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Research Article

Fractional Versions of Hermite-Hadamard, Fejér, and Schur Type Inequalities for Strongly Nonconvex Functions

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In modern world, most of the optimization problems are nonconvex which are neither convex nor concave. The objective of this research is to study a class of nonconvex functions, namely, strongly nonconvex functions. We establish inequalities of Hermite-Hadamard and Fejér type for strongly nonconvex functions in generalized sense. Moreover, we establish some fractional integral inequalities for strongly nonconvex functions in generalized sense in the setting of Riemann-Liouville integral operators.

1. Introduction

The integral and differential operators have remarkable impact on applied sciences, and the interest of researchers is increasing day by day in this research area [1, 2]. Consider a convex function $f : I \subset R \longrightarrow R$ defined on the interval $I \subset \mathbb{R}$ with $a, b \in I$ being constants and $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1}$$

is Hermite-Hadamard's (see [3, 4]).

The notion of convexity is very old, and it appears in Archimedes treatment of orbit length. Nowadays, convex geometry is a mathematical subject in its own right. There are several modern works on convexity that are for the studies of real analysis, linear algebra, geometry, and functional analysis. The theory of convexity helps us to solve many applied problems. In recent years, the theory of convex analysis gains huge attention of researchers due to its interesting applications in optimizations, geometry, and engineering [5, 6].

The present paper deals with a new class of convex functions and establishes inequalities of Hermite-Hadamard and Fejér. Moreover, we develop some fractional integral inequalities. See [7, 8] for more general inequalities via convexity of functions.

The classical definition of convex functions was given in [3]. Another concept which is used widely in convex analysis is p-convex sets and p-convexity (see [4]). By taking $p = 1$ in the above definition, we get classical notion of convexity. After that, the strongly convex with modulus $\mu > 0$ was introduced in [9]. And in [10], the notion of the strongly p-convex function had been introduced. The notion of generalized convex functions had been introduced in [11, 12].

Motivated by the above researches, [13] introduced the following class of functions.

The function f is strongly nonconvex in generalized sense if

$$f(tx^p + (1-t)y^p)^{1/p} \le f(y) + t\eta(f(x), f(y)) - \mu t(1-t)(y^p - x^p)^2 \tag{2}$$

holds for $t \in [0, 1]$.

Definition 1 (see [13, 14]). Consider $f \in L[a, b]$, then the RHS and the LHS Riemann-Liouville fractional integral (RL) of order $\alpha > 0$ with $b > a > 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, x > a,$$
$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (t-x)^{\alpha-1} f(t)dt, x < b, \quad (3)$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined as

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1}. \quad (4)$$

It is to be noted that $J^0_{\alpha+}f(x) = J^0_{b-}f(x) = f(x)$.

The Riemann integral is reduced to classical integral for $\alpha = 1$ [15–18].

The definition of strong p-convexity was studied in [13]. The aim of this paper is to establish the inequalities of Schur, Fejér, and Hermite-Hadamard type for the strongly nonconvex functions via RL fractional integrals.

2. Inequality of Hermite-Hadamard Type

In order to prove the inequality of Hermite-Hadamard type, the following lemma is very important.

Lemma 2 (see [19]). *Let p be any nonzero real number and α be any positive constant. Further consider an integrable function $w : A \longrightarrow \mathbb{R}$, where $A = [a, b] \subset (0,\infty)$ which is p-symmetric w.r.t. $[a^p + b^p/2]^{1/p}$; then, we have the following:*

(i) If $p > 0$,

$$J^{\alpha}_{a^p+}(wog)(b^p) = J^{\alpha}_{b^p-}(wog)(a^p) = \frac{1}{2}\left[J^{\alpha}_{a^p+}(wog)(b^p) + J^{\alpha}_{b^p-}(wog)(a^p)\right], \quad (5)$$

with $g(x) = x^{1/p}$, $x \in [a^p, b^p]$

(ii) If $p < 0$,

$$J^{\alpha}_{b^p+}(wog)(a^p) = J^{\alpha}_{a^p-}(wog)(a^p) = \frac{1}{2}\left[J^{\alpha}_{b^p+}(wog)(b^p) + J^{\alpha}_{a^p-}(wog)(b^p)\right], \quad (6)$$

with $g(x) = x^{1/p}$, $x \in [b^p, a^p]$

Theorem 3. *Let the strongly generalized p-convex function $f : I \longrightarrow \mathbb{R}$ with magnitude $\mu > 0$ and $\eta(\cdot)$ be bounded above in $f(I) \times f(I)$ y $f \in L[a, b]$. Then, if p is any positive real number, we have*

$$f\left(\frac{a^p + b^p}{2}\right) - M_\eta + \frac{\mu(b^p - a^p)^2}{6}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+3)}$$
$$\leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b^p - a^p)^{\alpha}}\left[J^{\alpha}_{a^p+b^p/2+}fog(b^p) + J^{\alpha}_{a^p+b^p/2-}fog(a^p)\right]$$
$$\leq \frac{f(a)+f(b)}{2} + \frac{\alpha M_\eta}{2(\alpha+1)} - \mu\frac{\alpha(\alpha+3)(b^p - a^p)^2}{4(\alpha+1)(\alpha+2)}. \quad (7)$$

Proof. We begin the proof by inserting $x = (ta^p + (1-t)b^p)^{1/p}$ and $y = (tb^p + (1-t)a^p)^{1/p}$

$$f\left[\left(\frac{x^p + y^p}{2}\right)\right]^{1/p} - \frac{M_\eta}{2} - \frac{\mu(x^p - y^p)^2}{12} \leq \frac{f(x)+f(y)}{2} + \frac{M_\eta}{2} - \frac{\mu(x^p - y^p)^2}{6}. \quad (8)$$

Take $x = [(ta^p + (1-t)b^p)]^{1/p}$ and $y = [(tb^p + (1-t)a^p)]^{1/p}$, then (8) yields

$$f\left[\left(\frac{a^p + b^p}{2}\right)\right]^{1/p} - \frac{M_\eta}{2} - \frac{\mu(2t-1)^2(b^p - a^p)^2}{12}$$
$$\leq \frac{1}{2}\left[f\left[(ta^p + (1-t)b^p)\right]^{1/p}\right] + \frac{1}{2}\left[f\left[(tb^p + (1-t)a^p)\right]^{1/p}\right] + \frac{M_\eta}{2} - \frac{\mu(2t-1)^2(b^p - a^p)^2}{6}. \quad (9)$$

