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Research Article

Solving an Integral Equation via Orthogonal Branciari Metric Spaces

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In this article, we present an orthogonal L-contraction mapping concepts and prove a fixed point theorem on orthogonal complete Branciari metric spaces. As an application, we apply our major results to solving integral equations.

1. Introduction

The general metric concept was introduced by Branciari [1] in 2000 and which is known as the Branciari metric. Later, many authors were interested to the Branciari metric space for extending the results of Branciari$\mathfrak{b}$-metric spaces (see [2–7]). The $\ddot{\Theta}$-contraction concept was introduced by Jleli and Samet [8] in 2014. It is based on some fixed point results [9, 10]. An orthogonality concept in metric spaces was introduced by Gordji et al. [11, 12]. Several authors proved the fixed point results in the generalized orthogonal metric space of Branciari metric spaces (BMS) [13–17]. The L-contraction concept was introduced by Cho [17] in 2018. In this article, we present the new concepts of L-contractive orthogonal mapping and prove fixed point theorems in an orthogonal complete Branciari metric space (OCBMS). We also give an example to our current results for using the integral equation solved, respectively.

2. Preliminaries

The basic definitions and results are required in the next section as follows.

Definition 1 (see [1]). Let P be a non-empty set and $\mathfrak{S} : P \times P \longrightarrow \mathbb{R}_+$ a mapping such that for all $\mathfrak{J}_1, \mathfrak{J}_2 \in P$ and all $\mathfrak{J}_3 \neq \mathfrak{J}_4 \in P/\{\mathfrak{J}_1, \mathfrak{J}_2\}$:

(BM1) $\mathfrak{S}(\mathfrak{J}_1, \mathfrak{J}_2) = 0$, iff $\mathfrak{J}_1 = \mathfrak{J}_2$

(BM2) $\mathfrak{S}(\mathfrak{J}_1, \mathfrak{J}_2) = \mathfrak{S}(\mathfrak{J}_2, \mathfrak{J}_1)$

(BM3) $\mathfrak{S}(\mathfrak{J}_1, \mathfrak{J}_2) \leq \mathfrak{S}(\mathfrak{J}_1, \mathfrak{J}_3) + \mathfrak{S}(\mathfrak{J}_3, \mathfrak{J}_4) + \mathfrak{S}(\mathfrak{J}_4, \mathfrak{J}_2)$.

The metric $\mathfrak{S}$ is called a Branciari metric, and the pair $(P, \mathfrak{S})$ is called a BMS.

Definition 2 (see [1]). Let $(P, \mathfrak{S})$ be a BMS. A self-map $H : P \longrightarrow P$ is called $\ddot{\Theta}$-contraction if there exist $\ddot{\Theta} \in \Gamma_{1,2,3}$ and $\nu \in (0,1)$ such that $\forall \mathfrak{J}_1, \mathfrak{J}_2 \in P$:

$$\mathfrak{S}(H\mathfrak{J}_1, H\mathfrak{J}_2) > 0 \Rightarrow \ddot{\Theta}(\mathfrak{S}(H\mathfrak{J}_1, H\mathfrak{J}_2)) \le \left[\ddot{\Theta}(\mathfrak{S}(\mathfrak{J}_1, \mathfrak{J}_2))\right]^{\nu}, \tag{1}$$

where $\Gamma_{1,2,3}$ is the family of all functions $\ddot{\Theta} : (0,\infty) \longrightarrow (0,\infty)$ which satisfy the following conditions:

$(\ddot{\Theta}_1)$ $\ddot{\Theta}$ is increasing

$(\ddot{\Theta}_2)$ For each sequence $\{\alpha_\imath\} \subset (0,\infty)$, $\lim_{\imath\longrightarrow\infty}\ddot{\Theta}(\alpha_\imath) = 1 \Leftrightarrow \lim_{\imath\longrightarrow\infty}\alpha_\imath = 0^{+.}$

$(\ddot{\Theta}_3)$ $\ddot{\Theta}$ is continuous.

Remark 3. We know that every $\ddot{\Theta}$-contraction mapping is continuous.

The following notes are subsequently adopted:

(1) $\Gamma_{1,2,3}$ is the class of all functions $\ddot{\Theta}$ which satisfy $[\ddot{\Theta}_1 - \ddot{\Theta}_3]$

Definition 4 (see [17]). Let $(P, \mathfrak{S})$ be a BMS. A mapping $H : P \longrightarrow P$ is called L -contraction with respect to $\varsigma \in L$ if there exists $\ddot{\Theta} \in \Gamma_{1,2,3}$ such that (for all $\mathfrak{J}_1, \mathfrak{J}_2 \in P$):

$$\mathfrak{S}(H\mathfrak{J}_1, H\mathfrak{J}_2) > 0 \Rightarrow \varsigma\left[\ddot{\Theta}(\mathfrak{S}(H\mathfrak{J}_1, H\mathfrak{J}_2)), \ddot{\Theta}(\mathfrak{S}(\mathfrak{J}_1, \mathfrak{J}_2))\right] \ge 1, \tag{2}$$

where L is the class of all functions $\varsigma : [1,\infty) \times [1,\infty) \longrightarrow \mathbb{R}$ in which the following conditions are satisfied (ς_1^*):

(ς_1^*) $\varsigma(1,1) = 1$

(ς_2^*) $\varsigma(\rho.\rho_1) < (\rho_1/\rho)$, for all $\rho, \rho_1 > 1$

(ς_3^*) If $\{\rho_\imath\}$ and $\{\rho_{1_\imath}\}$ are two sequence in $(1,\infty)$ with $\rho_\imath < \rho_{1_\imath}$, such that $\lim_{\imath\longrightarrow\infty}\rho_\imath = \lim_{\imath\longrightarrow\infty}\rho_{1_\imath} > 1$, then $\limsup_{\imath\longrightarrow\infty}\varsigma(\rho_\imath, \rho_{1_\imath}) < 1$.

