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Research Article

Parametric Marcinkiewicz Integral and Its Higher-Order Commutators on Variable Exponents Morrey-Herz Spaces

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In this article, we prove the boundedness of the parametric Marcinkiewicz integral and its higher-order commutators generated by BMO spaces on the variable Morrey-Herz space. All the results are new even when $\alpha(\cdot)$ is a constant.

1. Introduction

Throughout the entirety of this article, we assume that $n \geq 2$, $\mathbb{R}^n$ is the n-dimensional Euclidean space, and $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure $d\rho$. The function Ω is assumed to be homogeneous of degree zero on $\mathbb{R}^n$ with $\Omega \in L^1(\mathbb{S}^{n-1})$ and

$$\int_{\mathbb{S}^{n-1}} \Omega\left(x'\right) \mathrm{d}\rho\left(x'\right) = 0, \tag{1}$$

where $x' = x/|x|$ for any $x \in \mathbb{R}^n \setminus \{0\}$. For $\rho \in (0, n)$, the parametric Marcinkiewicz integral $\mathscr{M}_\Omega^\varrho$ of higher dimensions is defined as follows:

$$\mathscr{M}_\Omega^\varrho(f)(x) \coloneqq \left(\int_0^\infty \left| t^{-\varrho} \int_{|x-y| \leq t} \frac{\Omega(x-y) f(y)}{|x-y|^{n-\rho}} dy \right|^2 \frac{dt}{t}\right)^{1/2}. \tag{2}$$

Let B be a ball with a radius $\tau > 0$, and a center $x \in \mathbb{R}^n$. A locally integrable function Λ is said to be in the BMO space, if it satisfies

$$\|\Lambda\|_* \coloneqq \sup_B \frac{1}{|B|} \int_B |\Lambda(z) - \Lambda_B| dz < \infty, \tag{3}$$

where $\Lambda_B = |B|^{-1} \int_B \Lambda(t) dt$ and $|E|$ denotes the Lebesgue measure of the set E in $\mathbb{R}^n$. For $\Lambda \in \mathrm{BMO}, i \in \mathbb{N}$, the i-order commutator for the parametric Marcinkiewicz integral $\mathscr{M}_{\Omega,\Lambda^i}^\varrho$ is defined as follows:

$$\mathscr{M}_{\Omega,\Lambda^i}^\varrho(f)(x) \coloneqq \left(\int_0^\infty \left| t^{-\varrho} \int_{|x-y| \leq t} (\Lambda(x) - \Lambda(y))^i \frac{\Omega(x-y) f(y)}{|x-y|^{n-\rho}} dy \right|^2 \frac{dt}{t}\right)^{1/2}. \tag{4}$$

If $\rho = 1$ in (2), then the operator $\mathscr{M}_\Omega^\varrho$ is equivalent to the classical Marcinkiewicz function $\mathscr{M}_\Omega^1$, which was initially introduced by Stein [1] in 1958. When $\Omega \in Lip_\beta(\mathbb{S}^{n-1}), \beta \in (0, 1]$, Stein [1] demonstrated that $\mathscr{M}_\Omega^1$ is bounded on L^p for $p \in (1, 2]$. Subsequently, the authors of [2] established the L^p-boundedness of $\mathscr{M}_\Omega^1$ for every $p \in (1,\infty)$ when $\Omega \in \mathbb{C}^1(\mathbb{S}^{n-1})$. On the other hand, Calderón [3] proved that the commutator the Hilbert transform H generated by $\Lambda \in$ BMO, defined by $[\Lambda, T]f \coloneqq \Lambda T(f) - T(\Lambda f)$, is bounded on

$L^2(\mathbb{R}^n)$. Coifman et al. [4] arrived at the conclusion that the commutator, which was generated by the Calderón-Zygmund operator T and the $\Lambda \in \mathrm{BMO}$, is bounded on L^p for $p \in (1,\infty)$. Since then, the commutators of the Calderón-Zygmund operator have played an essential role in the study of the regularity of solutions to second-order elliptic, parabolic, and ultraparabolic partial differential equations, see for example [5–11]. Moreover, the boundedness of the commutators of various operators generated by a BMO function has been widely studied. Particularly, Torchinsky and Wang [12] studied the weighted L^p-boundedness of $\mathscr{M}^1_{\Omega,\Lambda^i}$, where $\mathscr{M}^1_{\Omega,\Lambda^i}$ is the i-order commutator of Marcinkiewicz integral. The authors of [13] studied the behaviour of the Hardy-Littlewood maximal operator and the action of commutators in generalized local Morrey spaces and generalized Morrey spaces. For further research works studying the commutators on different function spaces, we refer to [9, 14–21] and references therein.

The parametric Marcinkiewicz integral $\mathscr{M}^\varrho_\Omega$ was originally introduced by Hörmander in [22] where the author established the boundeness of $\mathscr{M}^\varrho_\Omega$ on L^p for $p \in (1,\infty)$ under the condition $\Omega \in Lip_\beta(\mathbb{S}^{n-1}), (\beta \in (0,1])$ and $\varrho > 0$. Shi and Jiang [23] investigated the weighted L^p-boundedness of $\mathscr{M}^\varrho_\Omega$ and $\mathscr{M}^\varrho_{\Omega,\Lambda^i}$. Since that time, the boundedness of the parametric Marcinkiewicz integral, as well as its related commutator, in several types of function spaces have attracted the attention of many researchers. Deringoz and Hasanov [24] considered the boundedness of the operator $\mathscr{M}^\varrho_\Omega$ on generalized Orlicz-Morrey spaces. On generalized weighted Morrey spaces, Deringoz [25] investigated the boundedness of rough parametric Marcinkiewicz integral $\mathscr{M}^\varrho_\Omega$ and its higher-order commutator $\mathscr{M}^\varrho_{\Omega,\Lambda^i}$. For more applications and recent developments on the research of the parametric Marcinkiewicz function, see [26–31].

In the last decades, the variable Lebesgue spaces have been intensively studied since the pioneering work of [32] by Kovácık and Rákosnık. Additionally, different studies on variable function spaces, such as variable exponents Fourier-Besov-Morrey spaces [33–35], variable exponents Fourier-Besov spaces [36, 37], variable exponent Morrey spaces [38], variable Bessel potential spaces [39, 40], and variable exponent Hardy spaces [41, 42], were developed due to their applications in the modeling of electrorheological fluids, PDEs with nonstandard growth, and image restoration. Recently, Izuki studied the Herz spaces $\dot{K}^\alpha_{p(\cdot),q}$ in [43, 44]. As a generalization, Izuki [45] introduced the variable Morrey-Herz spaces $M\dot{K}^{\alpha,\sigma}_{p(\cdot),q}$. In fact, the author of [45] found that vector-valued sublinear operators which satisfy a certain size condition are bounded on the variable Morrey-Herz spaces. Furthermore, Almeida and Drihem [46] enhanced the variable case of the Herz spaces $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}$ and established the boundedness results for a class of sublinear operators. Lu and Zhu [47] generalized Izuki's result for the $M\dot{K}^{\alpha(\cdot),\sigma}_{p(\cdot),q}$. For further information and applications, consult [48–54].

Inspired by the research mentioned above, the main goal of this article is to prove the boundedness of the rough parametric Marcinkiewicz integral and its higher-order commutators on the variable exponents Morrey-Herz spaces.

Henceforth, wherever the symbol C appears, it represents a positive constant whose value may vary but is independent of the basic variables. The expression $f \lesssim g$ denotes the existence of constant C such that $f \leq Cg$, and $f \approx g$ means that $f \lesssim g \lesssim f$. If no further instructions are provided, the symbol for any space denoted by $\mathscr{X}(\mathbb{R}^n)$ is represented by $\mathscr{X}$. For instance, $L^p(\mathbb{R}^n)$ is abbreviated as L^p.

2. Definitions and Preliminaries

In this section, we review some notations, definitions, and properties related to our work.

A variable exponent is a measurable function $p(\cdot)$: $\mathbb{R}^n \longrightarrow (0,\infty]$. For any variable exponent $p(\cdot)$, we set $p_- := \operatorname{essinf}\{p(x): x \in \mathbb{R}^n\}$ and $p_+ := \operatorname{esssup}\{p(x): x \in \mathbb{R}^n\}$. Define the sets $\mathscr{P}$ by

$$\mathscr{P} := \{p(x) \text{ is measurable function} : 1 < p_- \text{ and } p_+ < \infty\}. \quad (5)$$

Let $p(\cdot) \in \mathscr{P}$. The variable Lebesgue space $L^{p(\cdot)}$ consists of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^{p(\cdot)}} := \inf\left\{\vartheta \in (0,\infty): \wp_{p(\cdot)}(f(x)) \leq 1\right\} < \infty, \quad (6)$$

where

$$\wp_{p(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx. \quad (7)$$

It is obvious that the variable exponent Lebesgue norm has the following property $\| |f|^\beta \|_{L^{p(\cdot)}} = \|f\|^\beta_{L^{\beta p(\cdot)}}, \quad \beta \geq 1/p_-$.

