Research Article

Oscillatory and Asymptotic Behavior of Nonlinear Functional Dynamic Equations of Third Order

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The purpose of this work is to derive sufficient conditions for the oscillation of all solutions of the third-order functional dynamic equation

\[ \{p_2(\xi)\phi_{\gamma_2}(\{p_1(\xi)\phi_{\gamma_1}(y^{(\gamma)}(\xi))\}^\Delta)\}^\Delta + p(\xi)\phi_{\gamma_i}(y(g(\xi))) = 0, \]

on a time scale \( \mathbb{T} \). In addition, we present some Hille-type conditions for generalized third-order dynamic equations that improve and extend significant contributions reported in the literature without imposing time-scale restrictions. An example is given to demonstrate the essential results.

1. Introduction

Oscillatory criteria of solutions to dynamic equations on time scales are gaining interest due to their applications in engineering and natural sciences. Eventually, this kind of study aids in comprehending the geometric behavior of solutions. We are interested in the oscillatory and asymptotic behavior of the third-order functional dynamic equation in the form of

\[ \left\{ p_2(\xi)\phi_{\gamma_2} \left( \{ p_1(\xi)\phi_{\gamma_1}(y^{(\gamma)}(\xi)) \}^\Delta \right) \right\}^\Delta + p(\xi)\phi_{\gamma_i}(y(g(\xi))) = 0, \]

on an arbitrary time scale \( \mathbb{T} \) with \( \sup \mathbb{T} = \infty \), where \( \xi \in \mathbb{E}_0, \infty \) \( \mathbb{E}_0 \) \( \sup \mathbb{E}_0 = \xi_0 \), \( \xi_0 \in \mathbb{E} \), \( \phi_0(u) = |u|^\beta \sgn u \), \( \theta > 0 \), \( \gamma_1, \gamma_2, \beta > 0 \), \( p \in C_{rd}(\mathbb{T}, \mathbb{R}^+) \), and \( g \in C_{rd}(\mathbb{T}, \mathbb{R}) \) such that \( \lim_{\xi \to \infty} g(\xi) = \infty \) and \( g^{(\gamma)}(\xi) \geq 0 \) on \( \mathbb{T} \) and \( p_i \in C_{rd}(\mathbb{T}, \mathbb{R}^+) \), \( i = 1, 2 \), such that

\[ \int_{\xi_0}^{\infty} \frac{\Delta \tau}{\beta_i^{\gamma_i}(\tau)} = \infty, i = 1, 2. \]

By a solution of equation (1), we mean a nontrivial real-valued function \( y \in C_{rd}(T_y, \infty) \) for some \( T_y \geq \xi_0 \) for a positive constant \( \xi_0 \in \mathbb{T} \) such that \( p_1(\xi)\phi_{\gamma_1}(y^{(\gamma)}(\xi)) \), \( p_2(\xi)\phi_{\gamma_2}(\{ p_1(\xi)\phi_{\gamma_1}(y^{(\gamma)}(\xi)) \})^\Delta \in C_{rd}(T_y, \infty) \), and \( y(\xi) \) satisfies equation (1) on \( [T_y, \infty) \), where \( C_{rd} \) is the space of right-dense continuous functions. Solutions that disappear near infinity will not be considered. A solution \( y \) of (1) is oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is nonoscillatory. We presume that the
reader has a basic understanding of time scales and notation; see [1–4] for providing a great introduction to time scale calculus.

Hille [5] investigated the oscillatory behavior of second-order linear differential equation
\[ y''(\xi) + p(\xi)y(\xi) = 0, \]  
and shown that if
\[ \liminf_{\xi \to \infty} \int_{\xi}^{\infty} p(\tau) \, d\tau > \frac{1}{4}, \]
then every solution of (3) is oscillatory. The results in [6–13] generalized the Hille-type criterion for different forms of second-order dynamic equations. Regarding third-order dynamic equations, [14–18, 32] established several Hille-type oscillation criteria for different forms of third-order dynamic equations under some restrictive times, which ensure that the solutions are either oscillatory or nonoscillatory and converge to a finite limit under various restrictive conditions. Recently, Hassan et al. [19] improved the Hille-type criterion in [14–18, 32] for equation (1) with \( \beta = \gamma = \gamma_1 \gamma_2 \), see ([19], in Discussions and Conclusions) for a good comparison between these results. Some of these results in [19] are as follows.