Example 1 (see [17]). Let $\varsigma_\nu, \varsigma_\psi : [1,\infty) \times [1,\infty) \longrightarrow \mathbb{R}$ be two functions defined by

(a) $\varsigma_\nu(\rho, \rho_1) = {\rho_1}^{\nu}/\rho$, for all $\rho, \rho_1 \ge 1$, where $\nu \in (0,1)$

(b) $\varsigma_\psi(\rho, \rho_1) = \rho_1/\rho\psi(\rho_1)$, for all $\rho, \rho_1 \ge 1$, where $\psi : [1,\infty) \longrightarrow [1,\infty)$ is a lower semicontinuous and increasing function with $\psi^{-1}(\{1\}) = 1$

Then, $\varsigma_\nu, \varsigma_\psi \in L$.

Cho [17] proved the following theorem.

Theorem 5 (see [17]). *Let $(P, \mathfrak{S})$ be a complete BMS and $H : P \longrightarrow P$ an L-contraction mapping. Then, H has a unique fixed point.*

Remark 6. Let $\{\mathfrak{a}_\imath\}, \{\mathfrak{b}_\imath\}, \{\mathfrak{c}_\imath\}$ are sequences of $\mathbb{R}_+$ such that $\lim_{\imath\longrightarrow\infty}\mathfrak{a}_\imath = \mathfrak{a}$, $\lim_{\imath\longrightarrow\infty}\mathfrak{b}_\imath = \mathfrak{b}$ and $\lim_{\imath\longrightarrow\infty}\mathfrak{c}_\imath = \mathfrak{c}$. Then,

$$\begin{aligned}\lim_{\imath\longrightarrow\infty} \max\{\mathfrak{a}_\imath, \mathfrak{b}_\imath, \mathfrak{c}_\imath\} &= \max\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}\\ \lim_{\imath\longrightarrow\infty} \min\{\mathfrak{a}_\imath, \mathfrak{b}_\imath, \mathfrak{c}_\imath\} &= \min\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}.\end{aligned} \tag{3}$$

Lemma 7 (see [7]). *Let $\{\mathfrak{J}_{1_\imath}\}$ be a Cauchy sequence in a BMS $(P, \mathfrak{S})$ such that $\lim_{\imath\longrightarrow\infty}\mathfrak{S}(\mathfrak{J}_{1_\imath}, \mathfrak{J}_1) = 0$, for some $\mathfrak{J}_1 \in P$. Then, $\lim_{\imath\longrightarrow\infty}\mathfrak{S}(\mathfrak{J}_{1_\imath}, \mathfrak{J}_2) = \mathfrak{S}(\mathfrak{J}_1, \mathfrak{J}_2)$, for all $\mathfrak{J}_2 \in P$. In particular, $\{\mathfrak{J}_{1_\imath}\}$ diverge to $\mathfrak{J}_2$ if $\mathfrak{J}_2 \ne \mathfrak{J}_1$.*

Definition 8 (see [11]). Let $P \ne \phi$ and $\nabla \subseteq P \times P$ be a binary relation. If ∇ satisfies the following condition:

$$\exists \mathfrak{J}_{10} \in P : (\forall \mathfrak{J}_1 \in P, \mathfrak{J}_1 \nabla \mathfrak{J}_{10}) \text{ or } (\forall \mathfrak{J}_1 \in P, \mathfrak{J}_{10} \nabla \mathfrak{J}_1), \tag{4}$$

then it is called an orthogonal set. We denote this O-set by (P, ∇).

Example 2. Let $P = Z$ and define $\mathfrak{J}_2 \nabla \mathfrak{J}_1$ if there exists $\nu \in Z$ such that $\mathfrak{J}_2 = \nu\mathfrak{J}_1$. It is easy to see that $0\nabla\mathfrak{J}_1$ for all $\mathfrak{J}_1 \in Z$. Hence (P, ∇) is an O-set.

Example 3 (see [11]). A wheel graph $\mathscr{W}_\imath$ is a graph (see, for example, Figure 1) with $\imath$ vertices for each $\imath \ge 4$, a single vertex connect to all vertex to all vertices of an $(\imath - 1)$-cycle. Let P be the set of all vertices of $\mathscr{W}_\imath$ for each $\imath \ge 4$. Define $\mathfrak{J}_1 \nabla \mathfrak{J}_2$ if there is a connection from $\mathfrak{J}_1$ to $\mathfrak{J}_2$. Then, (P, ∇) is an O-set.

Definition 9 (see [11]). Let (P, ∇) be an O-set. A sequence $\{\mathfrak{J}_{1_\imath}\}$ is called an orthogonal sequence (shortly, O-sequence) if

$$(\forall i \in \mathbb{N}, \mathfrak{J}_{1_\imath} \nabla \mathfrak{J}_{1_{\imath+1}}) \text{ or } (\forall \imath \in \mathbb{N}, \mathfrak{J}_{1_{\imath+1}} \nabla \mathfrak{J}_{1_\imath}). \tag{5}$$

Definition 10 (see [11]). The triplet $(P,\nabla,\mathfrak{S})$ is called an orthogonal metric space if (P, ∇) is an O-set and $(P, \mathfrak{S})$ is a metric space.