Multiplying (9) by $t^{\alpha-1}$ and then integrating w.r.t. t over the interval $[0, 1/2]$,

$$\int_0^{1/2} f\left[\left(\frac{a^p + b^p}{2}\right)\right]^{1/p} t^{\alpha-1}dt - \int_0^{1/2}\frac{M_\eta}{2}t^{\alpha-1}dt - \frac{\mu(b^p - a^p)^2}{12}\int_0^{1/2}(2t-1)^2 t^{\alpha-1}dt \leq \frac{1}{2}\int_0^{1/2} t^{\alpha-1} f(ta^p + (1-t)b^p)dt$$
$$+ \frac{1}{2}\int_0^{1/2} t^{\alpha-1} f(tb^p + (1-t)a^p)dt + \int_0^{1/2}\frac{M_\eta}{2}t^{\alpha-1}dt - \frac{\mu(a^p - b^p)^2}{6}\int_0^{1/2}(2t-1)^2 t^{\alpha-1}dt, \quad (10)$$

$$f\left[\left(\frac{a^p + b^p}{2}\right)\right]^{1/p} - \frac{M_\eta}{2} + \frac{\mu(b^p - a^p)^2\Gamma(\alpha+1)}{6\Gamma(\alpha+3)}$$
$$\leq \frac{\Gamma(\alpha+1)2^{\alpha-1}}{(b^p - a^p)^{\alpha}}\left[J^{\alpha}_{a^p+b^p/2+}fog(b^p) + J^{\alpha}_{a^p+b^p/2-}fog(a^p)\right], \quad (11)$$

which is the left side of Theorem 3

Now, to obtain the left-hand side of Theorem 3, we have for $x = [(ta^p + (1-t)b^p)]^{1/p}$,

$$f\left[\left(ta^p + (1-t)b^p\right)\right]^{1/p} \le f(b) + t\eta(f(a), f(b)) - \mu t(1-t)\left(b^p - a^p\right)^2, \tag{12}$$

and for $y = [(tb^p + (1-t)a^p)]^{1/p}$,

$$f\left[\left(tb^p + (1-t)a^p\right)\right]^{1/p} \le f(a) + t\eta(f(b), f(a)) - \mu t(1-t)\left(b^p - a^p\right)^2. \tag{13}$$

Combining (12) and (13), we have

$$f\left[\left(ta^p + (1-t)b^p\right)\right]^{1/p} + f\left[\left(tb^p + (1-t)a^p\right)\right]^{1/p} \le f(a) + t\eta(f(a), f(b)) + f(b) + t\eta(f(b), f(a)) - 2\mu t(1-t)\left(b^p - a^p\right)^2. \tag{14}$$

Multiplying (14) by $2t^{\alpha-1}$ and then integrating w.r.t. t over the interval $[0, 1/2]$, we have

$$2\int_0^{1/2}\left[f\left[\left(ta^p + (1-t)b^p\right)\right]^{1/p}t^{\alpha-1} + f\left[\left(tb^p + (1-t)a^p\right)\right]^{1/p}t^{\alpha-1}\right] dt \le 2\int_0^{1/2}(f(a)+f(b))t^{\alpha-1}dt + 4M_\eta\int_0^{1/2}t^\alpha dt - 4\mu\left(b^p - a^p\right)^2 \int_0^{1/2} t(1-t)t^{\alpha-1}dt, \tag{15}$$

$$\frac{\Gamma(\alpha+1)2^{\alpha-1}}{\left(b^p - a^p\right)^\alpha}\left[J^\alpha_{a^p+b^p/2+}fog\left(b^p\right) + J^\alpha_{a^p+b^p/2-}fog\left(a^p\right)\right] \le \frac{f(a)+f(b)}{2} + \frac{\alpha M_\eta}{2(\alpha+1)} - \frac{\mu\alpha(\alpha+3)\left(b^p - a^p\right)^2}{4(\alpha+1)(\alpha+2)}. \tag{16}$$

Together (11) and (16) give the required result. □

Remark 4.

(i) Fixing $p = 1$ in Theorem 3 gives Hermite-Hadamard inequality in the sense of the strongly generalized convexity

(ii) Fixing $p = 1$ and $\mu = 0$ in Theorem 3, we obtain [20] (Theorem 2.1)

(iii) Fixing $\eta(x, y) = x - y$ and $\mu = 0$ in Theorem 3 yields [21] (Theorem 2.1)

(iv) Applying both (ii) and (iii) on Theorem 3, we obtain classical fractional version of H-H inequality

Definition 5 (see [22]). Let p be any nonzero real number; then, the function $w : [a, b] \longrightarrow \mathbb{R}$ is p-symmetric w.r.t. $[(a^p + b^p/2)]^{1/p}$ if $w(x) = w[(a^p + b^p - x^p)]^{1/p}$ for all $x \in [a, b]$.

Theorem 6 (inequality of Fejér type). *Suppose that f is a function as in Theorem 3 and an integrable, nonnegative function $w : [a, b] \longrightarrow R$ is symmetric w.r.t. $[(a^p + b^p/2)]^{1/p}$, then*

$$\frac{\Gamma(\alpha)}{2}f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}\left[J^\alpha_{a^p+}wog\left(b^p\right) + J^\alpha_{b^p-}wog\left(a^p\right)\right] - \frac{M_\eta\Gamma(\alpha)}{2}\left[J^\alpha_{a^p+}wog\left(b^p\right) + J^\alpha_{b^p-}wog\left(a^p\right)\right] + \frac{\mu}{2}\int_{a^p}^{b^p}\left(2x - b^p - a^p\right)^2\left(b^p - x\right)^{\alpha-1}wog(x)dx$$
$$\le \frac{\Gamma(\alpha)}{2}\left[J^\alpha_{a^p+}fwog\left(b^p\right) + J^\alpha_{b^p-}fwog\left(a^p\right)\right]\frac{f(a)+f(b)}{2}\frac{\Gamma(\alpha)}{2}\left[J^\alpha_{a^p+}wog\left(b^p\right) + J^\alpha_{b^p-}wog\left(a^p\right)\right] + \frac{M_\eta}{\left(b^p - a^p\right)}\int_{a^p}^{b^p}\left(b^p - x\right)^\alpha wog(x)dx - \mu\int_{a^p}^{b^p}\left(b^p - x\right)^2\left(x - a^p\right)wog(x)dx. \tag{17}$$

Proof. Setting $t = 1/2$ in (2),

$$f\left[\left(\frac{x^p+y^p}{2}\right)\right]^{1/p} \le f(y) + \frac{1}{2}\eta(f(x), f(y)) - \frac{\mu}{4}\left(y^p - x^p\right)^2. \tag{18}$$