**Theorem 1** (see [19]). Assume that \( \beta = \gamma = \gamma_1 \gamma_2 \), every solution of equation (1) is either oscillatory or convergent if
\[ (1) \]
\[ \liminfty_{\xi \to \infty} H_{i}^{(\gamma)}(\xi, \xi_{0}) \int_{\xi}^{\infty} \left( \frac{H_{i}(\gamma(\tau), \xi_{0})}{H_{i}(\sigma(\tau), \xi_{0})} \right)^{2} p(\tau) \, d\tau > \frac{y_{\gamma}^{2}}{p(\gamma^2 + 1)} |y_{\gamma}^{2}, \gamma \geq 1, \]  
(2)
\[ \liminfty_{\xi \to \infty} H_{i}^{(\gamma)}(\xi, \xi_{0}) \int_{\xi}^{\infty} \left( \frac{H_{i}(\gamma(\tau), \xi_{0})}{H_{i}(\sigma(\tau), \xi_{0})} \right)^{2} p(\tau) \, d\tau > \frac{y_{\gamma}^{2}}{\gamma_{2}^{2} - \gamma_{1}^{2} |y_{\gamma}^{2}, y_{\gamma}^{2} > 1, \]  
(3)
where
\[ \psi(\xi) = \min \{ \xi, g(\xi) \} \quad \text{and} \quad l = \liminfty_{\xi \to \infty} \frac{H_{i}(\xi, \xi_{0})}{H_{i}(\sigma(\xi), \xi_{0})}, \]
\[ H_{i}(v, u) = \psi_{\gamma_{i}, 1} \left( \int_{u}^{v} \psi_{\gamma_{i}, 1}(H_{i}(\gamma_{i}, \tau), u) \, d\tau \right), i = 1, 2, 3, \]  
(4)
with
\[ H_{0}(v, u) = \frac{1}{H_{0}^{(\gamma)}(v)}, p_{0} = y_{0} = 1. \]  
(5)

The reader is directed to papers [20–33] as well as the sources listed therein.

Contrary to [14–16, 19], we are concerned in this paper in deducing sufficient oscillation criteria that guarantee that all solutions of nonlinear third-order dynamic equation (1) are oscillatory when \( \beta = \gamma \) and \( g(\xi) \leq \xi \) without imposing restrictive on the time scales. This solves an open problem posed in [1] (Remark 3.3). Furthermore, we will propose certain Hille-type conditions for generalized third-order dynamic equation (1) for the cases \( \beta \geq \gamma, \beta \leq \gamma, g(\xi) \leq \xi, \) and \( g(\xi) \leq \xi \) that improve and extend relevant significant contributions reported in [14–16, 18, 19] without extra imposing time-scale constraints. All functional inequalities reported in this paper are considered to hold eventually, that is, for all sufficiently large \( \xi. \)

### 2. Main Results

Throughout this paper, we let
\[ \alpha = \{ lllb, \beta \geq \gamma, \beta \leq \gamma, \]
(6)
\[ y^{(\beta)}(\xi) = p(\xi) \phi_{1}(\gamma^{|i|}(\xi))^{\alpha}, i = 1, 2, \quad \text{with} y^{(i)}(0) = y. \]  
(7)

Before stating the main results, we will present some preliminary lemmas to aid in the proving of the main results.  

**Lemma 2.** Assume that equation (1) has a solution \( y \) such that
\[ (1) y^{(\beta)}(\xi) > 0, i = 0, 1, 2, \text{on} [\xi_{0}, \infty), \]
(8)
Then, for \( v \in (u, \infty) \),
\[ \phi_{i}(y(u)) \geq H_{i}^{(\gamma)}(u, v). \]  
(9)

**Proof.** Suppose, without losing generality, that \( y(g(\xi)) > 0 \) on \( [\xi_{0}, \infty) \). From (1) and (10), we have for \( [\xi_{0}, \infty) \),
\[ \left( y^{(\gamma)}(\xi) \right)^{\Delta} = -p(\xi) \phi_{1}(y(g(\xi))) < 0. \]  
(10)

From (13), we get for \( u \leq \tau \leq v \) and \( u, v \in (0, \infty) \),
\[ y^{(1)}(v) - y^{(1)}(u) = \int_{u}^{v} \left( y^{(2)}(\omega) \right)^{1/2} H_{0}(\omega, \tau) \, d\omega \geq \left( y^{(2)}(v) \right)^{1/2} \int_{u}^{v} H_{0}(\omega, \tau) \, d\omega \]
\[ = \left( y^{(2)}(v) \right)^{1/2} H_{0}(v, \tau). \]  
(11)
Therefore,
\[ n^{(1)}(\tau) \geq \left( y^{(2)}(v) \right)^{1/2} H_{0}(v, \tau). \]  
(12)
Suppose, without losing generality, that \( \frac{1}{2} \) is complete. So, from \( u \) to \( v \) yields that

\[
y(u) - y(v) \geq \left( \int_{u}^{v} \left( \frac{H_1(v, \tau)}{p_1(\tau)} \right)^{1/y_1} \Delta \tau \right) y^{[2]}(v).
\]

Thus, (12) holds for \( u, v \in \xi_0, \infty \). By dint of (13), there is a \( v \in (u, \infty) \) such that

\[
\left( \frac{y^{[1]}(v)}{H_1(v, u)} \right) > 0 \text{ for } v \in (u, \infty) \subset [\xi_0, \infty).
\]