Define the set $\mathscr{B}$ by

$$\mathscr{B} := \{p(x) \in \mathscr{P} : M_{\mathrm{HL}} \text{ is bounded on variable } L^p\}, \quad (8)$$

where M_{HL} stands for the Hardy-Littlewood maximal function, which is defined as follows:

$$(M_{\mathrm{HL}}f)(x) = \sup_{\substack{B \ni x,\\ B \subset \mathbb{R}^n}} \frac{1}{|B(x,\tau)|} \int_{B(x,\tau)} |f(z)| dz, f \in L^1_{\mathrm{loc}}. \quad (9)$$

Definition 1 (see [46]). Let $\Theta(\cdot)$ be real function on $\mathbb{R}^n$.

(i) If there exists a constant $C^{\log} > 0$ such that

$$|\Theta(x) - \Theta(0)| \leq \frac{C^{\log}}{\log(e + (1/|x|))}, \text{ for all } x \in \mathbb{R}^n, \quad (10)$$

then the function $\Theta(\cdot)$ is said to be a log-Hölder continuous at the origin (or has a log decay at the origin).

(ii) If there exist $\Theta_\infty \in (0,\infty)$ and a constant $C^{\log} > 0$ such that

$$|\Theta(x) - \Theta_\infty| \leq \frac{C^{\log}}{\log(e + |x|)}, \text{ for all } x \in \mathbb{R}^n, \quad (11)$$

then the function $\Theta(\cdot)$ is said to be a log-Hölder continuous at the infinity (or has a log decay at the infinity).

If $p(\cdot) \in \mathscr{P}$, then the following expression of Hölder's inequality is valid:

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq \left(1 + \frac{1}{p_-} - \frac{1}{p_+}\right) \|f\|_{L^{p(\cdot)}} \|g\|_{L^{p'(\cdot)}}. \quad (12)$$

See [55].

Here and hereafter, $p'(\cdot)$ denotes the conjugate exponent of $p(\cdot)$, i.e., $p'(\cdot) := p(\cdot)/(p(\cdot) - 1)$. It is well-known that if $p(\cdot)$ belongs to $\mathscr{B}$, then $p'(\cdot) \in \mathscr{B}$ (see [56]).

For any $\ell \in \mathbb{Z}$, let $B_\ell := \{x \in \mathbb{R}^n : |x| \leq 2^\ell\}$, $\mathfrak{R}_\ell := B_\ell \setminus B_{\ell-1}$, and denote by $\chi_\ell := \chi_{\mathfrak{R}_\ell}$ the characteristic function of $\mathfrak{R}_\ell$.

Definition 2 (see [46]). Let $q \in (0,\infty]$, $\alpha(\cdot) \in L^\infty$ and let $p(\cdot) \in \mathscr{P}$. The homogeneous variable Herz space $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}$ is defined as the set of all functions $f \in L^{p(\cdot)}_{loc}(\mathbb{R}^n \setminus \{0\})$ such that

$$\|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}} := \left(\sum_{\ell\in\mathbb{Z}} \left\|2^{\ell\alpha(\cdot)} f\chi_\ell\right\|^q_{L^{p(\cdot)}}\right)^{1/q} < \infty, \quad (13)$$

for $q < \infty$, and the usual modification should be made when $q = \infty$.

Definition 3 (see [47]). Let $\sigma \in [0,\infty)$, $\alpha(\cdot) \in L^\infty$, $q \in (0,\infty]$ and let $p(\cdot) \in \mathscr{P}$. The homogeneous variable Morrey-Herz space $M\dot{K}^{\alpha(\cdot),\sigma}_{q,p(\cdot)}$ is defined as the set of all functions $f \in L^{p(\cdot)}_{loc}(\mathbb{R}^n \setminus \{0\})$ such that

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\sigma}_{q,p(\cdot)}} := \sup_{\kappa\in\mathbb{Z}} 2^{-\kappa\sigma} \left(\sum_{\ell=-\infty}^{\kappa} \left\|2^{\ell\alpha(\cdot)} f\chi_\ell\right\|^q_{L^{p(\cdot)}}\right)^{1/q} < \infty, \quad (14)$$

for $q < \infty$, and the usual modification should be made when $q = \infty$.

Lemma 4 (see [44]). *Let $p(\cdot) \in \mathscr{B}$. Then, for any ball B in $\mathbb{R}^n$,*

$$\|\chi_B\|_{L^{p(\cdot)}} \|\chi_B\|_{L^{p'(\cdot)}} \leq C|B|. \quad (15)$$

Lemma 5 (see [44]). *Let $p_\flat(\cdot) \in \mathscr{B}$, $\flat = 1, 2$. Then, there are positive constants $\delta_{\flat 1}, \delta_{\flat 2} \in (0, 1)$, such that for any ball B in $\mathbb{R}^n$ and any measurable subset $S \subset B$,*

$$\frac{\|\chi_S\|_{L^{p'_\flat(\cdot)}}}{\|\chi_B\|_{L^{p'_\flat(\cdot)}}} \lesssim \left(\frac{|S|}{|B|}\right)^{\delta_{\flat 2}}, \text{ and } \frac{\|\chi_S\|_{L^{p_\flat(\cdot)}}}{\|\chi_B\|_{L^{p_\flat(\cdot)}}} \lesssim \left(\frac{|S|}{|B|}\right)^{\delta_{\flat 1}}. \quad (16)$$

Proposition 6 (see [47]). *Let $\alpha(\cdot) \in L^\infty$, $q \in (0,\infty)$, $\sigma \in [0,\infty)$, and let $p(\cdot) \in \mathscr{P}$. If the function $\alpha(\cdot)$ is log-Hölder continuous function both at origin and at infinity, then the following inequalities hold:*

$$\|f\|^q_{M\dot{K}^{\alpha(\cdot),\sigma}_{q,p(\cdot)}} \asymp \max\left\{\sup_{\substack{\kappa<0\\ \kappa\in\mathbb{Z}}} 2^{-\kappa\sigma q} \sum_{\ell=-\infty}^{\kappa} 2^{\ell q\alpha(0)} \|f\chi_\ell\|^q_{L^{p(\cdot)}}, \sup_{\substack{\kappa\geqslant 0\\ \kappa\in\mathbb{Z}}} 2^{-\kappa\sigma q} \sum_{\ell=-\infty}^{-1} 2^{\ell q\alpha(0)} \|f\chi_\ell\|^q_{L^{p(\cdot)}} + \sup_{\substack{\kappa\geqslant 0\\ \kappa\in\mathbb{Z}}} 2^{-\kappa\sigma q} \sum_{\ell=0}^{\kappa} 2^{\ell q\alpha_\infty} \|f\chi_\ell\|^q_{L^{p(\cdot)}}\right\}. \quad (17)$$

Lemma 7 (see [46]). *Let $\alpha(x) \in L^\infty$ and $\tau_1 > 0$. If the function $\alpha(x)$ is log-Hölder continuous both at origin and infinity, then the following inequality holds:*

$$\tau_1^{\alpha(x)} \lesssim \tau_2^{\alpha(y)} \times \begin{cases} \left(\dfrac{\tau_1}{\tau_2}\right)^{\alpha_+}, & 0 < \tau_2 \leqslant \dfrac{\tau_1}{2}, \\ 1, & \dfrac{\tau_1}{2} < \tau_2 \leqslant 2\tau_1, \\ \left(\dfrac{\tau_1}{\tau_2}\right)^{\alpha_-}, & \tau_2 > 2\tau_1, \end{cases} \quad (18)$$

for every $x \in B(0,\tau_1) \setminus B(0,\tau_1/2)$ and $y \in B(0,\tau_2) \setminus B(0,\tau_2/2)$.

Lemma 8 (see [56]). *Let $p(\cdot), q^*(x) \in \mathscr{P}$, and let $q \in (0,\infty)$ such that $(1/p(x)) = (1/q) + (1/q^*(x))$. Then, for any measurable functions $f \in L^{p(\cdot)}$ and $g \in L^{q^*(\cdot)}$,*

$$\|fg\|_{L^{p(\cdot)}} \lesssim \|f\|_{L^{q^*(\cdot)}} \|g\|_{L^q}. \quad (19)$$

Lemma 9 (see [57]). *Let i be a positive integer, $\Lambda \in BMO$ and let $p(\cdot) \in \mathscr{B}$. Then, there exists a positive C such that for all $\ell, j \in \mathbb{Z} (\ell > j)$,*

$$\sup_{B\subset\mathbb{R}^n} \frac{1}{\|\chi_B\|_{L^{p(\cdot)}}} \left\|(\Lambda - \Lambda_B)^i \chi_B\right\|_{L^{p(\cdot)}} \asymp \|\Lambda\|^i_*, \quad (20)$$

and

$$\left\|\left(\Lambda-\Lambda_{B_j}\right)^i\chi_{B_\ell}\right\|_{L^{p(\cdot)}}\lesssim(\ell-j)^i\|\Lambda\|_*^i\left\|\chi_{B_\ell}\right\|_{L^{p(\cdot)}}. \tag{21}$$

The main results of this article are as follows.

