Research Article

Parametric Marcinkiewicz Integral and Its Higher-Order Commutators on Variable Exponents Morrey-Herz Spaces

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In this article, we prove the boundedness of the parametric Marcinkiewicz integral and its higher-order commutators generated by BMO spaces on the variable Morrey-Herz space. All the results are new even when $\alpha(\cdot)$ is a constant.

1. Introduction

Throughout the entirety of this article, we assume that $n \geq 2$, $\mathbb{R}^n$ is the $n$-dimensional Euclidean space, and $S^{n-1}$ is the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure $d\rho$. The function $\Omega$ is assumed to be homogeneous of degree zero on $\mathbb{R}^n$ with $\Omega \in L^1(S^{n-1})$ and

$$\int_{S^{n-1}} \Omega(x') d\rho(x') = 0,$$  \hfill (1)

where $x' = x/|x|$ for any $x \in \mathbb{R}^n \setminus \{0\}$. For $\rho \in (0, n)$, the parametric Marcinkiewicz integral $\mathcal{M}_\Omega^\rho$ of higher dimensions is defined as follows:

$$\mathcal{M}_\Omega^\rho(f)(x) := \left( \int_0^\infty r^{\rho-1} \int_{|x-y| \leq r} \frac{\Omega(x-y)f(y)}{|x-y|^{n+\rho}} dy \frac{dt}{t} \right)^{1/2}.$$  \hfill (2)

Let $B$ be a ball with a radius $\tau > 0$, and a center $x \in \mathbb{R}^n$. A locally integrable function $\Lambda$ is said to be in the BMO space, if it satisfies

$$\|\Lambda\|_{\text{BMO}} = \sup_B \frac{1}{|B|} \int_B |\Lambda(x) - \Lambda_B| dz < \infty,$$  \hfill (3)

where $\Lambda_B = |B|^{-1} \int_B \Lambda(t) dt$ and $|E|$ denotes the Lebesgue measure of the set $E$ in $\mathbb{R}^n$. For $\Lambda \in \text{BMO}$, $i \in \mathbb{N}$, the $i$-order commutator for the parametric Marcinkiewicz integral $\mathcal{M}_\Omega^\rho$ is defined as follows:

$$\mathcal{M}_{\Omega,A}^\rho(f)(x) := \left( \int_0^\infty r^{\rho-1} \int_{|x-y| \leq r} (\Lambda(x) - \Lambda(y)) \Omega(x-y)f(y) \frac{dy}{|x-y|^{n+\rho}} \frac{dt}{t} \right)^{1/2}.$$  \hfill (4)

If $\rho = 1$ in (2), then the operator $\mathcal{M}_\Omega^1$ is equivalent to the classical Marcinkiewicz function $\mathcal{M}_\Omega$, which was initially introduced by Stein [1] in 1958. When $\Omega \in \text{Lip}_1(S^{n-1})$, $\beta \in (0, 1]$, Stein [1] demonstrated that $\mathcal{M}_\Omega^1$ is bounded on $L^p$ for $p \in (1, 2]$. Subsequently, the authors of [2] established the $L^p$-boundedness of $\mathcal{M}_\Omega$ for every $p \in (1, \infty)$ when $\Omega \in C^1(S^{n-1})$. On the other hand, Calderon [3] proved that the commutator the Hilbert transform $H$ generated by $\Lambda \in \text{BMO}$, defined by $[\Lambda, T]f = \Lambda Tf - T(\Lambda f)$, is bounded on...
Finally introduced by Hörmander in [22] where the author
studied the behaviour of the Hardy-Littlewood maximal
operator and the action of commutators in generalized
local Morrey spaces and generalized Morrey spaces. For
further research works studying the commutators on dif-
ferent function spaces, we refer to [9, 14–21] and refer-
ces therein.

The parametric Marcinkiewicz integral \( M_{\Omega}^{p} \) was orig-
inally introduced by Hörmander in [22] where the author
established the boundness of \( M_{\Omega}^{p} \) on \( L^p \) for \( p \in (1,\infty) \)
under the condition \( \Omega \in \text{Lip}_{1}^r(S^{n-1}) \). Since that time, the boundedness
of the parametric Marcinkiewicz integral, as well as its
related commutator, in several types of function spaces
attracted the attention of many researchers. Deringoz and
Hasanov [24] considered the boundness of the operator
\( M_{\Omega}^{p} \) on generalized Orlicz-Morrey spaces. On generalized
weighted Morrey spaces, Deringoz [25] investigated the boundness of the rough parametric Marcinkiewicz integral
\( M_{\Omega}^{p} \) and its higher-order commutator \( M_{\Omega}^{p,\Lambda} \). For more
applications and recent developments on the research of
the parametric Marcinkiewicz function, see [26–31].