Substitute $y = [(ta^p + (1-t)b^p)]^{1/p}$ and $x = [(tb^p + (1-t)a^p)]^{1/p}$ in (18),

$$f\left[\left(\frac{x^p+y^p}{2}\right)\right]^{1/p} \le f\left[\left(ta^p + (1-t)b^p\right)\right]^{1/p} + \frac{M_\eta}{2} - \frac{\mu}{4}(2t-1)^2\left(b^p - a^p\right)^2. \tag{19}$$

According to the given conditions of w, we have

$$w(x) = w\left[\left(a^p + b^p - x^p\right)\right]^{1/p}, \quad w\left[\left(ta^p + (1-t)b^p\right)\right]^{1/p} = w\left[\left(tb^p + (1-t)a^p\right)\right]^{1/p} \tag{20}$$

$\forall x, y \in [a, b]$. Multiplying (19) by $2t^{\alpha-1}w[(tb^p + (1-t)a^p)]^{1/p}$ and then integrating w.r.t. t over the interval $[0, 1]$,

$$\int_0^1 2t^{\alpha-1}w\left[\left(tb^p + (1-t)a^p\right)\right]^{1/p} \times f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p} \le \int_0^1 2t^{\alpha-1}w\left[\left(tb^p + (1-t)a^p\right)\right]^{1/p} \times f\left[\left(ta^p + (1-t)b^p\right)\right]^{1/p}dt + \frac{M_\eta}{2}\int_0^1 2t^{\alpha-1}w\left[\left(tb^p + (1-t)a^p\right)\right]^{1/p}dt$$

$$-\frac{\mu}{4}\int_0^1 2t^{\alpha-1}(2t-1)^2\left(b^p-a^p\right)^2\times w\left[\left(tb^p+(1-t)a^p\right)\right]^{1/p}dt, \tag{21}$$

$$\begin{aligned}
&f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}\int_{b^p}^{a^p}2\left(\frac{x-b^p}{a^p-a^p}\right)^{\alpha-1}w\left(x^{1/p}\right)\frac{dx}{\left(a^p-b^p\right)}\\
&\le\int_{b^p}^{a^p}2f\left(x^{1/p}\right)w\left(x^{1/p}\right)\left(\frac{x-b^p}{a^p-a^p}\right)^{\alpha-1}\frac{dx}{\left(a^p-b^p\right)}\\
&\quad+\int_{b^p}^{a^p}M_\eta\left(\frac{x-b^p}{a^p-a^p}\right)^{\alpha-1}w\left(x^{1/p}\right)\frac{dx}{\left(a^p-b^p\right)}\\
&\quad-\frac{\mu}{2}\int_{b^p}^{a^p}\left(2x-b^p-a^p\right)^2\left(\frac{x-b^p}{a^p-a^p}\right)^{\alpha-1}\times w\left(x^{1/p}\right)\frac{dx}{\left(a^p-b^p\right)},
\end{aligned} \tag{22}$$

$$\begin{aligned}
&f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}\int_{a^p}^{b^p}2\left(b^p-x\right)^{\alpha-1}w\left(x^{1/p}\right)\frac{dx}{\left(b^p-a^p\right)^\alpha}\\
&\le\int_{a^p}^{b^p}2f\left(x^{1/p}\right)w\left(x^{1/p}\right)\left(b^p-x\right)^{\alpha-1}\frac{dx}{\left(b^p-a^p\right)^\alpha}\\
&\quad+M_\eta\int_{a^p}^{b^p}\left(b^p-x\right)^{\alpha-1}w\left(x^{1/p}\right)\frac{dx}{\left(b^p-a^p\right)^\alpha}\\
&\quad-\frac{\mu}{2}\int_{b^p}^{a^p}\left(2x-b^p-a^p\right)^2\left(b^p-x\right)^{\alpha-1}\times w\left(x^{1/p}\right)\frac{dx}{\left(b^p-a^p\right)^\alpha}.
\end{aligned} \tag{23}$$

Let $g(x)=x^{1/p}$, then (23) becomes

$$\begin{aligned}
&f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}\frac{1}{\left(b^p-a^p\right)^\alpha}\int_{a^p}^{b^p}\left(b^p-x\right)^{\alpha-1}wog(x)dx\frac{2}{\left(b^p-a^p\right)^\alpha}\\
&\int_{a^p}^{b^p}fwog(x)\left(b^p-x\right)^{\alpha-1}dx+\frac{M_\eta}{\left(b^p-a^p\right)^\alpha}\int_{a^p}^{b^p}\left(b^p-x\right)^{\alpha-1}wog(x)dx\\
&\quad-\frac{\mu}{2\left(b^p-a^p\right)^\alpha}\int_{a^p}^{b^p}\left(2x-b^p-a^p\right)^2\times x\left(b^p-\right)^{\alpha-1}wog(x)dx,
\end{aligned} \tag{24}$$

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{2}f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}\left[J^\alpha_{a^p+}wog\left(b^p\right)+J^\alpha_{b^p-}wog\left(a^p\right)\right]\\
&-\frac{M_\eta}{2}\Gamma(\alpha)\left[J^\alpha_{a^p+}wog\left(b^p\right)+J^\alpha_{b^p-}wog\left(a^p\right)\right]\\
&+\frac{\mu}{2}\int_{a^p}^{b^p}\left(2x-b^p-a^p\right)^2\left(b^p-x\right)^{\alpha-1}wog(x)dx\\
&\le\frac{\Gamma(\alpha)}{2}\left[J^\alpha_{a^p+}fwog\left(b^p\right)+J^\alpha_{b^p-}fwog\left(a^p\right)\right].
\end{aligned} \tag{25}$$

Now, take $x=\left(ta^p+(1-t)b^p\right)\forall t\in[0,1]$, then by Def of f,

$$f\left[\left(ta^p+(1-t)b^p\right)\right]^{1/p}\le f(b)+t\eta(f(a),f(b))-\mu t(1-t)\left(b^p-a^p\right)^2. \tag{26}$$

