\right), \quad \forall k \geq 0. \end{cases} \tag{3}$$

This algorithm guarantees the convergence of the sequence $\{u^k\}$ defined by (3) if f is pseudomonotone. Since then, Korpelevich's algorithm has attracted so much attention by many scholars, who modified it in several different forms (see, e.g., [34, 44–47]). Especially, Vuong [31] proved that Korpelevich's extragradient method has weak convergence provided that f is sequentially weakly continuous and pseudomonotone.

A challenging task when devise efficient algorithms for solving variational inequalities is to avoid to compute the projection operators at each iteration because the computation of the projection operator may be very expensive. In this respect, Tseng [30] modified extragradient algorithm with the following form:

$$\begin{cases} u^0 \in H, \\ v^k = P_C\left(u^k - \lambda f\left(u^k\right)\right), \\ u^{k+1} = u^k + \lambda\left(f\left(u^k\right) - f\left(v^k\right)\right), \quad \forall k \geq 0. \end{cases} \tag{4}$$

Boț et al. [48] approach the solution of $\mathrm{VI}(C, f)$ from a continuous perspective by means of trajectories generated by the following dynamical system of forward-backward-forward type:

$$\begin{cases} u(0) = u^0, \\ v(t) = P_C(u(t) - \lambda f(u(t))), \\ \dot{u}(t) + u(t) = v(t) + \lambda(f(\mathrm{u}(t)) - f(v(t))), \end{cases} \tag{5}$$

where $\lambda > 0$ and $u^0 \in H$.

Note that (5) has its roots and the existence and uniqueness of the trajectory $x \in C^1([0,+\infty), H)$ generated by (5) has been obtained (see [49]). The explicit time discretization of the dynamical system (5) yields the following Tseng-type forward-backward-forward algorithm:

$$\begin{cases} u^0 \in H, \\ v^k = P_C\left(u^k - \lambda f\left(u^k\right)\right), \\ u^{k+1} = \mu_k\left(v^k + \lambda\left(f\left(u^k\right) - \left(v^k\right)\right)\right) + (1-\mu_k)u^k, \quad \forall k \geq 0. \end{cases} \tag{6}$$

Bot et al. ([48]) proved that the sequence $\{u^k\}$ generated by (6) converges weakly to an element in $\mathrm{Sol}(C, f)$ provided f is pseudomonotone and sequentially weakly continuous. On the other hand, for solving (1) and related problems, some self-techniques have been used to relax the step size without prior knowledge of the Lipschitz constant of the operator f (see [50–53]).

Let $\mathrm{Sol}^d(C, f)$ be the solution set of the dual variational inequality of (1), that is,

$$\mathrm{Sol}^d(C, f) \coloneqq \{u \in C \mid \langle f(x), x - u\rangle \geq 0, \forall x \in C\}. \tag{7}$$

Note that $\mathrm{Sol}^d(C, f)$ is closed convex. If C is convex and f is continuous, then $\mathrm{Sol}^d(C, f) \subset \mathrm{Sol}(C, f)$.

To prove the convergence of the sequence $\{u^k\}$, a common assumption $\mathrm{Sol}(C, f) \subset \mathrm{Sol}^d(C, f)$ has been used, that is,

$$\langle f(x), x - u\rangle \geq 0, \quad \forall u \in \mathrm{Sol}(C, f), \quad x \in C, \tag{8}$$

which is a direct consequence of the pseudomonotonicity of f. But this conclusion (that is, $\mathrm{Sol}(C, f) \subset \mathrm{Sol}^d(C, f)$) is false, if f is quasimonotone.

In this paper, we introduce a self-adaptive Tseng-type forward-backward-forward algorithm to solve quasimonotone variational inequalities (1). The algorithm is designed such that the step sizes are dynamically chosen and its convergence is guaranteed without prior knowledge of the Lipschitz constant of f. Moreover, we replace the sequential weak continuity imposed on f by a weaker condition. We show that the proposed algorithm converges weakly to a solution of quasimonotone variational inequalities under some additional conditions.

