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Research Article

Oscillatory and Asymptotic Behavior of Nonlinear Functional Dynamic Equations of Third Order

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The purpose of this work is to derive sufficient conditions for the oscillation of all solutions of the third-order functional dynamic equation $\{p_2(\xi)\phi_{\gamma_2}([p_1(\xi)\phi_{\gamma_1}(y^\Delta(\xi))]^\Delta)\}^\Delta + p(\xi)\phi_\beta(y(g(\xi))) = 0$, on a time scale $\mathbb{T}$. In addition, we present some Hille-type conditions for generalized third-order dynamic equations that improve and extend significant contributions reported in the literature without imposing time-scale restrictions. An example is given to demonstrate the essential results.

1. Introduction

Oscillatory criteria of solutions to dynamic equations on time scales are gaining interest due to their applications in engineering and natural sciences. Eventually, this kind of study aids in comprehending the geometric behavior of solutions. We are interested in the oscillatory and asymptotic behavior of the third-order functional dynamic equation in the form of

$$\left\{p_2(\xi)\phi_{\gamma_2}\left(\left[p_1(\xi)\phi_{\gamma_1}\left(y^\Delta(\xi)\right)\right]^\Delta\right)\right\}^\Delta + p(\xi)\phi_\beta(y(g(\xi))) = 0, \quad (1)$$

on an arbitrary time scale $\mathbb{T}$ with $\sup \mathbb{T} = \infty$, where $\xi \in \xi_0,\infty)_\mathbb{T} \coloneqq [\xi_0,\infty) \cap \mathbb{T}$, $\xi_0 \geq 0$, $\xi_0 \in \mathbb{T}$, $\phi_\theta(u) \coloneqq |u|^\theta \operatorname{sgn} u$, $\theta > 0$, $\gamma_1, \gamma_2, \beta > 0$, $p \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$, and $g \in C_{rd}(\mathbb{T}, \mathbb{T})$ such that $\lim_{\xi\longrightarrow\infty} g(\xi) = \infty$ and $g^\Delta(\xi) \geq 0$ on $\mathbb{T}$ and $p_i \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$, $i = 1, 2$, such that

$$\int_{\xi_0}^{\infty} \frac{\Delta\tau}{p_i^{1/\gamma_i}(\tau)} = \infty, i = 1, 2. \quad (2)$$

By a solution of equation (1), we mean a nontrivial real–valued function $y \in C^1_{rd}[T_y,\infty)_\mathbb{T}$ for some $T_y \geq \xi_0$ for a positive constant $\xi_0 \in \mathbb{T}$ such that $p_1(\xi)\phi_{\gamma_1}(y^\Delta(\xi))$, $p_2(\xi)\phi_{\gamma_2}([p_1(\xi)\phi_{\gamma_1}(y^\Delta(\xi))]^\Delta) \in C^1_{rd}[T_y,\infty)_\mathbb{T}$, and $y(\xi)$ satisfies equation (1) on $[T_y,\infty)_\mathbb{T}$, where C_{rd} is the space of right-dense continuous functions. Solutions that disappear near infinity will not be considered. A solution y of (1) is oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is nonoscillatory. We presume that the reader has a basic understanding of time scales and notation; see [1–4] for providing a great introduction to time scale calculus.

Hille [5] investigated the oscillatory behavior of second-order linear differential equation

$$y''(\xi) + p(\xi)y(\xi) = 0, \tag{3}$$

and shown that if

$$\liminf_{\xi \longrightarrow \infty} \xi \int_{\xi}^{\infty} p(\tau)\mathrm{d}\tau > \frac{1}{4}, \tag{4}$$

then every solution of (3) is oscillatory. The results in [6–13] generalized the Hille-type criterion for different forms of second-order dynamic equations. Regarding third-order dynamic equations, [14–18, 32] established several Hille-type oscillation criteria for different forms of third-order dynamic equations under some restrictive times, which ensure that the solutions are either oscillatory or nonoscillatory and converge to a finite limit under various restrictive conditions. Recently, Hassan et al. [19] improved the Hille-type criteria in [14–18, 32] for equation (1) with $\beta = \gamma := \gamma_1 \gamma_2$, see ([19], in Discussions and Conclusions) for a good comparison between these results. Some of these results in [19] are as follows.