Multiply on both sides of (26) by $2t^{\alpha-1}w\left[\left(tb^p+(1-t)a^p\right)\right]^{1/p}$ and then integrate w.r.t. t over the interval $[0,1]$,

$$\begin{aligned}
&\int_0^1 2f\left[\left(ta^p+(1-t)b^p\right)\right]^{1/p}t^{\alpha-1}w\left[\left(tb^p+(1-t)a^p\right)\right]^{1/p}dt\\
&\le\int_0^1 2t^{\alpha-1}w\left[\left(tb^p+(1-t)a^p\right)\right]^{1/p}f(b)dt\\
&\quad+\int_0^1 2t^\alpha M_\eta w\left[\left(tb^p+(1-t)a^p\right)\right]^{1/p}dt-2\mu\int_0^1 2t^\alpha(1-t)\left(b^p-a^p\right)^2\\
&\quad\times w\left[\left(tb^p+(1-t)a^p\right)\right]^{1/p}dt,
\end{aligned} \tag{27}$$

$$\begin{aligned}
&\int_{b^p}^{a^p}f\left(x^{1/p}\right)w\left(x^{1/p}\right)\left(\frac{x-b^p}{a^p-b^p}\right)^{\alpha-1}\frac{dx}{\left(a^p-b^p\right)}\\
&\le\int_{b^p}^{a^p}f(b)\left(\frac{x-b^p}{a^p-b^p}\right)^{\alpha-1}w\left(x^{1/p}\right)\frac{dx}{\left(a^p-b^p\right)}\\
&\quad+M_\eta\int_{b^p}^{a^p}\left(\frac{x-b^p}{a^p-b^p}\right)^{\alpha-1}w\left(x^{1/p}\right)\frac{dx}{\left(a^p-b^p\right)}\\
&\quad-\mu\int_{b^p}^{a^p}\left(b^p-x\right)^\alpha\left(x-a^p\right)w\left(x^{1/p}\frac{dx}{\left(b^p-a^p\right)^\alpha}\right).
\end{aligned} \tag{28}$$

Take $g(x)=x^{1/p}$ in (28), then we have

$$\begin{aligned}
&\int_{a^p}^{b^p}\left(b^p-x\right)^{\alpha-1}wfog(x)dx\le\int_{a^p}^{b^p}\left(b^p-x\right)^{\alpha-1}wog(x)f(b)dx\\
&+\frac{M_\eta}{\left(b^p-a^p\right)}\int_{a^p}^{b^p}\left(b^p\left(b^p-x\right)^\alpha wog(x)dx\right.\\
&-\mu\int_{a^p}^{b^p}\left(b^p\left(b^p-x\right)^\alpha\left(x-a^p\right)wog(x)dx.\right.
\end{aligned} \tag{29}$$

Similarly, we have

$$\begin{aligned}
&\int_{a^p}^{b^p}\left(b^p-x\right)^{\alpha-1}wfog(x)dx\le\int_{a^p}^{b^p}\left(b^p-x\right)^{\alpha-1}wog(x)f(a)dx\\
&+\frac{M_\eta}{\left(b^p-a^p\right)}\int_{a^p}^{b^p}\left(b^p\left(b^p-x\right)^\alpha wog(x)dx\right.\\
&-\mu\int_{a^p}^{b^p}\left(b^p\left(b^p-x\right)^\alpha\left(x-a^p\right)wog(x)dx,\right.
\end{aligned} \tag{30}$$

from definition of f by fixing $x=tb^p+(1-t)a^p$. Combining (29) and (30), we obtain

$$\begin{aligned}
&\int_{a^p}^{b^p}\left(b^p-x\right)^{\alpha-1}wfog(x)dx\le\frac{f(a)+f(b)}{2}\int_{a^p}^{b^p}\left(b^p-x\right)^{\alpha-1}wog(x)dx\\
&+\frac{M_\eta}{\left(b^p-a^p\right)}\int_{a^p}^{b^p}\left(b^p\left(b^p-x\right)^\alpha wog(x)dx-\mu\int_{a^p}^{b^p}\left(b^p-x\right)^\alpha\left(x-a^p\right)wog(x)dx,\right.
\end{aligned} \tag{31}$$

$$\frac{\Gamma(\alpha)}{2}\left[J^{\alpha}_{a^p+}fwog(b^p)+J^{\alpha}_{b^p-}fwog(a^p)\right]$$
$$\leq\frac{f(a)+f(b)}{2}\frac{\Gamma(\alpha)}{2}\left[J^{\alpha}_{a^p+}wog(b^p)+J^{\alpha}_{b^p-}wog(a^p)\right]$$
$$+\frac{M_\eta}{(b^p-a^p)}\int_{a^p}^{b^p}\left(b^p(b^p-x)^{\alpha}wog(x)dx\right.$$
$$-\mu\int_{a^p}^{b^p}(b^p-x)^{\alpha}(x-a^p)wog(x)dx. \tag{32}$$

Combining (32) and (25) completes the theorem (17). □

3. Fractional Integral Inequalities for Strongly Generalized p-Convex Function

Lemma 7. *Consider a differentiable function $f: I\subset(0,\infty)\longrightarrow R$ on I^o, with $f'\in L[a,b]$, where a, $b\in I$ and $a<b$. If $w:[a,b]\longrightarrow R$ is integrable, then*

$$f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p-a^p)^{\alpha}}\left[J^{\alpha}_{a^p+b^p/2+}fog(b^p)+J^{\alpha}_{a^p+b^p/2-}fog(a^p)\right]$$
$$-\frac{M_\eta}{2}+\frac{\mu(b^p-a^p)^2\Gamma(\alpha+1)}{2\Gamma(\alpha+3)}$$
$$=\frac{1}{2^{1-\alpha}(b^p-a^p)^{\alpha}}\left[\int_{a^p}^{a^p+b^p/2}(t-a^p)^{\alpha}(fog)'(t)-\int_{a^p+b^p/2}^{b^p}(b^p-t)^{\alpha}(fog)'(t)\right]dt$$
$$-\frac{M_\eta}{2}+\frac{\mu(b^p-a^p)^2\Gamma(\alpha+1)}{2\Gamma(\alpha+3)} \tag{33}$$

holds with $g(x)=x^{1/p}$.