Thus, (22) holds for \( u, v \in \xi_0, \infty \). The proof is now complete. \( \square \)

**Lemma 3.** Assume that equation (1) has a solution \( y \) such that

\[
y^{[i]}(0) > 0, i = 0, 1, 2, \text{ on } [\xi_0, \infty),
\]

then for \( v \in (u, \infty) \subset \xi_0, \infty \),

\[
\left( \frac{y^{[1]}(v)}{H_1(v, u)} \right) \geq 0,
\]

and

\[
\frac{\phi_{y_1^+}}{y^{[2]}(v)} > H_2^2(v, u),
\]

Thus, (21) holds for \( u, v \in \xi_0, \infty \). From (24), we have

\[
y^{[1]}(v) > \phi_{y_1^+} \left( \frac{y^{[2]}(v)}{H_1(v, u)} \right) \left( \frac{H_1(v, u)}{p_1(\tau)} \right)^{1/y_1} \Delta \tau.
\]

Replacing \( v \) by \( s \) in (25) and integrating with respect to \( u \) to \( v \) gives

\[
y(v) \geq \int_{u}^{v} \phi_{y_1^+} \left( \frac{y^{[2]}(\tau)}{H_1(\tau, u)} \right) \left( \frac{H_1(\tau, u)}{p_1(\tau)} \right)^{1/y_1} \Delta \tau
\]

This, (22) holds for \( u, v \in \xi_0, \infty \). By virtue of (24), there is a \( v \in (u, \infty) \) such that

\[
\left( \frac{y^{[1]}(v)}{H_1(v, u)} \right) \geq 0 \text{ for } v \in (u, \infty) \subset [\xi_0, \infty).
\]

Hence, for \( v \in (u, \infty) \),

\[
y(v) \geq \int_{u}^{v} \phi_{y_1^+} \left( \frac{y^{[1]}(\tau)}{H_1(\tau, u)} \right) \left( \frac{H_1(\tau, u)}{p_1(\tau)} \right)^{1/y_1} \Delta \tau
\]

Thus, (23) holds for \( v \in (u, \infty) \). The proof is now complete. \( \square \)

### 2.1. Asymptotic Behavior

In this subsection, we debate the asymptotic behavior of the solutions of equation (1) for both cases \( \beta \geq \gamma \) and \( \beta \leq \gamma \).

**Theorem 4.** Assume that \( l > 0 \) and for sufficiently large \( T \in [\xi_0, \infty) \),

\[
\liminf_{\xi \to \infty} H_1^2(\xi, T) \int_{\xi}^{\infty} \left( \frac{H_1^2(y(\xi), T)}{H_1^2(\xi, T)} \right)^{1/y_1} \rho(\xi) \Delta \xi > \frac{y^{[2]}_1}{p_1(\xi) + 1} y^{[2]}_1.
\]

If equation (1) has a nonoscillatory solution \( y(\xi) \), then \( y^{[i]}(\xi) \) and \( i = 0, 1, 2 \) converge.

**Proof.** Suppose, without losing generality, that \( y(\xi) \) and \( (g(\xi)) \) are eventually positive, by virtue of (1), we deduce that \( y^{[i]}(\xi) \) and \( i = 1, 2 \) are eventually of one sign and by (2), and we can easily see that \( y^{[i]}(\xi) \) is eventually positive, see [34], part (\( \mathcal{F} \)) of the proof of Theorem 4. Therefore, we consider the following two cases:
$(\mathcal{F})y^{[i]}(\xi)$ is eventually positive. In this case, there is a $\xi_1 \in [\xi_0, \infty)$ such that
\[
y^{[i]}(\xi) > 0, i = 0, 1, 2, \text{ and } (y^{[i]}(\xi))^\Delta < 0 \text{ on } [\xi_1, \infty). \tag{30}\]

Consider
\[
z(\xi) = \frac{y^{[2]}(\xi)}{(y^{[1]}(\xi))^\gamma}. \tag{31}\]

Hence,
\[
z^\Delta(\xi) = \frac{(y^{[2]}(\xi))^\Delta}{(y^{[1]}(\xi))^\gamma} + \frac{1}{(y^{[1]}(\xi))^\gamma}y^{[2]}(\sigma(\xi)) = \frac{(y^{[2]}(\xi))^\Delta}{(y^{[1]}(\xi))^\gamma} - \frac{(y^{[1]}(\xi))^\gamma}{(y^{[1]}(\xi))^\gamma}(y^{[1]}(\sigma(\xi)))^\gamma. \tag{32}\]