Theorem 10. *Suppose that $p(\cdot)\in\mathscr{B}$ and $\Omega\in L^s(\mathbb{S}^{n-1})$ with $s>(p')_+$ satisfying (1). Let $\varrho\in(0,n)$, $q\in(0,\infty]$, $\sigma>0$, and $\alpha(\cdot)$ be log-Hölder continuous both at the origin and at infinity, such that*

$$-n\delta_1+\sigma<\alpha_-\leq\alpha_+<\frac{-(n-1)}{s}+n\delta_2, \tag{22}$$

where $\delta_1,\delta_2\in(0,1)$ are the constants mentioned in Lemma 5. Then, the operator $\mathscr{M}_\Omega^\rho$ is bounded on $M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\sigma}$.

Theorem 11. *Suppose that $p(\cdot)\in\mathscr{B}$ and $\Omega\in L^s(\mathbb{S}^{n-1})$ with $s>(p')_+$ satisfies (1). Let $\varrho\in(0,n)$, $q\in(0,\infty]$, $b\in BMO$, $\sigma>0$, $i\in\mathbb{N}$ and $\alpha(\cdot)$ be log-Hölder continuous both at the origin and at infinity, such that*

$$-n\delta_1+\sigma<\alpha_-\leq\alpha_+<\frac{-(n-1)}{s}+n\delta_2, \tag{23}$$

where $\delta_1,\delta_2\in(0,1)$ are the constants mentioned in Lemma 5. Then, the operator $\mathscr{M}_{\Omega,\Lambda^i}^\varrho$ is bounded on $M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\sigma}$.

It is worth noting that if $\sigma=0$, then the variable Morrey-Herz space $M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),0}$ dates back to the variable Herz space $\dot{K}_{q,p(\cdot)}^{\alpha(\cdot)}$. Thus, by letting $\sigma=0$ in Theorems 10 and 11, we will get the following results on the variable exponents Herz spaces.

Corollary 12. *Suppose that $p(\cdot)\in\mathscr{B}$ and $\Omega\in L^s(\mathbb{S}^{n-1})$ with $s>(p')_+$ satisfies (1). Let $\rho\in(0,n)$, $q\in(0,\infty]$, and $\alpha(\cdot)$ be log-Hölder continuous both at the origin and at infinity, such that*

$$-n\delta_1<\alpha_-\leq\alpha_+<\frac{-(n-1)}{s}+n\delta_2, \tag{24}$$

where $\delta_1,\delta_2\in(0,1)$ are the constants mentioned in Lemma 5. Then, the operator $\mathscr{M}_\Omega^\varrho$ is bounded on $\dot{K}_{q,p(\cdot)}^{\alpha(\cdot)}$.

Corollary 13. *Suppose that $p(\cdot)\in\mathscr{B}$ and $\Omega\in L^s(\mathbb{S}^{n-1})$ with $s>(p')_+$ satisfies (1). Let $\varrho\in(0,n)$, $q\in(0,\infty]$, $b\in BMO$ and $i\in\mathbb{N}$ and $\alpha(\cdot)$ be log-Hölder continuous both at the origin and at infinity, such that*

$$-n\delta_1<\alpha_-\leq\alpha_+<\frac{-(n-1)}{s}+n\delta_2, \tag{25}$$

where $\delta_1,\delta_2\in(0,1)$ are the constants mentioned in Lemma 5. Then, the operator $\mathscr{M}_{\Omega,\Lambda^i}^\varrho$ is bounded on $\dot{K}_{q,p(\cdot)}^{\alpha(\cdot)}$.

Remark 14. If $\alpha(\cdot)$ is a constant function, i.e., $\alpha(\cdot)=\alpha$, then the results of Corollaries 12 and 13 can be founded in [30].

3. Proofs of Theorems 10 and 11

Proof of Theorem 10. Let $f\in M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\sigma}$. For any $j\in\mathbb{Z}$, let $f_j:=f\chi_j$, then we have

$$f(x)=\sum_{j=-\infty}^{\infty}f\chi_j(x)\coloneqq\sum_{j=\ell-1}^{\ell+1}f_j(x)+\sum_{j=-\infty}^{\ell-2}f_j(x)+\sum_{j=\ell+2}^{\infty}f_j(x). \tag{26}$$

□

Using the definition of $M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\sigma}$, we have

$$\begin{aligned}\|\mathscr{M}_\Omega^\varrho(f)\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\sigma}}^q&=\sup_{\kappa\in\mathbb{Z}}2^{-\kappa\sigma q}\sum_{\ell=-\infty}^{\kappa}\left\|2^{\ell\alpha(\cdot)}\left|\mathscr{M}_\Omega^\varrho(f)\right|\chi_\ell\right\|_{L^{p(\cdot)}}^q\\&\lesssim\sup_{\kappa\in\mathbb{Z}}2^{-\kappa\sigma q}\sum_{\ell=-\infty}^{\kappa}\left\|2^{\ell\alpha(\cdot)}\left(\sum_{j=\ell-1}^{\ell+1}\left|\mathscr{M}_\Omega^\varrho\left(f_j\right)\right|\right)\chi_\ell\right\|_{L^{p(\cdot)}}^q\\&\quad+\sup_{\kappa\in\mathbb{Z}}2^{-\kappa\sigma q}\sum_{\ell=-\infty}^{\kappa}\left\|2^{\ell\alpha(\cdot)}\left(\sum_{j=-\infty}^{\ell-2}\left|\mathscr{M}_\Omega^\varrho\left(f_j\right)\right|\right)\chi_\ell\right\|_{L^{p(\cdot)}}^q\\&\quad+\sup_{\kappa\in\mathbb{Z}}2^{-\kappa\sigma q}\sum_{\ell=-\infty}^{\kappa}\left\|2^{\ell\alpha(\cdot)}\left(\sum_{j=\ell+2}^{\infty}\left|\mathscr{M}_\Omega^\varrho\left(f_j\right)\right|\right)\chi_\ell\right\|_{L^{p(\cdot)}}^q\\&=:\mathfrak{T}_1+\mathfrak{T}_2+\mathfrak{T}_3.\end{aligned} \tag{27}$$

First, we estimate $\mathfrak{T}_1$. By Proposition 6, we obtain

$$\mathfrak{T}_1\simeq\max\left\{\sup_{\substack{\kappa<0\\\kappa\in\mathbb{Z}}}2^{-\kappa\sigma q}\sum_{\ell=-\infty}^{\kappa}\left\|2^{\ell\alpha(0)q}\left(\sum_{j=\ell-1}^{\ell+1}\left|\mathscr{M}_\Omega^\varrho\left(f_j\right)\right|\right)\chi_\ell\right\|_{L^{p(\cdot)}}^q,\ \sup_{\substack{\kappa\geq0\\\kappa\in\mathbb{Z}}}2^{-\kappa\sigma q}\sum_{\ell=-\infty}^{-1}\left\|2^{\ell\alpha(0)q}\left(\sum_{j=\ell-1}^{\ell+1}\left|\mathscr{M}_\Omega^\varrho\left(f_j\right)\right|\right)\chi_\ell\right\|_{L^{p(\cdot)}}^q+\sup_{\substack{\kappa\geq0\\\kappa\in\mathbb{Z}}}2^{-\kappa\sigma q}\sum_{\ell=0}^{\kappa}\left\|2^{\ell\alpha_\infty q}\left(\sum_{j=\ell-1}^{\ell+1}\left|\mathscr{M}_\Omega^\varrho\left(f_j\right)\right|\right)\chi_\ell\right\|_{L^{p(\cdot)}}^q\right\}. \tag{28}$$

Using the fact that $2^{\alpha(\cdot)} \asymp 2^{\alpha(0)}$, and the boundedness of $\mathscr{M}_{\Omega}^{\rho}$ on $L^{p(\cdot)}$(see [30]), we deduce

$$\mathfrak{T}_1 \lesssim \max\left\{\sup_{\substack{\kappa<0\\ \kappa\in\mathbb{Z}}} 2^{-\kappa\sigma q}\sum_{\ell=-\infty}^{\kappa}\left\|2^{\ell\alpha(0)}|f\chi_\ell|\right\|_{L^{p(\cdot)}}^q, \sup_{\substack{\kappa\geq0\\ \kappa\in\mathbb{Z}}} 2^{-\kappa\sigma q}\sum_{\ell=-\infty}^{-1}\left\|2^{\ell\alpha(0)}|f\chi_\ell|\right\|_{L^{p(\cdot)}}^q + \sup_{\substack{\kappa\geq0\\ \kappa\in\mathbb{Z}}} 2^{-\kappa\sigma q}\sum_{\ell=0}^{\kappa}\left\|2^{\ell\alpha_\infty}|f\chi_\ell|\right\|_{L^{p(\cdot)}}^q\right\} \lesssim \|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\sigma}}^q. \tag{29}$$