Definition 11 (see [11]). Let $(P,\nabla,\mathfrak{S})$ be an orthogonal metric space. Then, a mapping $H : P \longrightarrow P$ is said to be orthogonally continuous in $\mathfrak{J}_1 \in P$ if for each O-sequence $\{\mathfrak{J}_{1_\imath}\}$ in P with $\mathfrak{J}_{1_\imath} \longrightarrow \mathfrak{J}_1$ as $\imath \longrightarrow \infty$, we have $H(\mathfrak{J}_{1_\imath}) \longrightarrow H(\mathfrak{J}_1)$ as $\imath \longrightarrow \infty$. Also, H is said to be ∇-continuous on P if H is ∇-continuous in each $\mathfrak{J}_1 \in P$.

Definition 12 (see [11]). Let $(P,\nabla,\mathfrak{S})$ be an orthogonal metric space. Then, P is said to be an orthogonally complete, if every Cauchy O-sequence is convergent.

Definition 13 (see [11]). Let (P, ∇) be an O-set. A mapping $H : P \longrightarrow P$ is said to be ∇-preserving if $H\mathfrak{J}_1 \nabla H\mathfrak{J}_2$ whenever $\mathfrak{J}_1 \nabla \mathfrak{J}_2$. Also, $H : P \longrightarrow P$ is said to be weakly ∇-preserving if $H(\mathfrak{J}_1)\nabla H(\mathfrak{J}_2)$ or $H(\mathfrak{J}_2)\nabla H(\mathfrak{J}_1)$ whenever $\mathfrak{J}_1 \nabla \mathfrak{J}_2$ for all $\mathfrak{J}_1, \mathfrak{J}_2 \in P$.

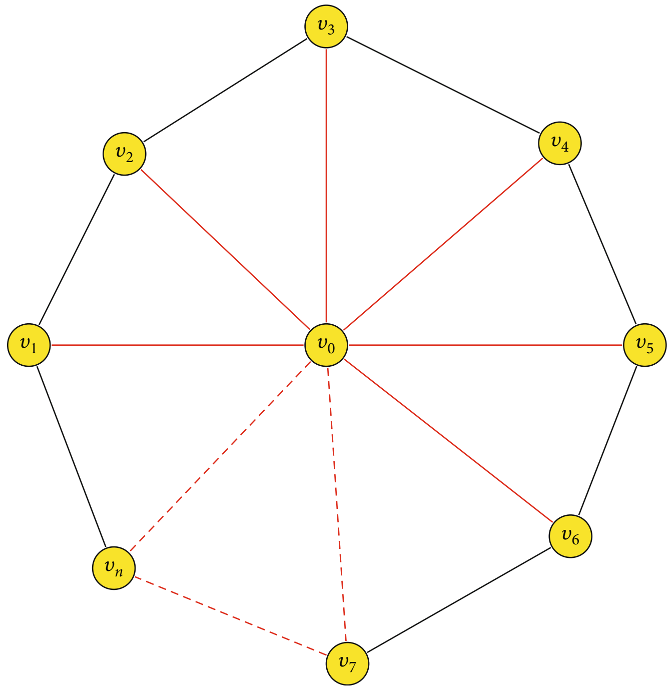

Figure 1: An image of a wheel graph.

3. Major Results

In this section, we present the generalized orthogonal L-contraction notion.

Definition 14. Let (P, ∇) be an O-set and $\mathfrak{S} : \mathrm{P} \times \mathrm{P} \longrightarrow \mathbb{R}_+$ a mapping such that for all $\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_3, \mathfrak{I}_4 \in \mathrm{P}$ and all $\mathfrak{I}_3 \neq \mathfrak{I}_4 \in \mathrm{P}/\{\mathfrak{I}_1, \mathfrak{I}_2\}$:

(OBM1) $\mathfrak{S}(\mathfrak{I}_1, \mathfrak{I}_2) = 0$, if and only if $\mathfrak{I}_1 = \mathfrak{I}_2$

(OBM2) $\mathfrak{S}(\mathfrak{I}_1, \mathfrak{I}_2) = \mathfrak{S}(\mathfrak{I}_2, \mathfrak{I}_1)$

(OBM3) $\mathfrak{S}(\mathfrak{I}_1, \mathfrak{I}_2) \leq \mathfrak{S}(\mathfrak{I}_2, \mathfrak{I}_3) + \mathfrak{S}(\mathfrak{I}_3, \mathfrak{I}_4) + \mathfrak{S}(\mathfrak{I}_4, \mathfrak{I}_2)$ for all $\mathfrak{I}_1 \nabla \mathfrak{I}_2, \mathfrak{I}_1 \nabla \mathfrak{I}_3, \mathfrak{I}_3 \nabla \mathfrak{I}_4, \mathfrak{I}_4 \nabla \mathfrak{I}_2$.

The metric $\mathfrak{S}$ is an orthogonal Branciari metric (shortly OBM), and the pair $(\mathrm{P},\nabla,\mathfrak{S})$ is an orthogonal BMS (shortly OCBMS).

Definition 15. Let $(\mathrm{P},\nabla,\mathfrak{S})$ be a OCBMS and $\mathrm{H} : \mathrm{P} \longrightarrow \mathrm{P}$. Then, H is said to be generalized orthogonal L-contraction with respect to $\varsigma \in \mathrm{L}$ if there exist $\ddot{\Theta} \in \Gamma_{1,2,3}$ such that

$$\forall \mathfrak{I}_1, \mathfrak{I}_2 \in \mathrm{P} \text{ with } \mathfrak{I}_1 \nabla \mathfrak{I}_2\ \mathfrak{S}(\mathrm{H}\mathfrak{I}_1, \mathrm{H}\mathfrak{I}_2) > 0 \Rightarrow \varsigma\left(\ddot{\Theta}(\mathfrak{S}(\mathrm{H}\mathfrak{I}_1, \mathrm{H}\mathfrak{I}_2)), \ddot{\Theta}(\mathfrak{S}(\mathfrak{I}_1, \mathfrak{I}_2))\right) \geq 1. \tag{6}$$

Theorem 16. *Let $(P,\nabla,\mathfrak{S})$ be a complete OCBMS with an orthogonal element $\mathfrak{I}_2$ and a self-mapping $H : P \longrightarrow P$. Suppose that there exist $\varsigma \in L$ and $l > 0$ such that the following conditions hold:*

(i) H is ∇-preserving;

(ii) H is generalized orthogonal L-contraction mapping;

(iii) H is ∇-continuous.