In the last decades, the variable Lebesgue spaces have
been intensively studied since the pioneering work of [32]
by Kováčik and Rákosník. Additionally, different studies
on variable function spaces, such as variable exponents
Fourier-Besov-Morrey spaces [33–35], variable exponents
Fourier-Besov spaces [36, 37], variable exponent Morrey
spaces [38], variable Bessel potential spaces [39, 40], and
variable exponent Hardy spaces [41, 42], were developed
due to their applications in the modeling of electro-
 rheological fluids, PDEs with nonstandard growth, and
image restoration. Recently, Izuki studied the Herz spaces
\( K_{P,1/q}^{\alpha} \) in [43, 44]. As a generalization, Izuki [45] intro-
duced the variable Morrey-Herz spaces \( \mathcal{M}_{P,1/q}^{\alpha} \). In fact, the author
of [45] found that vector-valued sublinear operators which
satisfy a certain size condition are bounded on the variable
Morrey-Herz spaces. Furthermore, Almeida and Drihem
[46] enhanced the variable case of the Herz spaces \( K_{P,1/q}^{\alpha} \)
and established the boundedness results for a class of sub-
linear operators. Lu and Zhu [47] generalized Izuki’s result
for the \( \mathcal{M}_{P,1/q}^{\alpha} \). For further information and applications,
consult [48–54].

Inspired by the research mentioned above, the main goal
of this article is to prove the boundedness of the rough para-
metric Marcinkiewicz integral and its higher-order com-
mutators on the variable exponents Morrey-Herz spaces.

Henceforth, wherever the symbol \( C \) appears, it repre-
sents a positive constant whose value may vary but is inde-
pendent of the basic variables. The expression \( f \lesssim g \)
denotes the existence of constant \( C \) such that \( f \leq Cg \), and \( f \approx g \)
means that \( f \leq g \approx f \). If no further instructions are pro-
vided, the symbol for any space denoted by \( \mathcal{X}(\mathbb{R}^n) \) is repre-
sented by \( \mathcal{X} \). For instance, \( L_i^p(\mathbb{R}^n) \) is abbreviated as \( L^p \).

2. Definitions and Preliminaries

In this section, we review some notations, definitions, and
properties related to our work.

A variable exponent is a measurable function \( p(\cdot): \mathbb{R}^n \rightarrow (0,\infty) \). For any variable exponent \( p(\cdot) \), we set \( p_- := \text{essinf} \{ p(x) : x \in \mathbb{R}^n \} \) and \( p_+ := \text{esssup} \{ p(x) : x \in \mathbb{R}^n \} \).

Define the sets \( \mathcal{P} \) by

\[ \mathcal{P} = \{ p(x) \text{ is measurable function : } 1 < p_- \text{ and } p_+ < \infty \} . \]

Let \( p(\cdot) \in \mathcal{P} \). The variable Lebesgue space \( L^{p(\cdot)} \) consists of all
measurable functions \( f \) on \( \mathbb{R}^n \) such that

\[ \| f \|_{L^{p(\cdot)}} = \inf \left\{ \delta \in (0,\infty) : \mathcal{P}(p(\cdot)f(x)) \leq 1 \right\} < \infty , \]

where

\[ \mathcal{P}(p(\cdot)f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)}dx . \]

It is obvious that the variable exponent Lebesgue norm
has the following property \( \| f \|_{L^{p(\cdot)}} \leq \| f \|_{L^{p(x)}} \), \( \beta \geq 1/p_- \).

Define the set \( \mathcal{B} \) by

\[ \mathcal{B} = \{ p(x) \in \mathcal{P} : M_{\text{HL}} \text{ is bounded on variable } L^p \} , \]

where \( M_{\text{HL}} \) stands for the Hardy-Littlewood maximal func-
tion, which is defined as follows:

\[ (M_{\text{HL}} f)(x) = \sup_{B \ni x} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(z)|dz , f \in L^1_{\text{loc}} . \]

Definition 1 (see [46]). Let \( \Theta(\cdot) \) be real function on \( \mathbb{R}^n \).

(i) If there exists a constant \( C_{\log} > 0 \) such that

\[ |\Theta(x) - \Theta(0)| \leq \frac{C_{\log}}{\log(1 + (1/|x|))} , \text{ for all } x \in \mathbb{R}^n , \]

then the function \( \Theta(\cdot) \) is said to be a log-Hölder continuous
at the origin (or has a log decay at the origin).
(ii) If there exist $\Theta_{\infty} \in (0, \infty)$ and a constant $C_{\log} > 0$ such that
\[
|\Theta(x) - \Theta_{\infty}| \leq \frac{C_{\log}}{\log (e + |x|)}, \text{ for all } x \in \mathbb{R}^n,
\]
then the function $\Theta(\cdot)$ is said to be a log-Hölder continuous at the infinity (or has a log decay at the infinity).

If $p(\cdot) \in \mathcal{P}$, then the following expression of Hölder's inequality is valid:
\[
\int_{\mathbb{R}^n} |f(x)g(x)| \, dx \leq \left(1 + \frac{1}{p_-} - \frac{1}{p_+}\right) \|f\|_{L^{p_-}(\mathbb{R}^n)} \|g\|_{L^{p_+}(\mathbb{R}^n)}.
\]

For any $p(\cdot) \in \mathcal{P}$, let $\Phi(x, y) := \Phi(\cdot)$ denote by $-\Phi(x, y)$.