For the term $\mathfrak{T}_2$, we firstly estimate $2^{\ell\alpha(x)}|\mathscr{M}_{\Omega}^{\varrho}(f_j)(x)|$.

$$\begin{aligned}
&2^{\ell\alpha(x)}\left|\mathscr{M}_{\Omega}^{\varrho}\left(f_j\right)(x)\right|\cdot\chi_\ell(x)\\
&\lesssim 2^{\ell\alpha(x)}\left(\int_0^{|x|}\left|t^{-\varrho}\int_{|x-y|\leq t}\frac{\Omega(x-y)f_j(y)}{|x-y|^{n-\varrho}}dy\right|^2\frac{dt}{t}\right)^{1/2}\cdot\chi_\ell(x)\\
&\quad+2^{\ell\alpha(x)}\left(\int_{|x|}^{\infty}\left|t^{-\varrho}\int_{|x-y|\leq t}\frac{\Omega(x-y)f_j(y)}{|x-y|^{n-\rho}}dy\right|^2\frac{dt}{t}\right)^{1/2}\\
&\quad\cdot\chi_\ell(x).
\end{aligned} \tag{30}$$

It is clear that if $x\in\mathfrak{R}_k$, $j+2\leq\ell$, and$y\in\mathfrak{R}_j$, then $|x-y| \asymp |x| \asymp 2^\ell$. Thus, for $\varrho\in(0,n)$, by the mean value theorem, we have

$$\left|\frac{1}{|x-y|^{2\varrho}}-\frac{1}{|x|^{2\varrho}}\right|\leq\frac{|y|}{|x-y|^{2\rho+1}}. \tag{31}$$

By (31) and the Minkowski's inequality, it follows that

$$\begin{aligned}
&2^{\ell\alpha(x)}\left|\mathscr{M}_{\Omega}^{\varrho}\left(f_j\right)(x)\right|\cdot\chi_\ell(x)\\
&\lesssim 2^{\ell\alpha(x)}\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|\left|f_j(y)\right|}{|x-y|^{n-\varrho}}\cdot\left(\int_{|x-y|}^{|x|}\frac{dt}{t^{2\rho+1}}\right)^{1/2}dy\cdot\chi_\ell(x)\\
&\quad+2^{\ell\alpha(x)}\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|\left|f_j(y)\right|}{|x-y|^{n-\varrho}}\cdot\left(\int_{|x|}^{\infty}\frac{dt}{t^{2\rho+1}}\right)^{1/2}dy\cdot\chi_\ell(x)\\
&\lesssim 2^{\ell\alpha(x)}\int_{\mathfrak{R}_j}\frac{|\Omega(x-y)|\left|f_j(y)\right|}{|x-y|^{n-\varrho}}\cdot\left|\frac{1}{|x-y|^{2\varrho}}-\frac{1}{|x|^{2\varrho}}\right|^{1/2}dy\\
&\quad\cdot\chi_\ell(x)+2^{\ell\alpha(x)}\int_{\mathfrak{R}_j}\frac{|\Omega(x-y)|\left|f_j(y)\right|}{|x-y|^{n-\varrho}}\cdot\frac{1}{|x|^{\varrho}}dy\cdot\chi_\ell(x)\\
&\lesssim 2^{\ell\alpha(x)}\int_{\mathfrak{R}_j}\frac{|\Omega(x-y)|\left|f_j(y)\right|}{|x-y|^{n-\varrho}}\cdot\frac{|y|^{1/2}}{|x-y|^{\rho+(1/2)}}dy\cdot\chi_\ell(x)\\
&\quad+2^{\ell\alpha(x)}\int_{\mathfrak{R}_j}\frac{|\Omega(x-y)|\left|f_j(y)\right|}{|x-y|^{n}}dy\cdot\chi_\ell(x)\\
&\lesssim\frac{2^{\ell\alpha(x)}}{|x|^n}\int_{\mathfrak{R}_j}|f(y)||\Omega(x-y)|dy\cdot\chi_\ell(x).
\end{aligned} \tag{32}$$

By Lemma 7, we deduce

$$\begin{aligned}
&2^{\ell\alpha(x)}\left|\mathscr{M}_{\Omega}^{\varrho}\left(f_j\right)(x)\right|\cdot\chi_\ell(x)\\
&\lesssim\frac{1}{|x|^n}\int_{\mathfrak{R}_j}2^{\ell\alpha(x)}|f(y)||\Omega(x-y)|dy\cdot\chi_\ell(x)\\
&\lesssim\frac{1}{|x|^n}\int_{\mathfrak{R}_j}2^{\ell\alpha(x)}2^{(j-j)\alpha(y)}|f(y)||\Omega(x-y)|dy\cdot\chi_\ell(x)\\
&\lesssim\frac{2^{(\ell-j)\alpha_+}}{|x|^n}\int_{\mathfrak{R}_j}2^{j\alpha(y)}|f(y)||\Omega(x-y)|dy\cdot\chi_\ell(x).
\end{aligned} \tag{33}$$

It follows from Hölder's inequality (12) that

$$\begin{aligned}
&2^{\ell\alpha(x)}\left|\mathscr{M}_{\Omega}^{\varrho}\left(f_j\right)(x)\right|\cdot\chi_\ell(x)\\
&\quad\lesssim 2^{(\ell-j)\alpha_+}|x|^{-n}\left\|\Omega(x-\cdot)\chi_j\right\|_{L^{p'(\cdot)}}\left\|2^{j\alpha(\cdot)}f\right\|_{L^{p(\cdot)}}\chi_\ell(x).
\end{aligned} \tag{34}$$

Since $s>(p')_+$, we can find a variable exponent $p^*(\cdot)>1$ such that $(1/p'(\cdot))=(1/s)+(1/p^*(\cdot))$, then by Lemma 8, it follows that

$$\begin{aligned}
\left\|\Omega(x-\cdot)\chi_j\right\|_{L^{p'(\cdot)}}&\lesssim\left\|\Omega(x-\cdot)\chi_j\right\|_{L^s}\left\|\chi_j\right\|_{L^{p^*(\cdot)}}\\
&\lesssim\|\Omega\|_{L^s(\mathbb{S}^{n-1})}2^{(\ell-j)((n-1)/s)}\left\|\chi_{B_j}\right\|_{L^{p'(\cdot)}},
\end{aligned} \tag{35}$$

where the last inequality (35) is based on the fact that $\|\chi_j\|_{L^{p^*(\cdot)}} \asymp |B_j|^{-1/s}\|\chi_j\|_{L^{p'(\cdot)}}$; see [18]. From (34) and (35), Lemmas 4 and 5, we can deduce

$$\begin{aligned}\left\|2^{\ell\alpha(\cdot)}\mathscr{M}_{\Omega}^{\varrho}\left(f_j\right)\chi_{\ell}\right\|_{L^{p(\cdot)}} &\lesssim \frac{1}{2^{\ell n}} \lesssim 2^{(\ell-j)(\alpha_+-((n-1)/s))}\left\|\chi_{B_j}\right\|_{L^{p'(\cdot)}}\left\|2^{j\alpha(\cdot)}f_j\right\|_{L^{p(\cdot)}}\left\|\chi_{B_\ell}\right\|_{L^{p(\cdot)}}\\ &\lesssim 2^{(\ell-j)(\alpha_+-((n-1)/s))}\left\|\chi_{B_j}\right\|_{L^{p'(\cdot)}}\left\|2^{j\alpha(\cdot)}f_j\right\|_{L^{p(\cdot)}}\frac{2^{\ell n}|B_\ell|}{\left\|\chi_{B_\ell}\right\|_{L^{p'(\cdot)}}}\\ &\lesssim 2^{(j-\ell)(n\delta_2+((n-1)/s)-\alpha_+)}\left\|2^{j\alpha(\cdot)}f_j\right\|_{L^{p(\cdot)}}.\end{aligned} \tag{36}$$

To estimate $\mathfrak{T}_2$, we need to consider two cases: $0<q\le 1$ and $1<q<\infty$.