2. Preliminaries

Let C be a nonempty convex and closed subset of a real Hilbert space H. Use " $\rightharpoonup$ " and "$\longrightarrow$" to denote weak convergence and strong convergence, respectively. Let $f : H \longrightarrow H$ be an operator. Recall that f is said to be

(i) strongly monotone if there exists a positive constant α such that

$$\langle f(u) - f(v), u - v\rangle \geq \alpha\|u - v\|^2, \quad \forall u, v \in H \tag{9}$$

(ii) α-inverse strongly monotone if there exists a positive constant α such that

$$\langle f(u) - f(v), u - v\rangle \geq \alpha\|f(u) - f(v)\|^2, \quad \forall u, v \in H \tag{10}$$

(iii) monotone if

$$\langle f(u) - f(v), u - v\rangle \geq 0, \quad \forall u, v \in H \tag{11}$$

(iv) pseudomonotone if

$$\langle f(v), u - v\rangle \geq 0 \text{ implies } \langle f(u), u - v\rangle \geq 0, \quad \forall u, v \in H \tag{12}$$

(v) quasimonotone if

$$\langle f(v), u - v\rangle > 0 \text{ implies } \langle f(u), u - v\rangle \geq 0, \quad \forall u, v \in H \tag{13}$$

It is easy to see that strongmonotonicity $\Rightarrow$ monotonicity $\Rightarrow$ pseudomonotonicity $\Rightarrow$ quasimonotonicity.

But the reverse assertions are not true in general.

Example 1 (see [50]). Let $H=\mathbb{R}^4$ and $C=\{(x_1,x_2,x_3,x_4)^T \in \mathbb{R}^4 : x_1-x_2-x_3\geq 1\}$. Let $f: C\longrightarrow \mathbb{R}^4$ be defined by $f(x)=(\|x\|^2+2)u$ for all $x\in C$, where $u=(1,-1,-1,0)^T$. Then, f is pseudomonotone on C. But f is not monotone on C.

Example 2 (see [33]). The function $f:\mathbb{R}\longrightarrow\mathbb{R}$ defined by $f(x)=x^2$ is quasimonotone on $\mathbb{R}$, but not pseudomonotone on $\mathbb{R}$.

An operator $f: H\longrightarrow H$ is said to be η-Lipschitz continuous if there exists a positive constant η such that

$$\|f(u)-f(v)\|\leq \eta\|u-v\|, \quad \forall u,v\in H. \tag{14}$$

If $\eta=1$, then f is said to be nonexpansive.

An operator $f: H\longrightarrow H$ is said to be sequentially weakly continuous if for given sequence $\{u^k\}$: $u^k\rightharpoonup u$ implies that $f(u^k)\rightharpoonup f(u)$.

For $\forall x\in H$, there exists a unique point in C, denoted by $P_C(x)$ satisfying

$$\|x-P_C(x)\|\leq\|y-x\|, \quad \forall y\in C. \tag{15}$$

Moreover, P_C has the following property:

$$\langle x-P_C(x), y-P_C(x)\rangle\leq 0, \quad \forall x\in H, \forall y\in C. \tag{16}$$

3. Main Results

In this section, we present our main results.

Let H be a real Hilbert space and $C\subset H$ a nonempty closed convex set. Assume that the following conditions are satisfied:

(C1) The operator f is quasimonotone on H.

(C2) The operator f is η-Lipschitz continuous on H.

(C3) $\mathrm{Sol}^d(C,f)\neq\varnothing$ and $\{u\in C: f(u)=0\}\setminus\mathrm{Sol}^d(C,f)$ is a finite set.

Assume that the operator f possesses the following property: for any given sequence $\{u^k\}\subset H$,

$$\left.\begin{array}{l} u^k\rightharpoonup u^\dagger\in H,\\ \liminf\limits_{k\longrightarrow+\infty}\left\|f\left(u^k\right)\right\|=0 \end{array}\right\}\text{imply that } f\left(u^\dagger\right)=0. \tag{17}$$

Remark 1. If the operator f is sequentially weakly continuous, then f satisfies the above property (17).

Next, we propose a self-adaptive Tseng-type forward-backward-forward algorithm for solving the quasimonotone variational inequality (1).

Remark 2. If $v^k=u^k$, that is, $u^k=P_C(u^k-\lambda_k f(u^k))$, then $u^k\in\mathrm{Sol}(C,f)$. In what follows, we assume that $v^k\neq u^k$. In this case, we can obtain an infinite sequence $\{u^k\}$ generated by Algorithm 1.