The homogeneous variable Herz space $K_{\alpha(\cdot)}^{p(\cdot)}$ is defined as the set of all functions $f \in L^{\sigma(\cdot)}_{\text{loc}} (\mathbb{R}^n \setminus \{0\})$ such that
\[
\|f\|_{K_{\alpha(\cdot)}^{p(\cdot)}} := \left(\sum_{k \in \mathbb{Z}} \|2^{k\alpha(\cdot)} f \chi_k\|_{L^{p(\cdot)}}^q\right)^{1/q} < \infty,
\]
for $q < \infty$, and the usual modification should be made when $q = \infty$.

**Definition 3** (see [47]). Let $\sigma \in (0, \infty), \alpha(\cdot) \in L^{\sigma(\cdot)}$, and $p(\cdot) \in \mathcal{P}$. The homogeneous variable Morrey-Herz space $M_{\alpha(\cdot)}^{p(\cdot)}$ is defined as the set of all functions $f \in L^{\sigma(\cdot)}_{\text{loc}} (\mathbb{R}^n \setminus \{0\})$ such that
\[
\|f\|_{M_{\alpha(\cdot)}^{p(\cdot)}} := \sup_{k \in \mathbb{Z}} \left(\sum_{\ell = \infty}^{k} \|2^{\ell\alpha(\cdot)} f \chi_\ell\|_{L^{p(\cdot)}}^q\right)^{1/q} < \infty,
\]
for $q < \infty$, and the usual modification should be made when $q = \infty$.

**Lemma 4** (see [44]). Let $p(\cdot) \in \mathcal{P}$. Then, for any ball $B$ in $\mathbb{R}^n$,
\[
\|X_B\|_{L^{p(\cdot)}} \leq C |B|.
\]

**Lemma 5** (see [44]). Let $p_q(\cdot) \in \mathcal{P}, b = 1, 2$. Then, there are positive constants $\delta_{b1}, \delta_{b2} \in (0, 1)$, such that for any ball $B$ in $\mathbb{R}^n$ and any measurable subset $S \subset B$,
\[
\|X_S\|_{L^{p(\cdot)}}^{\delta_{b1}} \leq \left(\frac{|S|}{|B|}\right)^{\delta_{b2}}, \text{ and } \|X_S\|_{L^{q(\cdot)}}^{\delta_{b1}} \leq \left(\frac{|S|}{|B|}\right)^{\delta_{b2}}.
\]

**Proposition 6** (see [47]). Let $\alpha(\cdot) \in L^{\infty}, q(\cdot) \in (0, \infty), \sigma(\cdot) \in [0, \infty)$, and let $p(\cdot) \in \mathcal{P}$. If the function $\alpha(\cdot)$ is log-Hölder continuous function both at origin and at infinity, then the following inequalities hold:
\[
\|f\|_{M_{\alpha(\cdot)}^{p(\cdot)}}^{\eta} := \max \left\{\sup_{k < 0} 2^{k\eta(\cdot)} \sum_{\ell = -\infty}^{k} \|2^{k\eta(\cdot)} f \chi_\ell\|_{L^{p(\cdot)}}^q, \sup_{k \geq 0} 2^{k\eta(\cdot)} \sum_{\ell = -\infty}^{k} \|2^{k\eta(\cdot)} f \chi_\ell\|_{L^{p(\cdot)}}^q, \sup_{k \geq 0} 2^{-\eta(\cdot)} \sum_{\ell = -\infty}^{k} 2^{\ell\eta(\cdot)} \|f \chi_\ell\|_{L^{p(\cdot)}}^q\right\}.
\]

**Lemma 7** (see [46]). Let $\alpha(x) \in L^{\infty}$ and $\tau_j > 0$. If the function $\alpha(x)$ is log-Hölder continuous both at origin and infinity, then the following inequality holds:
\[
\tau_1^a(\cdot) \leq \tau_2^a(\cdot) \times \begin{cases} \left(\frac{\tau_j}{\tau_k}\right)^{\alpha_1}, & 0 < \tau_2 \leq \frac{\tau_j}{2}, \\ 1, & \frac{\tau_j}{2} < \tau_2 \leq 2\tau_j, \\ \left(\frac{\tau_j}{\tau_k}\right)^{\alpha_2}, & \tau_2 > 2\tau_j, \end{cases}
\]
for every $x \in B(0, \tau_1) \setminus B(0, \tau_1/2)$ and $y \in B(0, \tau_2) \setminus B(0, \tau_2/2)$.

**Lemma 8** (see [56]). Let $p(\cdot), q^*(\cdot) \in \mathcal{P}$, and let $q \in (0, \infty)$ such that $(1/p(x)) = (1/q) + (1/q^*(x))$. Then, for any measurable functions $f \in L^{p(\cdot)}$ and $g \in L^{q^*(\cdot)}$,
\[
\|fg\|_{L^{q^*(\cdot)}} \leq \|f\|_{L^{p(\cdot)}} \|g\|_{L^{q^*(\cdot)}}.
\]

**Lemma 9** (see [57]). Let $i$ be a positive integer, $A \in \text{BMO}$ and let $p(\cdot) \in \mathcal{P}$. Then, there exists a positive constant $C$ such that for all $\ell, j \in \mathbb{Z}(\ell > j)$,
\[
\sup_{B \subset \mathbb{R}^n} \frac{1}{\|X_B\|_{L^{p(\cdot)}}} \|A - A_B\|_{X_B} \leq \|A\|_{\text{BMO}}.
\]
and
\[
\left\| \left( A - A_{k_i} \right)^{j} \chi_{B_i} \right\|_{L^p} \leq (\ell - j)\|A\|_{s}^{j} \chi_{B_i} \right\|_{L^p}. \tag{21}
\]

The main results of this article are as follows.