Case 1. $1<q<\infty$, by the fact that $n\delta_2-\alpha_++((n-1)/s)>0$, the Hölder's inequality, and the inequality (36), it follows that

$$\begin{aligned}\mathfrak{T}_2 &\lesssim \sup_{\kappa\in\mathbb{Z}}2^{-\kappa\sigma q}\sum_{\ell=-\infty}^{\kappa}\left(\sum_{j=-\infty}^{\ell-2}\left\|2^{\ell\alpha(\cdot)}\mathscr{M}_{\Omega}^{\varrho}\left(f_j\right)\chi_\ell\right\|_{L^{p(\cdot)}}\right)^q\\ &\lesssim \sup_{\kappa\in\mathbb{Z}}2^{-\kappa\sigma q}\sum_{\ell=-\infty}^{\kappa}\left(\sum_{j=-\infty}^{\ell-2}2^{q/2(j-\ell)(n\delta_2-\alpha_++((n-1)/s))}\left\|2^{j\alpha(\cdot)}f_j\right\|_{L^{p(\cdot)}}^q\right)\\ &\quad\cdot\left(\sum_{j=-\infty}^{\ell-2}2^{q'/2(j-\ell)(n\delta_2-\alpha_++((n-1)/s))}\right)^{q/q'}\\ &\lesssim \sup_{\kappa\in\mathbb{Z}}2^{-\kappa\sigma q}\sum_{j=-\infty}^{\kappa-2}\left\|2^{j\alpha(\cdot)}f_j\right\|_{L^{p(\cdot)}}^q\sum_{\ell=j+2}^{\kappa}2^{1/2(j-\ell)q(n\delta_2-\alpha_++((n-1)/s))}\\ &\lesssim \sup_{\kappa\in\mathbb{Z}}2^{-\kappa\sigma q}\sum_{j=-\infty}^{\kappa-2}\left\|2^{j\alpha(\cdot)}f_j\right\|_{L^{p(\cdot)}}^q \lesssim \|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\sigma}}^q.\end{aligned} \tag{37}$$

Case 2. $0<q\le 1$, we use the inequality

$$\left(\sum_{j=1}^{\infty}\aleph_j\right)^q \le \sum_{j=1}^{\infty}\aleph_j^q, \left(\aleph_1,\aleph_2,\cdots>0\right), \tag{38}$$

and obtain

$$\begin{aligned}\mathfrak{T}_2 &\lesssim \sup_{\kappa\in\mathbb{Z}}2^{-\kappa\sigma q}\sum_{\ell=-\infty}^{\kappa}\sum_{j=-\infty}^{\ell-2}\left\|2^{\ell\alpha(\cdot)}\mathscr{M}_{\Omega}^{\varrho}\left(f_j\right)\chi_\ell\right\|_{L^{p(\cdot)}}^q\\ &\lesssim \sup_{\kappa\in\mathbb{Z}}2^{-\kappa\sigma q}\sum_{\ell=-\infty}^{\kappa}\sum_{j=-\infty}^{\ell-2}2^{q(j-\ell)(n\delta_2-\alpha_++((n-1)/s))}\left\|2^{j\alpha(\cdot)}f_j\right\|_{L^{p(\cdot)}}^q\\ &\lesssim \sup_{\kappa\in\mathbb{Z}}2^{-\kappa\sigma q}\sum_{j=-\infty}^{\kappa-2}\left\|2^{j\alpha(\cdot)}f_j\right\|_{L^{p(\cdot)}}^q\sum_{\ell=j+2}^{\kappa}2^{(j-\ell)q(n\delta_2-\alpha_++((n-1)/s))}\\ &\lesssim \|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\sigma}}^q.\end{aligned} \tag{39}$$

For the term $\mathfrak{T}_3$, by applying Proposition 6, we can get

$$\begin{aligned}\mathfrak{T}_3 &\lesssim \max\left\{\sup_{\kappa<0,\kappa\in\mathbb{Z}}2^{-\kappa\sigma q}\sum_{\ell=-\infty}^{\kappa}\left\|2^{\ell\alpha(0)}\sum_{j=\ell+2}^{\infty}\mathscr{M}_{\Omega}^{\varrho}\left(f_j\right)\chi_\ell\right\|_{L^{p(\cdot)}}^q, \sup_{\kappa\ge 0,\kappa\in\mathbb{Z}}2^{-\kappa\sigma q}\sum_{\ell=-\infty}^{-1}\right.\\ &\quad\left.\cdot\left\|2^{\ell\alpha(0)}\sum_{j=\ell+2}^{\infty}\mathscr{M}_{\Omega}^{\varrho}\left(f_j\right)\chi_\ell\right\|_{L^{p(\cdot)}}^q + \sup_{\kappa\ge 0,\kappa\in\mathbb{Z}}2^{-\kappa\sigma q}\sum_{\ell=0}^{\kappa}\left\|2^{\ell\alpha_\infty}\sum_{j=\ell+2}^{\infty}\mathscr{M}_{\Omega}^{\varrho}\left(f_j\right)\chi_\ell\right\|_{L^{p(\cdot)}}^q\right\}\\ &=: \max\left(\mathfrak{T}_3^1,\mathfrak{T}_3^2\right).\end{aligned} \tag{40}$$

For $\mathfrak{T}_3$, it is clear that if $x\in\mathfrak{R}_\ell$, $j-2\ge\ell$, and $y\in\mathfrak{R}_j$, then $|x-y|\asymp|y|$. By (31), (12), and the Minkowski's inequality, we deduce that

$$\begin{aligned}\left|\mathscr{M}_{\Omega}^{\varrho}\left(f_j\right)(x)\right|\cdot\chi_\ell(x) &\lesssim \left(\int_0^{|y|}\left|t^{-\varrho}\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-\varrho}}f_j(y)dy\right|^2\frac{dt}{t}\right)^{1/2}\\ &\quad\cdot\chi_\ell(x)+\left(\int_{|y|}^{\infty}\left|t^{-\varrho}\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-\varrho}}f_j(y)dy\right|^2\frac{dt}{t}\right)^{1/2}\\ &\quad\cdot\chi_\ell(x)\lesssim\int_{\mathfrak{R}_j}\frac{\left|f_j(y)\right|\left|\Omega(x-y)\right|}{|x-y|^{n-\varrho}}\cdot\frac{|x|^{1/2}}{|x-y|^{\varrho+(1/2)}}\mathrm{d}y\\ &\quad\cdot\chi_\ell(x)+\int_{\mathfrak{R}_j}\frac{\left|f_j(y)\right|\left|\Omega(x-y)\right|}{|x-y|^n}\mathrm{d}y\cdot\chi_\ell(x)\\ &\lesssim\frac{1}{|y|^n}\int_{\mathfrak{R}_j}\left|f_j(y)\right||\Omega(x-y)|\mathrm{d}y\cdot\chi_\ell(x)\\ &\lesssim 2^{-nj}\left\|\Omega(x-\cdot)\chi_j\right\|_{L^{p'(\cdot)}}\left\|f_j\right\|_{L^{p(\cdot)}}\chi_\ell(x).\end{aligned} \tag{41}$$

Similar to (35), we conclude

$$\left\|\Omega(x-\cdot)\chi_j\right\|_{L^{p'(\cdot)}} \lesssim \left\|\Omega(x-\cdot)\chi_j\right\|_{L^s}\left\|\chi_j\right\|_{L^{p^*(\cdot)}} \lesssim \|\Omega\|_{L^s\left(\mathbb{S}^{n-1}\right)}\left\|\chi_{B_j}\right\|_{L^{p'(\cdot)}}. \tag{42}$$

From (42), Lemmas 4 and 5, it follows that

$$\begin{aligned}\left\|\sum_{j=\ell+2}^{\infty}\mathscr{M}_{\Omega}^{\varrho}\left(f_j\right)\chi_\ell\right\|_{L^{p(\cdot)}} &\lesssim \sum_{j=\ell+2}^{\infty}\left\|\mathscr{M}_{\Omega}^{\varrho}\left(f_j\right)\chi_\ell\right\|_{L^{p(\cdot)}}\\ &\lesssim \sum_{j=\ell+2}^{\infty}\frac{1}{2^{jn}}\left\|\chi_{B_j}\right\|_{L^{p'(\cdot)}}\left\|f_j\right\|_{L^{p(\cdot)}}\left\|\chi_{B_\ell}\right\|_{L^{p(\cdot)}}\\ &\lesssim \sum_{j=\ell+2}^{\infty}\left\|f_j\right\|_{L^{p(\cdot)}}\left\|\chi_{B_\ell}\right\|_{L^{p(\cdot)}}\frac{2^{-jn}|B_j|}{\left\|\chi_{B_j}\right\|_{L^{p(\cdot)}}}\\ &\lesssim \sum_{j=\ell+2}^{\infty}\left\|f_j\right\|_{L^{p(\cdot)}}2^{(\ell-j)n\delta_1}.\end{aligned} \tag{43}$$

To estimate $\mathfrak{T}_3^1$, we need to consider two cases below: $0<q\le 1$ and $1<q<\infty$.