(1) If $0 < \gamma_2 \leq 1$, by means of the Pötzsch chain rule and the definitions of $H_0(\xi, \xi_1)$ and $z(\xi)$, we obtain for $\xi \in [\xi_2, \infty)$
\[
z^\Delta(\xi) \leq \frac{(y^{[2]}(\xi))^\Delta}{(y^{[1]}(\xi))^\gamma} - \gamma_2 \frac{(y^{[1]}(\sigma(\xi)))^\gamma}{(y^{[1]}(\xi))^\gamma}z(\sigma(\xi)) \leq \gamma_2 H_0(\xi, \xi_1) \frac{(y^{[1]}(\xi))^\gamma}{(y^{[1]}(\xi))^\gamma}z(\sigma(\xi)). \tag{33}\]

Using the fact that $(y^{[1]}(\xi)/H_1(\xi, \xi_1))^\Delta < 0$ for $\xi \in [\xi_2, \infty)$, we get
\[
\frac{(y^{[1]}(\xi))^\gamma}{(y^{[1]}(\sigma(\xi)))^\gamma} \geq \frac{H_1(\xi, \xi_1)}{H_1(\sigma(\xi), \xi_1)} \tag{34}\]

Therefore, (33) becomes
\[
z^\Delta(\xi) \leq \frac{(y^{[2]}(\xi))^\Delta}{(y^{[1]}(\xi))^\gamma} - \gamma_2 H_0(\xi, \xi_1) \frac{(y^{[1]}(\xi))^\gamma}{(y^{[1]}(\sigma(\xi)))^\gamma}z(\sigma(\xi)). \tag{35}\]

Now, for any $\varepsilon > 0$, there exists a $\xi_3 \in [\xi_2, \infty)$ such that
\[
H_1(\xi, \xi_1) \geq 1 - \varepsilon \quad \text{and} \quad H_1^\gamma(\xi, \xi_1)z(\xi) \geq H - \varepsilon \text{ for } \xi \in [\xi_3, \infty). \tag{36}\]

where
\[
H = \liminf_{\xi \to \infty} H_1^\gamma(\xi, \xi_1)z(\xi), \quad 0 \leq H \leq 1. \tag{37}\]

Substituting (36) into (35), we get for $\xi \in [\xi_3, \infty)$
\[
z^\Delta(\xi) \leq \frac{(y^{[2]}(\xi))^\Delta}{(y^{[1]}(\xi))^\gamma} - \gamma_2 \frac{(y^{[1]}(\xi))^\gamma}{(y^{[1]}(\xi))^\gamma} - \gamma_2 H_0(\xi, \xi_1) \frac{(y^{[1]}(\xi))^\gamma}{(y^{[1]}(\xi))^\gamma}z(\sigma(\xi)) \leq \gamma_2 H_0(\xi, \xi_1) \frac{(y^{[1]}(\xi))^\gamma}{(y^{[1]}(\xi))^\gamma}z(\sigma(\xi)). \tag{38}\]

(2) If $\gamma_2 \geq 1$, by means of the Pötzsch chain rule and the definitions of $H_0(\xi, \xi_1)$ and $z(\xi)$, we obtain for $\xi \in [\xi_2, \infty)$
\[
z^\Delta(\xi) \leq \frac{(y^{[2]}(\xi))^\Delta}{(y^{[1]}(\xi))^\gamma} - \gamma_2 \frac{(y^{[1]}(\xi))^\gamma}{(y^{[1]}(\xi))^\gamma} - \gamma_2 H_0(\xi, \xi_1) \frac{(y^{[1]}(\xi))^\gamma}{(y^{[1]}(\xi))^\gamma}z(\sigma(\xi)) \leq \gamma_2 H_0(\xi, \xi_1) \frac{(y^{[1]}(\xi))^\gamma}{(y^{[1]}(\xi))^\gamma}z(\sigma(\xi)). \tag{39}\]

Substituting (36) into (39), we get for $\xi \in [\xi_3, \infty)$
\[
z^\Delta(\xi) \leq \frac{(y^{[2]}(\xi))^\Delta}{(y^{[1]}(\xi))^\gamma} - \gamma_2 \frac{(y^{[1]}(\xi))^\gamma}{(y^{[1]}(\xi))^\gamma} - \gamma_2 H_0(\xi, \xi_1) \frac{(y^{[1]}(\xi))^\gamma}{(y^{[1]}(\xi))^\gamma}z(\sigma(\xi)) \leq \gamma_2 H_0(\xi, \xi_1) \frac{(y^{[1]}(\xi))^\gamma}{(y^{[1]}(\xi))^\gamma}z(\sigma(\xi)). \tag{40}\]

Combining (38) with (40), we conclude that for $\gamma_2 > 0$ and $\xi \in [\xi_3, \infty)$
\[
z^\Delta(\xi) \leq \frac{(y^{[2]}(\xi))^\Delta}{(y^{[1]}(\xi))^\gamma} - (H - \varepsilon)^{1+\gamma_2} \frac{(y^{[1]}(\xi))^\gamma}{(y^{[1]}(\xi))^\gamma}z(\sigma(\xi)). \tag{41}\]
From (1), we see that
\[
z^{\xi}(\xi) \leq \left(\frac{y^{\xi}(g(\xi))}{y^{\xi}(\xi)}\right)^{\beta_y} p(\xi) - (H - e)^{\xi+1} (l - e)^{\gamma_y-1} \left(\frac{-1}{H^{\xi}(\xi, \xi)}\right)^{\lambda}.
\]
(42)