**Theorem 10.** Suppose that \( p(\cdot) \in \mathcal{B} \) and \( \Omega \in L^1(\mathbb{S}^{n-1}) \) with \( s > (p')_{+} \) satisfying (1). Let \( q \in (0, n), q \in (0, \infty), \alpha > 0, \) and \( \alpha(\cdot) \) be log-Hölder continuous both at the origin and at infinity, such that
\[
-n\delta_{1} + \sigma < \alpha_{-} \leq \alpha_{+} < \frac{-(n - 1)}{s} + n\delta_{2}, \tag{22}
\]
where \( \delta_{1}, \delta_{2} \in (0, 1) \) are the constants mentioned in Lemma 5. Then, the operator \( \mathcal{M}_{\Omega, A}^{q} \) is bounded on \( K_{q, p(\cdot)}^{(\alpha(\cdot))}. \)

**Theorem 11.** Suppose that \( p(\cdot) \in \mathcal{B} \) and \( \Omega \in L^1(\mathbb{S}^{n-1}) \) with \( s > (p')_{+} \) satisfying (1). Let \( q \in (0, n), q \in (0, \infty), b \in \text{BMO}, \sigma > 0, \) and \( \alpha(\cdot) \) be log-Hölder continuous both at the origin and at infinity, such that
\[
-n\delta_{1} + \sigma < \alpha_{-} \leq \alpha_{+} < \frac{-(n - 1)}{s} + n\delta_{2}, \tag{23}
\]
where \( \delta_{1}, \delta_{2} \in (0, 1) \) are the constants mentioned in Lemma 5. Then, the operator \( \mathcal{M}_{\Omega, A}^{q} \) is bounded on \( K_{q, p(\cdot)}^{(\alpha(\cdot))}. \)

It is worth noting that if \( \sigma = 0 \), then the variable Morrey-Herz space \( K_{q, p(\cdot)}^{(\alpha(\cdot))} \) dates back to the variable Herz space \( K_{q, p(\cdot)}^{(\alpha(\cdot))}. \) Thus, by letting \( \sigma = 0 \) in Theorems 10 and 11, we will get the following results on the variable exponents Herz spaces.

**Corollary 12.** Suppose that \( p(\cdot) \in \mathcal{B} \) and \( \Omega \in L^1(\mathbb{S}^{n-1}) \) with \( s > (p')_{+} \) satisfying (1). Let \( p \in (0, n), q \in (0, \infty), \) and \( \alpha(\cdot) \) be log-Hölder continuous both at the origin and at infinity, such that
\[
-n\delta_{1} < \alpha_{-} \leq \alpha_{+} < \frac{-(n - 1)}{s} + n\delta_{2}, \tag{24}
\]
where \( \delta_{1}, \delta_{2} \in (0, 1) \) are the constants mentioned in Lemma 5. Then, the operator \( \mathcal{M}_{\Omega}^{q} \) is bounded on \( K_{q, p(\cdot)}^{(\alpha(\cdot))}. \)

**Corollary 13.** Suppose that \( p(\cdot) \in \mathcal{B} \) and \( \Omega \in L^1(\mathbb{S}^{n-1}) \) with \( s > (p')_{+} \) satisfies (1). Let \( q \in (0, n), q \in (0, \infty), b \in \text{BMO} \) and \( i \in \mathbb{N} \) and \( \alpha(\cdot) \) be log-Hölder continuous both at the origin and at infinity, such that
\[
-n\delta_{1} < \alpha_{-} \leq \alpha_{+} < \frac{-(n - 1)}{s} + n\delta_{2}, \tag{25}
\]
where \( \delta_{1}, \delta_{2} \in (0, 1) \) are the constants mentioned in Lemma 5. Then, the operator \( \mathcal{M}_{\Omega, A}^{q} \) is bounded on \( K_{q, p(\cdot)}^{(\alpha(\cdot))}. \)

**Remark 14.** If \( \alpha(\cdot) \) is a constant function, i.e., \( \alpha(\cdot) = \alpha \), then the results of Corollaries 12 and 13 can be found in [30].