Remark 3. According to the definition (3.4) of $\{\lambda_k\}$, λ_k is monotonically decreasing and therefore converges. Set $\lim_{k\longrightarrow+\infty}\lambda_k=\tilde{\lambda}$. It is obvious that $\min\{\delta/\eta,\lambda_0\}\leq\tilde{\lambda}\leq\lambda_0$.

Next, we prove the convergence of the sequence $\{u^k\}$ generated by Algorithm 1.

Theorem 4. *Suppose that the conditions (C1)-(C3) and (17) are satisfied. Then, the sequence $\{u^k\}$ generated by Algorithm 1 converges weakly to a point in $\mathrm{Sol}(C,f)$.*

Proof. Let $x^*\in\mathrm{Sol}^d(C,f)$. Set $w^k=v^k+\lambda_k(f(u^k)-f(v^k)),\forall k\geq 0$. Then, we have

$$\begin{aligned}
\left\|w^k-x^*\right\|^2 &= \left\|v^k+\lambda_k\left(f\left(u^k\right)-f\left(v^k\right)\right)-x^*\right\|^2\\
&= \left\|v^k-x^*\right\|^2+2\lambda_k\left\langle f\left(u^k\right)-f\left(v^k\right), v^k-x^*\right\rangle\\
&\quad+\lambda_k^2\left\|f\left(u^k\right)-f\left(v^k\right)\right\|^2=\left\|u^k-x^*\right\|^2\\
&\quad+\left\|v^k-u^k\right\|^2+2\left\langle v^k-u^k, u^k-x^*\right\rangle\\
&\quad+2\lambda_k\left\langle f\left(u^k\right)-f\left(v^k\right), v^k-x^*\right\rangle\\
&\quad+\lambda_k^2\left\|f\left(u^k\right)-f\left(v^k\right)\right\|^2=\left\|u^k-x^*\right\|^2\\
&\quad-\left\|v^k-u^k\right\|^2+2\left\langle v^k-u^k, v^k-x^*\right\rangle\\
&\quad+2\lambda_k\left\langle f\left(u^k\right)-f\left(v^k\right), v^k-x^*\right\rangle\\
&\quad+\lambda_k^2\left\|f\left(u^k\right)-f\left(v^k\right)\right\|^2.
\end{aligned} \tag{18}$$

Since $x^*\in\mathrm{Sol}^d(C,f)\subset C$, from (16) and (3.2), we achieve $\langle u^k-\lambda_k f(u^k)-v^k, x^*-v^k\rangle\leq 0$. It follows that

$$\left\langle u^k-v^k, x^*-v^k\right\rangle\leq\lambda_k\left\langle f\left(u^k\right), x^*-v^k\right\rangle. \tag{19}$$

Using $v^k\in C$ and $x^*\in\mathrm{Sol}^d(C,f)$, we obtain

$$\left\langle f\left(v^k\right), v^k-x^*\right\rangle\geq 0. \tag{20}$$

By (18), (19), and (20), we receive

$$\begin{aligned}
\left\|w^k-x^*\right\|^2\leq{}&\left\|u^k-x^*\right\|^2-\left\|v^k-u^k\right\|^2+2\lambda_k\left\langle f\left(u^k\right), x^*-v^k\right\rangle\\
&+2\lambda_k\left\langle f\left(u^k\right)-f\left(v^k\right), v^k-x^*\right\rangle\\
&+\lambda_k^2\left\|f\left(u^k\right)-f\left(v^k\right)\right\|^2
\end{aligned}$$

Step 1. Let u^k and λ_k be given. Compute
$$v^k = P_c(u^k - \lambda_k f(u^k)).$$
Criterion: if $v^k = u^k$, then stop.
Step 2. Compute
$$u^{k+1} = \mu_k(v^k + \lambda_k(f(u^k) - f(v^k))) + (1-\mu_k)u^k,$$
$$\lambda_{k+1} = \begin{cases} \min\{(\delta\|u^k - v^k\|/\|f(u^k) - f(v^k)\|), \lambda_k\}, & f(u^k) \neq f(v^k), \\ \lambda_k, & \text{else}, \end{cases}$$
update k to $k+1$ and go to Step 1.