If \(\beta \geq \gamma\), by the fact that \(y^{\xi}(\xi) > 0\) for \(\xi \in \xi_{[1,\infty]}\), then
\[
\left(\frac{y^{\xi}(\xi)}{H^{\xi}(\xi, \xi)}\right)^{\beta_y-\gamma_y} \geq \left(\frac{y^{\xi}(\xi)}{H^{\xi}(\xi, \xi)}\right)^{\beta_y-\gamma_y} = k_1 > 0 \text{ for } \xi \geq \xi_3,
\]
(43)
whereas if \(\beta \leq \gamma\), by the fact that \((y^{\xi}(\xi)/H^{\xi}(\xi, \xi))^{\lambda} < 0\) for \(\xi \in \xi_{[1,\infty]}\), then
\[
\left(\frac{y^{\xi}(\xi)}{H^{\xi}(\xi, \xi)}\right)^{\beta_y-\gamma_y} \geq \left(\frac{y^{\xi}(\xi)}{H^{\xi}(\xi, \xi)}\right)^{\beta_y-\gamma_y} = k_2 > 0 \text{ for } \xi \geq \xi_3.
\]
(44)

Now, consider the case when \(g(\xi) \leq \xi\). From (23) and using the fact that \(y^{\xi}(\xi)/H^{\xi}(\xi, \xi)^{\lambda} < 0\), we deduce that
\[
y^{\xi}(g(\xi)) \geq \left(\frac{y^{\xi}(\xi)}{H^{\xi}(\xi, \xi)}\right)^{\lambda} H^{\xi}(g(\xi), \xi) \geq \left(\frac{y^{\xi}(\xi)}{H^{\xi}(\xi, \xi)}\right)^{\lambda} H^{\xi}(\xi, \xi) - H^{\xi}(g(\xi), \xi).
\]
(45)
Next consider the case when \(g(\xi) \geq \xi\). From (23) and using the fact that \(y^{\xi}(\xi) > 0\), we conclude that
\[
y^{\xi}(g(\xi)) \geq y^{\xi}(\xi) \geq \left(\frac{y^{\xi}(\xi)}{H^{\xi}(\xi, \xi)}\right)^{\lambda} H^{\xi}(\xi, \xi).
\]
(46)
By combining (45) with (46), we have
\[
\frac{y^{\xi}(g(\xi))}{y^{\xi}(\xi)} \geq \frac{H^{\xi}(\xi, \xi)}{H^{\xi}(\xi, \xi)} \text{ for } \xi \in \xi_{[1,\infty]},\]
(47)

From (43), (44), and (47), we get for \(\xi \in \xi_{[1,\infty]}\),
\[
\left(\frac{y^{\xi}(g(\xi))}{y^{\xi}(\xi)}\right)^{\beta_y} \left(\frac{y^{\xi}(\xi)}{H^{\xi}(\xi, \xi)}\right)^{\beta_y-\gamma_y} \geq k \left(\frac{H^{\xi}(\xi, \xi)}{H^{\xi}(\xi, \xi)}\right)^{\lambda y_y},
\]
(48)
where \(k := \min \{k_1, k_2\}\). Substituting (47) into (42), we obtain for \(\xi \in \xi_{[1,\infty]}\),
\[
z^{\xi}(\xi) \leq k \left(\frac{H^{\xi}(\xi, \xi)}{H^{\xi}(\xi, \xi)}\right)^{\lambda y_y} H^{\xi}(\xi, \xi) - (H - e)^{\xi+1} (l - e)^{\gamma_y-1} \left(\frac{-1}{H^{\xi}(\xi, \xi)}\right)^{\lambda}.
\]
(49)

Integrating (49) from \(\xi\) to \(v\), we have
\[
z(v) - z(\xi) \leq \int_{\xi}^{v} \left(\frac{H^{\xi}(\xi, \xi)}{H^{\xi}(\xi, \xi)}\right)^{\lambda y_y} p(\xi) - (H - e)^{\xi+1} (l - e)^{\gamma_y-1} \left(\frac{-1}{H^{\xi}(\xi, \xi)}\right)^{\lambda} d\xi.
\]
(50)

Taking into consideration that \(v > 0\) and passing to the limit as \(v \to \infty\), we conclude that
\[
k \int_{\xi}^{\infty} \left(\frac{H^{\xi}(\xi, \xi)}{H^{\xi}(\xi, \xi)}\right)^{\lambda y_y} p(\xi) d\xi \leq z(\xi) - (H - e)^{\xi+1} (l - e)^{\gamma_y-1} \left(\frac{-1}{H^{\xi}(\xi, \xi)}\right)^{\lambda}.
\]
(51)