Proof. Let $p>0$, and $x\in[a^p,b^p]$, then for generalized strongly p-convex function, we have

$$K=\frac{1}{2^{1-\alpha}(b^p-a^p)^{\alpha}}\int_{a^p}^{a^p+b^p/2}(t-a^p)^{\alpha}(fog)'(t)dt$$
$$-\frac{1}{2^{1-\alpha}(b^p-a^p)^{\alpha}}\int_{a^p+b^p/2}^{b^p}(b^p-t)^{\alpha}(fog)'(t)dt-\frac{M_\eta}{2}$$
$$+\frac{\mu(b^p-a^p)^2\Gamma(\alpha+1)}{2\Gamma(\alpha+3)}, \tag{34}$$

$$K=K_1-K_2-\frac{M_\eta}{2}+\frac{\mu(b^p-a^p)^2\Gamma(\alpha+!)}{2\Gamma(\alpha+3)}, \tag{35}$$

where

$$K_1=\frac{1}{2^{1-\alpha}(b^p-a^p)^{\alpha}}\int_{a^p}^{a^p+b^p/2}(t-a^p)^{\alpha}(fog)'(t)dt,$$
$$K_2=\frac{1}{2^{1-\alpha}(b^p-a^p)^{\alpha}}\int_{a^p+b^p/2}^{b^p}(b^p-t)^{\alpha}(fog)'(t)\Big]dt. \tag{36}$$

By integration by parts, we have

$$K_1=\frac{1}{2^{1-\alpha}(b^p-a^p)^{\alpha}}\left(\frac{b^p-a^p}{2}\right)^{\alpha}f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}$$
$$-\frac{\alpha}{2^{1-\alpha}(b^p-a^p)^{\alpha}}\int_{a^p}^{a^p+b^p/2}(t-a^p)^{\alpha-1}fog(t)dt$$
$$=\frac{1}{2}f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p-a^p)^{\alpha}\Gamma(\alpha)}$$
$$\times\int_{a^p}^{a^p+b^p/2}(t-a^p)^{\alpha-1}fog(t)dt, \tag{37}$$

$$K_2=-\frac{1}{2}f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}+\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p-a^p)^{\alpha}}\times J^{\alpha}_{a^p+b^p/2+}fog(b^p). \tag{38}$$

By combining (34), (37), and (38), we have (33). This completes the proof. □

Remark 8. Setting $\mu=0$ and $\eta=x-y$ in Lemma 7 gives us [21] (Lemma 2.1).

Theorem 9. *Let the function f be as in Theorem 3.1. If $|f'|$ is a strongly generalized p-convex function on $[a,b]$ for positive p and α, then*

$$\left|f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)-\frac{\Gamma(\alpha+1)}{2^{\alpha-1}(b^p-a^p)^{\alpha}}\left[J^{\alpha}_{a^p+b^p/2+}fog(b^p)+J^{\alpha}_{a^p+b^p/2-}fog(a^p)\right]\right.$$
$$\left.-\frac{M_\eta}{2}+\frac{\mu(b^p-a^p)^2\Gamma(\alpha+1)}{2\Gamma(\alpha+3)}\right|\leq\frac{b^p-a^p}{2^{1-\alpha}}\left[C_1(\alpha,p)|f'(b)|\right.$$
$$\left.+C_2(\alpha,p)\eta\left(|f'(b)|,|f'(a)|\right)-C_3(\alpha,p)\mu(b^p-a^p)^2\right]-\frac{M_\eta}{2}+\frac{\mu(b^p-a^p)^2\Gamma(\alpha+1)}{\Gamma(\alpha+3)}, \tag{39}$$

where

$$C_1(\alpha,p)=\int_0^{1/2}\frac{u^{\alpha}}{p(ua^p+(1-u)b^p)^{1-1/p}}du+\int_{1/2}^{1}\frac{(1-u)^{\alpha}}{p(ua^p+(1-u)b^p)^{1-1/p}}du,$$
$$C_2(\alpha,p)=\int_0^{1/2}\frac{u^{\alpha+1}}{p(ua^p+(1-u)b^p)^{1-1/p}}du+\int_{1/2}^{1}\frac{(1-u)^{\alpha}u}{p(ua^p+(1-u)b^p)^{1-1/p}}du,$$
$$C_3(\alpha,p)=\int_0^{1/2}\frac{u^{\alpha+1}(1-u)(b^p-a^p)^2}{p(ua^p+(1-u)b^p)^{1-1/p}}du+\int_{1/2}^{1}\frac{u^{\alpha+1}(1-u)(b^p-a^p)^2}{p(ua^p+(1-u)b^p)^{1-1/p}}du. \tag{40}$$

Proof. Theorem (3) gives

$$\begin{aligned}&\left|f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}-\frac{M_\eta}{2}+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+1)}{2\Gamma(\alpha+3)}\right.\\&\left.-\frac{\Gamma(\alpha+1)}{2^{\alpha-1}\left(b^p-a^p\right)^\alpha}\left[J^\alpha_{a^p+b^p/2+}fog\left(b^p\right)+J^\alpha_{a^p+b^p/2-}fog\left(a^p\right)\right]\right|\\&\leq\frac{1}{2^{1-\alpha}\left(b^p-a^p\right)^\alpha}\int_{a^p}^{a^p+b^p/2}\left(t-a^p\right)^\alpha f'\left(t^{1/p}\right)\frac{1}{t^{1-1/p}}dt\\&+\frac{1}{2^{1-\alpha}\left(b^p-a^p\right)^\alpha}\int_{a^p+b^p/2}^{b^p}\left(b^p-t\right)^\alpha f'\left(t^{1/p}\right)\frac{1}{t^{1-1/p}}dt-\frac{M_\eta}{2}\\&+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+1)}{2\Gamma(\alpha+3)}\leq\frac{b^p-a^p}{2^{1-\alpha}}\int_{a^p}^{a^p+b^p/2}\left(t-a^p\right)^\alpha f'\left(t^{1/p}\right)\frac{1}{t^{1-1/p}}dt\\&+\frac{b^p-a^p}{2^{1-\alpha}}\int_{a^p+b^p/2}^{b^p}\left(b^p-t\right)^\alpha f'\left(t^{1/p}\right)\frac{1}{t^{1-1/p}}dt-\frac{M_\eta}{2}+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+1)}{2\Gamma(\alpha+3)}.\end{aligned}\tag{41}$$

Setting $t=ua^p+(1-u)b^p$, $dt=(a^p-b^p)du$, we have

$$\begin{aligned}&\left|f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}-\frac{\Gamma(\alpha+1)}{2^{\alpha-1}\left(b^p-a^p\right)^\alpha}\left[J^\alpha_{a^p+b^p/2+}fog\left(b^p\right)\right.\right.\\&\left.\left.+J^\alpha_{a^p+b^p/2-}fog\left(a^p\right)\right]-\frac{M_\eta}{2}+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+1)}{\Gamma(\alpha+3)}\right|\\&\leq\frac{\left(b^p-a^p\right)}{2^{1-\alpha}}\int_0^{1/2}\frac{u^\alpha}{p\left(ua^p+(1-u)b^p\right)^{1-1/p}}\left|f'\left(ua^p+(1-u)b^p\right)^{1/p}\right|du\\&+\frac{\left(b^p-a^p\right)}{2^{1-\alpha}}\int_{1/2}^1\frac{(1-u)^\alpha}{p\left(ua^p+(1-u)b^p\right)^{1-1/p}}\left|f'\left(ua^p+(1-u)b^p\right)^{1/p}\right|du\\&-\frac{M_\eta}{2}+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+1)}{\Gamma(\alpha+3)}.\end{aligned}\tag{42}$$