Case 3. $0 < q \leq 1$, combining the above inequalities and using (38), we can obtain

$$\begin{aligned}\mathfrak{T}_3^1 &\lesssim \sup_{\kappa<0\kappa\in\mathbb{Z}} 2^{-\kappa\sigma q} \sum_{\ell=-\infty}^{\kappa} 2^{\ell\alpha(0)q} \sum_{j=\ell+2}^{\kappa-1} 2^{q(\ell-j)n\delta_1} \left\|f_j\right\|_{L^{p(\cdot)}}^q \\ &+ \sup_{\kappa<0\kappa\in\mathbb{Z}} 2^{-\kappa\sigma q} \sum_{\ell=-\infty}^{\kappa} 2^{\ell\alpha(0)q} \sum_{j=\kappa}^{\infty} 2^{q(\ell-j)n\delta_1} \left\|f_j\right\|_{L^{p(\cdot)}}^q \\ &=: \mathfrak{T}_3^{1'} + \mathfrak{T}_3^{1''}.\end{aligned} \tag{44}$$

For $\mathfrak{T}_3^{1'}$, in view of $\alpha(0) + n\delta_1 > \alpha_- + n\delta_1 > 0$, we get

$$\begin{aligned}\mathfrak{T}_3^{1'} &\lesssim \sup_{\kappa<0\kappa\in\mathbb{Z}} 2^{-\kappa\sigma q} \sum_{j=-\infty}^{\kappa-1} 2^{qj\alpha(0)} \left\|f_j\right\|_{L^{p(\cdot)}}^q \sum_{\ell=-\infty}^{j-2} 2^{q(\ell-j)(\alpha(0)+n\delta_1)} \\ &\lesssim \sup_{\kappa<0\kappa\in\mathbb{Z}} 2^{-\kappa\sigma q} \sum_{j=-\infty}^{\kappa-1} 2^{qj\alpha(0)} \left\|f_j\right\|_{L^{p(\cdot)}}^q \lesssim \|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\sigma}}^q.\end{aligned} \tag{45}$$

Now, let us deal with $\mathfrak{T}_3^{1''}$, noting that $\sigma - n\delta_1 - \alpha(0) < \sigma - n\delta_1 - \alpha_- < 0$, it follows that

$$\begin{aligned}\mathfrak{T}_3^{1''} &\lesssim \sup_{\kappa<0\kappa\in\mathbb{Z}} 2^{-\kappa\sigma q} \sum_{\ell=-\infty}^{\kappa} 2^{q\ell\alpha(0)} \sum_{j=\kappa}^{\infty} 2^{q(\ell-j)n\delta_1} 2^{-jq\alpha(0)} 2^{j\sigma q} 2^{-j\sigma q} \sum_{\mathfrak{I}=-\infty}^{j} 2^{\mathfrak{I}\alpha(0)q} \|f_{\mathfrak{I}}\|_{L^{p(\cdot)}}^q \\ &\lesssim \sup_{\kappa<0\kappa\in\mathbb{Z}} 2^{-\kappa\sigma q} \sum_{\ell=-\infty}^{\kappa} 2^{qk(\alpha(0)+n\delta_1)} \sum_{j=\kappa}^{\infty} 2^{jq(\sigma-n\delta_1-\alpha(0))} \|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\sigma}}^q \\ &\lesssim \|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\sigma}}.\end{aligned} \tag{46}$$

Case 4. $1 < q < \infty$, we have

$$\begin{aligned}\mathfrak{T}_3^1 &\lesssim \sup_{\kappa<0\kappa\in\mathbb{Z}} 2^{-\kappa\sigma q} \sum_{\ell=\infty}^{\kappa} 2^{\ell q\alpha(0)} \left(\sum_{j=\ell+2}^{\infty} 2^{(\ell-j)n\delta_1} \left\|f_j\right\|_{L^{p(\cdot)}}\right)^q \\ &\lesssim \sup_{\kappa<0\kappa\in\mathbb{Z}} 2^{-\kappa\sigma q} \sum_{\ell=-\infty}^{\kappa} 2^{\ell q\alpha(0)} \left(\sum_{j=\ell+2}^{\kappa} 2^{(\ell-j)n\delta_1} \left\|f_j\right\|_{L^{p(\cdot)}}\right)^q \\ &+ \sup_{\kappa<0\kappa\in\mathbb{Z}} 2^{-\kappa\sigma q} \sum_{\ell=-\infty}^{\kappa} 2^{\ell q\alpha(0)} \left(\sum_{j=\kappa+1}^{\infty} 2^{(\ell-j)n\delta_1} \left\|f_j\right\|_{L^{p(\cdot)}}\right)^q \\ &=: \mathfrak{T}_3^{1^\diamond} + \mathfrak{T}_3^{1^{\diamond\diamond}}.\end{aligned} \tag{47}$$

For $\mathfrak{T}_3^{1^\diamond}$, by the fact that $\alpha(0) + n\delta_1 > 0$ and using Hölder inequality, we infer that

$$\begin{aligned}\mathfrak{T}_3^{1^\diamond} &\lesssim \sup_{\kappa<0\kappa\in\mathbb{Z}} 2^{-\kappa\sigma q} \sum_{\ell=\infty}^{\kappa} \sum_{j=l+2}^{\kappa} 2^{qj\alpha(0)} \left\|f_j\right\|_{L^{p(\cdot)}}^q 2^{(q/2)(\ell-j)(\alpha(0)+n\delta_1)} \\ &\cdot \left(\sum_{j=l+2}^{\kappa} 2^{\left(q'/2\right)(\ell-j)(\alpha(0)+n\delta_1)}\right)^{q/q'} \\ &\lesssim \sup_{\kappa<0\kappa\in\mathbb{Z}} 2^{-\kappa\sigma q} \sum_{j=-\infty}^{\kappa} 2^{jq\alpha(0)} \left\|f_j\right\|_{L^{p(\cdot)}}^q \sum_{\ell=-\infty}^{j-2} 2^{(q/2)(\ell-j)(\alpha(0)+n\delta_1)} \\ &\lesssim \|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\sigma}}.\end{aligned} \tag{48}$$

For $\mathfrak{T}_3^{1^{\diamond\diamond}}$, from the inequality (36) and using the method as for $\mathfrak{T}_3^{1''}$, we obtain

$$\mathfrak{T}_3^{1^{\diamond\diamond}} \lesssim \|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\sigma}}^q. \tag{49}$$

By combining $\mathfrak{T}_3^{1'}$, $\mathfrak{T}_3^{1''}$, $\mathfrak{T}_3^{1^\diamond}$, and $\mathfrak{T}_3^{1^{\diamond\diamond}}$ estimates, we arrive at

$$\mathfrak{T}_3^1 \lesssim \|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\sigma}}^q. \tag{50}$$

By the similar method used in the estimate for $\mathfrak{T}_3^1$, it is not difficult to show that

$$\mathfrak{T}_3^2 \lesssim \|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\sigma}}^q. \tag{51}$$

Thus, we have

$$\left\|\mathscr{M}_\Omega^\varrho(f)\right\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\sigma}} \lesssim \mathfrak{T}_1 + \mathfrak{T}_2 + \mathfrak{T}_3 \lesssim \|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\sigma}}. \tag{52}$$

The proof for Theorem 10 is finished.

Proof of Theorem 11. Let $\Lambda \in \mathrm{BMO}$, $f \in M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\sigma}$. For any $j \in \mathbb{Z}$, let $f_j := f\chi_j$, then we have

$$f(x) = \sum_{j\in\mathbb{Z}} f\chi_j(x) := \sum_{j=\ell-1}^{\ell+1} f_j(x) + \sum_{j=-\infty}^{\ell-2} f_j(x) + \sum_{j=\ell+2}^{\infty} f_j(x). \tag{53}$$

□

Using the definition of $M\dot{K}^{\alpha(\cdot),\sigma}_{q,p(\cdot)}$, we have

$$\begin{aligned}\left\|\mathscr{M}^{\varrho}_{\Omega,\Lambda^i}(f)\right\|^q_{M\dot{K}^{\alpha(\cdot),\sigma}_{q,p(\cdot)}} &= \sup_{\kappa\in\mathbb{Z}} 2^{-\kappa\sigma q}\sum_{\ell=-\infty}^{\kappa}\left\|2^{\ell\alpha(\cdot)}\left|\mathscr{M}^{\varrho}_{\Omega,\Lambda^i}(f)\right|\chi_\ell\right\|^q_{L^{p(\cdot)}} \\ &\lesssim \sup_{\kappa\in\mathbb{Z}} 2^{-\kappa\sigma q}\sum_{\ell=-\infty}^{\kappa}\left\|2^{\ell\alpha(\cdot)}\left(\sum_{j=\ell-1}^{\ell+1}\left|\mathscr{M}^{\varrho}_{\Omega,\Lambda^i}\left(f_j\right)\right|\right)\chi_\ell\right\|^q_{L^{p(\cdot)}} \\ &\quad + \sup_{\kappa\in\mathbb{Z}} 2^{-\kappa\sigma q}\sum_{\ell=-\infty}^{\kappa}\left\|2^{\ell\alpha(\cdot)}\left(\sum_{j=-\infty}^{\ell-2}\left|\mathscr{M}^{\varrho}_{\Omega,\Lambda^i}\left(f_j\right)\right|\right)\chi_\ell\right\|^q_{L^{p(\cdot)}} \\ &\quad + \sup_{\kappa\in\mathbb{Z}} 2^{-\kappa\sigma q}\sum_{\ell=-\infty}^{\kappa}\left\|2^{\ell\alpha(\cdot)}\left(\sum_{j=\ell+2}^{\infty}\left|\mathscr{M}^{\varrho}_{\Omega,\Lambda^i}\left(f_j\right)\right|\right)\chi_\ell\right\|^q_{L^{p(\cdot)}} \\ &=: \mathfrak{Y}_1+\mathfrak{Y}_2+\mathfrak{Y}_3. \end{aligned} \tag{54}$$