Multiplying both sides of (51) by \(H^{\xi}(\xi, \xi)\), we get for \(\xi \in \xi_{[1,\infty]}\),
\[
k H^{\xi}(\xi, \xi) \int_{\xi}^{\infty} \left(\frac{H^{\xi}(\xi, \xi)}{H^{\xi}(\xi, \xi)}\right)^{\lambda y_y} p(\xi) d\xi \leq H^{\xi}(\xi, \xi) z(\xi) - (H - e)^{\xi+1} (l - e)^{\gamma_y-1}.
\]
(52)

Taking the liminf of both sides of the last inequality (52) as \(\xi \to \infty\), we obtain
\[
\liminf_{\xi \to \infty} H^{\xi}(\xi, \xi) \int_{\xi}^{\infty} \left(\frac{H^{\xi}(\xi, \xi)}{H^{\xi}(\xi, \xi)}\right)^{\lambda y_y} p(\xi) d\xi \leq H - (H - e)^{\xi+1} (l - e)^{\gamma_y-1}.
\]
(53)

By dint of the fact that \(k, e > 0\) is arbitrary, we deduce that
\[
\liminf_{\xi \to \infty} H^{\xi}(\xi, \xi) \int_{\xi}^{\infty} \left(\frac{H^{\xi}(\xi, \xi)}{H^{\xi}(\xi, \xi)}\right)^{\lambda y_y} p(\xi) d\xi \leq H - H^{\xi+1} (l - e)^{\gamma_y-1}.
\]
(54)

Setting
\[
X := H^{\xi+1} \gamma_{y-1}/\gamma_{y-1} \quad \text{and} \quad Y := \frac{\gamma_{y-1}}{1 + \gamma_{y-1}} Y_{\gamma_{y-1}} Y_{\gamma_{y-1}+1},
\]
(55)
and \(\lambda := \gamma_{y-1} + 1/\gamma_{y-1} > 1\). Then, using inequality (see [35])
\[
\lambda XY^{\lambda-1} - X^{\lambda} \leq (\lambda - 1) Y^{\lambda},
\]
(56)
we achieve
\[
H - H^{\xi+1} \gamma_{y-1} \gamma_{y-1} \leq \frac{\gamma_{y-1}}{p(\gamma_{y-1}+1)} (\gamma_{y-1} + 1)^{\gamma_{y-1}+1}.
\]
(57)
Thus, (54) becomes
\[
\liminf_{\xi \to -\infty} H_{1\xi}^\beta (\xi, \xi_i) \int_\xi^{y_{1\xi}} \left( \frac{H_2^\beta(y(r), \xi_i)}{H_1^\beta(r, \xi_i)} \right)^{1/y_{1\xi}} p(r) \Delta r \leq \frac{y_{1\xi}^{y_{1\xi}}}{p^{y_{1\xi}}(y_{2\xi} + 1)^{y_{2\xi} + 1}},
\]
(58)
as a result of which there is a contradiction with (29).
\((\mathcal{F}, \mathcal{F}) y^{[1]}(\xi)\) is eventually positive. In this case, there is a \(\xi_1 \in [\xi_0, \infty)_T\) such that
\[
(-1)^y y_{i}^{[1]}(\xi) > 0, i = 0, 1, 2, \text{ and } \left( y_{i}^{[2]}(\xi) \right) > 0 \text{ on } [\xi_1, \infty)_T.
\]
(59)
By virtue of (59), it is easy to see that \(y_{i}^{[1]}(\xi)\) and \(i = 0, 1, 2\) converge. This completes the proof.
\[\Box\]

**Remark 5.** [1] The deduction of Theorem 4 keeps intact if assumption (29) is replaced by
\[
\int_{\xi_0}^{\infty} \frac{H_2^\beta(\psi(r), T)}{H_1^\beta(r, T)} \frac{1}{\tau y_{1\xi}} p(r) \Delta r = \infty.
\]
(60)

2) If either
\[
\int_{\xi_0}^{\infty} \frac{1}{\tau y_{1\xi}} p(r) \Delta r = \infty, \quad \int_{\xi_0}^{\infty} \frac{1}{\tau y_{1\xi}} p(\omega) \Delta \omega = \infty
\]
or
\[
\int_{\xi_0}^{\infty} \left[ \frac{1}{\tau y_{1\xi}} p(\omega) \Delta \omega \right] \Delta \omega = \infty, \quad \int_{\xi_0}^{\infty} \frac{1}{\tau y_{1\xi}} p(\omega) \Delta \omega = \infty
\]
then nonoscillatory solutions of the investigated equation (1) are convergent to zero, see [35], [Theorem 4].