Algorithm 1: Let $\lambda_0 > 0$ and $\delta \in (0,1)$. Select the starting point $u^0 \in H$ and the sequence of relaxation parameters $\{\mu_k\}_{k\geq 0} \subset (0,1]$ satisfying $\liminf_{k\longrightarrow+\infty}\mu_k > 0$. Set $k=0$.

$$\begin{aligned} &= \left\|u^k - x^*\right\|^2 - \left\|v^k - u^k\right\|^2 + 2\lambda_k\left\langle f\left(v^k\right), x^* - v^k\right\rangle \\ &\quad + \lambda_k^2\left\|f\left(u^k\right) - f\left(v^k\right)\right\|^2 \\ &\leq \left\|u^k - x^*\right\|^2 - \left\|v^k - u^k\right\|^2 + \lambda_k^2\left\|f\left(u^k\right) - f\left(v^k\right)\right\|^2. \end{aligned} \tag{21}$$

From (3.4), we have $\|f(u^k) - f(v^k)\| \leq \delta/\lambda_{k+1}\|u^k - v^k\|$. This together with (21) implies that

$$\left\|w^k - x^*\right\|^2 \leq \left\|u^k - x^*\right\|^2 - \left(1 - \delta^2\frac{\lambda_k^2}{\lambda_{k+1}^2}\right)\left\|v^k - u^k\right\|^2. \tag{22}$$

Note that $\lim_{k\longrightarrow+\infty}\lambda_k/\lambda_{k+1} = 1$. So, there exists an integer K such that $(1-\delta^2(\lambda_k^2/\lambda_{k+1}^2)) > 0$ when $k \geq K$. Hence, from (22), we deduce $\|w^k - x^*\| \leq \|u^k - x^*\|$ when $k \geq K$.

In terms of (3.3), we get

$$\begin{aligned} \left\|u^{k+1} - x^*\right\| &= \left\|\mu_k\left(w^k - x^*\right) + (1-\mu_k)\left(u^k - x^*\right)\right\| \\ &\leq \mu_k\left\|w^k - x^*\right\| + (1-\mu_k)\left\|u^k - x^*\right\| \\ &\leq \left\|u^k - x^*\right\|. \end{aligned} \tag{23}$$

Thus, the sequence $\{\|u^k - x^*\|\}$ is monotonically decreasing and $\lim_{k\longrightarrow+\infty}\|u^k - x^*\|$ exists. So, the sequence $\{u^k\}$ is bounded.

By virtue of (22) and (23), we have

$$\begin{aligned} \|u^{k+1} - x^*\|^2 &\leq \mu_k\|w^k - x^*\|^2 + (1-\mu_k)\|u^k - x^*\|^2 \\ &\leq \|u^k - x^*\|^2 - \mu_k\left(1 - \delta^2\frac{\lambda_k^2}{\lambda_{k+1}^2}\right)\|v^k - u^k\|^2. \end{aligned} \tag{24}$$

It follows that

$$\mu_k\left(1 - \delta^2\frac{\lambda_k^2}{\lambda_{k+1}^2}\right)\|v^k - u^k\|^2 \leq \|u^k - x^*\|^2 - \|u^{k+1} - x^*\|^2. \tag{25}$$

Since $\lim_{k\longrightarrow+\infty}(1-\delta^2(\lambda_k^2/\lambda_{k+1}^2)) = 1-\delta^2 > 0$, $\liminf_{k\longrightarrow+\infty}\mu_k > 0$, and $\lim_{k\longrightarrow+\infty}\|u^k - x^*\|$ exist, it follows from (25) that

$$\lim_{k\longrightarrow+\infty}\left\|v^k - u^k\right\| = 0. \tag{26}$$

Since f is Lipschitz, from (26), we obtain

$$\lim_{k\longrightarrow+\infty}\left\|f\left(v^k\right) - f\left(u^k\right)\right\| = 0. \tag{27}$$

Thanks to (3.3), we derive

$$\left\|u^{k+1} - u^k\right\| \leq \mu_k\left\|u^k - v^k\right\| + \mu_k\lambda_k\left\|f\left(u^k\right) - f\left(v^k\right)\right\|. \tag{28}$$