Then, H has a unique fixed point.

Proof. Since (P, ∇) is an O-set,

$$\exists \mathfrak{I}_2 \in \mathrm{P} : (\forall \mathfrak{I}_1 \in \mathrm{P}, \mathfrak{I}_1 \nabla \mathfrak{I}_2) \text{ or } (\forall \mathfrak{I}_1 \in \mathrm{P}, \mathfrak{I}_2 \nabla \mathfrak{I}_1). \tag{7}$$

It follows that $\mathfrak{I}_2 \nabla \mathrm{H}\mathfrak{I}_2$ or $\mathrm{H}\mathfrak{I}_2 \nabla \mathfrak{I}_2$. Let

$$\mathfrak{I}_{11} = \mathrm{H}\mathfrak{I}_{10}, \mathfrak{I}_{12} = \mathrm{H}\mathfrak{I}_{11} = \mathrm{H}^2\mathfrak{I}_{10}, \cdots \cdots, \mathfrak{I}_{1_{\imath+1}} = \mathrm{H}\mathfrak{I}_{1_\imath} = \mathrm{H}^{\imath+1}\mathfrak{I}_{10}, \tag{8}$$

for all $\imath \in \mathbb{N} \cup \{0\}$. If $\mathfrak{I}_{1_{\imath_0}} = \mathfrak{I}_{1_{\imath_0+1}}$ for any $\imath_0 \in \mathbb{N} \cup \{0\}$, then it is clear that $\mathfrak{I}_{1_{\imath_0}}$ is a fixed point of H. Now, we consider $\mathfrak{I}_{1_{\imath_0}} \neq \mathfrak{I}_{1_{\imath_0+1}}$ for all $\imath_0 \in \mathbb{N} \cup \{0\}$. Since H is ∇-preserving, we have

$$\mathfrak{I}_{1_{\imath_0}} \nabla \mathfrak{I}_{1_{\imath_0+1}} \text{ or } \mathfrak{I}_{1_{\imath_0+1}} \nabla \mathfrak{I}_{1_{\imath_0}}, \tag{9}$$

for all $\imath_0 \in \mathbb{N} \cup \{0\}$. This implies $\{\mathfrak{I}_{1_\imath}\}$ is an O-sequence. Using contractive Condition (6) and (ς_2^*), we have

$$\begin{aligned} 1 &\leq \varsigma\left[\ddot{\Theta}(\mathfrak{S}(\mathrm{H}\mathfrak{I}_{1_{n-1}}, \mathrm{H}\mathfrak{I}_{1_\imath})), \ddot{\Theta}(\mathfrak{S}(\mathfrak{I}_{1_{\imath-1}}, \mathfrak{I}_{1_\imath}))\right] \\ &= \varsigma\left[\ddot{\Theta}(\mathfrak{S}(\mathfrak{I}_{1_\imath}, \mathfrak{I}_{1_{\imath+1}})), \ddot{\Theta}(\mathfrak{S}(\mathfrak{I}_{1_{\imath-1}}, \mathfrak{I}_{1_\imath})\right] \\ &< \frac{\ddot{\Theta}(\mathfrak{S}(\mathfrak{I}_{1_{\imath-1}}, \mathfrak{I}_{1_\imath}))}{\ddot{\Theta}(\mathfrak{S}(\mathfrak{I}_{1_\imath}, \mathfrak{I}_{1_{\imath+1}}))}, \end{aligned} \tag{10}$$

which is implies that

$$\ddot{\Theta}(\mathfrak{S}(\mathfrak{I}_{1_\imath}, \mathfrak{I}_{1_{\imath+1}})) < \ddot{\Theta}(\mathfrak{S}(\mathfrak{I}_{1_{\imath-1}}, \mathfrak{I}_{1_\imath}) \forall \imath \in \mathbb{N}. \tag{11}$$

Hence, inequality (11) becomes (in view of $(\ddot{\Theta}_1)$) that

$$\mathfrak{S}(\mathfrak{I}_{1_\imath}, \mathfrak{I}_{1_{\imath+1}}) < \mathfrak{S}(\mathfrak{I}_{1_{\imath-1}}, \mathfrak{I}_{1_\imath}), \forall \imath \in \mathbb{N}. \tag{12}$$

Therefore, the sequence $\{\mathfrak{S}(\mathfrak{I}_{1_{\imath-1}}, \mathfrak{I}_{1_\imath})\}$ is non-increasing and bounded below by 0. Then, $\ell \geq 0$ such that $\lim_{i \longrightarrow \infty} \mathfrak{S}(\mathfrak{I}_{1_{\imath-1}}, \mathfrak{I}_{1_\imath}) = \ell$. We can claim that $\ell \neq 0$, then

$$\lim_{\imath \longrightarrow \infty} \ddot{\Theta}(\mathfrak{S}(\mathfrak{I}_{1_{\imath-1}}, \mathfrak{I}_{1_\imath})) > 1. \tag{13}$$