2.2. Oscillation Criteria. In this subsection, we establish oscillation criteria of the solutions of equation (1) when \(\beta = y \text{ and } g(\xi) \leq \xi \text{ on } [\xi_0, \infty)_T\). This solves an open problem posed in [1].

**Theorem 6.** Assume that (29) and either
\[
\limsup_{\xi \to -\infty} \int_{g(\xi)}^{\xi} H_{1\xi}^\beta (g(\xi), g(\xi)) p(r) \Delta r > 1,
\]
(63)
or
\[
\limsup_{\xi \to -\infty} \int_{g(\xi)}^{\xi} \left( \frac{1}{\tau y_{1\xi}} p(\omega) \Delta \omega \right) \Delta \omega = \infty
\]
(64)
hold. Then, all solutions to equation (1) are oscillatory.

**Proof.** Suppose, without losing generality, that \(y(\xi) \text{ and } y(\xi)\) are eventually positive, by virtue of (1), we deduce that \(y_{i}^{[1]}(\xi) \text{ and } i = 1, 2 \text{ are eventually of one sign and by (2)}\), we can easily see that \(y_{1\xi}^{[2]}(\xi)\) is eventually positive. Therefore, we consider the following two cases:
\((\mathcal{F}, \mathcal{F}) y^{[1]}(\xi)\) is eventually positive. The same proof as in part (\(\mathcal{F}\)) of the proof of Theorem 4 hence is omitted.
\((\mathcal{F}, \mathcal{F}) y^{[1]}(\xi)\) is eventually positive. In this case, there is a \(\xi_1 \in [\xi_0, \infty)_T\) such that (59) for \(\xi \in [\xi_1, \infty)_T\).
Let (63) hold. Integrating (1) form \(g(\xi)\) to \(\xi\) gives
\[
y_{i}^{[2]}(g(\xi)) \geq y_{i}^{[2]}(g(\xi)) \frac{1}{g(\xi)} \int_{g(\xi)}^{\xi} p(r) \Delta r.
\]
(65)
By using Lemma 2 and setting \(u = g(r)\) and \(v = g(\xi)\), we obtain
\[
\frac{1}{p(\xi)} \int_{\xi}^{\xi} y_{i}^{[2]}(g(\xi)) \frac{1}{p(\xi)} \int_{\xi}^{\xi} p(r) \Delta r.
\]
(67)
We get a contradiction with (63) by taking the lim sup on both sides of the last inequality as \(\xi \to -\infty\).
Let (64) hold. Integrating (1) form \(\tau \to \xi\), we get
\[
y_{i}^{[2]}(\omega) \geq y_{i}^{[2]}(\xi) \frac{1}{p(\xi)} \int_{\xi}^{\xi} p(\omega) \Delta \omega \geq y_{i}^{[2]}(g(\xi)) \frac{1}{p(\xi)} \int_{\xi}^{\xi} p(\omega) \Delta \omega.
\]
(68)
Subsequently,
\[
\left( y_{i}^{[1]}(\omega) \right) \Delta \omega \geq y_{i}^{[2]}(g(\xi)) \frac{1}{p(\xi)} \int_{\xi}^{\xi} p(\omega) \Delta \omega.
\]
(69)
Integrating the last inequality form \(\tau \to \xi\), we conclude that
\[
y_{i}^{[1]}(\tau) \geq y_{i}^{[1]}(\xi) \frac{1}{p(\xi)} \int_{\xi}^{\xi} p(\omega) \Delta \omega \geq y_{i}^{[1]}(g(\xi)) \frac{1}{p(\xi)} \int_{\xi}^{\xi} p(\omega) \Delta \omega.
\]
(70)
As a result,
\[
y_{i}^{[1]}(\tau) \geq y_{i}^{[1]}(g(\xi)) \frac{1}{p(\xi)} \int_{\xi}^{\xi} p(\omega) \Delta \omega \geq y_{i}^{[1]}(g(\xi)) \frac{1}{p(\xi)} \int_{\xi}^{\xi} p(\omega) \Delta \omega.
\]
(71)
Integrating again the last inequality form \(g(\xi)\) to \(\xi\), we achieve
\[ y(g(\xi)) \geq y(g(\xi)) \int_{g(\xi)}^{\xi} \left( \frac{1}{p_1(\tau)} \right)^{\nu_1} \left( \frac{1}{p_2(\omega)} \right)^{\nu_2} p(s) \Delta s \Delta \tau. \] 

(72)

We get a contradiction with (64) by taking the lim sup on both sides of the last inequality as \( \xi \to \infty \). The proof is complete.

\[ \limsup_{\xi \to \infty} \int_{g(\xi)}^{\xi} H^2_{\xi}(g(\tau), g(\xi)) p(\tau) \Delta \tau = \limsup_{\xi \to \infty} \int_{g(\xi)}^{\xi} \frac{4^{11}}{72 \pi^2} \left( \frac{r^2 - \xi^2}{4} \right)^{4} \Delta \tau = \frac{664548}{35} + 64 \ln(2), \]