Based on (26)–(28), we deduce

$$\lim_{k\longrightarrow+\infty}\left\|u^{k+1} - u^k\right\| = 0. \tag{29}$$

According to (16) and (3.2), we have

$$\left\langle u^k - \lambda_k f\left(u^k\right) - v^k, x - v^k\right\rangle \leq 0, \quad \forall x \in C. \tag{30}$$

It follows that

$$\begin{aligned} &\frac{1}{\lambda_k}\left\langle u^k - v^k, x - v^k\right\rangle + \left\langle f\left(u^k\right), v^k - u^k\right\rangle \\ &\quad \leq \left\langle f\left(u^k\right), x - u^k\right\rangle, \quad \forall x \in C. \end{aligned} \tag{31}$$

Since $\{u^k\}$ is bounded, by (26), $\{v^k\}$ is also bounded. At the same time, using the Lipschitz continuity of f, $\{f(u^k\}$ is bounded. Combining (26), (27), and (31), we attain

$$\liminf_{k\longrightarrow+\infty}\left\langle f\left(u^k\right), x - u^k\right\rangle \geq 0, \quad \forall x \in C. \tag{32}$$

Since $\{u^k\}$ is bounded, there exists a subsequence $\{u^{k_i}\}$ of $\{u^k\}$ such that $u^{k_i} \rightharpoonup \widehat{u} \in C$ as $i \longrightarrow +\infty$. By virtue of

(32), we have

$$\liminf_{i\longrightarrow+\infty}\left\langle f\left(u^{k_i}\right), x-u^{k_i}\right\rangle \geq 0, \quad \forall x\in C. \tag{33}$$

Next, we consider two possible cases.

Case 1. $\liminf_{i\longrightarrow+\infty}\|f(u^{k_i})\|=0$. Since $u^{k_i}\rightharpoonup \widehat{u}$ and f satisfies (17), we deduce that $f(\widehat{u})=0$.

Case 2. $\liminf_{i\longrightarrow+\infty}\|f(u^{k_i})\|>0$. In this case, $\exists I_0>0$ such that $f(u^{k_i})\neq 0$ for all $i\geq I_0$. From (33), we obtain

$$\liminf_{i\longrightarrow+\infty}\left\langle \frac{f(u^{k_i})}{\|f(u^{k_i})\|}, x-u^{k_i}\right\rangle \geq 0, \quad \forall x\in C. \tag{34}$$

Choose a positive strictly decreasing sequence $\{\varepsilon_j\}$ such that $\varepsilon_j\longrightarrow 0$ as $j\longrightarrow+\infty$. Thanks to (34), there exists a strictly increasing subsequence $\{k_{i_j}\}$ with the property that $k_{i_j}\geq I_0$ and

$$\left\langle \frac{f\left(u^{k_{i_j}}\right)}{\left\|f\left(u^{k_{i_j}}\right)\right\|}, x-u^{k_{i_j}}\right\rangle+\varepsilon_j>0, \quad \forall j\geq 0. \tag{35}$$

It follows that

$$\left\langle f\left(u^{k_{i_j}}\right), x-u^{k_{i_j}}\right\rangle+\varepsilon_j\left\|f\left(u^{k_{i_j}}\right)\right\|>0, \quad \forall j\geq 0. \tag{36}$$

Set $y^j=f(u^{k_{i_j}})/\|f(u^{k_{i_j}})\|^2$ for all $j\geq 0$. Thus, we have $\langle f(u^{k_{i_j}}), y^j\rangle=1$ for each $j\geq 0$. From (36), we deduce

$$\left\langle f\left(u^{k_{i_j}}\right), x+\varepsilon_j\left\|f\left(u^{k_{i_j}}\right)\right\|y^j-u^{k_{i_j}}\right\rangle>0, \quad \forall j\geq 0. \tag{37}$$

Since f is quasimonotone on H, by (37), we get

$$\left\langle f\left(x+\varepsilon_j\left\|f\left(u^{k_{i_j}}\right)\right\|y^j\right), x+\varepsilon_j\left\|f\left(u^{k_{i_j}}\right)\right\|y^j-u^{k_{i_j}}\right\rangle\geq 0, \quad \forall j\geq 0. \tag{38}$$