**3. Proofs of Theorems 10 and 11**

**Proof of Theorem 10.** Let \( f \in MK_{q, p(\cdot)}^{(\alpha(\cdot))}. \) For any \( j \in \mathbb{Z} \), let \( f_{j} = f \chi_{j} \), then we have
\[
f(x) = \sum_{j = -\infty}^{\infty} f_{j}(x) = \sum_{j = -\infty}^{\ell_{1}} f_{j}(x) + \sum_{j = -\infty}^{\ell_{2}} f_{j}(x) + \sum_{j = -\infty}^{\ell_{3}} f_{j}(x). \tag{26}
\]
\( \square \)

Using the definition of \( MK_{q, p(\cdot)}^{(\alpha(\cdot))} \), we have
\[
\left\| \mathcal{M}_{\Omega, A}^{q}(f) \right\|_{MK_{q, p(\cdot)}^{(\alpha(\cdot))}} \leq \sup_{x \in \mathbb{Z}} \left\| 2^{(\alpha(\cdot))}(\mathcal{M}_{\Omega, A}^{q}(f)) x_{j} \right\|_{L^{q}(\mathbb{Z})} \left\| 2^{(\alpha(\cdot))}(\mathcal{M}_{\Omega, A}^{q}(f)) x_{j} \right\|_{L^{q}(\mathbb{Z})}
\leq \sum_{x \in \mathbb{Z}} \left\| 2^{(\alpha(\cdot))}(\mathcal{M}_{\Omega, A}^{q}(f)) x_{j} \right\|_{L^{q}(\mathbb{Z})} \left\| 2^{(\alpha(\cdot))}(\mathcal{M}_{\Omega, A}^{q}(f)) x_{j} \right\|_{L^{q}(\mathbb{Z})}.
\]

First, we estimate \( \mathcal{Z}_{1} \). By Proposition 6, we obtain
\[
\mathcal{Z}_{1} = \max \left\{ \sum_{k \geq 0}^{2^{-\kappa}} \sum_{k \in \mathbb{Z}} \left\| 2^{(\alpha(\cdot))}(\mathcal{M}_{\Omega, A}^{q}(f)) x_{j} \right\|_{L^{q}(\mathbb{Z})} \right\}.
\]
Using the fact that \(2^{(\ell-1)} > 2^{n(0)}\), and the boundedness of \(\mathcal{M}_\Omega^0\) on \(L^p(\Omega)\) (see [30]), we deduce

\[
\mathcal{T}_1 \leq \max \left\{ \sup_{\kappa > 0} \left( \sum_{l=0}^{k-1} \left\| 2^{f_{(\xi)}} |\mathcal{X}_\ell|_\mathcal{L}^p(\Omega) \right\|_{L^p(\Omega)}^{\frac{q}{2}} \right) + \sup_{\kappa > 0} \left( \sum_{l=0}^{k-1} \left\| 2^{f_{(\xi)}} |\mathcal{X}_\ell|_\mathcal{L}^p(\Omega) \right\|_{L^p(\Omega)}^{\frac{q}{2}} \right) \right\} \leq \left\| f \right\|_{L^p(\Omega)}^{\frac{q}{2}}. 
\]

For the term \(\mathcal{T}_2\), we firstly estimate \(2^{f_{(\xi)}} |\mathcal{M}_\Omega^0 (f_j)(x)|\).

\[
2^{f_{(\xi)}} |\mathcal{M}_\Omega^0 (f_j)(x)| \cdot \mathcal{X}_\ell(x) 
\leq 2^{f_{(\xi)}} \left( \int_0^{[x]} \left[ \frac{1}{|x-y|^{n-q}} \left| \frac{\Omega(x-y)f_j(y)}{|x-y|^{n-q}} \right|^2 \right]^{\frac{1}{2}} \int_{|x-y|<\ell} \mathcal{X}_\ell(x) \right) + 2^{f_{(\xi)}} \left( \int_0^{[x]} \frac{\Omega(x-y)f_j(y)}{|x-y|^{n-q}} \int_{|x-y|<\ell} \mathcal{X}_\ell(x) \right)
\]

\[
\leq 2^{f_{(\xi)}} \left( \int_0^{[x]} \int_{|x-y|<\ell} \mathcal{X}_\ell(x) \right) + 2^{f_{(\xi)}} \left( \int_0^{[x]} \int_{|x-y|<\ell} \mathcal{X}_\ell(x) \right).
\]

From (31) and the Minkowski’s inequality, it follows that

\[
2^{f_{(\xi)}} |\mathcal{M}_\Omega^0 (f_j)(x)| \cdot \mathcal{X}_\ell(x) 
\leq 2^{f_{(\xi)}} \left( \int_0^{[x]} \frac{\Omega(x-y)f_j(y)}{|x-y|^{n-q}} \int_{|x-y|<\ell} \mathcal{X}_\ell(x) \right) + 2^{f_{(\xi)}} \left( \int_0^{[x]} \frac{\Omega(x-y)f_j(y)}{|x-y|^{n-q}} \int_{|x-y|<\ell} \mathcal{X}_\ell(x) \right)
\]

\[
\leq 2^{f_{(\xi)}} \left( \int_0^{[x]} \int_{|x-y|<\ell} \mathcal{X}_\ell(x) \right) + 2^{f_{(\xi)}} \left( \int_0^{[x]} \int_{|x-y|<\ell} \mathcal{X}_\ell(x) \right).
\]

By Lemma 7, we deduce

\[
2^{f_{(\xi)}} |\mathcal{M}_\Omega^0 (f_j)(x)| \cdot \mathcal{X}_\ell(x) 
\leq 2^{f_{(\xi)}} \left( \int_0^{[x]} \int_{|x-y|<\ell} \mathcal{X}_\ell(x) \right) + 2^{f_{(\xi)}} \left( \int_0^{[x]} \int_{|x-y|<\ell} \mathcal{X}_\ell(x) \right).
\]