Since $|f'|$ is a strongly generalized p-convex function on $[a,b]$, we have

$$\left|f'\left(ua^p+(1-u)b^p\right)^{1/p}\right|\leq\left|f'(b)\right|+\mu\eta\left(\left|f'(a)\right|,\left|f'(b)\right|\right)-\mu(1-u)\left(b^p-a^p\right)^2.\tag{43}$$

After combining (48) and (43), we have

$$\begin{aligned}&\left|f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}-\frac{\Gamma(\alpha+1)}{2^{\alpha-1}\left(b^p-a^p\right)^\alpha}\left[J^\alpha_{a^p+b^p/2+}fog\left(b^p\right)\right.\right.\\&\left.\left.+J^\alpha_{a^p+b^p/2-}fog\left(a^p\right)\right]-\frac{M_\eta}{2}+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+1)}{\Gamma(\alpha+3)}\right|\\&\leq\frac{b^p-a^p}{2^{1-\alpha}}\int_0^{1/2}\frac{u^\alpha}{p\left(ua^p+(1-u)b^p\right)^{1-1/p}}\left(\left|f'(b)\right|+u\eta\left(\left|f'(a)\right|,\left|f'(b)\right|\right)\right.\\&\left.-\mu u(1-u)\left(b^p-a^p\right)^2\right)du+\frac{b^p-a^p}{2^{1-\alpha}}\int\frac{1}{2}^1\frac{(1-u)^\alpha}{p\left(ua^p+(1-u)b^p\right)^{1-1/p}}\\&\left(\left|f'(b)\right|+u\eta\left(\left|f'(a)\right|,\left|f'(b)\right|\right)-\mu u(1-u)\left(b^p-a^p\right)^2\right)du\\&-\frac{M_\eta}{2}+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+1)}{\Gamma(\alpha+3)},\left|f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}\right.\\&-\frac{\Gamma(\alpha+1)}{2^{\alpha-1}\left(b^p-a^p\right)^\alpha}\left[J^\alpha_{a^p+b^p/2+}fog\left(b^p\right)+J^\alpha_{a^p+b^p/2-}fog\left(a^p\right)\right]-\frac{M_\eta}{2}\\&\left.+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+1)}{\Gamma(\alpha+3)}\right|=\frac{b^p-a^p}{2^{1-\alpha}}\int_0^{1/2}\frac{u^\alpha}{p\left(ua^p+(1-u)b^p\right)^{1-1/p}}\left|f'(b)\right|\\&+\frac{b^p-a^p}{2^{1-\alpha}}\int_{1/2}^1\frac{(1-u)^\alpha}{p\left(ua^p+(1-u)b^p\right)^{1-1/p}}\left|f'(b)\right|du\\&+\frac{b^p-a^p}{2^{1-\alpha}}\int_0^{1/2}\frac{u^{\alpha+1}}{p\left(ua^p+(1-u)b^p\right)^{1-1/p}}\eta\left(\left|f'(a)\right|,\left|f'(b)\right|\right)\\&+\frac{b^p-a^p}{2^{1-\alpha}}\int_{1/2}^1\frac{(1-u)^\alpha u}{p\left(ua^p+(1-u)b^p\right)^{1-1/p}}\eta\left(\left|f'(a)\right|,\left|f'(b)\right|\right)du-\frac{M_\eta}{2}\\&+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+1)}{\Gamma(\alpha+3)}+\frac{b^p-a^p}{2^{1-\alpha}}\int_0^{1/2}\frac{u^{\alpha+1}(1-u)\left(b^p-a^p\right)^2\mu}{p\left(ua^p+(1-u)b^p\right)^{1-1/p}}\\&+\frac{b^p-a^p}{2^{1-\alpha}}\int_{1/2}^1\frac{u(1-u)^{\alpha+1}\mu}{p\left(ua^p+(1-u)b^p\right)^{1-1/p}},\left|f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\right.\\&-\frac{\Gamma(\alpha+1)}{2^{\alpha-1}\left(b^p-a^p\right)^\alpha}\left[J^\alpha_{a^p+b^p/2+}fog\left(b^p\right)+J^\alpha_{a^p+b^p/2-}fog\left(a^p\right)\right]-\frac{M_\eta}{2}\\&\left.+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+1)}{\Gamma(\alpha+3)}\right|\leq\frac{b^p-a^p}{2^{1-\alpha}}\left[C_1(\alpha,p)\left|f'(b)\right|\right.\\&\left.+C_2(\alpha,p)\eta\left(\left|f'(b)\right|,\left|f'(a)\right|\right)-C_3(\alpha,p)\mu\left(b^p-a^p\right)^2\right]-\frac{M_\eta}{2}\\&+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+1)}{\Gamma(\alpha+3)}.\end{aligned}\tag{44}$$

□

Remark 10. If one takes $\eta=x-y$ and $\mu=0$, then we get [21] (Theorem 2.2).

Theorem 11. *Let the function f be as in Theorem 3.1. If $|f'|$ is as in Theorem 9, then*

$$\begin{aligned}&\left|f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}-\frac{\Gamma(\alpha+1)}{2^{\alpha-1}\left(b^p-a^p\right)^\alpha}\left[J^\alpha_{a^p+b^p/2+}fog\left(b^p\right)+J^\alpha_{a^p+b^p/2-}fog\left(a^p\right)\right]\right.\\&\left.-M_\eta+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+1)}{6\Gamma(\alpha+3)}\right|\leq\left|\left(C_5(\alpha,p)\right)^{1-1/q}\left[C_5(\alpha,p)\left|f'(b)\right|\right.\right.\\&+C_6(\alpha,p)M_\eta-C_7(\alpha,p)\mu\left(b^p-a^p\right)^2+\left(C_8(\alpha,p)\right)^{1-1/q}\left[C_8\left|f'(b)\right|\right.\\&\left.\left.\left.+C_9M_\eta+C_{10}\mu\left(b^p-a^p\right)^2\right]\right]-M_\eta+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+1)}{6\Gamma(\alpha+3)}\right|,\end{aligned}\tag{45}$$

where

$$
\begin{aligned}
C_5(\alpha,p)&=\int_0^{1/2}\frac{u^{\alpha}}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}du,\\
C_6(\alpha,p)&=\int_0^{1/2}\frac{u^{\alpha+1}}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}du,\\
C_7(\alpha,p)&=\int_0^{1/2}\frac{u^{\alpha}(1-u)}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}du,\\
C_8(\alpha,p)&=\int_{1/2}^{1}\frac{(1-u)^{\alpha}}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}du,\\
C_9(\alpha,p)&=\int_{1/2}^{1}\frac{u(1-u)^{\alpha}}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}du,\\
C_{10}(\alpha,p)&=\int_{1/2}^{1}\frac{u(1-u)^{\alpha+1}}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}du.
\end{aligned}
\tag{46}
$$

