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Research Article

Fractional Sobolev Space on Time Scales and Its Application to a Fractional Boundary Value Problem on Time Scales

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By the concept of fractional derivative of Riemann-Liouville on time scales, we first introduce fractional Sobolev spaces, characterize them, define weak fractional derivatives, and show that they coincide with the Riemann-Liouville ones on time scales. Then, we prove equivalence of some norms in the introduced spaces and derive their completeness, reflexivity, separability, and some imbeddings. Finally, as an application, by constructing an appropriate variational setting, using fibering mapping and Nehari manifolds, the existence of weak solutions for a class of fractional boundary value problems on time scales is studied, and a result of the existence of weak solutions for this problem is obtained.

1. Introduction

The Sobolev space theory was developed by the Soviet mathematician S.L. Sobolev in the 1930s. It was created for the needs of studying modern theories of differential equations and studying many problems in the fields related to mathematical analysis. It has become a basic content in mathematics. In order to study the existence of solutions of differential and difference equations under a unified framework, papers [1–3] study some Sobolev space theories on time scales.

In the past few decades, fractional calculus and fractional differential equations have attracted widespread attention in the field of differential equations, as well as in applied mathematics and science. In addition to true mathematical interest and curiosity, this trend is also driven by interesting scientific and engineering applications that have produced fractional differential equation models to better describe (time) memory effects and (space) nonlocal phenomena [4–9]. It is the rise of these applications that give new vitality to the field of fractional calculus and fractional differential equations and call for further research in this field.

In order to unify the discrete analysis and continuous analysis, Hilger [10] proposed the time scale theory and established its related basic theory [11, 12]. So far, the study of time scale theory has attracted worldwide attention. It has been widely used in engineering, physics, economics, population dynamics, cybernetics, and other fields [13–17].

As far as we know, no one has studied the fractional Sobolev space and its properties on time scales through the Riemann-Liouville derivative. In order to fill this gap, the main purpose of this article is to establish the fractional Sobolev space on time scales via the Riemann-Liouville derivative and to study its basic properties. Then, as an application of our new theory, we study the solvability of a class of fractional boundary value problems on time scales.

2. Preliminaries

In this section, we briefly collect some basic known notations, definitions, and results that will be used later.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real set $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$. Throughout this paper, we denote by $\mathbb{T}$ a time scale. We will use the following notations: $J_{\mathbb{R}}^0 = [a, b)$, $J_{\mathbb{R}} = [a, b]$, $J^0 = J_{\mathbb{R}}^0 \cap \mathbb{T}$, $J = J_{\mathbb{R}} \cap \mathbb{T}$, $J^k = [a, \rho(b)] \cap \mathbb{T}$.

Definition 1 (see [18]). For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \longrightarrow \mathbb{T}$ by $\sigma(t) := \inf \{s \in \mathbb{T} : s > t\}$, while the backward jump operator $\rho : \mathbb{T} \longrightarrow \mathbb{T}$ is defined by $\rho(t) \coloneqq \sup \{s \in \mathbb{T} : s < t\}$.

Remark 2 (see [18]).

(1) In Definition 1, we put $\inf \varnothing = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup \varnothing = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum t), where $\varnothing$ denotes the empty set.

(2) If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated.

(3) If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, we say that t is right-dense, while if $t > \inf \mathbb{T}$ and $\rho(t) = t$, we say that t is left-dense. Points that are right-dense and left-dense at the same time are called dense

(4) The graininess function $\mu : \mathbb{T} \longrightarrow [0,\infty)$ is defined by $\mu(t) \coloneqq \sigma(t) - t$.

(5) The derivative makes use of the set $\mathbb{T}^k$, which is derived from the time scale $\mathbb{T}$ as follows: if $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^k \coloneqq \mathbb{T} \setminus \{M\}$; otherwise, $\mathbb{T}^k \coloneqq \mathbb{T}$.

Definition 3 [18]. Assume that $f : \mathbb{T} \longrightarrow \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then, we define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s) \right| \le \varepsilon |\sigma(t) - s|, \tag{1}$$

for all $s \in U$. We call $f^{\Delta}(t)$ the delta (or Hilger) derivative of f at t. Moreover, we say that f is delta (or Hilger) differentiable (or in short: differentiable) on $\mathbb{T}^k$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^k$. The function $f^{\Delta} : \mathbb{T}^k \longrightarrow \mathbb{R}$ is then called the (delta) derivative of f on $\mathbb{T}^k$.

Definition 4 (see [18]). A function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \longrightarrow \mathbb{R}$ will be denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$. The set of functions $f : \mathbb{T} \longrightarrow \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by $C^1_{rd} = C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 5 (see [19]). Let J denote a closed bounded interval in $\mathbb{T}$. A function $F : J \longrightarrow \mathbb{R}$ is called a delta antiderivative of function $f : J \longrightarrow \mathbb{R}$ provided F is continuous on J, delta differentiable at J, and $F^{\Delta}(t) = f(t)$ for all $t \in J$. Then, we define the Δ-integral of f from a to b by $\int_a^b f(t)\Delta t \coloneqq F(b) - F(a)$.

Proposition 6 (see [20]). *f is an increasing continuous function on J. If F is the extension of f to the real interval $J_{\mathbb{R}}$ given by*

$$F(s) \coloneqq \begin{cases} f(s), & \text{if } s \in \mathbb{T}, \\ f(t), & \text{if } s \in (t, \sigma(t)) \notin \mathbb{T}, \end{cases} \tag{2}$$

then $\int_a^b f(t)\Delta t \le \int_a^b F(t)dt$.

Definition 7 (see [19]) (fractional integral on time scales). Suppose h is an integrable function on J. Let $0 < \alpha \le 1$. The left fractional integral of order α of h is defined by

$${}_a^{\mathbb{T}}I_t^{\alpha} h(t) \coloneqq \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\Delta s. \tag{3}$$

The right fractional integral of order α of h is defined by

$${}_t^{\mathbb{T}}I_b^{\alpha} h(t) \coloneqq \int_t^b \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} h(s)\Delta s, \tag{4}$$

where Γ is the gamma function.

Definition 8 (see [19]) (Riemann-Liouville fractional derivative on time scales). Let $t \in \mathbb{T}$, $0 < \alpha \le 1$, and $h : \mathbb{T} \longrightarrow \mathbb{R}$. The left Riemann-Liouville fractional derivative of order α of h is defined by

$${}_a^{\mathbb{T}}D_t^{\alpha} h(t) \coloneqq \left({}_a^{\mathbb{T}}I_t^{1-\alpha} h(t)\right)^{\Delta} = \frac{1}{\Gamma(1-\alpha)} \left(\int_a^t (t-s)^{-\alpha} h(s)\Delta s\right)^{\Delta}. \tag{5}$$

The right Riemann-Liouville fractional derivative of order α of h is defined by

$${}_a^{\mathbb{T}}D_t^{\alpha} h(t) \coloneqq \left({}_t^{\mathbb{T}}I_b^{1-\alpha} h(t)\right)^{\Delta} = \frac{-1}{\Gamma(1-\alpha)} \left(\int_t^b (t-s)^{-\alpha} h(s)\Delta s\right)^{\Delta}. \tag{6}$$

Proposition 9 (see [19]). *Let $0 < \alpha \le 1$, we have ${}_a^{\mathbb{T}}D_t^{\alpha} = \Delta \circ {}_a^{\mathbb{T}}I_t^{1-\alpha}$.*

Proposition 10 (see [19]). *For any function h that is integrable on J, the Riemann-Liouville Δ-fractional integral satisfies ${}_v^{\mathbb{T}}I_t^{\alpha} \circ {}_a^{\mathbb{T}}I_t^{\beta} = {}_a^{\mathbb{T}}I_t^{\alpha+\beta}$ for $\alpha > 0$ and $\beta > 0$.*

Proposition 11 (see [19]). *For any function h that is integrable on J, one has ${}_a^{\mathbb{T}}D_t^{\alpha} \circ {}_a^{\mathbb{T}}I_t^{\alpha} h = h$.*

Corollary 12 (see [19]). *For $0 < \alpha \le 1$, we have ${}_a^{\mathbb{T}}D_t^{\alpha} \circ {}_a^{\mathbb{T}}D_t^{-\alpha} = Id$ and ${}_a^{\mathbb{T}}I_t^{-\alpha} \circ {}_a^{\mathbb{T}}I_t^{\alpha} = Id$, where Id denotes the identity operator.*

Theorem 13 (see [19]). *Let $f \in C(J)$ and $\alpha > 0$, then $f \in {}_{a}^{\mathbb{T}}I_{t}^{\alpha}(J)$ iff*

$$ {}_{a}^{\mathbb{T}}I_{t}^{1-\alpha} f \in C^{1}(J), \quad \left({}_{a}^{\mathbb{T}}I_{t}^{1-\alpha} f(t) \right)|_{t=a} = 0. \tag{7}$$

Theorem 14 (see [19]). *Let $\alpha > 0$ and $f \in C(J)$ satisfy the condition in Theorem 13. Then,*

$$\left({}_{a}^{\mathbb{T}}I_{t}^{\alpha} \circ {}_{a}^{\mathbb{T}}D_{t}^{\alpha} \right)(f) = f. \tag{8}$$

Theorem 15 (see [2]). *A function $f : J \longrightarrow \mathbb{R}^{N}$ is absolutely continuous on J iff f is Δ -differentiable Δ-a.e. on J^{0} and*

$$f(t) = f(a) + \int_{[a,t)_{\mathbb{T}}} f^{\Delta}(s)\Delta s, \quad \forall t \in J. \tag{9}$$

Theorem 16 (see [21]). *A function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is absolutely continuous on $\mathbb{T}$ iff the following conditions are satisfied:*

(i) f is Δ-differentiable Δ-a.e. on J^{0} and $f^{\Delta} \in L^{1}(\mathbb{T})$

(ii) The equality

$$f(t) = f(a) + \int_{[a,t)_{\mathbb{T}}} f^{\Delta}(s)\Delta s, \tag{10}$$

holds for every $t \in \mathbb{T}$.

Theorem 17 (see [22]). *A function $q : J_{\mathbb{R}} \longrightarrow \mathbb{R}^{m}$ is absolutely continuous iff there exist a constant $c \in \mathbb{R}^{m}$ and a function $\varphi \in L^{1}$ such that*

$$q(t) = c + \left(I_{a^{+}}^{1} \varphi \right)(t), \quad t \in J_{\mathbb{R}}. \tag{11}$$

In this case, we have $q(a) = c$ and $q'(t) = \varphi(t)$, $t \in J_{\mathbb{R}}$ a.e.

Theorem 18 (see [2]) (integral representation). *Let $\alpha \in (0, 1)$ and $q \in L^{1}$. Then, q has a left-sided Riemann-Liouville derivative $D_{a^{+}}^{\alpha} q$ of order α iff there exist a constant $c \in \mathbb{R}^{m}$ and a function $\varphi \in L^{1}$ such that*

$$q(t) = \frac{1}{\Gamma(\alpha)} \frac{c}{(t-a)^{1-\alpha}} + \left(I_{a^{+}}^{\alpha} \varphi \right)(t), \quad t \in J_{\mathbb{R}} \text{ a.e.} \tag{12}$$

In this case, we have $I_{a^{+}}^{1-\alpha} q(a) = c$ and $(D_{a^{+}}^{\alpha} q)(t) = \varphi(t)$, $t \in J_{\mathbb{R}}$ a.e.

Theorem 19 (see [23]). *Let $\alpha > 0$, $p, q \geq 1$, and $1/p + 1/q \leq 1 + \alpha$, where $p \neq 1$ and $q \neq 1$ in the case when $1/p + 1/q = 1 + \alpha$. Moreover, let*

$$\begin{aligned} {}_{a}^{\mathbb{T}}I_{t}^{\alpha}(L^{p}) &:= \left\{ f : f = {}_{a}^{\mathbb{T}}I_{t}^{\alpha} g, g \in L^{p}(J) \right\}, \\ {}_{t}^{\mathbb{T}}I_{b}^{\alpha}(L^{p}) &:= \left\{ f : f = {}_{t}^{\mathbb{T}}I_{b}^{\alpha} g, g \in L^{p}(J) \right\}, \end{aligned} \tag{13}$$

then the following integration by part formulas hold.

(a) If $\varphi \in L^{p}(J)$ and $\psi \in L^{q}(J)$, then

$$\int_{J^{0}} \varphi(t) \left({}_{a}^{\mathbb{T}}I_{t}^{\alpha} \psi \right)(t)\Delta t = \int_{J^{0}} \psi(t) \left({}_{t}^{\mathbb{T}}I_{t}^{\alpha} \varphi \right)(t)\Delta t. \tag{14}$$

(b) If $g \in {}_{t}^{\mathbb{T}}I_{b}^{\alpha}(L^{p})$ and $f \in {}_{t}^{\mathbb{T}}I_{b}^{\alpha}(L^{p})$, then

$$\int_{J^{0}} g(t) \left({}_{a}^{\mathbb{T}}D_{t}^{\alpha} f \right)(t)\Delta t = \int_{J^{0}} f(t) \left({}_{t}^{\mathbb{T}}D_{b}^{\alpha} g \right)(t)\Delta t. \tag{15}$$

Lemma 20 (see [1]). *Let $f \in L_{\Delta}^{1}(J^{0})$. Then, the following*

$$\int_{J^{0}} \left(f \cdot \varphi^{\Delta} \right)(s)\Delta s = 0, \quad \text{for every } \varphi \in C_{0,rd}^{1}\left(J^{k}\right), \tag{16}$$

holds iff there exists a constant $c \in \mathbb{R}$ such that

$$f \equiv c\, \Delta - a.e. \text{ on } J^{0}. \tag{17}$$

Definition 21 (see [1]). Let $p \in \overline{\mathbb{R}}$ be such that $p \geq 1$ and $u : J \longrightarrow \overline{\mathbb{R}}$. Say that u belongs to $W_{\Delta}^{1,p}(J)$ iff $u \in L_{\Delta}^{p}(J^{0})$ and there exists $g : J^{k} \longrightarrow \overline{\mathbb{R}}$ such that $g \in L_{\Delta}^{p}(J^{0})$ and

$$\int_{J^{0}} \left(u \cdot \varphi^{\Delta} \right)(s)\Delta s = -\int_{J^{0}} (g \cdot \varphi^{\sigma})(s)\Delta s, \quad \forall \varphi \in C_{0,\mathrm{rd}}^{1}\left(J^{k}\right), \tag{18}$$

with

$$C_{0,\mathrm{rd}}^{1}\left(J^{k}\right) := \left\{ f : J \longrightarrow \mathbb{R} : f \in C_{\mathrm{rd}}^{1}\left(J^{k}\right), f(a) = f(b) \right\}, \tag{19}$$

where $C_{\mathrm{rd}}^{1}(J^{k})$ is the set of all continuous functions on J such that they are Δ-differential on J^{k} and their Δ-derivatives are rd-continuous on J^{k}.

Theorem 22 (see [1]). *Let $p \in \overline{\mathbb{R}}$ be such that $p \geq 1$. Then, the set $L_{\Delta}^{p}(J^{0})$ is a Banach space together with the norm defined for every $f \in L_{\Delta}^{p}(J^{0})$ as*

$$\|f\|_{L_{\Delta}^{p}} := \begin{cases} \left[\int_{J^{0}} |f|^{p}(s)\Delta s \right]^{1/p}, & \text{if } p \in \mathbb{R}, \\ \inf \left\{ C \in \mathbb{R} : |f| \leq C\, \Delta - a.e. \text{on } J^{0} \right\}, & \text{if } p = +\infty. \end{cases} \tag{20}$$

Moreover, $L_{\Delta}^{2}(J^{0})$ is a Hilbert space together with the *inner product given for every* $(f, g) \in L^2_\Delta(J^0) \times L^2_\Delta(J^0)$ *by*

$$(f, g)_{L^2_\Delta} := \int_{J^0} f(s) \cdot g(s) \Delta s. \tag{21}$$

Theorem 23 (see [24]). *Fractional integration operators are bounded in* $L^p(J_{\mathbb{R}})$*, i.e., the following estimate*

$$\| I^\alpha_{a^+} \varphi \|_{L^p(a,b)} \leq \frac{(b-a)^{Re\alpha}}{Re\alpha |\Gamma(\alpha)|} \|\varphi\|_{L^p(J_{\mathbb{R}})}, \quad Re\alpha > 0, \tag{22}$$

holds.