Observe that $\lim_{j\longrightarrow+\infty}\varepsilon_j\|f(u^{k_{i_j}})\|\|y^j\|=\lim_{j\longrightarrow+\infty}\varepsilon_j=0$. Since f is Lipschitz continuous, $f(x+\varepsilon_j\|f(u^{k_{i_j}})\|y^j)\longrightarrow f(x)$ as $j\longrightarrow+\infty$. Thus, taking the limit as $j\longrightarrow+\infty$ in (38), we obtain that

$$\langle f(x), x-\widehat{u}\rangle\geq 0, \quad \forall x\in C. \tag{39}$$

So, $\widehat{u}\in \mathrm{Sol}^d(C,f)$.

Next, we prove $\{u^k\}$ has finite weak cluster points in $\mathrm{Sol}(C,f)$. First, we show that $\{u^k\}$ has at most one weak cluster point in $\mathrm{Sol}^d(C,f)$. Let $\widehat{u}\in\mathrm{Sol}^d(C,f)$ and $\tilde{u}\in\mathrm{Sol}^d(C,f)$ be two distinct weak cluster points of $\{u^k\}$. There exist two sequences $\{u^{k_i}\}$ and $\{u^{k_j}\}$ of $\{u^k\}$ satisfying $u^{k_i}\rightharpoonup\widehat{u}$ as $i\longrightarrow+\infty$ and $u^{k_j}\rightharpoonup\tilde{u}$ as $j\longrightarrow+\infty$. Note that for all $k\geq 0$,

$$2\left\langle u^k, \widehat{u}-\tilde{u}\right\rangle=\|u^k-\tilde{u}\|^2-\|u^k-\widehat{u}\|^2+\|\widehat{u}\|^2-\|\tilde{u}\|^2. \tag{40}$$

Since $\lim_{k\longrightarrow+\infty}\|u^k-\widehat{u}\|$ and $\lim_{k\longrightarrow+\infty}\|u^k-\tilde{u}\|$ exist, by (40), we conclude that $\lim_{k\longrightarrow+\infty}\langle u^k,\widehat{u}-\tilde{u}\rangle$ exists, denoted by l. Thus,

$$l=\lim_{i\longrightarrow+\infty}\left\langle u^{k_i},\widehat{u}-\tilde{u}\right\rangle=\lim_{j\longrightarrow+\infty}\left\langle u^{k_j},\widehat{u}-\tilde{u}\right\rangle. \tag{41}$$

Since $u^{k_i}\rightharpoonup\widehat{u}$ and $u^{k_j}\rightharpoonup\tilde{u}$, from (41), we have

$$l=\langle\widehat{u},\widehat{u}-\tilde{u}\rangle=\langle\tilde{u},\widehat{u}-\tilde{u}\rangle, \tag{42}$$

which implies that $\|\widehat{u}-\tilde{u}\|^2=0$ and hence, $\widehat{u}=\tilde{u}$. Therefore, $\{u^k\}$ has at most one weak cluster point in $\mathrm{Sol}^d(C,f)$. By the condition (C3), $\{u\in C, f(u)=0\}\setminus\mathrm{Sol}^d(C,f)$ is a finite set. Therefore, $\{u^k\}$ has finite weak cluster points in $\mathrm{Sol}(C,f)$.

Let $p_1,p_2,\cdots,p_t$ be the finite weak cluster points of $\{u^k\}$ in $\mathrm{Sol}(C,f)$. Set $I=\{1,2,\cdots,t\}$ and

$$\sigma=\min\left\{\frac{\|p_n-p_m\|}{3}, n,m\in I, n\neq m\right\}. \tag{43}$$

Taking any weak cluster point $p_n, n\in I$, there exists a subsequence $\{u_n^{k_i}\}$ of $\{u^k\}$ such that $u_n^{k_i}\rightharpoonup p_n$ as $i\longrightarrow+\infty$. Then, we have

$$\lim_{i\longrightarrow+\infty}\left\langle u_n^{k_i},\frac{p_n-p_m}{\|p_n-p_m\|}\right\rangle=\left\langle p_n,\frac{p_n-p_m}{\|p_n-p_m\|}\right\rangle, \quad \forall m\in I. \tag{44}$$