Setting $\rho_\imath = \ddot{\Theta}(\mathfrak{S}(\mathfrak{I}_{1_\imath}, \mathfrak{I}_{1_{\imath+1}}))$ and $\rho_{1_\imath} = \ddot{\Theta}(\mathfrak{S}(\mathfrak{I}_{1_{\imath-1}}, \mathfrak{I}_{1_\imath}))$. In view of (11),(13), and $(\ddot{\Theta}_3)$, we have $\lim_{\imath \longrightarrow \infty} \rho_\imath = \lim_{\imath \longrightarrow \infty} \rho_{1_\imath} > 1$ and $\rho_\imath < \rho_{1_\imath}$, for all $\imath \in \mathbb{N}$. Therefore, applying the condition (ς_3^*), we deduce

$$1 \leq \limsup_{\imath \longrightarrow \infty} \varsigma(\rho_\imath, \rho_{1_\imath}) < 1, \tag{14}$$

which is a contradiction, and therefore

$$\lim_{\imath \longrightarrow \infty} \mathfrak{S}(\mathfrak{I}_{1_{\imath-1}}, \mathfrak{I}_{1_\imath}) = 0. \tag{15}$$

Now, we consider $\mathfrak{I}_{1_\jmath} = \mathfrak{I}_{1_\imath}$, for some $\jmath > \imath$. Then, also $\mathfrak{I}_{1_{\jmath+1}} = \mathfrak{I}_{1_{\imath+1}}$. Using (11), we get

$$\begin{aligned} \ddot{\Theta}\left(\mathfrak{S}\left(\mathfrak{I}_{1_\jmath}, \mathfrak{I}_{1_{\jmath+1}}\right)\right) &< \ddot{\Theta}\left(\mathfrak{S}\left(\mathfrak{I}_{1_{\jmath-1}}, \mathfrak{I}_{1_\jmath}\right)\right) \\ &< \ddot{\Theta}\left(\mathfrak{S}\left(\mathfrak{I}_{1_{\jmath-2}}, \mathfrak{I}_{1_{\jmath-1}}\right)\right) \\ &< \cdots < \ddot{\Theta}(\mathfrak{S}(\mathfrak{I}_{1_\imath}, \mathfrak{I}_{1_{\imath+1}})) \\ &= \ddot{\Theta}\left(\mathfrak{S}\left(\mathfrak{I}_{1_\jmath}, \mathfrak{I}_{1_{\jmath+1}}\right)\right), \end{aligned} \tag{16}$$

which is a contradiction. Hence, we conclude that $\mathfrak{I}_{1_\jmath} \neq \mathfrak{I}_{1_\imath}, \forall \imath \neq \jmath$. □

Next, we show that $\{\mathfrak{I}_{1_\imath}\}$ is a Cauchy sequence in $(\mathrm{P}, \nabla, \mathfrak{S})$. On the contrary, assume that it is not Cauchy, then there exists an $\epsilon > 0$ for which we can find two subsequences $\{\mathfrak{I}_{1_{\imath_\nu}}\}$ and $\{\mathfrak{I}_{1_{\jmath_\nu}}\}$ of $\{\mathfrak{I}_{1_\imath}\}$ such that $\imath_\nu > \jmath_\nu \geq \nu$, for all $\nu \in \mathrm{N}$ and

$$\mathfrak{S}\left(\mathfrak{I}_{1_{\jmath_\nu}}, \mathfrak{I}_{1_{\imath_\nu}}\right) \geq \epsilon. \tag{17}$$

Suppose that i_ν is the least integer exceeding $\jmath_\nu$ satisfying inequality (17). Then,

$$\mathfrak{S}\left(\mathfrak{I}_{1_{\imath_\nu}}, \mathfrak{I}_{1_{\imath_\nu - 1}}\right) < \epsilon. \tag{18}$$

Using (17), (18), and the triangular inequality, we get

$$\begin{aligned} \epsilon &< \mathfrak{S}\left(\mathfrak{I}_{1_{\jmath_\nu}}, \mathfrak{I}_{1_{\imath_\nu}}\right) \\ &\leq \mathfrak{S}\left(\mathfrak{I}_{1_{\jmath_\nu}}, \mathfrak{I}_{1_{\imath_\nu - 2}}\right) + \mathfrak{S}\left(\mathfrak{I}_{1_{\imath_\nu - 2}}, \mathfrak{I}_{1_{\imath_\nu - 1}}\right) + \mathfrak{S}\left(\mathfrak{I}_{1_{\imath_\nu - 1}}, \mathfrak{I}_{1_{\imath_\nu}}\right) \\ &< \epsilon + \mathfrak{S}\left(\mathfrak{I}_{1_{\imath_\nu - 2}}, \mathfrak{I}_{1_{\imath_\nu - 1}}\right) + \mathfrak{S}\left(\mathfrak{I}_{1_{\imath_\nu - 1}}, \mathfrak{I}_{1_{\imath_\nu}}\right). \end{aligned} \tag{19}$$

As $\nu \longrightarrow \infty$,

$$\lim_{\nu \longrightarrow \infty} \mathfrak{S}\left(\mathfrak{I}_{1_{\jmath_\nu}}, \mathfrak{I}_{1_{\imath_\nu}}\right) = \epsilon. \tag{20}$$

Employing the triangular inequality once again, we get

$$\begin{aligned} \mathfrak{S}\left(\mathfrak{I}_{1_{\jmath_\nu}}, \mathfrak{I}_{1_{\imath_\nu}}\right) &\leq \mathfrak{S}\left(\mathfrak{I}_{1_{\jmath_\nu}}, \mathfrak{I}_{1_{\jmath_\nu - 1}}\right) + \mathfrak{S}\left(\mathfrak{I}_{1_{\jmath_\nu - 1}}, \mathfrak{I}_{1_{\imath_\nu - 1}}\right) \\ &\quad + \mathfrak{S}\left(\mathfrak{I}_{1_{\imath_\nu - 1}}, \mathfrak{I}_{1_{\imath_\nu}}\right) \\ &\leq 2\mathfrak{S}\left(\mathfrak{I}_{1_{\jmath_\nu}}, \mathfrak{I}_{1_{\jmath_\nu - 1}}\right) + \mathfrak{S}\left(\mathfrak{I}_{1_{\jmath_\nu}}, \mathfrak{I}_{1_{\imath_\nu}}\right) \\ &\quad + 2\mathfrak{S}\left(\mathfrak{I}_{1_{\imath_\nu - 1}}, \mathfrak{I}_{1_{\imath_\nu}}\right). \end{aligned} \tag{21}$$