Proposition 24 (see [1]). *Suppose* $p \in \overline{\mathbb{R}}$ *and* $p \geq 1$*. Let* $p' \in \overline{\mathbb{R}}$ *be such that* $1/p' + 1/p' = 1$*. Then, if* $f \in L^p_\Delta(J^0)$ *and* $g \in L^{p'}_\Delta(J^0)$*, then* $f \cdot g \in L^1_\Delta(J^0)$ *and*

$$\|f \cdot g\|_{L^1_\Delta} \leq \|f\|_{L^p_\Delta} \cdot \|g\|_{L^{p'}_\Delta}. \tag{23}$$

This expression is called Hölder's inequality and Cauchy-Schwarz's inequality whenever $p = 2$*.*

Theorem 25 (see [25]) (the first mean value theorem). *Let* f *and* g *be bounded and integrable functions on* J*, and let* g *be nonnegative (or nonpositive) on* J*. Let us set*

$$m = \inf \left\{ f(t) : t \in J^0 \right\}, \quad M = \sup \left\{ f(t) : t \in J^0 \right\}. \tag{24}$$

Then, there exists a real number Λ *satisfying the inequalities* $m \leq \Lambda \leq M$ *such that*

$$\int_a^b f(t) g(t) \Delta t = \Lambda \int_a^b g(t) \Delta t. \tag{25}$$

Corollary 26 (see [25]). *Let* f *be an integrable function on* J *and let* m *and* M *be the infimum and supremum, respectively, of* f *on* J^0*. Then, there exists a number* Λ *between* m *and* M *such that* $\int_a^b f(t) \Delta t = \Lambda (b-a)$*.*

Theorem 27 (see [25]). *Let* f *be a function defined on* J *and let* $c \in \mathbb{T}$ *with* $a < c < b$*. If* f *is* Δ*-integrable from* a *to* c *and from* c *to* b*, then* f *is* Δ*-integrable from* a *to* b *and*

$$\int_a^b f(t) \Delta t = \int_a^c f(t) \Delta t + \int_c^b f(t) \Delta t. \tag{26}$$

Lemma 28 (see [26]) (a time scale version of the Arzelà-Ascoli theorem). *Let* X *be a subset of* $C(J, \mathbb{R})$ *satisfying the following conditions:*

(i) X *is bounded*

(ii) For any given $\varepsilon > 0$*, there exists* $\delta > 0$ *such that* $t_1, t_2 \in J$*,* $|t_1 - t_2| < \delta$ *implies* $|f(t_1) - f(t_2)| < \varepsilon$ *for all* $f \in X$

Then, X *is relatively compact.*

3. Fractional Sobolev Spaces on Time Scales and Their Properties

In this section, we present and prove some lemmas, propositions, and theorems, which are of utmost significance for our main results.

In the following, let $0 < a < b$. Inspired by Theorems 15–18, we give the following definition.

Definition 29. Let $0 < \alpha \leq 1$. By $AC^{\alpha,1}_{\Delta,a^+}(J, \mathbb{R}^N)$, we denote the set of all functions $f : J \longrightarrow \mathbb{R}^N$ that have the representation

$$f(t) = \frac{1}{\Gamma(\alpha)} \frac{c}{(t-a)^{1-\alpha}} + {}^{\mathbb{T}}_a I^\alpha_t \varphi(t), \quad t \in J \, \Delta - \text{a.e.}, \tag{27}$$

with $c \in \mathbb{R}^N$ and $\varphi \in L^1_\Delta$.

Then, we have the following result.

Theorem 30. *Let* $0 < \alpha \leq 1$ *and* $f \in L^1_\Delta$*. Then, function* f *has the left Riemann-Liouville derivative* ${}^T_a D^\alpha_t f$ *of order* α *on the interval* J *iff* $f \in AC^{\alpha,1}_{\Delta,a^+}(J, \mathbb{R}^N)$*; that is,* f *has the representation (27). In such a case,*

$$\left({}^{\mathbb{T}}_a I^{1-\alpha}_t f \right)(a) = c, \quad \left({}^{\mathbb{T}}_a D^\alpha_t f \right)(t) = \varphi(t), \quad t \in J \quad \Delta\text{-a.e.} \tag{28}$$

Proof. Let us assume that $f \in L^1_\Delta$ has a left-sided Riemann-Liouville derivative ${}^{\mathbb{T}}_a D^\alpha_t f$. This means that ${}^{\mathbb{T}}_a I^{1-\alpha}_t f$ is (identified to) an absolutely continuous function. From the integral representation of Theorems 15 and 17, there exist a constant $c \in \mathbb{R}^N$ and a function $\varphi \in L^1_\Delta$ such that

$$\left({}^{\mathbb{T}}_a I^{1-\alpha}_t f \right)(t) = c + \left({}^{\mathbb{T}}_a I^1_t \varphi \right)(t), \quad t \in J, \tag{29}$$

with ${}^{\mathbb{T}}_a I^{1-\alpha}_t f(a) = c$ and $(({}^{\mathbb{T}}_a I^{1-\alpha}_t f)(t))^\Delta = {}^{\mathbb{T}}_a D^\alpha_t f(t) = \varphi(t)$, $t \in J$ $\Delta -$ a.e.

By Proposition 10 and applying ${}^{\mathbb{T}}_a I^\alpha_t$ to (29), we obtain

$$\left({}^{\mathbb{T}}_a I^1_t f \right)(t) = \left({}^{\mathbb{T}}_a I^\alpha_t c \right)(t) + \left({}^{\mathbb{T}}_a I^1_t {}^{\mathbb{T}}_a I^\alpha_t \varphi \right)(t), \quad t \in J \quad \Delta\text{-a.e.} \tag{30}$$

The result follows from the Δ-differentiability of (30).

Conversely, let us assume that (27) holds true. From Proposition 10 and applying ${}^T_a I^{1-a}_t$ to (27), we obtain

$$\left({}^{\mathbb{T}}_a I^{1-\alpha}_t f \right)(t) = c + \left({}^{\mathbb{T}}_a I^1_t \varphi \right)(t), \quad t \in J \quad \Delta\text{-a.e.}, \tag{31}$$

and then, $({}^{\mathbb{T}}_a I^{1-\alpha}_t f)$ has an absolutely continuous representation. Further, f has a left-sided Riemann-Liouville derivative ${}^{\mathbb{T}}_a D^\alpha_t f$. This completes the proof. □

Remark 31.

(i) By $AC^{\alpha,p}_{\Delta,a^+}$ $(1 \le p<\infty)$, we denote the set of all functions $f : J \longrightarrow \mathbb{R}^N$ possessing representation (27) with $c \in \mathbb{R}^N$ and $\varphi \in L^p_\Delta$

(ii) It is easy to see that Theorem 30 implies that for any $1 \le p < \infty$, f has the left Riemann-Liouville derivative ${}^{\mathbb{T}}_{a}D^{\alpha}_{t}f \in L^p_\Delta$ iff $f \in AC^{\alpha,p}_{\Delta,a^+}$; that is, f has the representation (27) with $\varphi \in L^p_\Delta$

Definition 32. Let $0 < \alpha \le 1$ and let $1 \le p < \infty$. By the left Sobolev space of order α, we will mean the set $W^{\alpha,p}_{\Delta,a^+} = W^{\alpha,p}_{\Delta,a^+}(J, \mathbb{R}^N)$ given by

$$W^{\alpha,p}_{\Delta,a^+} := \left\{ u \in L^p_\Delta ; \exists g \in L^p_\Delta, \forall \varphi \in C^\infty_{c,\mathrm{rd}} \text{ such that} \int_{J^0} u(t) \cdot {}^{\mathbb{T}}_{t}D^{\alpha}_{b}\varphi(t)\Delta t = \int_{J^0} g(t) \cdot \varphi(t)\Delta t \right\}. \tag{32}$$

Remark 33. A function g given in Definition 32 will be called the weak left fractional derivative of order $0 < \alpha \le 1$ of u; let us denote it by ${}^{\mathbb{T}}u^{\alpha}_{a^+}$. The uniqueness of this weak derivative follows from 1.

We have the following characterization of $W^{\alpha,p}_{\Delta,a^+}$.

Theorem 34. *If $0 < \alpha \le 1$ and $1 \le p < \infty$, then $W^{\alpha,p}_{\Delta,a^+} = AC^{\alpha,p}_{\Delta,a^+} \cap L^p_\Delta$.*

Proof. On the one hand, if $u \in AC^{\alpha,p}_{\Delta,a^+} \cap L^p_\Delta$, then from Theorem 30, it follows that u has derivative ${}^{\mathbb{T}}_{a}D^{\alpha}_{t}u \in L^p_\Delta$. Theorem 19 implies that

$$\int_{J^0} u(t){}^{\mathbb{T}}_{t}D^{\alpha}_{b}\varphi(t)\Delta t = \int_{J^0} \left({}^{\mathbb{T}}_{a}D^{\alpha}_{t}u\right)(t)\varphi(t)\Delta t, \tag{33}$$

for any $\varphi \in C^\infty_{c,\mathrm{rd}}$. So, $u \in W^{\alpha,p}_{\Delta,a^+}$ with ${}^{\mathbb{T}}u^{\alpha}_{a^+} = g = {}^{\mathbb{T}}_{a}D^{\alpha}_{t}u \in L^p_\Delta$.

On the other hand, if $u \in W^{\alpha,p}_{\Delta,a^+}$, then $u \in L^p_\Delta$, and there exists a function $g \in L^p_\Delta$ such that

$$\int_{J^0} u(t){}^{\mathbb{T}}_{t}D^{\alpha}_{b}\varphi(t)\Delta t = \int_{J^0} g(t)\varphi(t)\Delta t, \tag{34}$$

for any $\varphi \in C^\infty_{c,\mathrm{rd}}$. To show that $u \in AC^{\alpha,p}_{\Delta,a^+} \cap L^p_\Delta$, it suffices to check (Theorem 30 and definition of $AC^{\alpha,p}_{\Delta,a^+}$) that u possesses the left Riemann-Liouville derivative of order α, which belongs to L^p_Δ; that is, ${}^{\mathbb{T}}_{a}I^{1-\alpha}_{t}u$ is absolutely continuous on J and its delta derivative of α order (existing Δ-a.e. on J) belongs to L^p_Δ.

In fact, let $\varphi \in C^\infty_{c,\mathrm{rd}}$, then $\varphi \in {}^{\mathbb{T}}_{t}D^{\alpha}_{b}(C_{\mathrm{rd}})$ and ${}^{\mathbb{T}}_{t}D^{\alpha}_{b}\varphi = -({}^{\mathbb{T}}_{t}I^{1-\alpha}_{b})^{\Delta}$. From Theorem 19, it follows that

$$\begin{aligned}\int_{J^0} u(t){}^{\mathbb{T}}_{t}D^{\alpha}_{b}\varphi(t)\Delta t &= \int_{J^0} u(t)\left(-{}^{\mathbb{T}}_{t}I^{1-\alpha}_{b}\varphi\right)^{\Delta}(t)\Delta t \\ &= \int_{J^0} \left({}^{\mathbb{T}}_{a}D^{1-\alpha}_{t}{}^{\mathbb{T}}_{a}I^{1-\alpha}_{t}u\right)(t)\left(-{}^{\mathbb{T}}_{t}I^{1-\alpha}_{b}\varphi\right)^{\Delta}(t)\Delta t \\ &= \int_{J^0} \left({}^{\mathbb{T}}_{a}I^{1-\alpha}_{t}u\right)(t)(-\varphi)^{\Delta}(t)\Delta t \\ &= -\int_{J^0} \left({}^{\mathbb{T}}_{a}I^{1-\alpha}_{t}u\right)(t)\varphi^{\Delta}(t)\Delta t.\end{aligned} \tag{35}$$

In view of (34) and (35), we get

$$\int_{J^0} \left({}^{\mathbb{T}}_{a}I^{1-\alpha}_{t}u\right)(t)\varphi^{\Delta}(t)\Delta t = -\int_{J^0} g(t)\varphi(t)\Delta t, \tag{36}$$

for any $\varphi \in C^\infty_{c,\mathrm{rd}}$. So, ${}^{\mathbb{T}}_{a}I^{1-\alpha}_{t}u \in W^{1,p}_{\Delta,a^+}$. Consequently, ${}^{\mathbb{T}}_{a}I^{1-\alpha}_{t}u$ is absolutely continuous and its delta derivative is equal Δ-a.e. on $[a, b]_{\mathbb{T}}$ to $g \in L^p_\Delta$. The proof is complete. □

From the proof of Theorem 34 and the uniqueness of the weak fractional derivative, the following theorem follows.

Theorem 35. *If $0 < \alpha \le 1$ and $1 \le p < \infty$, then the weak left fractional derivative ${}^{\mathbb{T}}u^{\alpha}_{a^+}$ of a function $u \in W^{\alpha,p}_{\Delta,a^+}$ coincides with its left Riemann-Liouville fractional derivative ${}^{\mathbb{T}}_{a}D^{\alpha}_{t}u$ Δ-a.e. on J.*

Remark 36.

(1) If $0 < \alpha \le 1$ and $(1-\alpha)p < 1$, then $AC^{\alpha,p}_{\Delta,a^+} \subset L^p_\Delta$ and, consequently,

$$W^{\alpha,p}_{\Delta,a^+} = AC^{\alpha,p}_{\Delta,a^+} \cap L^p_\Delta = AC^{\alpha,p}_{\Delta,a^+}. \tag{37}$$

(2) If $0 < \alpha \le 1$ and $(1-\alpha)p \ge 1$, then $W^{\alpha,p}_{\Delta,a^+} = AC^{\alpha,p}_{\Delta,a^+} \cap L^p_\Delta$ is the set of all functions belonging to $AC^{\alpha,p}_{\Delta,a^+}$ that satisfy the condition $({}^{\mathbb{T}}_{a}I^{1-\alpha}_{t}f)(a) = 0$

By using the definition of $W^{\alpha,p}_{\Delta,a^+}$ with $0 < \alpha \le 1$ and Theorem 35, one can easily prove the following result.

Theorem 37. *Let $0 < \alpha \le 1$, $1 \le p < \infty$, and $u \in L^p_\Delta$. Then, $u \in W^{\alpha,p}_{\Delta,a^+}$ iff there exists a function $g \in L^p_\Delta$ such that*

$$\int_{J^0} u(t){}^{\mathbb{T}}_{t}D^{\alpha}_{b}\varphi(t)\Delta t = \int_{J^0} g(t)\varphi(t)\Delta t, \quad \varphi \in C^\infty_{c,rd}. \tag{38}$$

In such a case, there exists the left Riemann-Liouville derivative ${}^{\mathbb{T}}_{a}D^{\alpha}_{t}u$ of u and $g = {}^{\mathbb{T}}_{a}D^{\alpha}_{t}u$.

Remark 38. Function g will be called the weak left fractional derivative of $u \in W^{\alpha,p}_{\Delta,a^+}$ of order α. Its uniqueness follows from [1]. From the above theorem, it follows that it coincides with an appropriate Riemann-Liouville derivative.

Let us fix $0 < \alpha \leq 1$ and consider in the space $W^{\alpha,p}_{\Delta,a^+}$ a norm $\|\cdot\|_{W^{\alpha,p}_{\Delta,a^+}}$ given by

$$\|u\|^p_{W^{\alpha,p}_{\Delta,a^+}} = \|u\|^p_{L^p_\Delta} + \left\|{}^{\mathbb{T}}_a D^\alpha_t u\right\|^p_{L^p_\Delta}, \quad u \in W^{\alpha,p}_{\Delta,a^+}. \tag{39}$$

(Here $\|\cdot\|^p_{L_\Delta}$ denotes the delta norm in L^p_Δ (Theorem 22)).

Lemma 39. *Let $0 < \alpha \leq 1$ and $1 \leq p < \infty$, then*

$$\left\|{}^{\mathbb{T}}_a I^\alpha_t \varphi\right\|^p_{L^p_\Delta} \leq K^p \|\varphi\|^p_{L^p_\Delta}, \tag{40}$$

where $K = (b-a)^\alpha / \Gamma(\alpha+1)$. That is to say, the fractional integration operator is bounded in L^p_Δ.

Proof. The conclusion follows from Theorem 23, Proposition 24, and Proposition 6. The proof is complete. □

Theorem 40. *If $0 < \alpha \leq 1$, then the norm $\|\cdot\|_{W^{\alpha,p}_{\Delta,a^+}}$ is equivalent to the norm $\|\cdot\|_{a,W^{\alpha,p}_{\Delta,a^+}}$ given by*

$$\|u\|^p_{a,W^{\alpha,p}_{\Delta,a^+}} = \left|{}^{\mathbb{T}}_a I^{1-\alpha}_t u(a)\right|^p + \left\|{}^{\mathbb{T}}_a D^\alpha_t u\right\|^p_{L^p_\Delta}, \quad u \in W^{\alpha,p}_{\Delta,a^+}. \tag{41}$$

Proof.