Proof. Let $p>0$:

$$
\begin{aligned}
&\left|f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}-M_\eta+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+1)}{6\Gamma(\alpha+3)}\right.\\
&\left.-\frac{\Gamma(\alpha+1)}{2^{\alpha-1}\left(b^p-a^p\right)^{\alpha}}\left[J^{\alpha}_{a^p+b^p/2+}fog\left(b^p\right)+J^{\alpha}_{a^p+b^p/2-}fog\left(a^p\right)\right]\right|\\
&\le\frac{1}{2^{1-\alpha}\left(b^p-a^p\right)^{\alpha}}\int_{a^p}^{a^p+b^p/2}\left(t-a^p\right)^{\alpha}f'\left(t^{1/p}\right)\frac{1}{t^{1-1/p}}dt\\
&+\frac{1}{2^{1-\alpha}\left(b^p-a^p\right)^{\alpha}}\int_{a^p+b^p/2}^{b^p}\left(b^p-t\right)^{\alpha}f'\left(t^{1/p}\right)\frac{1}{t^{1-1/p}}dt-M_\eta\\
&+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+1)}{6\Gamma(\alpha+3)}\le\frac{b^p-a^p}{2^{1-\alpha}}\int_{a^p}^{a^p+b^p/2}\left(t-a^p\right)^{\alpha}f'\left(t^{1/p}\right)\frac{1}{t^{1-1/p}}dt\\
&+\frac{b^p-a^p}{2^{1-\alpha}}\int_{a^p+b^p/2}^{b^p}\left(b^p-t\right)^{\alpha}f'\left(t^{1/p}\right)\frac{1}{t^{1-1/p}}dt-M_\eta+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+1)}{6\Gamma(\alpha+3)}.
\end{aligned}
\tag{47}
$$

Setting $t=ua^p+(1-u)b^p$, $dt=(a^p-b^p)du$, we have

$$
\begin{aligned}
&\left|f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}-\frac{\Gamma(\alpha+1)}{2^{\alpha-1}\left(b^p-a^p\right)^{\alpha}}\left[J^{\alpha}_{a^p+b^p/2+}fog\left(b^p\right)\right.\right.\\
&\left.\left.+J^{\alpha}_{a^p+b^p/2-}fog\left(a^p\right)\right]-M_\eta+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+1)}{6\Gamma(\alpha+3)}\right|\\
&\le\frac{\left(b^p-a^p\right)}{2^{1-\alpha}}\int_0^{1/2}\frac{u^{\alpha}}{p\left[\left(ua^p+(1-u)b^p\right)\right]^{1-1/p}}\left|f'\left[\left(ua^p+(1-u)b^p\right)\right]^{1/p}\right|du\\
&+\frac{\left(b^p-a^p\right)}{2^{1-\alpha}}\times\int_{1/2}^{1}\frac{(1-u)^{\alpha}}{p\left[\left(ua^p+(1-u)b^p\right)\right]^{1-1/p}}\left|f'\left[\left(ua^p+(1-u)b^p\right)\right]^{1/p}\right|du\\
&-M_\eta+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+1)}{6\Gamma(\alpha+3)}.
\end{aligned}
\tag{48}
$$

Using the inequality of power mean the definition of $|f'|^q$,

$$
\begin{aligned}
&\left|f\left[\left(\frac{a^p+b^p}{2}\right)\right]^{1/p}-\frac{\Gamma(\alpha+1)}{2^{\alpha-1}\left(b^p-a^p\right)^{\alpha}}\left[J^{\alpha}_{a^p+b^p/2+}fog\left(b^p\right)+J^{\alpha}_{a^p+b^p/2-}fog\left(a^p\right)\right]\right.\\
&\left.-M_\eta+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+1)}{6\Gamma(\alpha+3)}\right|\le\frac{b^p-a^p}{2^{1-\alpha}}\left[\left(\int_0^{1/2}\frac{u^{\alpha}}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}\right)^{1-1/q}\right]\\
&\times\left[\left(\int_0^{1/2}\frac{u^{\alpha}}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}\left|f'\left(\left[ua^p+(1-u)b^p\right]^{1/p}\right)\right|^q du\right)^{1/q}\right]\\
&+\frac{b^p-a^p}{2^{1-\alpha}}\left[\left(\int_{1/2}^{1}\frac{(1-u)^{\alpha}}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}\right)^{1-1/q}\right]\\
&\times\left[\left(\int_{1/2}^{1}\frac{(1-u)^{\alpha}}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}\left|f'\left(\left[ua^p+(1-u)b^p\right]^{1/p}\right)\right|^q du\right)^{1/q}\right]\\
&\le\left[\frac{b^p-a^p}{2^{1-\alpha}}\left(\int_0^{1/2}\frac{u^{\alpha}}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}\right)^{1-1/q}\right]\\
&\times\left[\left(\int_0^{1/2}\frac{u^{\alpha}}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}du\right)\left|f'(b)\right|\right.\\
&\left.+\int_0^{1/2}\frac{u^{\alpha+1}M_\eta}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}du\right]\\
&+\left[\frac{b^p-a^p}{2^{1-\alpha}}\left(\int_0^{1/2}\frac{u^{\alpha}}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}\right)^{1-1/q}\right]\\
&\times\left[\int_0^{1/2}\frac{u^{\alpha+1}(1-u)\left(b^p-a^p\right)^2}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}du\right]\\
&+\left[\frac{b^p-a^p}{2^{1-\alpha}}\left(\int_{1/2}^{1}\frac{u(1-u)^{\alpha}}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}\right)^{1-1/q}\right]\\
&\times\left[\left(\int_{1/2}^{1}\frac{(1-u)^{\alpha}}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}du\right)\left|f'(b)\right|\right.\\
&\left.+\int_0^{1/2}\frac{(1-u)^{\alpha}uM_\eta}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}du\right]\\
&+\left[\frac{b^p-a^p}{2^{1-\alpha}}\left(\int_{1/2}^{1}\frac{(1-u)^{\alpha}}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}\right)^{1-1/q}\right]\\
&\times\left[\int_{1/2}^{1}\frac{(1-u)^{\alpha+1}u\left(b^p-a^p\right)^2}{p\left[ua^p+(1-u)b^p\right]^{1-1/p}}du\right]\\
&=\left|\left[\left(C_5(\alpha,p)\right)^{1-1/q}\left[C_5(\alpha,p)\right|\left[f'(b)\right|\right.\right.\\
&+C_6(\alpha,p)M_\eta-C_7(\alpha,p)\mu\left(b^p-a^p\right)^2\Big]\\
&+\Big[C_8(\alpha,p))^{1-1/q}\Big[C_8(\alpha,p)\left|f'(b)\right|+C_9(\alpha,p)M_\eta
\end{aligned}
$$