On letting $\nu \longrightarrow \infty$ and using (15) as well as (20) we get

$$\lim_{\nu \longrightarrow \infty} \mathfrak{S}\left(\mathfrak{I}_{1_{\jmath_\nu - 1}}, \mathfrak{I}_{1_{\imath_\nu - 1}}\right) = \epsilon. \tag{22}$$

Now, using (6) and (ς_2^*), we obtain

$$\begin{aligned} 1 &\leq \varsigma\left[\ddot{\Theta}\left(\mathfrak{S}\left(\mathrm{H}\mathfrak{I}_{1_{\jmath_\nu - 1}}, \mathrm{H}\mathfrak{I}_{1_{\imath_\nu - 1}}\right)\right), \ddot{\Theta}\left(\mathfrak{S}\left(\mathfrak{I}_{1_{\jmath_\nu - 1}}, \mathfrak{I}_{1_{\imath_\nu - 1}}\right)\right)\right] \\ &= \left[\ddot{\Theta}\left(\mathfrak{S}\left(\mathfrak{I}_{1_{\jmath_\nu}}, \mathfrak{I}_{1_{\imath_\nu}}\right)\right), \ddot{\Theta}\left(\mathfrak{S}\left(\mathfrak{I}_{1_{\jmath_\nu - 1}}, \mathfrak{I}_{1_{\imath_\nu - 1}}\right)\right)\right] \\ &< \frac{\ddot{\Theta}\left(\mathfrak{S}\left(\mathfrak{I}_{1_{\jmath_\nu - 1}}, \mathfrak{I}_{1_{\imath_\nu - 1}}\right)\right)}{\ddot{\Theta}\left(\mathfrak{S}\left(\mathfrak{I}_{1_{\jmath_\nu}}, \mathfrak{I}_{1_{\imath_\nu}}\right)\right)}. \end{aligned} \tag{23}$$

Consequently, we deduce that

$$\ddot{\Theta}\left(\mathfrak{S}\left(\mathfrak{I}_{1_{\jmath_\nu}}, \mathfrak{I}_{1_{\imath_\nu}}\right)\right) < \ddot{\Theta}\left(\mathfrak{S}\left(\mathfrak{I}_{1_{\jmath_\nu - 1}}, \mathfrak{I}_{1_{\imath_\nu - 1}}\right)\right), \forall \nu \in \mathrm{N}. \tag{24}$$

Let $\rho_\nu = \ddot{\Theta}(\mathfrak{S}(\mathfrak{I}_{1_{\jmath_\nu}}, \mathfrak{I}_{1_{\imath_\nu}}))$ and $\rho_{1\nu} = \ddot{\Theta}(\mathfrak{S}(\mathfrak{I}_{1_{\jmath_\nu - 1}}, \mathfrak{I}_{1_{\imath_\nu - 1}}))$. Then, in view of Remark 6 and (24), we have $\lim_{\nu \longrightarrow \infty} \rho_\nu = \lim_{\nu \longrightarrow \infty} \rho_{1\nu} > 1$ and $\rho_\nu < \rho_{1\nu}, \forall \nu \in \mathrm{N}$. So, on using ς_3^*, we obtain

$$1 \leq \limsup_{\nu \longrightarrow \infty} \varsigma(\rho_\nu, \rho_{1\nu}) < 1, \tag{25}$$

which is a reductio ad absurdum. Therefore, $\{\mathfrak{J}_{1_\imath}\}$ must be a Cauchy sequence in $(\mathrm{P},\nabla,\mathfrak{S})$. Since $(\mathrm{P},\nabla,\mathfrak{S})$ is a complete, then there exists $\nu_1 \in \mathrm{P}$ such that $\lim_{\imath\longrightarrow\infty}\mathfrak{J}_{1_\imath} = \nu_1$, then,

$$\lim_{\imath\longrightarrow\infty}\mathfrak{S}(\mathfrak{J}_{1_\imath},\nu_1)=0. \tag{26}$$

As H is continuous, then we get that (due to (26))

$$\lim_{\imath\longrightarrow\infty}\mathfrak{S}(\mathfrak{J}_{1_{\imath+1}},\mathrm{H}\nu_1)=\lim_{\imath\longrightarrow\infty}\mathfrak{S}(\mathrm{H}\mathfrak{J}_{1_\imath},\mathrm{H}\nu_1)=0. \tag{27}$$

Using Lemma 7, we have $\nu_1 = \mathrm{H}\nu_1$ that is, ν_1 is a fixed point of H. On the contrary, assume that there are two fixed points such that $\mathfrak{S}(\nu_1,\mathfrak{J}_3)=\mathfrak{S}(\mathrm{H}\nu_1,\mathrm{H}\mathfrak{J}_3)>0$. From (6), since H is preserving, $\forall \mathrm{H}\nu_1\nabla\mathrm{H}\mathfrak{J}_3$ we have

$$\begin{aligned}1 &\le \varsigma\left[\ddot{\Theta}(\mathfrak{S}(\mathrm{H}\nu_1,\mathrm{H}\mathfrak{J}_3)),\ddot{\Theta}(\mathfrak{S}(\nu_1,\mathfrak{J}_3))\right]\\ &=\varsigma\left[\ddot{\Theta}(\mathfrak{S}(\nu_1,\mathfrak{J}_3)),\ddot{\Theta}(\mathfrak{S}(\nu_1,\mathfrak{J}_3))\right]\\ &<\frac{\ddot{\Theta}(\mathfrak{S}(\nu_1,\mathfrak{J}_3))}{\ddot{\Theta}(\mathfrak{S}(\nu_1,\mathfrak{J}_3))}.\end{aligned} \tag{28}$$

This is implies that

$$\ddot{\Theta}(\mathfrak{S}(\nu_1,\mathfrak{J}_3))<\ddot{\Theta}(\mathfrak{S}(\nu_1,\mathfrak{J}_3)), \tag{29}$$

which is a reductio ad absurdum. Then, H has a unique fixed point.