(1) Assume that $(1-\alpha)p < 1$. On the one hand, in view of Remarks 31 and 36, for $u \in W^{\alpha,p}_{\Delta,a^+}$, we can write it as

$$u(t) = \frac{1}{\Gamma(\alpha)} \frac{c}{(t-a)^{1-\alpha}} + {}^{\mathbb{T}}_a I^\alpha_t \varphi(t), \tag{42}$$

with $c \in \mathbb{R}^N$ and $\varphi \in L^p_\Delta$. Since $(t-a)^{(\alpha-1)p}$ is an increasing monotone function, by using Proposition 6, we can write that $\int_{J^0} (t-a)^{(\alpha-1)p} \Delta t \leq \int_{J^0_{\mathbb{R}}} (t-a)^{(\alpha-1)p} dt$. And taking into account Lemma 39, we have

$$\begin{aligned}
\|u\|^p_{L^p_\Delta} &= \int_{J^0} \left| \frac{1}{\Gamma(\alpha)} \frac{c}{(t-a)^{1-\alpha}} + {}^{\mathbb{T}}_a I^\alpha_t \varphi(t) \right|^p \Delta t \\
&\leq 2^{p-1} \left(\frac{|c|^p}{\Gamma^p(\alpha)} \int_{J^0} (t-a)^{(\alpha-1)p} \Delta t \left\|{}^{\mathbb{T}}_a I^\alpha_t \varphi\right\|^p_{L^p_\Delta} \right) \\
&\leq 2^{p-1} \left(\frac{|c|^p}{\Gamma^p(\alpha)} \int_{J^0} (t-a)^{(\alpha-1)p} dt \left\|{}^{\mathbb{T}}_a I^\alpha_t \varphi\right\|^p_{L^p_\Delta} \right) \\
&\leq 2^{p-1} \left(\frac{|c|^p}{\Gamma^p(\alpha)} 1(\alpha-1)p + 1(b-a)^{(\alpha-1)p+1} + K^p \|\varphi\|^p_{L^p_\Delta} \right),
\end{aligned} \tag{43}$$

where K comes from Lemma 39. Noting that $c = {}^{\mathbb{T}}_a I^{1-\alpha}_t u(a)$, $\varphi = {}^{\mathbb{T}}_a D^\alpha_t u$, one can obtain

$$\begin{aligned}
\|u\|^p_{L^p_\Delta} &\leq L_{\alpha,0} \left(|c|^p + \|\varphi\|^p_{L^p_\Delta} \right) \\
&\leq L_{\alpha,0} \left(\left|{}^{\mathbb{T}}_a I^{1-\alpha}_t u(a)\right|^p + \left\|{}^{\mathbb{T}}_a D^\alpha_t u\right\|^p_{L^p_\Delta} \right) \\
&= L_{\alpha,0} \|u\|^p_{a,W^{\alpha,p}_{\Delta,a^+}},
\end{aligned} \tag{44}$$

where

$$L_{\alpha,0} = 2^{p-1} \left(\frac{(b-a)^{1-(1-\alpha)p}}{\Gamma^p(\alpha)(1-(1-\alpha)p)} + K^p \right). \tag{45}$$

Consequently,

$$\|u\|^p_{W^{\alpha,p}_{\Delta,a^+}} = \|u\|^p_{L^p_\Delta} + \left\|{}^{\mathbb{T}}_a D^\alpha_t u\right\|^p_{L^p_\Delta} \leq L_{\alpha,1} \|u\|^p_{a,W^{\alpha,p}_{\Delta,a^+}}, \tag{46}$$

where $L_{\alpha,1} = L_{\alpha,0} + 1$.

On the other hand, we will prove that there exists a constant $M_{\alpha,1}$ such that

$$\|u\|^p_{a,W^{\alpha,p}_{\Delta,a^+}} \leq M_{\alpha,1} \|u\|^p_{W^{\alpha,p}_{\Delta,a^+}}, u \in W^{\alpha,p}_{\Delta,a^+}. \tag{47}$$

Indeed, let $u \in W^{\alpha,p}_{\Delta,a^+}$ and consider coordinate functions $({}^{\mathbb{T}}_a I^{1-\alpha}_t u)^i$ of $({}^{\mathbb{T}}_a I^{1-\alpha}_t u)$ with $i \in \{1, \cdots, N\}$. Lemma 39, Theorem 25, and Corollary 26 imply that there exist constants

$$\Lambda_i \in \left[\inf_{t \in [a,b)_{\mathbb{T}}} \left({}^{\mathbb{T}}_a I^{1-\alpha}_t u\right)^i(t), \sup_{t \in [a,b)_{\mathbb{T}}} \left({}^{\mathbb{T}}_a I^{1-\alpha}_t u\right)^i(t) \right], \quad (i = 1, 2, \cdots, N), \tag{48}$$

such that

$$\Lambda_i = \frac{1}{b-a} \int_a^b \left({}^{\mathbb{T}}_a I^{1-\alpha}_t u\right)^i(s) \Delta s. \tag{49}$$

Hence, for a fixed $t_0 \in J^0$, if $({}^{\mathbb{T}}_a I^{1-\alpha}_t u)^i(t_0) \neq 0$ for all $i = 1, 2, \cdots, N$, then we can take constants θ_i such that

$$\theta_i \left({}^{\mathbb{T}}_a I^{1-\alpha}_t u\right)^i(t_0) = \Lambda_i = \frac{1}{b-a} \int_a^b \left({}^{\mathbb{T}}_a I^{1-\alpha}_t u\right)^i(s) \Delta s. \tag{50}$$

Therefore, we have

$$\left({}^{\mathbb{T}}_a I^{1-\alpha}_t u\right)^i(t_0) = \frac{\theta_i}{b-a} \int_a^b \left({}^{\mathbb{T}}_a I^{1-\alpha}_t u\right)^i(s) \Delta s. \tag{25}$$

From the absolute continuity (Theorem 16) of $({}^{\mathbb{T}}_a I^{1-\alpha}_t u)^i$, it follows that

$$\left({}^{\mathbb{T}}_a I^{1-\alpha}_t u\right)^i(t) = \left({}^{\mathbb{T}}_a I^{1-\alpha}_t u\right)^i(t_0) + \int_{[t_0,t)_{\mathbb{T}}} \left[\left({}^{\mathbb{T}}_a I^{1-\alpha}_t u\right)^i(s) \right]^\Delta \Delta s, \tag{52}$$

for any $t \in J$. Consequently, combining with Proposition 9 and Lemma 39, we see that

$$\begin{aligned} \left| \left({}_a^{\mathbb{T}}I_t^{1-\alpha}u\right)^i(t) \right| &= \left| \left({}_a^{\mathbb{T}}I_t^{1-\alpha}u\right)^i(t_0) + \int_{[t_0,t)_{\mathbb{T}}} \left[\left({}_a^{\mathbb{T}}I_t^{1-\alpha}u\right)^i(s) \right]^{\Delta} \Delta s \right| \\ &\le \frac{|\theta_i|}{b-a} \left\| {}_a^{\mathbb{T}}I_t^{1-\alpha}u \right\|_{L_\Delta^1} + \int_{[t_0,t)_{\mathbb{T}}} \left| \left({}_a^{\mathbb{T}}D_t^{\alpha}u\right)(s) \right| \Delta s \\ &\le \frac{|\theta_i|}{b-a} \left\| {}_a^{\mathbb{T}}I_t^{1-\alpha}u \right\|_{L_\Delta^1} + \left\| {}_a^{\mathbb{T}}D_t^{\alpha}u \right\|_{L_\Delta^1} \\ &\le \frac{|\theta_i|}{b-a} \frac{(b-a)^{1-\alpha}}{\Gamma(2-\alpha)} \|u\|_{L_\Delta^1} + \left\| {}_a^{\mathbb{T}}D_t^{\alpha}u \right\|_{L_\Delta^1}, \end{aligned} \tag{53}$$

for $t \in J$. In particular,

$$\left| \left({}_a^{\mathbb{T}}I_t^{1-\alpha}u\right)^i(t) \right| \le \frac{|\theta_i|}{b-a} \frac{(b-a)^{1-\alpha}}{\Gamma(2-\alpha)} \|u\|_{L_\Delta^1} + \left\| {}_a^{\mathbb{T}}D_t^{\alpha}u \right\|_{L_\Delta^1}. \tag{54}$$

So,

$$\begin{aligned} \left| \left({}_a^{\mathbb{T}}I_t^{1-\alpha}u\right)(a) \right| &\le N \left(\frac{|\theta|(b-a)^{-\alpha}}{\Gamma(2-\alpha)} + 1 \right) \left(\|u\|_{L_\Delta^1} + \left\| {}_a^{\mathbb{T}}D_t^{\alpha}u \right\|_{L_\Delta^1} \right) \\ &\le N M_{\alpha,0} (b-a)^{p-1/p} \left(\|u\|_{L_\Delta^p} + \left\| {}_a^{\mathbb{T}}D_t^{\alpha}u \right\|_{L_\Delta^p} \right), \end{aligned} \tag{55}$$

where $|\theta| = \max_{i \in \{1,2,\cdots,N\}} |\theta_i|$ and $M_{\alpha,0} = |\theta|(b-a)^{-\alpha}/\Gamma(2-\alpha) + 1$. Thus,

$$\left| \left({}_a^{\mathbb{T}}I_t^{1-\alpha}u\right)(a) \right|^p \le N^p M_{\alpha,0}^p (b-a)^{p-1} 2^{p-1} \left(\|u\|_{L_\Delta^p}^p + \left\| {}_a^{\mathbb{T}}D_t^{\alpha}u \right\|_{L_\Delta^p}^p \right), \tag{56}$$

and, consequently,

$$\begin{aligned} \|u\|_{a,W_{\Delta,a^+}^{\alpha,p}}^p &= \left| \left({}_a^{\mathbb{T}}I_t^{1-\alpha}u\right)(a) \right|^p + \left\| {}_a^{\mathbb{T}}D_t^{\alpha}u \right\|_{L_\Delta^p}^p \\ &\le \left(N^p M_{\alpha,0}^p (b-a)^{p-1} 2^{p-1} + 1 \right) \left(\|u\|_{L_\Delta^p}^p + \left\| {}_a^{\mathbb{T}}D_t^{\alpha}u \right\|_{L_\Delta^p}^p \right) \\ &= M_{\alpha,1} \|u\|_{a,W_{\Delta,a^+}^{\alpha,p}}^p, \end{aligned} \tag{57}$$

where $M_{\alpha,1} = N^p M_{\alpha,0}^p (b-a)^{p-1} 2^{p-1} + 1$.

If $\left({}_a^{\mathbb{T}}I_t^{1-\alpha}u\right)^i(t_0) = 0$ for i belongs to some subset of $\{1, 2, \cdots, N\}$, from the above argument process, one can easily see that there exists a constant $M_{\alpha,1}$ such that (32) holds.

(2) When $(1-\alpha)p \ge 1$, then (Remark 36) $W_{\Delta,a^+}^{\alpha,p} = AC_{\Delta,a^+}^{\alpha,p} \cap L_\Delta^p$ is the set of all functions that belong to $AC_{\Delta,a^+}^{\alpha,p}$ that satisfy the condition $\left({}_a^{\mathbb{T}}I_t^{1-\alpha}u\right)(a) = 0$. Hence, in the same way as in the case of $(1-\alpha)p < 1$ (putting $c = 0$), we obtain the inequality

$$\|u\|_{W_{\Delta,a^+}^{\alpha,p}}^p \le L_{\alpha,1} \|u\|_{a,W_{\Delta,a^+}^{\alpha,p}}^p, \quad \text{with some } L_{\alpha,1} > 0. \tag{58}$$

The inequality,

$$\|u\|_{a,W_{\Delta,a^+}^{\alpha,p}}^p \le M_{\alpha,1} \|u\|_{W_{\Delta,a^+}^{\alpha,p}}^p, \quad \text{with some } M_{\alpha,1} > 0, \tag{59}$$

is obvious (it is sufficient to put $M_{\alpha,1} = 1$ and use the fact that $\left({}_a^{\mathbb{T}}I_t^{1-\alpha}u\right)(a) = 0$).

The proof is complete. □

Now, we are in a position to prove some basic properties of the space $W_{\Delta,a^+}^{\alpha,p}$.

Theorem 41. *The space $W_{\Delta,a^+}^{\alpha,p}$ is complete with respect to each of the norms $\|\cdot\|_{W_{\Delta,a^+}^{\alpha,p}}$ and $\|\cdot\|_{a,W_{\Delta,a^+}^{\alpha,p}}$ for any $0 < \alpha \le 1$, $1 \le p < \infty$.*

Proof. In view of Theorem 40, we only need to show that $W_{\Delta,a^+}^{\alpha,p}$ with the norm $\|\cdot\|_{a,W_{\Delta,a^+}^{\alpha,p}}$ is complete. Let $\{u_k\} \subset W_{\Delta,a^+}^{\alpha,p}$ be a Cauchy sequence with respect to this norm. So, the sequences $\{{}_a^{\mathbb{T}}I_t^{1-\alpha}u_k(a)\}$ and $\{{}_a^{\mathbb{T}}D_t^{\alpha}u_k\}$ are Cauchy sequences in $\mathbb{R}^N$ and L_Δ^p, respectively.

Let $c \in \mathbb{R}^N$ and $\varphi \in L_\Delta^p$ be the limits of the above two sequences in $\mathbb{R}^N$ and L_Δ^p, respectively. Then, the function

$$u(t) = \frac{c}{\Gamma(\alpha)} (t-a)^{\alpha-1} + {}_a^{\mathbb{T}}I_t^{\alpha}\varphi(t), \quad t \in J\ \Delta-\text{a.e.}, \tag{60}$$

belongs to $W_{\Delta,a^+}^{\alpha,p}$ and it is the limit of $\{u_k\}$ in $W_{\Delta,a^+}^{\alpha,p}$ with respect to $\|\cdot\|_{a,W_{\Delta,a^+}^{\alpha,p}}$. The proof is complete. □

The proof method of the following two theorems is inspired by the method used in the proof of Proposition 8.1 (b) and (c) in [27].

Theorem 42. *The space $W_{\Delta,a^+}^{\alpha,p}$ is reflexive with respect to the norm $\|\cdot\|_{W_{\Delta,a^+}^{\alpha,p}}$ for any $0 < \alpha \le 1$ and $1 < p < \infty$.*

Proof. Let us consider $W_{\Delta,a^+}^{\alpha,p}$ with the norm $\|\cdot\|_{W_{\Delta,a^+}^{\alpha,p}}$ and define a mapping

$$\lambda : W_{\Delta,a^+}^{\alpha,p} \ni u \longmapsto \left(u, {}_a^{\mathbb{T}}D_t^{\alpha}u\right) \in L_\Delta^p \times L_\Delta^p. \tag{61}$$

It is obvious that

$$\|u\|_{W_{\Delta,a^+}^{\alpha,p}} = \|\lambda u\|_{L_\Delta^p \times L_\Delta^p}, \tag{62}$$

where

$$\|\lambda u\|_{L_\Delta^p \times L_\Delta^p} = \left(\sum_{i=1}^{2} \left\| (\lambda u)_i \right\|_{L_\Delta^p}^p \right)^{1/p}, \quad \lambda u = \left(u, {}_a^{\mathbb{T}}D_t^{\alpha}u\right) \in L_\Delta^p \times L_\Delta^p, \tag{63}$$

which means that the operator $\lambda : u \mapsto \left(u, {}_a^{\mathbb{T}}D_t^{\alpha}u\right)$ is an isometric isomorphic mapping and the space $W_{\Delta,a^+}^{\alpha,p}$ is isometric isomorphic to the space $\Omega = \{(u, {}_a^{\mathbb{T}}D_t^{\alpha}u) : \forall u \in W_{\Delta,a^+}^{\alpha,p}\}$, which is a closed subset of $L_\Delta^p \times L_\Delta^p$ as $W_{\Delta,a^+}^{\alpha,p}$ is closed.

Since L^p_Δ is reflexive, the Cartesian product space $L^p_\Delta \times L^p_\Delta$ is also a reflexive space with respect to the norm $\|v\|_{L^p_\Delta \times L^p_\Delta} = (\sum_{i=1}^2 \|v_i\|^p_{L^p_\Delta})^{1/p}$, where $v = (v_1, v_2) \in L^p_\Delta \times L^p_\Delta$.