Observe that $\forall m\neq n$,

$$\begin{aligned}\left\langle p_n,\frac{p_n-p_m}{\|p_n-p_m\|}\right\rangle&=\frac{\|p_n-p_m\|}{2}+\frac{\|p_n\|^2-\|p_m\|^2}{2\|p_n-p_m\|}\\&>\sigma+\frac{\|p_n\|^2-\|p_m\|^2}{2\|p_n-p_m\|}.\end{aligned} \tag{45}$$

According to (44) and (45), there exists a large enough positive integer $n(i)$ such that when $i\geq n(i)$,

$$u_n^{k_i}\in\left\{x:\left\langle x,\frac{p_n-p_m}{\|p_n-p_m\|}\right\rangle>\sigma+\frac{\|p_n\|^2-\|p_m\|^2}{2\|p_n-p_m\|}\right\}, \quad m\in I, m\neq n. \tag{46}$$

Set

$$R_n=\bigcap_{m=1,m\neq n}^{t}\left\{x:\left\langle x,\frac{p_n-p_m}{\|p_n-p_m\|}\right\rangle>\sigma+\frac{\|p_n\|^2-\|p_m\|^2}{2\|p_n-p_m\|}\right\}. \tag{47}$$

Step 1. Let u^k and λ_k be given. Compute
$v^k = P_C(u^k - \lambda_k f(u^k))$.
Criterion: if $v^k = u^k$, then stop.
Step 2. Compute
$u^{k+1} = v^k + \lambda_k(f(u^k) - f(v^k))$,
$$\lambda_{k+1} = \begin{cases} \min\{(\delta\|u^k - v^k\|/\|f(u^k) - f(v^k)\|), \lambda_k\}, & f(u^k) \neq f(v^k), \\ \lambda_k, & \text{else}, \end{cases}$$
update k to $k+1$ and go to Step 1.

Algorithm 2: Let $\lambda_0 > 0$ and $\delta \in (0, 1)$. Select the starting point $u^0 \in H$. Set $k = 0$.

In the light of (46) and (47), we have $u_n^{k_i} \in R_n$ when $i \geq \max\{n(i), n \in I\}$.

Now, we show that $u^k \in \bigcup_{n=1}^t R_n$ for a large enough k. Assume that there exists a subsequence $\{u^{k_j}\}$ of $\{u^k\}$ such that $u^{k_j} \notin \bigcup_{n=1}^t R_n$. By the boundedness of $\{u^{k_j}\}$, there exists a subsequence of $\{u^{k_j}\}$ convergent weakly to $p^\dagger$. Without loss of generality, we still denote the subsequence as $\{u^{k_j}\}$. According to assumptions, $u^{k_j} \notin \bigcup_{n=1}^t R_n$, so $u^{k_j} \notin R_n$ for any $n \in I$. Therefore, there exists a subsequence $\{u^{k_{j_s}}\}$ of $\{u^{k_j}\}$ such that $\forall s \geq 0$,

$$u^{k_{j_s}} \notin \left\{x : \left\langle x, \frac{p_n - p_m}{\|p_n - p_m\|}\right\rangle > \sigma + \frac{\|p_n\|^2 - \|p_m\|^2}{2\|p_n - p_m\|}\right\}, \quad m \in I, m \neq n. \tag{48}$$

Thus,

$$p^\dagger \notin \left\{x : \left\langle x, \frac{p_n - p_m}{\|p_n - p_m\|}\right\rangle > \sigma + \frac{\|p_n\|^2 - \|p_m\|^2}{2\|p_n - p_m\|}\right\}, \quad m \in I, m \neq n, \tag{49}$$

which implies that $p^\dagger \neq p_n, n \in I$. This is impossible. So, for a large enough positive integer K_0, $u^k \in \bigcup_{n=1}^t R_n$ when $k \geq K_0$.