Let us first estimate $\mathfrak{Y}_1$. From Proposition 6 and the boundedness of $\mathscr{M}^{\rho}_{\Omega}$ on $L^{p(\cdot)}$(see [30]), and using the similar methods as that for $\mathfrak{T}_1$, it is not difficult to see that

$$\begin{aligned}\mathfrak{Y}_1 &\approx \max\Bigg\{\sup_{\substack{\kappa<0\\ \kappa\in\mathbb{Z}}} 2^{-\kappa\sigma q}\sum_{\ell=-\infty}^{\kappa}2^{\ell\alpha(0)q}\left\|\left(\sum_{j=\ell-1}^{\ell+1}\left|\mathscr{M}^{\varrho}_{\Omega,\Lambda^i}\left(f_j\right)\right|\right)\chi_\ell\right\|^q_{L^{p(\cdot)}}, \sup_{\substack{\kappa\ge0\\ \kappa\in\mathbb{Z}}} 2^{-\kappa\sigma q}\sum_{\ell=-\infty}^{-1}2^{\ell\alpha(0)q} \\ &\quad\cdot\left\|\left(\sum_{j=\ell-1}^{\ell+1}\left|\mathscr{M}^{\varrho}_{\Omega,\Lambda^i}\left(f_j\right)\right|\right)\chi_\ell\right\|^q_{L^{p(\cdot)}} + \sup_{\substack{\kappa\ge0\\ \kappa\in\mathbb{Z}}} 2^{-\kappa\sigma q}\sum_{\ell=0}^{\kappa}2^{\ell\alpha_\infty q}\left\|\left(\sum_{j=\ell-1}^{\ell+1}\left|\mathscr{M}^{\varrho}_{\Omega,\Lambda^i}\left(f_j\right)\right|\right)\chi_\ell\right\|^q_{L^{p(\cdot)}}\Bigg\} \\ &\lesssim \|\Lambda\|^i_*\max\Bigg\{\sup_{\substack{\kappa<0\\ \kappa\in\mathbb{Z}}} 2^{-\kappa\sigma q}\sum_{\ell=-\infty}^{\kappa}\left\|2^{\ell\alpha(0)}|f\chi_\ell|\right\|^q_{L^{p(\cdot)}}, \sup_{\substack{\kappa\ge0\\ \kappa\in\mathbb{Z}}} 2^{-\kappa\sigma q}\sum_{\ell=-\infty}^{-1}\left\|2^{\ell\alpha(0)}|f\chi_\ell|\right\|^q_{L^{p(\cdot)}} \\ &\quad + \sup_{\substack{\kappa\ge0\\ \kappa\in\mathbb{Z}}} 2^{-\kappa\sigma q}\sum_{\ell=0}^{\kappa}\left\|2^{\ell\alpha_\infty}|f\chi_\ell|\right\|^q_{L^{p(\cdot)}}\Bigg\} \lesssim \|\Lambda\|^i_*\|f\|^q_{M\dot{K}^{\alpha(\cdot),\sigma}_{q,p(\cdot)}}. \end{aligned} \tag{55}$$

Now, let us turn to the estimates of $\mathfrak{Y}_2$. We consider $2^{\ell\alpha(x)}|\mathscr{M}^{\varrho}_{\Omega,\Lambda}(f_j)(x)|$

$$\begin{aligned} 2^{\ell\alpha(x)}&\left|\mathscr{M}^{\varrho}_{\Omega,\Lambda}\left(f_j\right)(x)\right|\cdot\chi_\ell(x) \\ &\lesssim 2^{\ell\alpha(x)}\left(\int_0^{|x|}\left|t^{-\varrho}\int_{|x-y|\le t}(\Lambda(x)-\Lambda(y))^i\frac{\Omega(x-y)}{|x-y|^{n-\rho}}f_j(y)dy\right|^2\frac{dt}{t}\right)^{1/2} \\ &\quad\cdot\chi_\ell(x)+2^{\ell\alpha(x)}\left(\int_{|x|}^{\infty}\left|t^{-\varrho}\int_{|x-y|\le t}(\Lambda(x)-\Lambda(y))^i\frac{\Omega(x-y)}{|x-y|^{n-\rho}}f_j(y)dy\right|^2\frac{dt}{t}\right)^{1/2} \\ &\quad\cdot\chi_\ell(x). \end{aligned} \tag{56}$$

It is clear that if $x\in\mathfrak{R}_\ell$, $j+2\le\ell$, and $y\in\mathfrak{R}_j$, then $|x-y|\asymp|x|$. Thus, for $\rho\in(0,n)$, we use the Minkowski's inequality to get

$$\begin{aligned} 2^{\ell\alpha(x)}&\left|\mathscr{M}^{\varrho}_{\Omega,\Lambda}\left(f_j\right)(x)\right|\cdot\chi_\ell(x) \\ &\lesssim 2^{\ell\alpha(x)}\int_{\mathbb{R}^n}|\Lambda(x)-\Lambda(y)|^i\frac{|\Omega(x-y)|\left|f_j(y)\right|}{|x-y|^{n-\rho}} \\ &\quad\cdot\left(\int_{|x-y|}^{|x|}\frac{dt}{t^{2\rho+1}}\right)^{1/2}dy\cdot\chi_\ell(x) \\ &\quad+2^{\ell\alpha(x)}\int_{\mathbb{R}^n}|\Lambda(x)-\Lambda(y)|^i\frac{|\Omega(x-y)|\left|f_j(y)\right|}{|x-y|^{n-\rho}} \\ &\quad\cdot\left(\int_{|x|}^{\infty}\frac{dt}{t^{2\rho+1}}\right)^{1/2}dy\cdot\chi_\ell(x) \\ &\lesssim 2^{\ell\alpha(x)}\int_{\mathfrak{R}_j}|\Lambda(x)-\Lambda(y)|^i\frac{|\Omega(x-y)|\left|f_j(y)\right|}{|x-y|^{n-\rho}} \\ &\quad\cdot\frac{|y|^{1/2}}{|x-y|^{\rho+(1/2)}}dy\cdot\chi_\ell(x) \\ &\quad+2^{\ell\alpha(x)}\int_{\mathfrak{R}_j}|\Lambda(x)-\Lambda(y)|^i\frac{|\Omega(x-y)|\left|f_j(y)\right|}{|x-y|^{n}}dy\cdot\chi_\ell(x) \\ &\lesssim\frac{2^{\ell\alpha(x)}}{|x|^n}\int_{\mathfrak{R}_j}|\Omega(x-y)||\Lambda(x)-\Lambda(y)|^i\left|f_j(y)\right|\mathrm{d}y\cdot\chi_\ell(x). \end{aligned} \tag{57}$$

Using Lemma 7 and inequality (12), it follows that

$$\begin{aligned} 2^{\ell\alpha(x)}&\left|\mathscr{M}^{\varrho}_{\Omega,\Lambda^i}\left(f_j\right)(x)\right|\cdot\chi_\ell(x) \\ &\lesssim\frac{1}{|x|^n}\int_{\mathfrak{R}_j}2^{\ell\alpha(x)}|\Lambda(x)-\Lambda(y)|^i|\Omega(x-y)||f(y)|dy\cdot\chi_\ell(x) \\ &\lesssim\frac{1}{|x|^n}\int_{\mathfrak{R}_j}2^{\ell\alpha(x)}2^{(j-j)\alpha(y)}|\Lambda(x)-\Lambda(y)|^i|\Omega(x-y)||f(y)|dy \\ &\quad\cdot\chi_\ell(x)\frac{2^{(\ell-j)\alpha_+}}{|x|^n}\left(\left|\Lambda(x)-\Lambda_{B_j}\right|^i\int_{\mathfrak{R}_j}2^{j\alpha(y)}|\Omega(x-y)||f(y)|dy\right. \\ &\quad\left.+\int_{\mathfrak{R}_j}2^{j\alpha(y)}|\Omega(x-y)|\left|\Lambda(y)-\Lambda_{B_j}\right|^i|f(y)|dy\right)\cdot\chi_\ell(x). \end{aligned} \tag{58}$$