Thus, $W^{\alpha,p}_{\Delta,a^+}$ is reflexive with respect to the norm $\|\cdot\|_{W^{\alpha,p}_{\Delta,a^+}}$. The proof is complete. □

Theorem 43. *The space $W^{\alpha,p}_{\Delta,a^+}$ is separable with respect to the norm $\|\cdot\|_{W^{\alpha,p}_{\Delta,a^+}}$ for any $0 < \alpha \le 1$ and $1 \le p < \infty$.*

Proof. Let us consider $W^{\alpha,p}_{\Delta,a^+}$ with the norm $\|\cdot\|_{W^{\alpha,p}_{\Delta,a^+}}$ and the mapping λ defined in the proof of Theorem 42. Obviously, $\lambda(W^{\alpha,p}_{\Delta,a^+})$ is separable as a subset of separable space $L^p_\Delta \times L^p_\Delta$. Since λ is the isometry, $W^{\alpha,p}_{\Delta,a^+}$ is also separable with respect to the norm $\|\cdot\|_{W^{\alpha,p}_{\Delta,a^+}}$. The proof is complete. □

Theorem 44. *Let $1 \le s \le r \le t < \infty$, $u \in L^s_\Delta(J^0) \cap L^t_\Delta(J^0)$, then $u \in L^r_\Delta(J^0)$ and*

$$\|u\|_{L^r_\Delta} \le \|u\|^\theta_{L^s_\Delta} \|u\|^{1-\theta}_{L^t_\Delta}, \tag{64}$$

where $\theta \in [0, 1]$ with $1/r = \theta/s + (1-\theta)/t$.

Proof. We will divide the proof into the following three major cases.

(i) When $r = s$, we can take $\theta = 1$, the conclusion is evident

(ii) When $r = t$, we can take $\theta = 0$, the conclusion is obvious

(iii) Let $1 \le s < r < t < \infty$

In this case, if there exist $m, n > 0$ such that $r = s/m + t/n$, then

$$|u|^r = |u|^{s/m} \cdot |u|^{t/n}. \tag{65}$$

In view of $u \in L^s_\Delta(J^0) \cap L^t_\Delta(J^0)$, we have

$$\begin{aligned} \int_{J^0} \left(|u|^{s/m}\right)^m \Delta x &= \int_{J^0} |u|^s \Delta x < +\infty, \\ \int_{J^0} \left(|u|^{t/n}\right)^n \Delta x &= \int_{J^0} |u|^t \Delta x < +\infty. \end{aligned} \tag{66}$$

Hence, we obtain that

$$|u|^{s/m} \in L^m_\Delta(J^0), |u|^{t/n} \in L^n_\Delta(J^0). \tag{67}$$

Therefore, when m, n satisfy the following conditions

$$\begin{cases} m, n > 0, \\ \dfrac{s}{m} + \dfrac{t}{n} = r, \\ \dfrac{1}{m} + \dfrac{1}{n} = 1, \end{cases} \tag{68}$$

that is to say,

$$m = \frac{t-s}{t-r}, \quad n = \frac{t-s}{r-s}, \tag{69}$$

by Proposition 2.6 in P_4 from [1], one obtains

$$\begin{aligned} \int_{J^0} |u|^r \Delta x &= \int_{J^0} |u|^{s/m} \cdot |u|^{t/n} \Delta x \\ &\le \left[\int_{J^0} \left(|u|^{s/m}\right)^m \Delta x\right]^{1/m} \cdot \left[\int_{J^0} \left(|u|^{t/n}\right)^n \Delta x\right]^{1/n} \\ &= \left(\int_{J^0} |u|^s \Delta x\right)^{1/m} \cdot \left(\int_{J^0} |u|^t \Delta x\right)^{1/n} \\ &= \|u\|^{s/m}_{L^s_\Delta} \cdot \|u\|^{t/n}_{L^t_\Delta} < \infty, \end{aligned} \tag{70}$$

so $u \in L^r_\Delta(J^0)$ and

$$\|u\|^r_{L^r_\Delta} = \int_{J^0} |u|^r \Delta x \le \|u\|^{s/m}_{L^s_\Delta} \cdot \|u\|^{t/n}_{L^t_\Delta}. \tag{71}$$

Let $\theta = s/rm$, then $\theta \in (0, 1)$, $t/m = 1 - \theta$, $\theta/s + (1-\theta)/t = 1/rm + 1/rn = 1/r$, and hence,

$$\|u\|_{L^r_\Delta} \le \|u\|^\theta_{L^s_\Delta} \|u\|^{1-\theta}_{L^t_\Delta}. \tag{72}$$

The proof is complete. □

Proposition 45. *Let $0 < \alpha \le 1$ and $1 < p < \infty$. For all $u \in W^{\alpha,p}_{\Delta,a^+}$, if $1 - \alpha \ge 1/p$ or $\alpha > 1/p$, then*

$$\|u\|_{L^p_\Delta} \le \frac{b^\alpha}{\Gamma(\alpha+1)} \left\| {}_a^{\mathbb{T}}D^\alpha_t u \right\|_{L^p_\Delta}, \tag{73}$$

if $\alpha > 1/p$ and $1/p + 1/q = 1$, then

$$\|u\|_\infty \le \frac{b^{\alpha - 1/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}} \left\| {}_a^{\mathbb{T}}D^\alpha_t u \right\|_{L^p_\Delta}. \tag{74}$$

Proof. In view of Remark 36 and Theorem 14, in order to prove inequalities (73) and (74), we only need to prove that

$$\left\| {}_a^{\mathbb{T}}I^\alpha_t \left({}_a^{\mathbb{T}}D^\alpha_t u\right) \right\|_{L^p_\Delta} \le \frac{b^\alpha}{\Gamma(\alpha+1)} \left\| {}_a^{\mathbb{T}}D^\alpha_t u \right\|_{L^p_\Delta}, \tag{75}$$

for $1-\alpha \geq 1/p$ or $\alpha > 1/p$, and that

$$\left\| {}_a^{\mathbb{T}}I_t^{\alpha}\left({}_a^{\mathbb{T}}D_t^{\alpha}u\right)\right\|_{\infty} \leq \frac{b^{\alpha-1/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}}\left\| {}_a^{\mathbb{T}}D_t^{\alpha}u\right\|_{L_{\Delta}^{p}}, \quad (76)$$

for $\alpha > 1/p$ and $1/p + 1/q = 1$.

Note that ${}_a^{\mathbb{T}}D_t^{\alpha}u \in L_{\Delta}^{p}(J, \mathbb{R}^N)$, the inequality (75) follows from Lemma 39 directly.

We are now in a position to prove (76). For $\alpha > 1/p$, choose q such that $1/p + 1/q = 1$. For all $u \in W_{\Delta,a^+}^{\alpha,p}$, since $(t-s)^{(\alpha-1)q}$ is an increasing monotone function, by using Proposition 6, we find that $\int_a^t (t-s)^{(\alpha-1)q}\Delta s \leq \int_a^t (t-s)^{(\alpha-1)q} ds$. Taking into account Proposition 24, we have

$$\begin{aligned}
\left| {}_a^{\mathbb{T}}I_t^{\alpha}\left({}_a^{\mathbb{T}}D_t^{\alpha}u(t)\right)\right| &= \frac{1}{\Gamma(\alpha)}\left|\int_a^t (t-s)^{\alpha-1}\, {}_a^{\mathbb{T}}D_t^{\alpha}u(s)\Delta s\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\left(\int_a^t (t-s)^{(\alpha-1)q}\Delta s\right)^{1/q}\|{}_a^{\mathbb{T}}D_t^{\alpha}u\|_{L_{\Delta}^{p}} \\
&\leq \frac{1}{\Gamma(\alpha)}\left(\int_a^t (t-s)^{(\alpha-1)q}ds\right)^{1/q}\|{}_a^{\mathbb{T}}D_t^{\alpha}u\|_{L_{\Delta}^{p}} \quad (77)\\
&\leq \frac{b^{1/q+\alpha-1}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}}\left\| {}_a^{\mathbb{T}}D_t^{\alpha}u\right\|_{L_{\Delta}^{p}} \\
&= \frac{b^{\alpha-1/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}}\left\| {}_a^{\mathbb{T}}D_t^{\alpha}u\right\|_{L_{\Delta}^{p}}.
\end{aligned}$$

The proof is complete. □

Remark 46.

(i) According to (73), we can consider $W_{\Delta,a^+}^{\alpha,p}$ with respect to the norm

$$\|u\|_{W_{\Delta,a^+}^{\alpha,p}}^{p} = \|{}_a^{\mathbb{T}}D_t^{\alpha}u\|_{L_{\Delta}^{p}}^{p} = \left(\int_{J^0}\left|{}_a^{\mathbb{T}}D_t^{\alpha}u(t)\right|^p\Delta t\right)^{1/p}, \quad (78)$$

in the following analysis.

(ii) It follows from (73) and (74) that $W_{\Delta,a^+}^{\alpha,p}$ is continuously immersed into $C(J, \mathbb{R}^N)$ with the natural norm $\|\cdot\|_{\infty}$

Proposition 47. *Let $0 < \alpha \leq 1$ and $1 < p < \infty$. Assume that $\alpha > 1/p$ and the sequence $\{u_k\} \subset W_{\Delta,a^+}^{\alpha,p}$ converges weakly to u in $W_{\Delta,a^+}^{\alpha,p}$. Then, $u_k \longrightarrow u$ in $C(J, \mathbb{R}^N)$, i.e., $\|u-u_k\|_{\infty} = 0$, as $k \longrightarrow \infty$.*

Proof. If $\alpha > 1/p$, then by (74) and (78), the injection of $W_{\Delta,a^+}^{\alpha,p}$ into $C(J, \mathbb{R}^N)$, with its natural norm $\|\cdot\|_{\infty}$, is continuous, i.e., $u_k \longrightarrow u$ in $W_{\Delta,a^+}^{\alpha,p}$, then $u_k \longrightarrow u$ in $C(J, \mathbb{R}^N)$.

Since $u_k \rightharpoonup u$ in $W_{\Delta,a^+}^{\alpha,p}$, it follows that $u_k \rightharpoonup u$ in $C(J, \mathbb{R}^N)$. In fact, for any $h \in (C(J, \mathbb{R}^N))^*$, if $u_k \longrightarrow u$ in $W_{\Delta,a^+}^{\alpha,p}$, then $u_k \longrightarrow u$ in $C(J, \mathbb{R}^N)$, and thus, $h(u_k) \longrightarrow h(u)$. Therefore, $h \in (W_{\Delta,a^+}^{\alpha,p})^*$, which means that $(C(J, \mathbb{R}^N))^* \subset (W_{\Delta,a^+}^{\alpha,p})^*$. Hence, if $u_k \rightharpoonup u$ in $W_{\Delta,a^+}^{\alpha,p}$, then for any $h \in (C(J, \mathbb{R}^N))^*$, we have $h \in (W_{\Delta,a^+}^{\alpha,p})^*$, and thus, $h(u_k) \longrightarrow h(u)$, i.e., $u_k \rightharpoonup u$ in $C(J, \mathbb{R}^N)$.

By the Banach-Steinhaus theorem, $\{u_k\}$ is bounded in $W_{\Delta,a^+}^{\alpha,p}$ and, hence, in $C(J, \mathbb{R}^N)$. Now, we prove that the sequence $\{u_k\}$ is equicontinuous. Let $1/p + 1/q = 1$ and $t_1, t_2 \in J$ with $t_1 \leq t_2$, for all $f \in L_{\Delta}^{p}(J, \mathbb{R}^N)$, by using Proposition 24, Proposition 6, and Theorem 27, and noting $\alpha > 1/p$, we have

$$\begin{aligned}
&\left| {}_a^{\mathbb{T}}I_{t_1}^{\alpha}f(t_1) - {}_a^{\mathbb{T}}I_{t_2}^{\alpha}f(t_2)\right| \\
&= \frac{1}{\Gamma(\alpha)}\left|\int_a^{t_1}(t_1-s)^{\alpha-1}f(s)\Delta s - \int_a^{t_2}(t_2-s)^{\alpha-1}f(s)\Delta s\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\left|\int_a^{t_1}(t_1-s)^{\alpha-1}f(s)\Delta s - \int_a^{t_1}(t_2-s)^{\alpha-1}f(s)\Delta s\right| \\
&\quad + \frac{1}{\Gamma(\alpha)}\left|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f(s)\Delta s\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_a^{t_1}\left((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\right)|f(s)|\Delta s \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|f(s)|\Delta s \\
&\leq \frac{1}{\Gamma(\alpha)}\left(\int_a^{t_1}\left((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\right)^q\Delta s\right)^{1/q}\|f\|_{L_{\Delta}^{p}} \\
&\quad + \frac{1}{\Gamma(\alpha)}\left(\int_{t_1}^{t_2}(t_2-s)^{(\alpha-1)q}\Delta s\right)^{1/q}\|f\|_{L_{\Delta}^{p}} \\
&\leq \frac{1}{\Gamma(\alpha)}\left(\int_a^{t_1}\left((t_1-s)^{(\alpha-1)q} - (t_2-s)^{(\alpha-1)q}\right)\Delta s\right)^{1/q}\|f\|_{L_{\Delta}^{p}} \\
&\quad + \frac{1}{\Gamma(\alpha)}\left(\int_{t_1}^{t_2}(t_2-s)^{(\alpha-1)q}\Delta s\right)^{1/q}\|f\|_{L_{\Delta}^{p}} \\
&\leq \frac{1}{\Gamma(\alpha)}\left(\int_a^{t_1}\left((t_1-s)^{(\alpha-1)q} - (t_2-s)^{(\alpha-1)q}\right)ds\right)^{1/q}\|f\|_{L_{\Delta}^{p}} \\
&\quad + \frac{1}{\Gamma(\alpha)}\left(\int_{t_1}^{t_2}(t_2-s)^{(\alpha-1)q}ds\right)^{1/q}\|f\|_{L_{\Delta}^{p}} \\
&= \frac{\|f\|_{L_{\Delta}^{p}}}{\Gamma(\alpha)(1+(\alpha-1)q)^{1/q}}\left(t_1^{(\alpha-1)q+1} - t_2^{(\alpha-1)q+1}\right. \\
&\quad \left. + (t_2-t_1)^{(\alpha-1)q+1}\right)^{1/q} \\
&\quad + \frac{\|f\|_{L_{\Delta}^{p}}}{\Gamma(\alpha)(1+(\alpha-1)q)^{1/q}}\left((t_2-t_1)^{(\alpha-1)q+1}\right)^{1/q} \\
&\leq \frac{2\|f\|_{L_{\Delta}^{p}}}{\Gamma(\alpha)(1+(\alpha-1)q)^{1/q}}(t_2-t_1)^{\alpha-1+1/q} \\
&= \frac{2\|f\|_{L_{\Delta}^{p}}}{\Gamma(\alpha)(1+(\alpha-1)q)^{1/q}}(t_2-t_1)^{\alpha-1/p}.
\end{aligned} \quad (79)$$

Therefore, the sequence $\{u_k\}$ is equicontinuous since, for $t_1, t_2 \in J$, $t_1 \le t_2$, by applying (79) and (78), we have

$$\begin{aligned}|u_k(t_1) - u_k(t_2)| &= \left| {}_a^{\mathbb{T}}I_{t_1}^{\alpha}\left({}_a^{\mathbb{T}}D_{t_1}^{\alpha}u_k(t_1)\right) - {}_a^{\mathbb{T}}I_{t_2}^{\alpha}\left({}_a^{\mathbb{T}}D_{t_2}^{\alpha}u_k(t_2)\right)\right| \\ &\le \frac{2(t_2-t_1)^{\alpha-1/p}}{\Gamma(\alpha)(1+(\alpha-1)q)^{1/q}}\|{}_a^{\mathbb{T}}D_t^{\alpha}u_k\|_{L_\Delta^p} \\ &= \frac{2(t_2-t_1)^{\alpha-1/p}}{\Gamma(\alpha)(1+(\alpha-1)q)^{1/q}}\|{}_a^{\mathbb{T}}D_t^{\alpha}u_k\|_{L_\Delta^p} \\ &\le \frac{2(t_2-t_1)^{\alpha-1/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}}\left\|{}_a^{\mathbb{T}}D_t^{\alpha}u\right\|_{L_\Delta^p} \\ &= \frac{2(t_2-t_1)^{\alpha-1/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}}\|u_k\|_{W_{\Delta,a^+}^{\alpha,p}} \\ &\le C(t_2-t_1)^{\alpha-1/p},\end{aligned} \tag{80}$$

where $1/p + 1/q = 1$ and $C \in \mathbb{R}^+$ is a constant. By the Arzelà-Ascoli theorem on time scales (Lemma 28), $\{u_k\}$ is relatively compact in $C(J, \mathbb{R}^N)$. By the uniqueness of the weak limit in $C(J, \mathbb{R}^N)$, every uniformly convergent subsequence of $\{u_k\}$ converges uniformly on J to u. The proof is complete. □

Remark 48. It follows from Proposition 47 that $W_{\Delta,a^+}^{\alpha,p}$ is compactly immersed into $C(J, \mathbb{R}^N)$ with the natural norm $\|\cdot\|_\infty$.