It follows from Hölder’s inequality (12) that

\[
2^{f_{(\xi)}} |\mathcal{M}_\Omega^0 (f_j)(x)| \cdot \mathcal{X}_\ell(x) 
\leq 2^{f_{(\xi)}} \left( \int_0^{[x]} \int_{|x-y|<\ell} \mathcal{X}_\ell(x) \right) + 2^{f_{(\xi)}} \left( \int_0^{[x]} \int_{|x-y|<\ell} \mathcal{X}_\ell(x) \right).
\]

Since \(s > (p')_+\), we can find a variable exponent \(p^+(\cdot) > 1\) such that \((1/p'+(\cdot)) = (1/s) + 1/p^+(\cdot))\), then by Lemma 8, it follows that

\[
\left\| \Omega(x-\cdot) \mathcal{X}_\ell \right\|_{L^{p^+}(\Omega)} \leq \left\| \Omega(x-\cdot) \mathcal{X}_\ell \right\|_{L^{p^+}(\Omega)} \left\| \mathcal{X}_\ell \right\|_{L^{p^+}(\Omega)} \leq \left\| \Omega(x-\cdot) \mathcal{X}_\ell \right\|_{L^{p^+}(\Omega)} \left\| \mathcal{X}_\ell \right\|_{L^{p^+}(\Omega)}.
\]

where the last inequality (35) is based on the fact that \(\left\| \mathcal{X}_\ell \right\|_{L^{p^+}(\Omega)} = \left\| \mathcal{X}_\ell \right\|_{L^{p^+}(\Omega)}\); see [18]. From (34) and (35),
Lemma 4 and 5, we can deduce
\[
\left\| 2^{j_1n_1(t)} \mathcal{M}_f^0(f_j) X_i \right\|_{L^p(t)} \leq \frac{1}{2^{j_1}} \leq 2^{(e-j)(a_j^* - ((n-1)|a|))} \left\| X_B \right\|_{L^p(t)} \left\| 2^{j_1n_1(t)} f_j \right\|_{L^p(t)} \left\| X_B \right\|_{L^p(t)} \\
\leq 2^{(e-j)(a_j^* - ((n-1)|a|))} \left\| X_B \right\|_{L^p(t)} \left\| 2^{j_1n_1(t)} f_j \right\|_{L^p(t)} \left\| 2^{e-j} B_i \right\| \right\|_{L^p(t)} \\
\leq 2^{(e-j)(n_2^* + (n-1)|a|, a_j^*)} \left\| 2^{j_1n_1(t)} f_j \right\|_{L^p(t)}.
\]

(36)

To estimate \( \mathfrak{I}_2 \), we need to consider two cases: \( 0 < q \leq 1 \) and \( 1 < q < \infty \).

Case 1. \( 1 < q < \infty \), by the fact that \( n \delta_2 - a + ((n-1)|a|) > 0 \), the Hölder’s inequality, and the inequality (36), it follows that
\[
\left\| 2^{j_1n_1(t)} \mathcal{M}_f^0(f_j) X_i \right\|_{L^p(t)} \leq \sup_{k \in \mathbb{Z}} 2^{-kq} \left\| \mathcal{M}_f^0(f_j) X_i \right\|_{L^p(t)} \right\|_{L^p(t)} \\
\leq \sup_{k \in \mathbb{Z}} 2^{-kq} \left\| 2^{j_1n_1(t)} f_j \right\|_{L^p(t)} \left\| 2^{j_1n_1(t)} X_i \right\|_{L^p(t)} \\
\leq \sup_{k \in \mathbb{Z}} 2^{-kq} \left\| 2^{j_1n_1(t)} f_j \right\|_{L^p(t)} \left\| 2^{2j_1n_1(t)} X_i \right\|_{L^p(t)} \\
\leq \sup_{k \in \mathbb{Z}} 2^{-kq} \sum_{j=-\infty}^{\infty} \left\| 2^{j_1n_1(t)} f_j \right\|_{L^p(t)} \left\| 2^{2j_1n_1(t)} X_i \right\|_{L^p(t)} \\
\leq \left\| f \right\|_{M^q_{\mathfrak{I}_2}^e(t)}.
\]

(37)

Similarly to (35), we conclude
\[
\left\| \mathcal{M}_f^0(f_j) X_i \right\|_{L^p(t)} \leq \left\| f \right\|_{M^q_{\mathfrak{I}_2}^e(t)}.
\]

(42)

For the term \( \mathfrak{I}_3 \), by applying Proposition 6, we can get
\[
\mathfrak{I}_3 \leq \sup_{k \in \mathbb{Z}} 2^{-q} \left\| \mathcal{M}_f^0(f_j) X_i \right\|_{L^p(t)} = \left\| \mathcal{M}_f^0(f_j) X_i \right\|_{L^p(t)} \\
\leq \sup_{k \in \mathbb{Z}} 2^{-q} \left\| \mathcal{M}_f^0(f_j) X_i \right\|_{L^p(t)} = \left\| \mathcal{M}_f^0(f_j) X_i \right\|_{L^p(t)} \\
= \max\left( \mathfrak{I}_4^1, \mathfrak{I}_4^2 \right).
\]

(40)