Next, we show that $\{u^k\}$ has a unique weak cluster point in $\mathrm{Sol}(C, f)$. First, from (29), there exists a positive integer $K_1 \geq K_0$ such that $\|u^{k+1} - u^k\| < \sigma$ for all $k \geq K_1$. Assume that $\{u^k\}$ has at least two weak cluster points in $\mathrm{Sol}(C, f)$. Then, there exists $\widehat{K} \geq K_1$ such that $u^{\widehat{K}} \in R_n$ and $u^{\widehat{K}+1} \in R_m$, where $n, m \in I$ and $t \geq 2$, that is,

$$\begin{aligned} u^{\widehat{K}} \in R_n &= \bigcap_{m=1, m\neq n}^{t} \left\{x : \left\langle x, \frac{p_n - p_m}{\|p_n - p_m\|}\right\rangle > \sigma + \frac{\|p_n\|^2 - \|p_m\|^2}{2\|p_n - p_m\|}\right\}, \\ u^{\widehat{K}+1} \in R_m &= \bigcap_{n=1, n\neq m}^{t} \left\{x : \left\langle x, \frac{p_m - p_n}{\|p_m - p_n\|}\right\rangle > \sigma + \frac{\|p_m\|^2 - \|p_n\|^2}{2\|p_m - p_n\|}\right\}. \end{aligned} \tag{50}$$

Therefore,

$$\left\langle u^{\widehat{K}}, \frac{p_n - p_m}{\|p_n - p_m\|}\right\rangle > \sigma + \frac{\|p_n\|^2 - \|p_m\|^2}{2\|p_n - p_m\|}, \tag{51}$$

$$\left\langle u^{\widehat{K}+1}, \frac{p_m - p_n}{\|p_m - p_n\|}\right\rangle > \sigma + \frac{\|p_m\|^2 - \|p_n\|^2}{2\|p_m - p_n\|}. \tag{52}$$

Combining (51) and (52), we achieve

$$\left\langle u^{\widehat{K}} - u^{\widehat{K}+1}, \frac{p_n - p_m}{\|p_n - p_m\|}\right\rangle > 2\sigma. \tag{53}$$

At the same time, we have

$$\left\| u^{\widehat{K}+1} - u^{\widehat{K}} \right\| < \sigma. \tag{54}$$

Based on (53) and (54), we deduce

$$2\sigma < \left\langle u^{\widehat{K}} - u^{\widehat{K}+1}, \frac{p_n - p_m}{\|p_n - p_m\|}\right\rangle \leq \left\| u^{\widehat{K}} - u^{\widehat{K}+1} \right\| < \sigma. \tag{55}$$

This leads to a contradiction. Thus, $\{u^k\}$ has a unique weak cluster point in $\mathrm{Sol}(C, f)$. Therefore, $\{u^k\}$ converges weakly to a point in $\mathrm{Sol}(C, f)$. This completes the proof.□

□

Corollary 5. *Suppose that the conditions (C1)-(C3) and (17) are satisfied. Then, the sequence $\{u^k\}$ generated by Algorithm 2 converges weakly to a point in $\mathrm{Sol}(C, f)$.*

Remark 6. If f is pseudomonotone, then Theorem 4 and Corollary 5 hold.

Remark 7. If the operator f is sequentially weakly continuous and also satisfies conditions (C1)-(C3), then Theorem 4 and Corollary 5 still hold.

Remark 8. Our main purpose is to solve (1); hence, a natural condition is $\mathrm{Sol}(C, f) \neq \varnothing$. In order to prove our main theorem, we assume that $\mathrm{Sol}^d(C, f) \neq \varnothing$. Note that $\mathrm{Sol}^d(C, f) \subset \mathrm{Sol}(C, f)$. This means that even if $\mathrm{Sol}(C, f) \neq \varnothing$, $\mathrm{Sol}^d(C, f) \neq \varnothing$ does not necessarily hold. A question is under what conditions $\mathrm{Sol}^d(C,f) \neq \varnothing$ holds. In fact, we have the following results:

(i) If f is pseudomonotone on C and $\mathrm{Sol}(C,f) \neq \varnothing$, then $\mathrm{Sol}^d(C,f) \neq \varnothing$

(ii) If f is quasimonotone on C, int $C \neq \varnothing$ and $\{u \mid f(u) = 0\} \neq \varnothing$, then $\mathrm{Sol}^d(C,f) \neq \varnothing$ (see [35])

Data Availability

No data were used to support this study.

Conflicts of Interest

All authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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