Theorem 49. *Let $1 < p < \infty$, $1/p < \alpha \le 1$, $1/p + 1/q = 1$, $L : J \times \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$, $(t, x, y) \mapsto L(t, x, y)$ satisfies the following:*

(i) For each $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$, $L(t, x, y)$ is Δ-measurable in t

(ii) For Δ-almost every $t \in J$, $L(t, x, y)$ is continuously differentiable in (x, y)

If there exist $m_1 \in C(\mathbb{R}^+, \mathbb{R}^+)$, $m_2 \in L_\Delta^1(J, \mathbb{R}^+)$, and $m_3 \in L_\Delta^q(J, \mathbb{R}^+)$, $1 < q < \infty$, such that, for Δ-a.e. $t \in J$ and every $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$, one has

$$\begin{aligned}|L(t,x,y)| &\le m_1(|x|)\left(m_2(t) + |y|^p\right), \\ |D_xL(t,x,y)| &\le m_1(|x|)\left(m_2(t) + |y|^p\right), \\ |D_yL(t,x,y)| &\le m_1(|x|)\left(m_3(t) + |y|^{p-1}\right).\end{aligned} \tag{81}$$

Then, the functional χ defined by

$$\chi(u) = \int_{J^0} L\left(t, u(t), {}_a^{\mathbb{T}}D_t^{\alpha}u(t)\right)\Delta t, \tag{82}$$

is continuously differentiable on $W_{\Delta,a^+}^{\alpha,p}$, and $\forall u, v \in W_{\Delta,a^+}^{\alpha,p}$, one has

$$\begin{aligned}\left\langle \chi'(u), v\right\rangle = \int_{J^0} &\left[\left(D_xL\left(t, u(t), {}_a^{\mathbb{T}}D_t^{\alpha}u(t), v(t)\right)\right.\right. \\ &\left.\left. + \left(D_yL\left(t, u(t), {}_a^{\mathbb{T}}D_t^{\alpha}u(t), {}_a^{\mathbb{T}}D_t^{\alpha}v(t)\right)\right)\right]\Delta t.\end{aligned} \tag{83}$$

Proof. It suffices to prove that χ has, at every point u, a directional derivative $\chi'(u) \in (W_{\Delta,a^+}^{\alpha,p})^*$ given by (84) and that the mapping

$$\chi' : W_{\Delta,a^+}^{\alpha,p} \ni u \mapsto \chi'(u) \in \left(W_{\Delta,a^+}^{\alpha,p}\right)^*, \tag{84}$$

is continuous. The rest proof is similar to the proof of Theorem 1.4 in [28]. We will omit it here. The proof is complete. □

4. An Application

As an application of the concepts we introduced and the results obtained in Section 3, in this section, we will use critical point theory to study the solvability of a class of boundary value problems on time scales. More precisely, our goal is to study the following fractional nonlinear Dirichlet problem on time scale $\mathbb{T}$:

$$\begin{cases} {}_t^{\mathbb{T}}D_b^{\alpha}\left(\left|{}_a^{\mathbb{T}}D_t^{\alpha}u(t)\right|^{p-2}{}_a^{\mathbb{T}}D_t^{\alpha}u(t)\right) = \nabla F(t,u(t)) + \sigma\omega(t)|u(t)|^{q-2}u(t), & \Delta\text{-a.e.}\, t \in J, \\ u(\alpha) = u(b) = 0, \end{cases} \tag{85}$$

where ${}_t^{\mathbb{T}}D_b^{\alpha}$ and ${}_a^{\mathbb{T}}D_t^{\alpha}$ are the right and the left Riemann-Liouville fractional derivative operators of order α defined on $\mathbb{T}$, respectively, $\nabla F(t, u)$ is the gradient of $F(t, u)$ at u and $F \in C(J \times \mathbb{R}^N, \mathbb{R})$ is homogeneous of degree r, σ is a positive parameter, $\omega \in C(J)$, $1 < r < p < q$ and $1/p < \alpha < 1$.

We make the following assumptions:

$(H_1) F : J \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is homogeneous of degree r, that is,

$$F(t, su) = s^r F(t, u)(s > 0), \tag{86}$$

for all $t \in J$, $u \in \mathbb{R}^N$;

$(H_2) F^{\pm}(t, u) = \max(\pm F(t, u), 0) \ne 0$ for all $u \ne 0$.

By (H_1), $F(t, u)$, we have

$$u\nabla F(x, u) = s^r F(x, u), \tag{87}$$

$$|F(x, u)| \le K|u|^r, \tag{88}$$

for some constant $K > 0$.

Our main results are as follows.

Theorem 50. *Let $1/p < \alpha < 1$, $1 < r < p < q$ and suppose that $F(t, u)$ satisfies the conditions (H_1) and (H_2). Then, there exists $\sigma_0 > 0$ such that for all $\sigma \in (0, \sigma_0)$, (85) has at least two nontrivial solutions.*

There have been many results using critical point theory to study boundary value problems of fractional differential equations [29–35] and dynamic equations on time scales [36–40], but the results of using critical point theory to study boundary value problems of fractional dynamic equations on time scales are still rare [3]. This section will explain that critical point theory is an effective way to deal with the existence of solutions of (85) on time scales.

We will use the famous Nehari manifold and fibering map theory to prove our main results.

We say that $u \in W^{\alpha,p}_{\Delta,a^+}$ is a solution to the problem (85), if u satisfies the following equality:

$$\begin{aligned}&\int_{J^0}\left|{}_a^{\mathbb{T}}D_t^{\alpha}u(t)\right|^{p-2}\left({}_a^{\mathbb{T}}D_t^{\alpha}u(t),{}_a^{\mathbb{T}}D_t^{\alpha}v(t)\right)\Delta t-\int_{J^0}(\nabla F(t,u(t)),v(t))\Delta t\\&-\sigma\int_{J^0}\omega(t)|u(t)|^{q-2}(u(t),v(t))\Delta t=0,\quad \forall v\in W^{\alpha,p}_{\Delta,a^+}.\end{aligned} \tag{89}$$

As a result, associated to the problem (85), we define the functional

$$\begin{aligned}J_\sigma(u)&=\frac{1}{p}\|u\|^p-\int_{J^0}F(t,u(t))\Delta t-\frac{\sigma}{q}\int_{J^0}\omega(t)|u(t)|^q\Delta t\\&=J(u)-H(u)-M_\sigma(u),\end{aligned} \tag{90}$$

where

$$\begin{aligned}J(u)&=\frac{1}{p}\int_{J^0}\left|{}_a^{\mathbb{T}}D_t^{\alpha}u(t)\right|^p\Delta t, H(u)=\int_{J^0}F(t,u(t))\Delta t,\\M_\sigma(u)&=\frac{\sigma}{q}\int_{J^0}\omega(t)|u(t)|^q\Delta t.\end{aligned} \tag{91}$$

We need to show that the following lemma holds.

Lemma 51.

(i) The functional J_σ is well defined on $W^{\alpha,p}_{\Delta,a^+}$

(ii) The functional J_σ is of class $C^1(W^{\alpha,p}_{\Delta,a^+},\mathbb{R})$, and for all $u, v \in W^{\alpha,p}_{\Delta,a^+}$, we have

$$\begin{aligned}\left\langle J'_\sigma(u),v\right\rangle&=\int_{J^0}\left|{}_a^{\mathbb{T}}D_t^{\alpha}u(t)\right|^{p-2}\left({}_a^{\mathbb{T}}D_t^{\alpha}u(t),{}_a^{\mathbb{T}}D_t^{\alpha}v(t)\right)\Delta t\\&\quad-\int_{J^0}(\nabla F(t,u(t)),v(t))\Delta t\\&\quad-\sigma\int_{J^0}\omega(t)|u(t)|^{q-2}(u(t),v(t))\Delta t\\&=\left\langle J'(u),v\right\rangle-\left\langle H'(u),v\right\rangle-\left\langle M'_\sigma(u),v\right\rangle,\end{aligned} \tag{92}$$

where

$$\begin{aligned}\left\langle J'(u),v\right\rangle&=\int_{J^0}\left|{}_a^{\mathbb{T}}D_t^{\alpha}u(t)\right|^{p-2}\left({}_a^{\mathbb{T}}D_t^{\alpha}u(t),{}_a^{\mathbb{T}}D_t^{\alpha}v(t)\right)\Delta t,\\\left\langle H'(u),v\right\rangle&=\int_{J^0}(\nabla F(t,u(t)),v(t))\Delta t,\\\left\langle M'_\sigma(u),v\right\rangle&=\sigma\int_{J^0}\omega(t)|u(t)|^{q-2}(u(t),v(t))\Delta t.\end{aligned} \tag{93}$$

Proof.

(i) From (33) in Proposition 45, (87), (88), and the equivalent norm, we obtain

$$\begin{aligned}J_\sigma(u)&=\frac{1}{p}\|u\|^p-\int_{J^0}F(t,u(t))\Delta t-\frac{\sigma}{q}\int_{J^0}\omega(t)|u(t)|^q\Delta t\\&\le\frac{1}{p}\|u\|^p+\int_{J^0}|F(t,u(t))|\Delta t+\frac{\sigma}{q}\max_{t\in J}\omega(t)\int_{J^0}|u(t)|^q\Delta t\\&\le\frac{1}{p}\|u\|^p+K\int_{J^0}\left\|u(t)\right|^r\Delta t+\frac{\sigma}{q}\|\omega\|_\infty\|u\|^q_{L^q_\Delta}\\&\le\frac{1}{p}\|u\|^p+K\|u\|^r_{L^r_\Delta}+\frac{\sigma}{q}\|\omega\|_\infty\frac{b^\alpha q}{\Gamma^q(\alpha+1)}\left\|{}_a^{\mathbb{T}}D_t^{\alpha}u\right\|^q_{L^q_\Delta}\\&\le\frac{1}{p}\|u\|^p+K\frac{b^\alpha r}{\Gamma^r(\alpha+1)}\left\|{}_a^{\mathbb{T}}D_t^{\alpha}u\right\|^r_{L^r_\Delta}\\&\quad+\frac{\sigma}{q}\|\omega\|_\infty\frac{b^\alpha q}{\Gamma^q(\alpha+1)}\left\|{}_a^{\mathbb{T}}D_t^{\alpha}u\right\|^q_{L^q_\Delta}\\&\le\frac{1}{p}\|u\|^p+c_1\|u\|^r+c_2\|u\|^q,\end{aligned} \tag{94}$$

which implies that J_σ is well defined on $W^{\alpha,p}_{\Delta,a^+}$.

(ii) Let

$$\Pi(u)=\frac{1}{p}\left|{}_a^{\mathbb{T}}D_t^{\alpha}u\right|^p-F(t,u(t))-\frac{\sigma}{q}\omega(t)|u|^q. \tag{95}$$

Then, we can easily show that for all $u, v \in W^{\alpha,p}_{\Delta,a^+}$ and for Δ-a.e. $t \in J$,

$$\begin{aligned}&\lim_{s\longrightarrow 0}\frac{\Pi(u(t)+sv(t))-\Pi(u(t))}{s}\\&\quad=\lim_{s\longrightarrow 0}\frac{1}{s}\left\{\frac{1}{p}\left|{}_a^{\mathbb{T}}D_t^{\alpha}(u+sv)\right|^p-F(t,u+sv)-\frac{\sigma}{q}\omega(t)|u+sv|^q\right.\\&\qquad\left.-\frac{1}{p}\left|{}_a^{\mathbb{T}}D_t^{\alpha}u\right|^p+F(t,u(t))+\frac{\sigma}{q}\omega(t)|u|^q\right\}\\&\quad=\left|{}_a^{\mathbb{T}}D_t^{\alpha}u(t)\right|^{p-2}\left({}_a^{\mathbb{T}}D_t^{\alpha}u(t),{}_a^{\mathbb{T}}D_t^{\alpha}v(t)\right)-(\nabla F(t,u(t)),v(t))\\&\qquad-\sigma\omega(t)|u(t)|^{q-2}(u(t),v(t)).\end{aligned} \tag{96}$$

Hence, in view of the Lagrange mean value theorem, (87) and (42), there exists a real number κ such that $|\kappa| \leq |s|$ and

$$\begin{aligned}
&\frac{\Pi(u(t)+sv(t))-\Pi(u(t))}{s}\\
&\quad=\left|{}_{a}^{\mathbb{T}}D_{t}^{\alpha}(u+\kappa v)\right|^{p-2}\left({}_{a}^{\mathbb{T}}D_{t}^{\alpha}(u+\kappa v),{}_{a}^{\mathbb{T}}D_{t}^{\alpha}v(t)\right)\\
&\qquad-(\nabla F(t,(u+\kappa v)),v)-\sigma\omega(t)|(u+\kappa v)|^{q-2}((u+\kappa v),v)\\
&\quad\leq\left|{}_{a}^{\mathbb{T}}D_{t}^{\alpha}(u+\kappa v)\right|^{p-1}\left|{}_{a}^{\mathbb{T}}D_{t}^{\alpha}v\right|-(\nabla F(t,(u+\kappa v)),v)\\
&\qquad-\sigma\omega(t)|u+\kappa v|^{q-1}|v|\leq\left|{}_{a}^{\mathbb{T}}D_{t}^{\alpha}(u+\kappa v)\right|^{p-1}\left|{}_{a}^{\mathbb{T}}D_{t}^{\alpha}v\right|\\
&\qquad-\frac{r}{u+\kappa v}|F(t,(u+\kappa v))|v+\sigma|\omega(t)||u+\kappa v|^{q-1}|v|\\
&\quad\leq\left|{}_{a}^{T}D_{t}^{\alpha}(u+\kappa v)\right|^{p-1}\left|{}_{a}^{T}D_{t}^{\alpha}v\right|-\frac{r}{u+\kappa v}K|u+\kappa v|^{r}v\\
&\qquad+\sigma|\omega(t)||u+\kappa v|^{q-1}|v|\leq\left|{}_{a}^{T}D_{t}^{\alpha}(u+\kappa v)\right|^{p-1}\left|{}_{a}^{T}D_{t}^{\alpha}v\right|\\
&\qquad-rK|u+\kappa v|)|^{r-1}|v|+\sigma|\omega(t)||u+\kappa v|^{q-1}|v|\\
&\quad\leq\left|{}_{a}^{T}D_{t}^{\alpha}u\right|^{p-1}\left|{}_{a}^{T}D_{t}^{\alpha}v\right|+\left|{}_{a}^{T}D_{t}^{\alpha}v\right|^{p}+rK|u|^{r-1}|v|+rK|v|^{r}\\
&\qquad+\sigma|\omega(t)||u|^{q-1}|v|+\sigma|\omega(t)||v|^{q}.
\end{aligned}\tag{97}$$

On the other hand, in view of Hölder inequality on time scales, we see that

$$\begin{aligned}
&\int_{J^0}\left|{}_{a}^{\mathbb{T}}D_{t}^{\alpha}u(t)\right|^{p-1}\left|{}_{a}^{\mathbb{T}}D_{t}^{\alpha}v(t)\right|\Delta t\\
&\quad\leq\left(\int_{J^0}\left[\left|{}_{a}^{\mathbb{T}}D_{t}^{\alpha}u(t)\right|^{p-1}\right]^{p/(p-1)}\Delta t\right)^{(p-1)/p}\left(\int_{J^0}\left|{}_{a}^{\mathbb{T}}D_{t}^{\alpha}v(t)\right|^{p}\Delta t\right)^{1/p}\\
&\quad=\left\|\left|{}_{a}^{\mathbb{T}}D_{t}^{\alpha}u\right|^{p-1}\right\|_{L_{\Delta}^{p/(p-1)}}\cdot\left\|{}_{a}^{\mathbb{T}}D_{t}^{\alpha}v\right\|_{L_{\Delta}^{p}},\\
&\int_{J^0}|u(t)|^{\varsigma-1}\|v(t)\,|\Delta t\\
&\quad\leq\left(\int_{J^0}\left[|u(t)|^{\varsigma-1}\right]^{\varsigma/(\varsigma-1)}\Delta t\right)^{(p-1)/p}\left(\int_{J^0}|v(t)|^{p}\Delta t\right)^{1/p}\\
&\quad=\||u|^{\varsigma-1}\|_{L_{\Delta}^{\varsigma/(\varsigma-1)}}\cdot\|v\|_{L_{\Delta}^{\varsigma}}
\end{aligned}\tag{98}$$

for $\varsigma=r$ or $\varsigma=q$. Because ω is bounded, then, from the above inequalities, we conclude that the expression (97) is in $L_{\Delta}^{1}(J)$.