For \( \mathfrak{I}_3 \), it is clear that if \( x \in \mathfrak{I}_j, j - \ell \geq \ell, \) and \( y \in \mathfrak{I}_j \), then \( |x - y| = |y| \). By (31), (12), and the Minkowski’s inequality, we deduce that
\[
\left\| \mathcal{M}_f^0(f_j) X_i \right\|_{L^p(t)} \leq \left\| f \right\|_{M^q_{\mathfrak{I}_2}^e(t)}.
\]

(41)

From (42), Lemmas 4 and 5, it follows that
\[
\left\| \sum_{j=\ell}^{\infty} \mathcal{M}_f^0(f_j) X_i \right\|_{L^p(t)} \leq \sum_{j=\ell}^{\infty} \left\| \mathcal{M}_f^0(f_j) X_i \right\|_{L^p(t)} \leq \left\| f \right\|_{M^q_{\mathfrak{I}_2}^e(t)}.
\]

(43)

To estimate \( \mathfrak{I}_4^1 \), we need to consider two cases below: \( 0 < q \leq 1 \) and \( 1 < q < \infty \).
Case 3. $0 < q \leq 1$, combining the above inequalities and using (38), we can obtain

$$
\mathcal{T}_3^I \leq \sup_{\kappa < 0; \kappa \in \mathbb{Z}} 2^{-\sigma q} \sum_{\ell = -\infty}^{\infty} 2^{\sigma \alpha \ell(0)} \sum_{j = \ell + 2}^{\infty} 2^{\sigma j(j-n\delta_1)} \|f_j\|_{L^q}^q + \sup_{\kappa < 0; \kappa \in \mathbb{Z}} 2^{-\sigma q} \sum_{\ell = -\infty}^{\infty} 2^{\sigma \alpha \ell(0)} \sum_{j = \kappa}^{\infty} 2^{\sigma j(j-n\delta_1)} \|f_j\|_{L^q}^q
$$

$$
= \mathcal{T}_3^{II} + \mathcal{T}_3^{IV}.
$$

(44)

For $\mathcal{T}_3^{II}$, in view of $\alpha(0) + n\delta_1 > \alpha_+ + n\delta_1 > 0$, we get

$$
\mathcal{T}_3^{II} \leq \sup_{\kappa < 0; \kappa \in \mathbb{Z}} 2^{-\sigma q} \sum_{\ell = -\infty}^{\infty} 2^{\sigma \alpha \ell(0)} \|f_j\|_{L^q}^q \leq \|f\|_{\text{BMO}}^q.
$$

(45)

Now, let us deal with $\mathcal{T}_3^{IV}$, noting that $\sigma - n\delta_1 - \alpha(0) < \sigma - n\delta_1 - \alpha_+ < 0$, it follows that

$$
\mathcal{T}_3^{IV} \leq \sup_{\kappa < 0; \kappa \in \mathbb{Z}} 2^{-\sigma q} \sum_{\ell = -\infty}^{\infty} 2^{\sigma \alpha \ell(0)} \sum_{j = \ell + 2}^{\infty} 2^{\sigma j(j-n\delta_1)} \|f_j\|_{L^q}^q \leq \|f\|_{\text{BMO}}^q.
$$

(46)

Case 4. $1 < q < \infty$, we have

$$
\mathcal{T}_3^I \leq \sup_{\kappa < 0; \kappa \in \mathbb{Z}} 2^{-\sigma q} \sum_{\ell = -\infty}^{\infty} 2^{\sigma \alpha \ell(0)} \left( \sum_{j = \ell + 2}^{\infty} 2^{\sigma j(j-n\delta_1)} \|f_j\|_{L^q}^q \right)
$$

$$
\leq \sup_{\kappa < 0; \kappa \in \mathbb{Z}} 2^{-\sigma q} \sum_{\ell = -\infty}^{\infty} 2^{\sigma \alpha \ell(0)} \left( \sum_{j = \ell + 2}^{\infty} 2^{\sigma j(j-n\delta_1)} \|f_j\|_{L^q}^q \right)
$$

$$
+ \sup_{\kappa < 0; \kappa \in \mathbb{Z}} 2^{-\sigma q} \sum_{\ell = -\infty}^{\infty} 2^{\sigma \alpha \ell(0)} \left( \sum_{j = \kappa}^{\infty} 2^{\sigma j(j-n\delta_1)} \|f_j\|_{L^q}^q \right)
$$

$$
= \mathcal{T}_3^{IV} + \mathcal{T}_3^{III}.
$$

(47)

For $\mathcal{T}_3^{III}$, by the fact that $\alpha(0) + n\delta_1 > 0$ and using Hölder inequality, we infer that

$$
\mathcal{T}_3^{III} \leq \sup_{\kappa < 0; \kappa \in \mathbb{Z}} 2^{-\sigma q} \sum_{\ell = -\infty}^{\infty} 2^{\sigma \alpha \ell(0)} \left( \sum_{j = \ell + 2}^{\infty} 2^{\sigma j(j-n\delta_1)} \|f_j\|_{L^q}^q \right)
$$

$$
\leq \sup_{\kappa < 0; \kappa \in \mathbb{Z}} 2^{-\sigma q} \sum_{\ell = -\infty}^{\infty} 2^{\sigma \alpha \ell(0)} \left( \sum_{j = \ell + 2}^{\infty} 2^{\sigma j(j-n\delta_1)} \|f_j\|_{L^q}^q \right)
$$

$$
\leq \|f\|_{\text{BMO}}^q.
$$

(48)