Example 4. Let $\mathrm{P}=\mathrm{E}\cup\mathrm{G}$, where $\mathrm{E}=[0,2]$ and $\mathrm{G}=\{(1/i): n=2,3,4,5\}$. Define the binary relation ∇ on P by $\mathfrak{J}_1\nabla\mathfrak{J}_2$ if $\mathfrak{J}_1,\mathfrak{J}_2\ge 0$. Define a mapping $\mathfrak{S}:\mathrm{P}\times\mathrm{P}\longrightarrow[0,\infty)$ defined by $\mathfrak{S}(\mathfrak{J}_1,\mathfrak{J}_2)=|\mathfrak{J}_1-\mathfrak{J}_2|$, for all $\mathfrak{J}_1,\mathfrak{J}_2\in\mathrm{P}$.

It is easy to see that $(\mathrm{P},\nabla,\mathfrak{S})$ is an orthogonal complete BMS. Let $\mathrm{H}:\mathrm{P}\longrightarrow\mathrm{P}$ be defined as $\mathrm{H}\mathfrak{J}_1=\mathfrak{J}_1/6$ for all $\mathfrak{J}_1\in\mathrm{P}$. Clearly, H is an orthogonal preserving and orthogonal continuous. Observe that H is an L-contraction with respect to $\varsigma:[1,\infty)\times[1,\infty)\longrightarrow\mathbb{R}$, where

$$\varsigma_\nu(\rho,\rho_1)=\frac{{\rho_1}^\nu}{\rho},\forall\rho,\rho_1\in[1,\infty),\nu\in(0,1), \tag{30}$$

and $\ddot{\Theta}:(0,\infty)\longrightarrow(1,\infty)$ such that $\ddot{\Theta}(\rho)=\mathrm{e}^\rho,\forall\rho>0$.

Let $\mathfrak{J}_1,\mathfrak{J}_2\in\mathrm{P}$; then,

$$\begin{aligned}&\varsigma\left[\ddot{\Theta}(\mathfrak{S}(\mathrm{H}\mathfrak{J}_1,\mathrm{H}\mathfrak{J}_2)),\ddot{\Theta}(\mathfrak{S}(\mathfrak{J}_1,\mathfrak{J}_2))\right]\\ &=\frac{\left[\ddot{\Theta}(\mathfrak{S}(\mathfrak{J}_1,\mathfrak{J}_2))\right]^\nu}{\ddot{\Theta}(\mathfrak{S}(\mathrm{H}\mathfrak{J}_1,\mathrm{H}\mathfrak{J}_2))}=\frac{\mathrm{e}^{\nu|\mathfrak{J}_1-\mathfrak{J}_2|}}{\mathrm{e}^{|\mathfrak{J}_1-\mathfrak{J}_2|/6}}\ge 1.\end{aligned} \tag{31}$$

Hence, all the hypotheses of Theorem 16 are satisfied, and $\mathfrak{J}_1=0$ is the unique fixed point of H.

4. Applications

As an application of Theorem 16, we find an existence and uniqueness of the solution of the following integral equation:

$$\mathfrak{J}_1(\rho)=\mathfrak{g}(\rho)+\int_0^{\mathfrak{a}}\jmath(\rho,\nu_1)\mathfrak{f}(\nu_1,\mathfrak{J}_1(\nu_1))d\nu_1,\quad \rho\in[0,\mathfrak{a}],\mathfrak{a}>0. \tag{32}$$

Let $\mathscr{U}=\mathscr{C}([0,\mathfrak{a}],R)$ be the set of real continuous functions defined on $[0,\mathfrak{a}]$ and the mapping $\mathrm{H}:\mathscr{U}\longrightarrow\mathscr{U}$ defined by

$$\mathrm{H}(\mathfrak{J}_1(\rho))=\mathfrak{g}(\rho)+\int_0^{\mathfrak{a}}\jmath(\rho,\nu_1)\mathfrak{f}(\nu_1,\mathfrak{J}_1(\nu_1))d\nu_1,\quad \rho\in[0,\mathfrak{a}]. \tag{33}$$

Obviously, $\mathfrak{J}_1(\rho)$ is a solution of integral Equation (32) iff $\mathfrak{J}_1(\rho)$ is a fixed point of H.

Theorem 17. *Suppose that*

(R1) The mappings $\jmath:[0,\mathfrak{a}]\times\mathbb{R}\longrightarrow[0,\infty)$, $\mathfrak{f}:[0,\mathfrak{a}]\times\mathbb{R}\longrightarrow\mathbb{R}$, and $\mathfrak{g}:[0,\mathfrak{a}]\longrightarrow\mathbb{R}$ are continuous functions;

(R2) there exist $\tau>0$ and $\nu\in(0,1)$ such that

$$|\mathfrak{f}(\nu_1,\mathfrak{J}_1(\nu_1))-\mathfrak{f}(\nu_1,\mathfrak{J}_2(\nu_1))|\le\nu|\mathfrak{J}_1(\nu_1)-\mathfrak{J}_2(\nu_1)|; \tag{34}$$

(R3) $\int_0^{\mathfrak{a}}\jmath(\rho,\nu_1)d\nu_1\le 1$.