As a result, in view of the dominated convergence theorem on time scales, one gets

$$\begin{aligned}
\lim_{s\longrightarrow0}\frac{J_{\sigma}(u(t)+sv(t))-J_{\sigma}(u(t))}{s}
&=\int_{J^0}\left|{}_{a}^{\mathbb{T}}D_{t}^{\alpha}u(t)\right|^{p-2}\left({}_{a}^{\mathbb{T}}D_{t}^{\alpha}u(t),{}_{a}^{\mathbb{T}}D_{t}^{\alpha}v(t)\right)\Delta t\\
&\quad-\int_{J^0}(\nabla F(t,u(t)),v(t))\Delta t\\
&\quad-\sigma\int_{J^0}\omega(t)|u(t)|^{q-2}(u(t),v(t))\Delta t.
\end{aligned}\tag{99}$$

That is to say, J_{σ} is Gâteaux differentiable.

In what follows, we prove that the Gâteaux derivative of J_{σ} is continuous.

First, we verify that $H':W_{\Delta,a^+}^{\alpha,p}\longrightarrow(W_{\Delta,a^+}^{\alpha,p})^*$ is continuous. Taking into account (88), we have

$$\begin{aligned}
&\int_{J^0}|\nabla F(t,u_n(t))-\nabla F(t,u(t))|^2\Delta t\\
&\quad\leq\int_{J^0}(|\nabla F(t,u_n(t))|+|\nabla F(t,u(t))\,|\,)^2\Delta t\\
&\quad\leq2\int_{J^0}\left(|\nabla F(t,u_n(t))|^2+|\nabla F(t,u(t))|^2\right)\Delta t\\
&\quad\leq2r^2K^2\int_{J^0}\left(|u_n(t)|^{2(r-1)}+|u(t)|^{2(r-1)}\right)\Delta t\\
&\quad\leq2r^2K^2\left(\|u_n\|_{\infty}^{2(r-1)}+\|u\|_{\infty}^{2(r-1)}\right)(b-a),
\end{aligned}\tag{100}$$

which combining with $u_n\longrightarrow u$ in L_{Δ}^{2} and Lebesgue's dominated convergence theorem on time scales leads to

$$\left(\int_{J^0}|\nabla F(t,u_n(t))-\nabla F(t,u(t))|^2\Delta t\right)^{1/2}\longrightarrow0,\quad n\longrightarrow\infty.\tag{101}$$

Namely,

$$\nabla F(t,u_n)\longrightarrow\nabla F(t,u),\quad\text{in }L_{\Delta}^{2}\left(J,\mathbb{R}^N\right).\tag{102}$$

Let $\{u_n\}$, $u\in W_{\Delta,a^+}^{\alpha,p}$ such that $u_n\rightharpoonup u$ in $W_{\Delta,a^+}^{\alpha,p}$ $(n\longrightarrow\infty)$. Using the Hölder inequality on time scales and (73) in Proposition 45, we can obtain

$$\begin{aligned}
&\left\|H'(u_n)-H'(u)\right\|_{(W_{\Delta,a^+}^{\alpha,p})^*}\\
&\quad=\sup_{v\in W_{\Delta,a^+}^{\alpha,p},\|v\|=1}\left|\left(H'(u_n)-H'(u)\right)v\right|\\
&\quad=\sup_{v\in W_{\Delta,a^+}^{\alpha,p},\|v\|=1}\left|\int_{J^0}(\nabla F(t,u_n(t))-\nabla F(t,u(t)))v(t)\Delta t\right|\\
&\quad\leq\sup_{v\in W_{\Delta,a^+}^{\alpha,p},\|v\|=1}\|\nabla F(\cdot,u_n(\cdot))-\nabla F(\cdot,u(\cdot))\|_{L_{\Delta}^{2}}\|v(\cdot)\|_{L_{\Delta}^{2}}\\
&\quad\leq\sup_{v\in W_{\Delta,a^+}^{\alpha,p},\|v\|=1}\|\nabla F(\cdot,u_n(\cdot))-\nabla F(\cdot,u(\cdot))\|_{L_{\Delta}^{2}}\frac{b^{\alpha}}{\Gamma(\alpha+1)}\left\|{}_{a}^{\mathbb{T}}D_{t}^{\alpha}v\right\|_{L_{\Delta}^{2}}\\
&\quad=C\sup_{v\in W_{\Delta,a^+}^{\alpha,p},\|v\|=1}\|\nabla F(\cdot,u_n(\cdot))-\nabla F(\cdot,u(\cdot))\|_{L_{\Delta}^{2}}\longrightarrow0.
\end{aligned}\tag{103}$$

So, $H'(u)$ is continuous.

Next, we will prove $J\in C^1(W_{\Delta,a^+}^{\alpha,p},\mathbb{R})$. For any given u, $v\in W_{\Delta,a^+}^{\alpha,p}$, by the Hölder inequality on time scales, we have

$$\begin{aligned}\left|\left\langle J'(u),v\right\rangle\right| &= \left|\int_{J^0}\left|{}_a^{\mathbb{T}}D_t^{\alpha}u(t)\right|^{p-2}\left({}_a^{\mathbb{T}}D_t^{\alpha}u(t),{}_a^{\mathbb{T}}D_t^{\alpha}v(t)\right)\Delta t\right| \\ &\leq \int_{J^0}\left|{}_a^{\mathbb{T}}D_t^{\alpha}u(t)\right|^{p-1}\left|{}_a^{\mathbb{T}}D_t^{\alpha}v(t)\right|\Delta t \\ &\leq \left(\int_{J^0}\left(\left|{}_a^{\mathbb{T}}D_t^{\alpha}u(t)\right|^{p-1}\right)^{p/(p-1)}\Delta t\right)^{(p-1)/p} \\ &\quad\cdot\left(\int_{J^0}\left|{}_a^{\mathbb{T}}D_t^{\alpha}v(t)\right|^{p}\Delta t\right)^{1/p} \\ &= \left(\int_{J^0}\left|{}_a^{\mathbb{T}}D_t^{\alpha}u(t)\right|^{p}\Delta t\right)^{(p-1)/p}\|v\| = \|u\|^{p-1}\|v\|.\end{aligned} \tag{104}$$

That is, $J'(u)$ is bounded. It is obvious that $J'(u)$ is linear. Hence, for any $u \in W_{\Delta,a^+}^{\alpha,p}$, $J'(u) \in (W_{\Delta,a^+}^{\alpha,p})^*$.

Define $g: W_{\Delta,a^+}^{\alpha,p} \longrightarrow L_{\Delta}^{p'}(J)$, $g(u) = |{}_a^{\mathbb{T}}D_t^{\alpha}u|^{p-2}\,{}_a^{\mathbb{T}}D_t^{\alpha}u$, $\forall u \in W_{\Delta,a^+}^{\alpha,p}$, where $1/p + 1/p' = 1$. Now, it is time for us to demonstrate g is continuous in the following two cases:

(1) If $p \in (2,\infty)$, then, for $u, v \in W_{\Delta,a^+}^{\alpha,p}$, using Hölder inequality on time scales, we can deduce that

$$\begin{aligned}&\int_{J^0}|g(u)-g(v)|^{p'}\Delta t \\ &= \int_{J^0}\left||{}_a^{\mathbb{T}}D_t^{\alpha}u|^{p-2}\,{}_a^{\mathbb{T}}D_t^{\alpha}u - |{}_a^{\mathbb{T}}D_t^{\alpha}v|^{p-2}\,{}_a^{\mathbb{T}}D_t^{\alpha}v\right|^{p'}\Delta t \\ &\leq \beta\int_{J^0}\left|{}_a^{\mathbb{T}}D_t^{\alpha}u - {}_a^{\mathbb{T}}D_t^{\alpha}v\right|^{p'}\left(|{}_a^{\mathbb{T}}D_t^{\alpha}u| + |{}_a^{\mathbb{T}}D_t^{\alpha}v|\right)^{p'(p-2)}\Delta t \\ &\leq \beta\left[\int_{J^0}\left(\left|{}_a^{\mathbb{T}}D_t^{\alpha}u - {}_a^{\mathbb{T}}D_t^{\alpha}v\right|^{p'}\right)^{p/p'}\Delta t\right]^{p'/p} \\ &\quad\times\left[\int_{J^0}\left(\left(|{}_a^{\mathbb{T}}D_t^{\alpha}u| + |{}_a^{\mathbb{T}}D_t^{\alpha}v|\right)^{p'(p-2)}\right)^{p/p'(p-2)}\Delta t\right]^{p'(p-2)/p} \\ &= \beta\|{}_a^{\mathbb{T}}D_t^{\alpha}u - {}_a^{\mathbb{T}}D_t^{\alpha}v\|_{L_\Delta^p}^{p'}\,\||{}_a^{\mathbb{T}}D_t^{\alpha}u| + |{}_a^{\mathbb{T}}D_t^{\alpha}v|\|_{L_\Delta^p}^{p'(p-2)} \\ &\leq \bar{C}\|u-v\|^{p'}(\|u\|+\|v\|)^{p'(p-2)}.\end{aligned} \tag{105}$$

(2) If $p \in (1,2]$, then, for $u, v \in W_{\Delta,a^+}^{\alpha,p}$, we have

$$\begin{aligned}&\int_{J^0}|g(u)-g(v)|^{p'}\Delta t \\ &= \int_{J^0}\left||{}_a^{\mathbb{T}}D_t^{\alpha}u|^{p-2}\,{}_a^{\mathbb{T}}D_t^{\alpha}u - |{}_a^{\mathbb{T}}D_t^{\alpha}v|^{p-2}\,{}_a^{\mathbb{T}}D_t^{\alpha}v\right|^{p'}\Delta t \\ &\leq \beta\int_{J^0}\left|{}_a^{\mathbb{T}}D_t^{\alpha}u - {}_a^{\mathbb{T}}D_t^{\alpha}v\right|^{p'(p-1)}\Delta t \leq \bar{C}_1\|u-v\|.\end{aligned} \tag{106}$$

Consequently, when $p > 1$, g is continuous.

Now, for $u, v \in W_{\Delta,a^+}^{\alpha,p}$, we will show that

$$\|J'(u) - J'(v)\|_{(W_{\Delta,a^+}^{\alpha,p})^*} \leq k\|g(u)-g(v)\|_{L_\Delta^{p'}}. \tag{107}$$

In fact, for $u, v \in W_{\Delta,a^+}^{\alpha,p}$, by Hölder inequality on time scales, we have

$$\begin{aligned}&\|J'(u) - J'(v)\|_{(W_{\Delta,a^+}^{\alpha,p})^*} \\ &= \sup_{\varphi\in W_{\Delta,a^+}^{\alpha,p},\|\varphi\|=1}\left|\left\langle J'(u)-J'(v),v\right\rangle\right| \\ &= \sup_{\varphi\in W_{\Delta,a^+}^{\alpha,p},\|\varphi\|=1}\left|\int_{J^0}\left|{}_a^{\mathbb{T}}D_t^{\alpha}u(t)\right|^{p-2}\left({}_a^{\mathbb{T}}D_t^{\alpha}u(t),{}_a^{\mathbb{T}}D_t^{\alpha}\varphi(t)\right)\Delta t\right. \\ &\quad\left.-\int_{J^0}\left|{}_a^{\mathbb{T}}D_t^{\alpha}v(t)\right|^{p-2}\left({}_a^{\mathbb{T}}D_t^{\alpha}v(t),{}_a^{\mathbb{T}}D_t^{\alpha}\varphi(t)\right)\Delta t\right| \\ &= \sup_{\varphi\in W_{\Delta,a^+}^{\alpha,p},\|\varphi\|=1}\left|\int_{J^0}\left(g(u)-g(v),{}_a^{\mathbb{T}}D_t^{\alpha}\varphi(t)\right)\Delta t\right| \\ &\leq \sup_{\varphi\in W_{\Delta,a^+}^{\alpha,p},\|\varphi\|=1}\int_{J^0}|g(u)-g(v)|\left|{}_a^{\mathbb{T}}D_t^{\alpha}\varphi(t)\right|\Delta t \\ &\leq \sup_{\varphi\in W_{\Delta,a^+}^{\alpha,p},\|\varphi\|=1}\left(\int_{J^0}|g(u)-g(v)|^{p'}\Delta t\right)^{1/p'}\left(\int_{J^0}\left|{}_a^{\mathbb{T}}D_t^{\alpha}\varphi\right|^{p}\Delta t\right)^{1/p} \\ &\leq \|g(u)-g(v)\|_{L_\Delta^{p'}}\|\varphi\| = \|g(u)-g(v)\|_{L_\Delta^{p'}}.\end{aligned} \tag{108}$$

Combining with the continuity of g, we see that $J \in C^1(W_{\Delta,a^+}^{\alpha,p},\mathbb{R})$.

In conclusion, (ii) is proven. The proof is complete. □

We deduce from Lemma 51 and (87) that

$$\begin{aligned}\left\langle J'_\sigma(u),u\right\rangle &= \int_{J^0}\left|{}_a^{\mathbb{T}}D_t^{\alpha}u(t)\right|^{p}\Delta t - r\int_{J^0}F(t,u(t))\Delta t \\ &\quad-\sigma\int_{J^0}\omega(t)|u(t)|^q\Delta t.\end{aligned} \tag{109}$$

It is easy to see that the energy functional J_σ is not bounded below on the space $W_{\Delta,a^+}^{\alpha,p}$, but it is bounded below on a suitable subset of $W_{\Delta,a^+}^{\alpha,p}$. In order to study problem (85), we define the constraint set

$$\mathcal{N}_\sigma := \left\{u \in W_{\Delta,a^+}^{\alpha,p}\setminus\{0\}: \left\langle J'_\sigma(u),u\right\rangle = 0\right\}. \tag{110}$$

Note that $\mathcal{N}_\sigma$ contains every nonzero solution of (85), and $u \in \mathcal{N}_\sigma$ if and only if

$$\|u\|^p - r\int_{J^0}F(t,u(t))\Delta t - \sigma\int_{J^0}\omega(t)|u(t)|^q\Delta t = 0. \tag{111}$$

In order to get the existence of solutions, we decompose $\mathcal{N}_\sigma$ into three parts: corresponding to local minima, local maxima, and points of inflection are Δ-measurable sets defined as follows:

$$\mathcal{N}_\sigma^+ := \left\{ u \in \mathcal{N}_\sigma : (p-1)\|u\|^p - r(r-1)\int_{J^0} F(t,u(t))\Delta t - \sigma(q-1)\int_{J^0} \omega(t)|u(t)|^q \Delta t > 0 \right\},$$

$$\mathcal{N}_\sigma^- := \left\{ u \in \mathcal{N}_\sigma : (p-1)\|u\|^p - r(r-1)\int_{J^0} F(t,u(t))\Delta t - \sigma(q-1)\int_{J^0} \omega(t)|u(t)|^q \Delta t < 0 \right\},$$

$$\mathcal{N}_\sigma^0 := \left\{ u \in \mathcal{N}_\sigma : (p-1)\|u\|^p - r(r-1)\int_{J^0} F(t,u(t))\Delta t - \sigma(q-1)\int_{J^0} \omega(t)|u(t)|^q \Delta t = 0 \right\}. \tag{112}$$

Next, we give some important attributes of $\mathcal{N}_\sigma^+$, $\mathcal{N}_\sigma^-$ and $\mathcal{N}_\sigma^0$. Let $\bar{p}$ be such that $1/p + 1/\bar{p} = 1$ and put

$$\eta_0 = \frac{(p-r)\Gamma^q(\alpha)((\alpha-1)\bar{p}+1)^{q/p}}{(q-r)\|\omega\|_\infty b^{1+q(\alpha-1/p)}} \cdot \left(\frac{(q-p)\Gamma^r(\alpha)((\alpha-1)\bar{p}+1)^{r/p}}{Kr(q-r)b^{1+r(\alpha-1/p)}} \right)^{(q-p)/(p-r)}. \tag{113}$$

Then, we have the following crucial result.