For $\mathcal{T}_3^{IV}$, from the inequality (36) and using the method as for $\mathcal{T}_3^{III}$, we obtain

$$
\mathcal{T}_3^{IV} \leq \|f\|_{\text{BMO}}^q.
$$

(49)

By combining $\mathcal{T}_3^{I}$, $\mathcal{T}_3^{IV}$, $\mathcal{T}_3^{III}$, and $\mathcal{T}_3^{IV}$ estimates, we arrive at

$$
\mathcal{T}_3^I \leq \|f\|_{\text{BMO}}^q.
$$

(50)

By the similar method used in the estimate for $\mathcal{T}_3^I$, it is not difficult to show that

$$
\mathcal{T}_3^I \leq \|f\|_{\text{BMO}}^q.
$$

(51)

Thus, we have

$$
\|M^0_{\Omega}(f)\|_{\text{BMO}}^q \leq \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 \leq \|f\|_{\text{BMO}}^q.
$$

(52)

The proof for Theorem 10 is finished.

Proof of Theorem 11. Let $A \in \text{BMO}, f \in \text{MK}_{\mathbb{Z}}^{a(\cdot)}$. For any $j \in \mathbb{Z}$, let $f_j := f \chi_j$, then we have

$$
f(x) = \sum_{j \in \mathbb{Z}} f_j(x) = \sum_{j \in \mathbb{Z}} f_j(x) + \sum_{j = -\infty}^{\ell_1} f_j(x) + \sum_{j = \ell_1 + 2}^{\ell_2} f_j(x) + \sum_{j = \ell_2 + 2}^{\infty} f_j(x).
$$

(53)
Using the definition of $M^{\alpha(n), \sigma}_{\Omega(p)}$, we have

\[
\left\| M^{\alpha(n), \sigma}_{\Omega(p)}(f) \right\|_{L^q(\Omega)} \leq \sup_{x \in \Omega} \sum_{k=0}^n \left( \frac{\left\| M^{\alpha(n), \sigma}_{\Omega(p)}(f) \right\|_{L^q(\Omega)}}{k!} \right) \chi_k \left( \frac{x}{k!} \right)
\]

Let us first estimate $\Psi_1$. From Proposition 6 and the boundedness of $M^{\alpha(n), \sigma}_{\Omega(p)}$ on $L^p(\Omega)$ (see [30]), and using the similar methods as that for $\mathcal{T}_1$, it is not difficult to see that

\[
\Psi_1 \leq \max \left\{ \sup_{x \in \Omega} \sum_{k=0}^n \left( \frac{\left\| M^{\alpha(n), \sigma}_{\Omega(p)}(f) \right\|_{L^q(\Omega)}}{k!} \right) \chi_k \left( \frac{x}{k!} \right), \sup_{x \in \Omega} \sum_{k=0}^n \left( \frac{\left\| M^{\alpha(n), \sigma}_{\Omega(p)}(f) \right\|_{L^q(\Omega)}}{k!} \right) \chi_k \left( \frac{x}{k!} \right) \right\}
\]

\[
= \left\| \Lambda \mathcal{T}_1 \right\|_{L^1(\Omega)} \leq \left\| \Lambda \right\|_{L^1(\Omega)} \left\| \mathcal{T}_1 \right\|_{L^1(\Omega)}
\]

Now, let us turn to the estimates of $\Psi_2$. We consider

\[
2^{\alpha(n)} \left\| M^{\alpha(n), \sigma}_{\Omega(p)}(f) \right\|_{L^q(\Omega)} \leq \left\| \Lambda \mathcal{T}_2 \right\|_{L^1(\Omega)} \leq \left\| \Lambda \right\|_{L^1(\Omega)} \left\| \mathcal{T}_2 \right\|_{L^1(\Omega)}
\]

It is clear that if $x \in \mathcal{R}_\ell$, and $y \in \mathcal{R}_\ell$, then $|x - y| = |x|$. Thus, for $\rho \in (0, n)$, we use the Minkowski's inequality to get

\[
2^{\alpha(n)} \left\| M^{\alpha(n), \sigma}_{\Omega(p)}(f) \right\|_{L^q(\Omega)} \leq \left\| \Lambda \mathcal{T}_3 \right\|_{L^1(\Omega)} \leq \left\| \Lambda \right\|_{L^1(\Omega)} \left\| \mathcal{T}_3 \right\|_{L^1(\Omega)}
\]
Summing up the estimates of $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$, we conclude that

$$\left\| M_{\Omega}^0 (f) \right\|_{MK_{\alpha}^{\ell}(\mathbb{R}^d)} \leq \mathcal{P}_1 + \mathcal{P}_2 + \left\| \Lambda_{\alpha}^0 \right\|_{MK_{\alpha}^{\ell}(\mathbb{R}^d)}. \quad (64)$$

The proof for Theorem 11 is finished.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no competing interests.

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**References**