$$+C_{10}(\alpha,p)\mu\left(b^p-a^p\right)^2]]-M_\eta+\frac{\mu\left(b^p-a^p\right)^2\Gamma(\alpha+1)}{6\Gamma(\alpha+3)}|. \quad (49)$$

This completes the proof. □

Remark 12.

(i) Setting $p=1$ and $\mu=0, \eta=xy$ in Theorem 11 gives H-H type inequality for convex functions

(ii) Setting $p=1$ and $\alpha=1, \mu=0, \eta=xy$ in Theorem 11 gives H-H inequality for convex functions

4. Conclusion

In this paper, we established inequalities of Hermite-Hadamard and Fejér type for strongly generalized p-convex functions. We also established some fractional integral inequalities for this class of function in the setting of RL fractional integrals. We also related our results with the existing results and proved that by fixing involved parameters, we get many previous results.

Data Availability

The data required for this research is included within this paper.

Conflicts of Interest

The authors do not have any competing interests.

Authors' Contributions

All authors contributed equally in this paper.

References

[1] I. Abbas Baloch and Y. M. Chu, "Petrović-type inequalities for harmonic -convex functions," *Journal of Function Spaces*, vol. 2020, 7 pages, 2020.

[2] R. S. Ali, S. Mubeen, S. Ali, G. Rahman, J. Younis, and A. Ali, "Generalized Hermite–Hadamard-type integral inequalities forh-Godunova–Levin functions," *Journal of Function Spaces*, vol. 2022, 12 pages, 2022.

[3] J. Hadamard, "Etude sur les properietes des functions entries et en particulier d'une function consideree par Riemann," *Journal De Mathématiques Pures Et Appliquées*, vol. 58, pp. 171–215, 1893.

[4] S. Sezer, Z. Eken, G. Tinaztepe, and G. Adilov, "p–convex functions and some of their properties," *Numerical Functional Analysis and Optimization*, vol. 42, no. 4, pp. 443–459, 2021.

[5] M. B. Khan, H. M. Srivastava, P. O. Mohammed, K. Nonlaopon, and Y. S. Hamed, "Some new Jensen, Schur and Hermite-Hadamard inequalities for log convex fuzzy interval-valued functions," *AIMS Math*, vol. 7, no. 3, pp. 4338–4358, 2022.

[6] J. Nasir, S. Qaisar, S. I. Butt, K. A. Khan, and R. M. Mabela, "Some Simpson's Riemann–Liouville fractional integral inequalities with applications to special functions," *Journal of Function Spaces*, vol. 2022, 12 pages, 2022.

[7] Y. Yue, G. Farid, A. K. Demirel et al., "Hadamard and Fejér–Hadamard inequalities for generalized $ k $-fractional integrals involving further extension of Mittag-Leffler function," *AIMS Mathematics*, vol. 7, no. 1, pp. 681–703, 2021.

[8] L. Chen, W. Nazeer, F. Ali, T. Botmart, and S. Mehfooz, "Some midpoint inequalities forη-convex function via weighted fractional integrals," *Journal of Function Spaces*, vol. 2022, 12 pages, 2022.

[9] N. Merentes and Nikodem, "Remarks on strongly convex functions," *Aequationes Mathematicae*, vol. 80, no. 1-2, pp. 193–199, 2010.

[10] H. Angulo, J. Giménez, A. M. Moros, and K. Nikodem, "On strongly h–convex functions," *Annals of Functional Analysis*, vol. 2, no. 2, pp. 85–91, 2011.

[11] S. Abramovich, "Convexity, subadditivity and generalized Jensen's inequality," *Annals of Functional Analysis*, vol. 4, no. 2, pp. 183–194, 2013.

[12] M. R. Delavar and S. S. Dragomir, "On η- convexity," *Mathematical Inequalities & Applications*, vol. 20, no. 1, pp. 203–216, 2017.

[13] R. Ger and K. Nikodem, "Strongly convex functions of higher order," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 2, pp. 661–665, 2011.

[14] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[15] B. Meftah, A. Souahi, and A. Souahi, "Cebysev inequalities for co-ordinated QC-convex and (s, QC)-convex," *Engineering and Applied Science Letters*, vol. 4, no. 1, pp. 14–20, 2021.

[16] M. Iftikhar, A. Qayyum, S. Fahad, and M. Arslan, "A new version of Ostrowski type integral inequalities for different differentiable mapping," *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 353–359, 2021.

[17] M. E. Omaba, L. O. Omenyi, and L. O. Omenyi, "Generalized fractional Hadamard type inequalities for (Qs)-class functions of the second kind," *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 270–278, 2021.

[18] A. Adu-Sackey, F. T. Oduro, F. T. Oduro, and G. O. Fosu, "Inequalities approach in determination of convergence of recurrence sequences," *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 65–72, 2021.

[19] M. U. Awan, M. A. Noor, K. I. Noor, and F. Safdar, "On strongly generalized convex functions," *Univerzitet u Nišu*, vol. 31, no. 18, pp. 5783–5790, 2017.

[20] T. Ali, M. A. Khan, and Y. Khurshidi, "Hermite-Hadamard inequality for fractional integrals via eta-convex functions," *Acta Mathematica Universitatis Comenianae*, vol. 86, no. 1, pp. 153–164, 2017.

[21] M. Kunt and I. Iscan, "Hermite-Hadamard type inequalities for p-convex functions via fractional integrals," *Moroccan Journal of Pure and Applied Analysis*, vol. 3, no. 1, pp. 22–34, 2017.

[22] M. Kunt and I. Iscan, "Hermite-Hadamard-Fejer type inequalities for p-convex functions," *Arab Journal of Mathematical Sciences*, vol. 23, no. 2, pp. 215–230, 2017.