Lemma 52. *If $\sigma \in (0,\eta_0)$, then $\mathcal{N}_\sigma^0 = \varnothing$.*

Proof. We proceed by contradiction to show that $\mathcal{N}_\sigma^0 = \varnothing$ for all $\sigma \in (0,\eta_0)$. If there exists $u_0 \in \mathcal{N}_\sigma^0$, then, in view of (111), we get

$$(p-r)\|u_0\|^p - \sigma(q-r)\int_{J^0} \omega(t)|u_0(t)|^q \Delta t = 0, \tag{114}$$

$$(q-p)\|u_0\|^p - r(q-r)\int_{J^0} F(t,u_0(t))\Delta t = 0. \tag{115}$$

By Proposition 45 and (114), we have

$$\begin{aligned}(p-r)\|u_0\|^p &= \sigma(q-r)\int_{J^0} \omega(t)|u_0(t)|^q \Delta t \\ &\le \sigma(q-r)\|\omega\|_\infty b\|u_0\|_\infty^q \\ &\le b\sigma(q-r)\|\omega\|_\infty \frac{b^{(\alpha-1/p)q}}{\Gamma^q(\alpha)((\alpha-1)\bar{p}+1)^{q/\bar{p}}} \left\| {}_a^{\mathbb{T}}D_t^\alpha u_0 \right\|_{L_\Delta^p}^q \\ &= \sigma(q-r)\|\omega\|_\infty \frac{b^{1+(\alpha-1/p)q}}{\Gamma^q(\alpha)((\alpha-1)\bar{p}+1)^{q/\bar{p}}} \|u_0\|^q,\end{aligned} \tag{116}$$

which implies that

$$\|u_0\| \ge \left(\frac{(p-r)\Gamma^q(\alpha)((\alpha-1)\bar{p}+1)^{q/\bar{p}}}{\sigma(q-r)\|\omega\|_\infty b^{1+q(\alpha-1/p)}} \right)^{1/(q-p)}. \tag{117}$$

Moreover, combining with Proposition 45, (88), and (115), one has

$$\begin{aligned}(q-p)\|u_0\|^p &= r(q-r)\int_{J^0} F(t,u_0(t))\Delta t \\ &\le r(q-r)\int_{J^0} K|u_0|^r \Delta t \\ &\le Kr(q-r)\|u_0\|_\infty^r b \\ &\le Kr(q-r)b \frac{b^{(\alpha-1/p)r}}{\Gamma^r(\alpha)((\alpha-1)\bar{p}+1)^{r/\bar{p}}} \left\| {}_a^{\mathbb{T}}D_t^\alpha u_0 \right\|_{L_\Delta^p}^r \\ &= Kr(q-r) \frac{b^{1+(\alpha-1/p)r}}{\Gamma^r(\alpha)((\alpha-1)\bar{p}+1)^{r/\bar{p}}} \|u_0\|^r.\end{aligned} \tag{118}$$

Hence,

$$\|u_0\| \le \left(\frac{Kr(q-r)b^{1+(\alpha-1/p)r}}{(q-p)\Gamma^r(\alpha)((\alpha-1)\bar{p}+1)^{r/\bar{p}}} \right)^{1/(p-r)}. \tag{119}$$

It follows from (47) that

$$\sigma \ge \frac{(p-r)\Gamma^q(\alpha)((\alpha-1)\bar{p}+1)^{q/p}}{\|u_0\|^{q-p}(q-r)\|\omega\|_\infty b^{1+q(\alpha-1/p)}}, \tag{120}$$

and that

$$\|u_0\|^{1/(q-p)} \ge \left(\frac{(q-p)\Gamma^r(\alpha)((\alpha-1)\bar{p}+1)^{r/\bar{p}}}{Kr(q-r)b^{1+(\alpha-1/p)r}} \right)^{(q-p)/(p-r)}. \tag{121}$$

Combining with (120) and (120), we gain that

$$\sigma \ge \frac{(p-r)\Gamma^q(\alpha)((\alpha-1)\bar{p}+1)^{q/p}}{(q-r)\|\omega\|_\infty b^{1+q(\alpha-1/p)}} \cdot \left(\frac{(q-p)\Gamma^r(\alpha)((\alpha-1)\bar{p}+1)^{r/\bar{p}}}{Kr(q-r)b^{1+(\alpha-1/p)r}} \right)^{(q-p)/(p-r)} = \eta_0. \tag{122}$$

Namely, $\sigma \ge \eta_0$, which leads to a contradiction. This completes the proof of Lemma 52. □

Lemma 53. *If $\sigma \in (0,\eta_0)$, then J_σ is coercive and bounded below on $\mathcal{N}_\sigma$.*

Proof. Let $u \in \mathcal{N}_\sigma$, then, by (88) and Proposition 45, we gain

$$\begin{aligned}\int_{J^0} F(t,u(t))\Delta t &\le K\int_{J^0} |u(t)|^r \Delta t \le K\|u\|_\infty^r b \\ &\le \frac{Kb^{1+r(\alpha-1/p)}}{\Gamma^r(\alpha)((\alpha-1)\bar{p}+1)^{r/\bar{p}}} \left\| {}_a^{\mathbb{T}}D_t^\alpha u \right\|_{L_\Delta^p}^r \\ &= \frac{Kb^{1+r(\alpha-1/p)}}{\Gamma^r(\alpha)((\alpha-1)\bar{p}+1)^{r/\bar{p}}} \|u\|^r. \end{aligned} \tag{123}$$

Hence, in view of (111), one gets

$$\begin{aligned} J_\sigma(u) &= \frac{1}{p}\|u\|^p - \int_{J^0} F(t,u(t))\Delta t - \frac{\sigma}{q}\int_{J^0} \omega(t)|u(t)|^q \Delta t \\ &= \frac{q-p}{qp}\|u\|^p - \int_{J^0} F(t,u(t))\Delta t + \frac{r}{q}\int_{J^0} F(t,u(t))\Delta t \\ &= \frac{q-p}{qp}\|u\|^p - \frac{q-r}{q}\int_{J^0} F(t,u(t))\Delta t \\ &\ge \frac{q-p}{qp}\|u\|^p - \frac{K(q-r)b^{1+r(\alpha-1/p)}}{q\Gamma^r(\alpha)((\alpha-1)\bar{p}+1)^{r/\bar{p}}}\|u\|^r. \end{aligned} \tag{124}$$

Since $r < p < q$, J_σ is coercive and bounded below on $\mathcal{N}_\sigma$. The proof of Lemma 53 is now completed. □

Now as we know, the Nehari manifold is closely related to the behavior of the functions $\Theta_u : [0,\infty) \longrightarrow \mathbb{R}$ defined as

$$\Theta_u(s) = J_\sigma(su). \tag{125}$$

Such maps are called fibering maps. For $u \in W_{\Delta,a^+}^{\alpha,p}$, we define

$$\begin{aligned} \Theta_u(s) &= J_\sigma(su) \\ &= \frac{1}{p}\|su\|^p - \int_{J^0} F(t,u(t))\Delta t - \frac{\sigma}{q}\int_{J^0} \omega(t)|su(t)|^q \Delta t \\ &= \frac{s^p}{p}\|u\|^p - s^r\int_{J^0} F(t,u(t))\Delta t - \sigma\frac{s^q}{q}\int_{J^0} \omega(t)|u(t)|^q \Delta t. \end{aligned} \tag{126}$$

Then, one obtains

$$\Theta_u'(s) = s^{p-1}\|u\|^p - rs^{r-1}\int_{J^0} F(t,u(t))\Delta t - \sigma s^{q-1}\int_{J^0} \omega(t)|u(t)|^q \Delta t, \tag{127}$$

$$\begin{aligned} \Theta''_u(s) = {} & (p-1)s^{p-2}\|u\|^p - r(r-1)s^{r-2}\int_{J^0} F(t,u(t))\Delta t \\ & - \sigma(q-1)s^{q-2}\int_{J^0} \omega(t)|u(t)|^q \Delta t. \end{aligned} \tag{128}$$

Then, it is obvious to see that $su \in \mathcal{N}_\sigma$ iff $\Theta'_u(s) = 0$, and in particular, $u \in \mathcal{N}_\sigma$ iff $\Theta'_u(1) = 0$.

Before using fiber mapping to study the behavior of Nehari manifolds, let us introduce some symbols.

$$\begin{aligned} \mathcal{F}^\pm &= \left\{ u \in W_{\Delta,a^+}^{\alpha,p} \setminus \{0\} : \int_{J^0} F(t,u(t))\Delta t \gtrless 0 \right\}, \\ \mathcal{F}^0 &= \left\{ u \in W_{\Delta,a^+}^{\alpha,p} \setminus \{0\} : \int_{J^0} F(t,u(t))\Delta t = 0 \right\}, \\ \Pi^\pm &= \left\{ u \in W_{\Delta,a^+}^{\alpha,p} \setminus \{0\} : \int_{J^0} \omega(t)|u(t)|^q \Delta t \gtrless 0 \right\}, \\ \Pi^0 &= \left\{ u \in W_{\Delta,a^+}^{\alpha,p} \setminus \{0\} : \int_{J^0} \omega(t)|u(t)|^q \Delta t = 0 \right\}. \end{aligned} \tag{129}$$

We will study the fibering map Θ_u according to the signs of $\int_{J^0}\omega(t)|u(t)|^q\Delta t$ and $\int_{J^0}F(t,u(t))\Delta t$. To this end, let us define $\rho_u : [0,\infty) \longrightarrow \mathbb{R}$ by setting

$$\rho_u(s) = s^{p-r}\|u\|^p - \sigma s^{q-r}\int_{J^0} \omega(t)|u(t)|^q \Delta t. \tag{130}$$

Hence, for $s > 0$, one gets

$$\begin{aligned} \Theta'_u(s) &= s^{p-1}\|u\|^p - rs^{r-1}\int_{J^0} F(t,u(t))\Delta t - \sigma s^{q-1}\int_{J^0} \omega(t)|u(t)|^q \Delta t \\ &= s^{r-1}\left(\rho_u(s) - r\int_{J^0} F(t,u(t))\Delta t\right), \end{aligned} \tag{131}$$

which implies that $su \in \mathcal{N}_\sigma$ iff s is a solution of the following equation:

$$\rho_u(s) = r\int_{J^0} F(t,u(t))\Delta t. \tag{132}$$

Furthermore, obviously, $\rho_u(0) = 0$ and

$$\rho'_u(s) = (p-r)s^{p-r-1}\|u\|^p - \sigma(q-r)s^{q-r-1}\int_{J^0} \omega(t)|u(t)|^q \Delta t. \tag{133}$$

Lemma 54. *If $u \in \mathcal{F}^{\underline{0}} \cap \Pi^{\underline{0}}$, then Θ_u has no critical point.*

Proof. In this case, $\Theta_u(0) = 0$ and $\Theta'_u(s) > 0$ for all $s > 0$, which yields that Θ_u is strictly increasing and hence has no critical point. The proof is complete. □

Lemma 55. *If $u \in \mathcal{F}^{\underline{0}} \cap \Pi^+$, then Θ_u possesses a unique critical point that corresponds to a global maximum point. Moreover, there exists $s_0 > 0$ such that $s_0 \in \mathcal{N}_\sigma^-$ and $J_\sigma(s_0u) > 0$.*

Proof. In this case, there exists a unique $\bar{s} \in (0,\infty)$ such that $\rho'_u(\bar{s}) = 0$. In addition, $\rho'_u(s) > 0$ for $s \in (0,\bar{s})$ and $\rho'_u(s) < 0$ for $s \in (\bar{s},\infty)$. Note that $\rho_u(0) = 0$ and $\rho_u(s) \longrightarrow -\infty$ as $s \longrightarrow \infty$. So, for $u \in \mathcal{F}^-$, there exists a unique s_0 such that $\rho_u(s_0) = \int_{J^0}F(t,u(t))\Delta t$. Consequently, according to (131), we get $\Theta'_u(s) > 0$ for $0 < s < s_0$, and $\Theta'_u(s) < 0$ for $s > s_0$. That is, Θ_u is increasing on $(0, s_0)$ and decreasing on (s_0, ∞). Therefore, Θ_u has exactly one critical point at s_0, which is a global maximum point. Thus, by (128), $s_0 u \in \mathcal{N}_\sigma^-$. The proof is complete. □

Lemma 56. *If $u \in \mathcal{F}^+ \cap \Pi^0$, then Θ_u possesses a unique critical point that corresponds to a global minimum point. Moreover, there exists $s_1 > 0$ such that $s_1 \in \mathcal{N}_\sigma^+$ and $J_\sigma(s_1 u) < 0$.*

Proof. In this case, it is easy to see that $\rho_u(0) = 0$ and $\rho'_u(s) > 0$ for all $s > 0$, which implies that ρ_u is strictly increasing. Since $u \in \mathcal{F}^+$, there exists a unique $s_1 > 0$ such that $\rho_u(s_1) = \int_{J^0} F(t, u(t))\Delta t$. This implies that Θ_u is decreasing on $(0, s_1)$, increasing on (s_1, ∞) and $\Theta'_u(s_1) = 0$. Thus, Θ_u has exactly one critical point corresponding to global minimum point. Hence, $s_1 u \in \mathcal{N}_\sigma^+$. Moreover, since $J_\sigma(0) = 0$, then we have $J_\sigma(s_1 u) < 0$. The proof is complete. □

Lemma 57. *If $u \in \mathcal{F}^+ \cap \Pi^+$, then there exists $\eta_1 > 0$ such that for $\sigma \in (0, \eta_1)$, Θ_u has a positive value and Θ_u has exactly two critical points that correspond to the local minimum and local maximum. Moreover, there exists $s_2 > 0$ such that $s_2 \in \mathcal{N}_\sigma^+$ and $J_\sigma(s_2 u) < 0$.*

Proof. Let $u \in W^{\alpha,p}_{\Delta,a^+}$. As in above, we define

$$\Xi_u(s) = \frac{s^p}{p}\|u\|^p - \sigma\frac{s^q}{q}\int_{J^0}\omega(t)|u(t)|^q\Delta t. \tag{134}$$

Then,

$$\Xi'_u(s) = s^{p-1}\|u\|^p - \sigma s^{q-1}\int_{J^0}\omega(t)|u(t)|^q\Delta t. \tag{135}$$

Let $\Xi'_u(s) = 0$, we have

$$\tilde{s} = \left(\frac{\|u\|^p}{\sigma\int_{J^0}\omega(t)|u(t)|^q\Delta t}\right)^{1/(q-p)}, \tag{136}$$

which is the maximum value point of Ξ_u. Moreover, one has

$$\begin{aligned}
\Xi_u(\tilde{s}) &= \frac{\tilde{s}^p}{p}\|u\|^p - \sigma\frac{\tilde{s}^q}{q}\int_{J^0}\omega(t)|u(t)|^q\Delta t \\
&= \tilde{s}^p\left(\frac{1}{p}\|u\|^p - \sigma\frac{\tilde{s}^{q-p}}{q}\int_{J^0}\omega(t)|u(t)|^q\Delta t\right) \\
&= \tilde{s}^p\left(\frac{1}{p}\|u\|^p - \sigma\frac{\left(\|u\|^p/\sigma\int_{J^0}\omega(t)|u(t)|^q\Delta t\right)^{(q-p)/(q-p)}}{q}\right. \\
&\quad \left.\cdot\int_{J^0}\omega(t)|u(t)|^q\Delta t\right) \\
&= \tilde{s}^p\left(\frac{1}{p}\|u\|^p - \frac{1}{q}\|u\|^p\right) = \tilde{s}^p\|u\|^p\left(\frac{1}{p} - \frac{1}{q}\right) \\
&= \left[\left(\frac{\|u\|^p}{\sigma\int_{J^0}\omega(t)|u(t)|^q\Delta t}\right)^{1/(q-p)}\|u\|^{(q-p)/(q-p)}\right]^p\left(\frac{1}{p} - \frac{1}{q}\right) \\
&= \left(\frac{1}{p} - \frac{1}{q}\right)\left(\frac{\|u\|^{p+q-p}}{\sigma\int_{J^0}\omega(t)|u(t)|^q\Delta t}\right)^{p/(q-p)} \\
&= \left(\frac{1}{p} - \frac{1}{q}\right)\left(\frac{\|u\|^q}{\sigma\int_{J^0}\omega(t)|u(t)|^q\Delta t}\right)^{p/(q-p)},
\end{aligned}$$

$$\begin{aligned}
\Xi''_u(\tilde{s}) &= (p-1)\tilde{s}^{p-2}\|u\|^p - \sigma(q-1)\tilde{s}^{q-2}\int_{J^0}\omega(t)|u(t)|^q\Delta t \\
&= \tilde{s}^{p-2}\left[(p-1)\|u\|^p - \sigma(q-1)\tilde{s}^{q-p}\int_{J^0}\omega(t)|u(t)|^q\Delta t\right] \\
&= \tilde{s}^{p-2}\left[(p-1)\|u\|^p - \sigma(q-1)\left(\frac{\|u\|^p}{\sigma\int_{J^0}\omega(t)|u(t)|^q\Delta t}\right)^{(q-p)/(q-p)}\right. \\
&\quad \left.\cdot\int_{J^0}\omega(t)|u(t)|^q\Delta t\right] \\
&= \tilde{s}^{p-2}\left[(p-1)\|u\|^p - (q-1)\|u\|^p\right] = \tilde{s}^{p-2}\|u\|^p(p-q) \\
&= (p-q)\left(\frac{\|u\|^p}{\sigma\int_{J^0}\omega(t)|u(t)|^q\Delta t}\right)^{(p-2)/(q-p)}\|u\|^p \\
&= (p-q)\frac{\|u\|^{p(q-2)/(q-p)}}{\left(\sigma\int_{J^0}\omega(t)|u(t)|^q\Delta t\right)^{(p-2)/(q-p)}} < 0.
\end{aligned} \tag{137}$$