Applying Hölder's inequality (12), the inequality (35), and Lemmas 4–8, we obtain

$$\begin{aligned} &\left\|2^{\ell\alpha(\cdot)}\left|\mathscr{M}^{\varrho}_{\Omega,\Lambda^i}\left(f_j\right)(x)\right|\chi_\ell\right\|_{L^{p(\cdot)}} \\ &\lesssim|x|^{-n}2^{(\ell-j)\alpha_+}\left(\left\|\left(\Lambda-\Lambda_{B_j}\right)^i\chi_\ell\right\|_{L^{p(\cdot)}}\left\|\Omega(x-\cdot)\chi_j\right\|_{L^{p'(\cdot)}}\right. \\ &\quad\left.+\left\|\left(\Lambda_{B_j}-\Lambda\right)^i\Omega(x-\cdot)\chi_j\right\|_{L^{p'(\cdot)}}\|\chi_\ell\|_{L^{p(\cdot)}}\right)\left\|2^{j\alpha(\cdot)}f_j\right\|_{L^{p(\cdot)}} \\ &\lesssim|x|^{-n}2^{(\ell-j)\alpha_+}\left(\left\|\left(\Lambda-\Lambda_{B_j}\right)^i\chi_\ell\right\|_{L^{p(\cdot)}}\left\|\Omega(x-\cdot)\chi_j\right\|_{L^{s}}\left\|\chi_j\right\|_{L^{p^*(\cdot)}}\right. \end{aligned}$$

$$
\begin{aligned}
&+\left\|\left(\Lambda_{B_j}-\Lambda\right)^i\chi_j\right\|_{L^{p^*(\cdot)}}\left\|\Omega(x-\cdot)\chi_j\right\|_{L^s}\|\chi_\ell\|_{L^{p(\cdot)}}\Big)\left\|2^{j\alpha(\cdot)}f_j\right\|_{L^{p(\cdot)}}\\
&\lesssim |x|^{-n}2^{(\ell-j)\alpha_+}\Big(\left\|\left(\Lambda-\Lambda_{B_j}\right)^i\chi_\ell\right\|_{L^{p(\cdot)}}\left\|\Omega(x-\cdot)\chi_j\right\|_{L^s}\left\|\chi_j\right\|_{L^{p^*(\cdot)}}\\
&\quad+\|\Lambda\|_*^i\left\|\chi_j\right\|_{L^{p^*(\cdot)}}\left\|\Omega(x-\cdot)\chi_j\right\|_{L^s}\|\chi_\ell\|_{L^{p(\cdot)}}\Big)\\
&\quad\times\left\|2^{j\alpha(\cdot)}f_j\right\|_{L^{p(\cdot)}}2^{-\ell n}2^{(\ell-j)(\alpha_++((n-1)/s))}\\
&\quad\cdot\left((\ell-j)^i\|\Lambda\|_*^i\|\chi_\ell\|_{L^{p(\cdot)}}+\|\Lambda\|_*^i\|\chi_\ell\|_{L^{p(\cdot)}}\right)\|\chi_\ell\|_{L^{p'(\cdot)}}\left\|2^{j\alpha(\cdot)}f_j\right\|_{L^{p(\cdot)}}\\
&\lesssim 2^{(\ell-j)(\alpha_++((n-1)/s)-n\delta_2)}(\ell-j)^i\left\|2^{j\alpha(\cdot)}f_j\right\|_{L^{p(\cdot)}}\|\Lambda\|_*^i.
\end{aligned}
\tag{59}
$$

Hence, combining the above estimate and using the same approach as the one used for estimating $\mathfrak{T}_2$, we conclude that

$$
\mathfrak{Y}_2\lesssim\|\Lambda\|_*^i\|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\sigma}}. \tag{60}
$$

Finally, we estimate $\mathfrak{Y}_3$. It is clear that if $x\in\mathfrak{R}_\ell$, $j+2\le\ell$, and$y\in\mathfrak{R}_j$, then $|x-y|\asymp|y|\asymp 2^{jn}$. By (31), the Minkowski's inequality, and the inequality (12), we deduce, for $\rho\in(0,n)$,

$$
\begin{aligned}
&\left|\mathscr{M}^{\mathrm{Q}}_{\Omega,\Lambda^i}\left(f_j\right)(x)\right|\cdot\chi_\ell(x)\\
&\le\left(\int_0^{|y|}\left|t^{-\rho}\int_{|x-y|\le t}(\Lambda(x)-\Lambda(y))^i\frac{\Omega(x-y)}{|x-y|^{n-\rho}}f_j(y)dy\right|^2\frac{dt}{t}\right)^{1/2}\\
&\quad\cdot\chi_\ell(x)+\left(\int_{|y|}^{\infty}\left|t^{-\rho}\int_{|x-y|\le t}(\Lambda(x)-\Lambda(y))^i\frac{\Omega(x-y)}{|x-y|^{n-\rho}}f_j(y)dy\right|^2\frac{dt}{t}\right)^{1/2}\cdot\chi_\ell(x)\\
&\lesssim\int_{\mathfrak{R}_j}|\Lambda(x)-\Lambda(y)|^i\frac{|\Omega(x-y)|\left|f_j(y)\right|}{|x-y|^{n-\rho}}\cdot\frac{|x|^{1/2}}{|x-y|^{\rho+(1/2)}}dy\cdot\chi_\ell(x)\\
&\quad+\int_{\mathfrak{R}_j}|\Lambda(x)-\Lambda(y)|^i\frac{|\Omega(x-y)|\left|f_j(y)\right|}{|x-y|^{n}}dy\cdot\chi_\ell(x)\\
&\lesssim|y|^{-n}\int_{\mathfrak{R}_j}|\Lambda(x)-\Lambda(y)|^i|\Omega(x-y)|\left|f_j(y)\right|dy\cdot\chi_\ell(x)\\
&\lesssim|y|^{-n}\left(\left|\left(\Lambda-\Lambda_{B_j}\right)^i\right|\left\|\Omega(x-\cdot)\chi_j\right\|_{L^{p'(\cdot)}}+\left\|\left(\Lambda_{B_j}-\Lambda\right)\right)^i\Omega(x-\cdot)\chi_j\right\|_{L^{p'(\cdot)}}\Big)\chi_\ell(x)\left\|f_j\right\|_{L^{p(\cdot)}}.
\end{aligned}
\tag{61}
$$

From this, Lemmas 4–8 and (38), we deduce

$$
\begin{aligned}
&\left\|\mathscr{M}^{\mathrm{Q}}_{\Omega,\Lambda^i}\left(f_j\right)(x)\chi_\ell\right\|_{L^{p(\cdot)}}\\
&\lesssim|y|^{-n}\Big(\left\|\left(\Lambda-\Lambda_{B_j}\right)^i\chi_\ell\right\|_{L^{p(\cdot)}}\left\|\Omega(x-\cdot)\chi_j\right\|_{L^{p'(\cdot)}}\\
&\quad+\left\|\left(\Lambda_{B_j}-\Lambda\right)\right)^i\Omega(x-\cdot)\chi_j\right\|_{L^{p'(\cdot)}}\|\chi_\ell\|_{L^{p(\cdot)}}\Big)\left\|f_j\right\|_{L^{p(\cdot)}}\\
&\lesssim|y|^{-n}\|\Omega\|_{L^s\left(\mathbb{S}^{n-1}\right)}\left((j-\ell)^i\|\chi_\ell\|_{L^{p(\cdot)}}+\|\Lambda\|_*^i\|\chi_\ell\|_{L^{p(\cdot)}}\right)\left\|\chi_j\right\|_{L^{p(\cdot)}}\left\|f_j\right\|_{L^{p(\cdot)}}\\
&\lesssim\|\Omega\|_{L^s\left(\mathbb{S}^{n-1}\right)}\left\|f_j\right\|_{L^{p(\cdot)}}\|\Lambda\|_*^i2^{(l-j)n\delta_1}(j-\ell)^i.
\end{aligned}
\tag{62}
$$

Thus, combining the above estimates and using the same approach as for the $\mathfrak{T}_2$ estimate, we deduce that

$$
\mathfrak{Y}_2\lesssim\|\Lambda\|_*^i\|f\|^q_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\sigma}}. \tag{63}
$$

Summing up the estimates of $\mathfrak{Y}_1$, $\mathfrak{Y}_2$, and $\mathfrak{Y}_3$, we conclude that

$$
\left\|\mathscr{M}^{\mathrm{Q}}_{\Omega,\Lambda^i}(f)\right\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\sigma}}\lesssim\mathfrak{Y}_1+\mathfrak{Y}_2+\mathfrak{Y}_3\lesssim\|\Lambda\|_*^i\|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\sigma}}. \tag{64}
$$

The proof for Theorem 11 is finished.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

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