Then, the integral Equation (32) has a unique solution in $\mathscr{U}$.

Proof. Define the orthogonality relation ∇ on $\mathscr{U}$ by

$$\begin{aligned}\mathfrak{J}_1\nabla\mathfrak{J}_2\Leftrightarrow\mathfrak{J}_1(\rho)\mathfrak{J}_2(\rho)\ge\mathfrak{J}_1(\rho)\text{ or }\mathfrak{J}_1(\rho)\mathfrak{J}_2(\rho)\\ \ge\mathfrak{J}_2(\rho),\forall\rho\in[0,\mathfrak{a}].\end{aligned} \tag{35}$$

Define $\mathfrak{S}:\mathscr{U}\times\mathscr{U}\longrightarrow[0,\infty)$ given by

$$\mathfrak{S}(\mathfrak{J}_1,\mathfrak{J}_2)=\sup_{\rho\in[0,\mathfrak{a}]}|\mathfrak{J}_1(\rho)-\mathfrak{J}_2(\rho)|, \tag{36}$$

for all $\mathfrak{J}_1,\mathfrak{J}_2\in\mathscr{U}$. It is easy to see that $(\mathscr{U},\nabla,\mathfrak{S})$ is complete orthogonal BMS. For each $\mathfrak{J}_1,\mathfrak{J}_2\in\mathscr{U}$ with $\mathfrak{J}_1\nabla\mathfrak{J}_2$ and $\rho\in[0,\mathfrak{a}]$, we have

$$\mathrm{H}(\mathfrak{J}_1(\rho))=\mathfrak{g}(\rho)+\int_0^{\mathfrak{a}}j(\rho,\nu_1)\mathfrak{f}(\nu_1,\mathfrak{J}_1(\nu_1))d\nu_1\ge 1. \tag{37}$$

Accordingly, $[(\mathrm{H}\mathfrak{J}_1)(\rho)][(\mathrm{H}\mathfrak{J}_2)(\rho)]\ge(\mathrm{H}\mathfrak{J}_2)(\rho)$ and so $(\mathrm{H}\mathfrak{J}_1)(\rho)\nabla(\mathrm{H}\mathfrak{J}_2)(\rho)$. Then, H is ∇-preserving. Let $\mathfrak{J}_1$,

$\mathfrak{I}_2 \in \mathscr{U}$ with $\mathfrak{I}_1 \nabla \mathfrak{I}_2$. Suppose that $\mathrm{H}(\mathfrak{I}_1) \neq \mathrm{H}(\mathfrak{I}_2)$. For each $\rho \in [0, \mathfrak{a}]$, we have

$$\begin{aligned} |\mathrm{H}\mathfrak{I}_1(\rho) - \mathrm{H}\mathfrak{I}_2(\rho)| &= \left| \int_0^{\mathfrak{a}} j(\rho, \nu_1)[\mathfrak{f}(\nu_1, \mathfrak{I}_1(\nu_1)) - \mathfrak{f}(\nu_1, \mathfrak{I}_2(\nu_1))]\mathfrak{d}\mathfrak{u} \right| \\ &\leq \int_0^{\mathfrak{a}} j(\rho, \nu_1)\nu|\mathfrak{I}_1(\nu_1) - \mathfrak{I}_2(\nu_1)|\mathfrak{d}\mathfrak{u} \\ &\leq \nu \int_0^{\mathfrak{a}} j(\rho, \nu_1 \mathfrak{d}\mathfrak{u}) \sup_{\mathfrak{u} \in [0,\mathfrak{a}]} |\mathfrak{I}_1(\mathfrak{u}) - \mathfrak{I}_2(\mathfrak{u})| \leq \nu \mathfrak{S}(\mathfrak{I}_1, \mathfrak{I}_2). \end{aligned} \tag{38}$$

Thus,

$$\mathfrak{S}(\mathrm{H}\mathfrak{I}_1, \mathrm{H}\mathfrak{I}_2) \leq \nu \mathfrak{S}(\mathfrak{I}_1, \mathfrak{I}_2), \tag{39}$$

which implies that

$$e^{\mathfrak{S}(\mathrm{H}\mathfrak{I}_1, \mathrm{H}\mathfrak{I}_2)} \leq e^{\nu \mathfrak{S}(\mathfrak{I}_1, \mathfrak{I}_2)}, \tag{40}$$

for each $\mathfrak{I}_1, \mathfrak{I}_2 \in \mathscr{U}$. We consider L-simulation mapping $\varsigma : [1,\infty) \times [1,\infty) \longrightarrow \mathbb{R}$, where

$$\varsigma_\nu(\rho, \rho_1) = \frac{{\rho_1}^\nu}{\rho}, \forall \rho, \rho_1 \in [1,\infty), \nu \in (0, 1), \tag{41}$$

and $\ddot{\Theta} : (0,\infty) \longrightarrow (1,\infty)$ such that $\ddot{\Theta}(\rho) = e^\rho, \forall \rho > 0$. Then,

$$\varsigma\left[\ddot{\Theta}(\mathfrak{S}(\mathrm{H}\mathfrak{I}_1, \mathrm{H}\mathfrak{I}_2)), \ddot{\Theta}(\mathfrak{S}(\mathfrak{I}_1, \mathfrak{I}_2))\right] \geq 1. \tag{42}$$

Hence, all the conditions of Theorem 16 are fulfilled. Therefore, the integral equation has a unique solution. □

5. Conclusion

In this article, we proved the fixed point theorems for orthogonal L-contraction mapping on orthogonal complete BMS.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors equally contributed to this work. All authors read and approved the final manuscript.

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