In consideration of Proposition 45, we deduce that

$$\begin{aligned}
\Xi_u(\tilde{s}) &= \left(\frac{1}{p} - \frac{1}{q}\right)\left(\frac{\|u\|^q}{\sigma\int_{J^0}\omega(t)|u(t)|^q\Delta t}\right)^{p/(q-p)} \\
&\geq \frac{q-p}{qp}\left(\frac{\|u\|^q}{\sigma\|\omega\|_\infty\|u\|_\infty^q b}\right)^{p/(q-p)} \\
&\geq \frac{q-p}{qp}\left(\frac{\|u\|^q}{b\sigma\|\omega\|_\infty\left(b^{(\alpha-1/p)q}/\Gamma^q(\alpha)((\alpha-1)\bar{p}+1)^{q/p}\right)\|u\|^q}\right)^{p/(q-p)} \\
&= \frac{q-p}{qp}\left(\frac{\Gamma^q(\alpha)((\alpha-1)\bar{p}+1)^{q/\bar{p}}}{\sigma\|\omega\|_\infty b^{1+q(\alpha-1/p)}}\right)^{p/(q-p)} := \delta,
\end{aligned} \tag{138}$$

which is independent of u. We now prove that there exists $\eta_1 > 0$ such that $\Theta'_u(\tilde{s}) > 0$. Taking (16) and Proposition 45 into consideration, one obtains

$$\begin{aligned}\tilde{s}^r\int_{J^0}F(t,u(t))\Delta t &\le \tilde{s}^r\int_{J^0}K|u(t)|^r\Delta t\\ &\le \tilde{s}^r K\|u\|_\infty^r b\\ &\le \left(\frac{\|u\|^p}{\sigma\int_{J^0}g(t)|u(t)|^q\Delta t}\right)^{r/(q-p)} K\frac{b^{1+(\alpha-1/p)r}}{\Gamma^r(\alpha)((\alpha-1)\bar{p}+1)^{r/\bar{p}}}\|u\|^r\\ &= \left(\frac{\|u\|^p}{\sigma\int_{J^0}g(t)|u(t)|^q\Delta t}\right)^{r/(q-p)} \frac{Kb^{1+(\alpha-1/p)r}}{\Gamma^r(\alpha)((\alpha-1)\bar{p}+1)^{r/\bar{p}}}\\ &\quad\times\left(\|u\|^{r\cdot(q-p)/r}\right)^{r/(q-p)}\\ &= \frac{Kb^{1+(\alpha-1/p)r}}{\Gamma^r(\alpha)((\alpha-1)\bar{p}+1)^{r/\bar{p}}}\left(\frac{\|u\|^q}{\sigma\int_{J^0}g(t)|u(t)|^q\Delta t}\right)^{r/(q-p)}\\ &= \frac{Kb^{1+(\alpha-1/p)r}}{\Gamma^r(\alpha)((\alpha-1)\bar{p}+1)^{r/\bar{p}}}\left(\frac{pq}{q-p}\right)^{r/p}(M_u(\tilde{s}))^{r/p}.\end{aligned} \tag{139}$$

Hence, we have

$$\begin{aligned}\Theta_u(\tilde{s}) &= \Xi_u(\tilde{s}) - \tilde{s}^r\int_{J^0}F(t,u(t))\Delta t\\ &\ge \Xi_u(\tilde{s}) - \frac{Kb^{1+(\alpha-1/p)r}}{\Gamma^r(\alpha)((\alpha-1)\bar{p}+1)^{r/\bar{p}}}\left(\frac{pq}{q-p}\right)^{r/p}(\Xi_u(\tilde{s}))^{r/p}\\ &\ge \delta - \delta^{r/p}\frac{Kb^{1+(\alpha-1/p)r}}{\Gamma^r(\alpha)((\alpha-1)\bar{p}+1)^{r/\bar{p}}}\left(\frac{pq}{q-p}\right)^{r/p}\\ &= \delta^{r/p+(p-r)/p} - \delta^{r/p}\frac{Kb^{1+(\alpha-1/p)r}}{\Gamma^r(\alpha)((\alpha-1)\bar{p}+1)^{r/\bar{p}}}\left(\frac{pq}{q-p}\right)^{r/p}\\ &= \delta^{r/p}\left(\delta^{(p-r)/p} - \frac{Kb^{1+(\alpha-1/p)r}}{\Gamma^r(\alpha)((\alpha-1)\bar{p}+1)^{r/\bar{p}}}\left(\frac{pq}{q-p}\right)^{r/p}\right) > 0,\end{aligned} \tag{140}$$

for $0 < \sigma < \eta_1$, where δ is the constant given in (138) and

$$\eta_1 = \frac{q-p}{qp}\frac{\Gamma^q(\alpha)((\alpha-1)\bar{p}+1)^{q/\bar{p}}}{\|\omega\|_\infty b^{1+q(\alpha-1/p)}}\cdot\left[\frac{\Gamma^r(\alpha)((\alpha-1)\bar{p}+1)^{r/\bar{p}}}{Kb^{1+r(\alpha-1/p)}}\cdot\left(\frac{q-p}{pq}\right)^{r/p}\right]^{(q-p)/(p-r)}. \tag{141}$$

The same arguments used in the proof of Lemma 55 show that Θ_u has exactly two critical points which correspond to the local minimum and local maximum. Furthermore, there exists $s_2 > 0$ such that $s_2 u \in \mathcal{N}_\sigma^+$ and $J_\sigma(s_2 u) < 0$. The proof of Lemma 57 is now completed. □

From now on, we define σ_0 by

$$\sigma_0 = \min(\eta_0, \eta_1). \tag{142}$$

Note that if $0 < \sigma < \sigma_0$, then all the above related lemmas are true.

Lemma 58. *Let u be a local minimizer for J_σ on subsets $\mathcal{N}_\sigma^+$ or $\mathcal{N}_\sigma^-$ of $\mathcal{N}_\sigma$ such that $u \notin \mathcal{N}_\sigma^0$. Then, u is a critical of J_σ.*

Proof. Since u is a minimizer for J_σ under the constraint

$$I_\sigma(u) \coloneqq \left\langle J'_\sigma(u), u\right\rangle. \tag{143}$$

Then, applying the theory of Lagrange multipliers, we get the existence of $\eta \in \mathbb{R}$ such that

$$J'_\sigma(u) = \eta I'_\sigma(u). \tag{144}$$

Therefore, one has

$$\left\langle J'_\sigma(u), u\right\rangle = \eta\left\langle I'_\sigma(u), u\right\rangle = \eta\Theta''_u(1) = 0, \tag{145}$$

but $u \notin \mathcal{N}_\sigma^0$ and so $\Theta''_u(1) \ne 0$. Hence, $\eta = 0$, which gives the proof of Lemma 58. The proof is complete. □

In the following, we assume that $1/2 < \alpha < 1$ and $1 < r < p < q$. Let σ_0 be the constant given by (56). Then, the proof of Theorem 50 is based on the following two propositions.

Proposition 59. *Suppose that assumptions of Theorem 50 are satisfied. Then, for all $0 < \sigma < \sigma_0$, J_σ achieves its minimum on $\mathcal{N}_\sigma^+$.*

Proof. In view of $\sigma \in (0, \sigma_0)$ and Lemma 53, we have J_σ which is bounded below on $\mathcal{N}_\sigma$ and also on $\mathcal{N}_\sigma^+$. Therefore, there exists a minimizing sequence $\{u_k\} \subset \mathcal{N}_\sigma^+$ such that

$$\lim_{k\longrightarrow\infty} J_\sigma(u_k) = \inf_{u\in\mathcal{N}_\sigma^+} J_\sigma(u). \tag{146}$$

As J_σ is coercive on $\mathcal{N}_\sigma$, $\{u_k\}$ is a bounded sequence in $W_{\Delta,a^+}^{\alpha,p}$ up to a subsequence; there is $\{u_k\} \subset W_{\Delta,a^+}^{\alpha,p}$ such that $u_k u_\sigma$ weakly in $W_{\Delta,a^+}^{\alpha,p}$.

Let $u \in W_{\Delta,a^+}^{\alpha,p}$ such that $\int_{J^0} F(t,u(t))\Delta t > 0$. So, using Lemmas 56 and 57, there is $s_1 > 0$ such that $s_1 u \in \mathcal{N}_\sigma^+$ and $J_\sigma(u) < 0$. Therefore, $\inf_{u\in\mathcal{N}_\sigma^+} J_\sigma(u) < 0$.

Because of $\{u_k\} \subset \mathcal{N}_\sigma$, we have

$$J_\sigma(u_k) = \left(\frac{1}{p} + \frac{1}{q}\right)\|u_k\|^p - \left(1 + \frac{r}{q}\right)\int_{J^0} F(t,u_k(t))\Delta t, \tag{147}$$

which yields that

$$\left(1 + \frac{r}{q}\right)\int_{J^0} F(t,u_k(t))\Delta t = \left(\frac{1}{p} + \frac{1}{q}\right)\|u_k\|^p - J_\sigma(u_k). \tag{148}$$

Letting k go to infinity in the above equation, we obtain

$$\int_{J^0} F(t,u_\sigma(t))\Delta t > 0. \tag{149}$$

Now, we declare that $u_k \longrightarrow u_\sigma$ strongly in $W^{\alpha,p}_{\Delta,a^+}$. Otherwise, we have

$$\|u_\sigma\|^p < \liminf_{k\longrightarrow\infty}\|u_k\|^p. \tag{150}$$

Since $\Theta'_{u_\sigma}(s_1)=0$, it follows from (150) that $\Theta'_{u_k}(s_1)>0$ for sufficiently large k. Hence, we must have $s_1>1$.

However, $s_1u_\sigma \in \mathcal{N}^+_\sigma$, and so,

$$J_\sigma(s_1u_\sigma) < J_\sigma(u_\sigma) \le \lim_{k\longrightarrow\infty} J_\sigma(u_k) = \inf_{u\in\mathcal{N}^+_\sigma} J_\sigma(u), \tag{151}$$

which gives a contradiction. Thus, $u_k \longrightarrow u_\sigma$ strongly in $W^{\alpha,p}_{\Delta,a^+}$; as a consequence, $u_\sigma \in \mathcal{N}_\sigma = \mathcal{N}^+_\sigma \cup \mathcal{N}^0_\sigma$. In addition, it is easy to check by contradiction that $u_\sigma \in \mathcal{N}^+_\sigma$. Therefore, from (149), u_σ is a nontrivial solution of (85). The proof is complete. □

Proposition 60. *Suppose that assumptions of Theorem 50 are satisfied. Then, for all $0<\sigma<\sigma_0$, J_σ achieves its minimum on $\mathcal{N}^-_\sigma$.*

Proof. Let $u\in\mathcal{N}^-_\sigma$. Hence, by the result of Lemma 55, we obtain the existence of $\eta_1>0$ such that $J_\sigma(u)\ge\eta_1$. Therefore, there is a minimizing sequence $\{v_k\}\subset\mathcal{N}^-_\sigma$ such that

$$\lim_{k\longrightarrow\infty} J_\sigma(v_k) = \inf_{u\in\mathcal{N}^-_\sigma} J_\sigma(u) > 0. \tag{152}$$

Furthermore, in view of Lemma 53, we know that J_σ is coercive, so $\{v_k\}$ is a bounded sequence in $W^{\alpha,p}_{\Delta,a^+}$ up to a subsequence, there is $\{v_k\}\subset W^{\alpha,p}_{\Delta,a^+}$ such that $v_k \rightharpoonup v_\sigma$ weakly in $W^{\alpha,p}_{\Delta,a^+}$.

Because of $\{v_k\}\subset\mathcal{N}_\sigma$, then we have

$$\sigma\left(\frac{1}{r}-\frac{1}{q}\right)\int_{J^0}\omega(t)|v_k(t)|^q\Delta t = J_\sigma(v_k) - \left(\frac{1}{p}-\frac{1}{r}\right)\|v_k\|^p. \tag{153}$$

Letting k go to infinity in (153), it follows from (152) that

$$\int_{J^0}\omega(t)|v_k(t)|^q\Delta t > 0. \tag{154}$$

Therefore, $v_\sigma\in\Pi^+$, and so, Θ_{v_σ} has a global maximum at some point $\tilde{s}$. Consequently, $\tilde{s}v_\sigma\in\mathcal{N}^-_\sigma$.

On the other hand, $v_k\in\mathcal{N}^-_\sigma$ implies that 1 is a global maximum point for Θ_{v_k}, i.e.,

$$J_\sigma(\tilde{s}u_k) = \Theta_{v_k}(\tilde{s}) \le \Theta_{v_k}(1) = J_\sigma(v_k). \tag{155}$$

Now, as in the proof of Proposition 59, we assert that $v_k \longrightarrow v_\sigma$ in $W^{\alpha,p}_{\Delta,a^+}$. Assuming it is not true, then

$$\|v_\sigma\|^p < \liminf_{k\longrightarrow\infty}\|v_k\|^p. \tag{156}$$

It follows from 4.23 that

$$\begin{aligned} J_\sigma(\tilde{s}v_\sigma) &= \frac{\tilde{s}^p}{p}\|v_\sigma\|^p - \tilde{s}^r\int_{J^0}F(t,v_\sigma(t))\Delta t - \sigma\frac{\tilde{s}^q}{q}\int_{J^0}\omega(t)|v_\sigma(t)|^q\Delta t \\ &< \liminf_{k\longrightarrow\infty}\left(\frac{\tilde{s}^p}{p}\|v_k\|^p - \tilde{s}^r\int_{J^0}F(t,v_k(t))\Delta t \right.\\ &\quad \left. - \sigma\frac{\tilde{s}^q}{q}\int_{J^0}\omega(t)|v_k(t)|^q\Delta t\right) \\ &\le \lim_{k\longrightarrow\infty} J_\sigma(\tilde{s}v_k) = \Theta_{v_k}(\tilde{s}) \le \inf_{u\in\mathcal{N}^-_\sigma} J_\sigma(v_k) \\ &= \Theta_{v_k}(1) = \inf_{u\in\mathcal{N}^-_\sigma} J_\sigma(u), \end{aligned} \tag{157}$$

which gives a contradiction. Therefore, $v_k \longrightarrow v_\sigma$, and so, $v_\sigma\in\mathcal{N}^-_\sigma\cup\mathcal{N}^0_\sigma$.

Using Lemma 52, we have $\mathcal{N}^0_\sigma=\varnothing$, so v_σ is a minimizer for J_σ on $\mathcal{N}^-_\sigma$.

On the other hand, by (22), v_σ is a nontrivial solution of (1). Finally, since $\mathcal{N}^-_\sigma\cap\mathcal{N}^+_\sigma=\varnothing$, u_σ and v_σ are distinct. That is, the result of Theorem 50 holds true. The proof is complete. □

5. Conclusions

In this paper, we introduced a class of fractional Sobolev spaces via the fractional derivative of Riemann-Liouville on time scales and obtain some of their basic properties. As an application, we use critical point theory to study the solvability of a class of fractional boundary value problems on time scales. The results and methods in this paper can also be used to study the solvability of other boundary value problems on time scales. At present, the concept of fractional derivatives in different meanings is constantly being proposed. Therefore, studying the theory and application of fractional Sobolev spaces on time scales in other meanings is our future direction.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Authors' Contributions

The authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

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