(74)

\[ \liminf_{\xi \to \infty} H^2_{\xi}(\xi, \xi_0) \int_{\xi}^{\xi_0} \left( \frac{H^2_{\xi}(\psi(\tau), T)}{H^2_{\xi}(\tau, T)} \right)^{\nu_1} p(\tau) \Delta \tau = \liminf_{\xi \to \infty} \left( \frac{\xi^2 - 1}{\xi^2} \right)^{2} \int_{\xi}^{\xi_0} \left( \frac{2(4)^5 (r^2/4 - 1)^4}{r^2 - 1} \right)^{2} \frac{1}{r^2} \Delta \tau = 16. \]

Therefore, the conditions (63) and (29) are satisfied. Then, all solutions to equation (73) are oscillatory, according to Theorem 6.

### 3. Conclusions

(1) The important point to note here is that the proposed results in Theorem 6 are new for third-order dynamic equation (1) and also, these results, in a special case, have answered an open problem stated by [1] (Remark 3.3), which is concerned with studying the sufficient conditions that guarantee that all solutions of third-order delay dynamic equations oscillate.

(2) In comparison to the results in the cited papers [14–16, 18], Hille-type criteria for dynamic equation (1) in the cases \( \beta \geq \gamma, \beta \leq \gamma, g(\xi) \leq T, \) and \( g(\xi) \geq \xi \) have been developed, and the results in this study are a significant improvement; for more details, see ([19], see the following details)

(i) If \( \beta = \gamma \) and \( 0 < \gamma_2 < 1 \), then condition (29) becomes

\[ \liminf_{\xi \to \infty} H^2_{\xi}(\xi, T) \int_{\xi}^{\xi_0} \left( \frac{H^2_{\xi}(\psi(\tau), T)}{H^2_{\xi}(\tau, T)} \right)^{\nu_2} p(\tau) \Delta \tau > \frac{\gamma^2_{\nu_2}}{p(\xi_{\xi_0}^{\nu_2})/(\gamma_2 + 1)^{\nu_2}}. \]

(75)

By dint of

\[ H^2_{\xi}(\xi, T) \int_{\xi}^{\xi_0} \left( \frac{H^2_{\xi}(\psi(\tau), T)}{H^2_{\xi}(\tau, T)} \right)^{\nu_2} p(\tau) \Delta \tau \geq H^2_{\xi}(\xi, T) \int_{\xi}^{\xi_0} \left( \frac{H^2_{\xi}(\psi(\tau), T)}{H^2_{\xi}(\tau, T)} \right)^{\nu_1} p(\tau) \Delta \tau, \]

(76)

Hille-type criterion (75) improves (5);

(ii) Condition (29) reduces to (6) in the case where \( \beta = \gamma = \gamma_2 = 1 \)

(ii) The asymptotic behavior of solutions is viable to dynamic equation (1) for both \( g(\xi) \leq \xi \) and \( g(\xi) \geq \xi \), whereas the oscillation conditions are viable to dynamic equation (1) for \( g(\xi) \leq \xi \). As a result, oscillations can be ensured by a delay in equations.

(4) The results presented here are for equation (1) on an unbounded above arbitrary time scale; therefore, they are applicable to different of time scales.

(5) It would be interesting to define Hille-type criteria for the third-order dynamic equation (1) under non-canonical assumptions

\[ \int_{\xi}^{\xi_0} \frac{\Delta \tau}{p(\xi_{\xi_0}^{\nu_1})} < \infty, i = 1, 2. \]

### Data Availability

The numerical data used to support the findings of this study are included in the article.

### Conflicts of Interest

The authors declare that they have no competing interests. There are no any nonfinancial competing interests (political, personal, religious, ideological, academic, intellectual, commercial, or any other) to declare in relation to this manuscript.

### Authors’ Contributions

Hassan oversaw the study and help inspection. All the authors carried out the main results of this article and drafted the manuscript and read and approved the final manuscript.

Example 7. Consider the third-order delay dynamic equation

\[ \left\{ \frac{1}{4^{2}} \frac{\psi_2\left( \left[ \frac{1}{2} \left( \frac{\psi_2\left( \psi_2(\xi) \right) \right)^2 \right] \right) ^{\nu_2}}{\xi^2 \psi_2\left( \left( \psi_2(\xi) \right) ^2 \right) } + \frac{1}{\xi^{\nu_1}} \psi_2\left( \left( \psi_2(\xi) \right) ^2 \right) \right\} \]

(73)

It is obvious that condition (2) is fulfilled. Now,

\[ \int_{\xi}^{\xi_0} \frac{\Delta \tau}{p(\xi_{\xi_0}^{\nu_1})} < \infty, i = 1, 2. \]
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References