Theorem 1 (see [19]). *Assume that $\beta = \gamma := \gamma_1\gamma_2$, every solution of equation (1) is either oscillatory or convergent if*

(1)

$$\liminf_{\xi \longrightarrow \infty} H_1^{\gamma_2}(\xi, \xi_0) \int_{\xi}^{\infty} \left(\frac{H_2(\psi(\tau), \xi_0)}{H_1(\sigma(\tau), \xi_0)} \right)^{\gamma_2} p(\tau)\Delta\tau > \frac{\gamma_2^{\gamma_2}}{l^{\gamma_2(1-\gamma_2)}(\gamma_2 + 1)^{\gamma_2+1}}, 0 < \gamma_2 \leq 1. \tag{5}$$

(2)

$$\liminf_{\xi \longrightarrow \infty} H_1^{\gamma_2}(\xi, \xi_0) \int_{\xi}^{\infty} \left(\frac{H_2(\psi(\tau), \xi_0)}{H_1(\tau, \xi_0)} \right)^{\gamma_2} p(\tau)\Delta\tau > \frac{\gamma_2^{\gamma_2}}{l^{\gamma_2(\gamma_2-1)}(\gamma_2 + 1)^{\gamma_2+1}}, \gamma_2 \geq 1, \tag{6}$$

where

$$\psi(\xi) := \min\{\xi, g(\xi)\} \quad \text{and} \quad l := \liminf_{\xi \longrightarrow \infty} \frac{H_1(\xi, \xi_0)}{H_1(\sigma(\xi), \xi_0)},$$
$$H_i(\nu, u) := \phi_{\gamma_{i-1}} \left(\int_u^{\nu} \phi_{\gamma_{i-1}}^{-1} \left(\frac{H_{i-1}(\tau, u)}{p_{i-1}(\tau)} \right) \Delta\tau \right), i = 1, 2, 3, \tag{7}$$

with

$$H_0(\nu, u) := \frac{1}{p_2^{1/\gamma_2}(\nu)}, p_0 = \gamma_0 = 1. \tag{8}$$

The reader is directed to papers [20–33] as well as the sources listed therein.

Contrary to [14–16, 19], we are concerned in this paper in deducing sufficient oscillation criteria that guarantee that all solutions of nonlinear third-order dynamic equation (1) are oscillatory when $\beta = \gamma$ and $g(\xi) \leq \xi$ without imposing restrictive on the time scales. This solves an open problem posed in [1] (Remark 3.3). Furthermore, we will propose certain Hille-type conditions for generalized third-order dynamic equation (1) for the cases $\beta \geq \gamma, \beta \leq \gamma, g(\xi) \leq \xi$, and $g(\xi) \leq \xi$ that improve and extend relevant significant contributions reported in [14–16, 18, 19] without extra imposing time-scale constraints. All functional inequalities reported in this paper are considered to hold eventually, that is, for all sufficiently large ξ.

2. Main Results

Throughout this paper, we let

$$\alpha := \{lll\beta, \beta \geq \gamma, \gamma, \beta \leq \gamma, \tag{9}$$

$$y^{[i]}(\xi) := p_i(\xi)\phi_{\gamma_i}\left(\left[y^{[i-1]}(\xi) \right]^{\Delta} \right), i = 1, 2, \quad \text{with} y^{[0]}(\xi) = y. \tag{10}$$

Before stating the main results, we will present some preliminary lemmas to aid in the proving of the main results.

Lemma 2. *Assume that equation (1) has a solution y such that*

$$(-1)^i y^{[i]}(\xi) > 0, i = 0, 1, 2, on\ [\xi_0, \infty)_{\mathbb{T}}. \tag{11}$$

Then, for $\nu \in u, \infty)_{\mathbb{T}} \subseteq \xi_0, \infty)_{\mathbb{T}}$,

$$\frac{\phi_{\gamma}(y(u))}{y^{[2]}(\nu)} \geq H_2^{\gamma_2}(u, \nu). \tag{12}$$

Proof. Suppose, without losing generality, that $y(g(\xi)) > 0$ on $[\xi_0, \infty)_{\mathbb{T}}$. From (1) and (10), we have for $[\xi_0, \infty)_{\mathbb{T}}$,

$$\left(y^{[2]}(\xi) \right)^{\Delta} = -p(\xi)\phi_{\beta}(y(g(\xi))) < 0. \tag{13}$$

From (13), we get for $u \leq \tau \leq \nu$ and $u, \tau, \nu \in [\xi_0, \infty)_{\mathbb{T}}$,

$$y^{[1]}(\nu) - y^{[1]}(\tau) = \int_{\tau}^{\nu} \left(y^{[2]}(\omega) \right)^{1/\gamma 2} H_0(\omega, \tau)\Delta\omega \geq \left(y^{[2]}(\nu) \right)^{1/\gamma_2} \int_{\tau}^{\nu} H_0(\omega, \tau)\Delta\omega$$
$$= \left(y^{[2]}(\nu) \right)^{1/\gamma_2} H_1(\nu, \tau). \tag{14}$$

Therefore,

$$-y^{[1]}(\tau) \geq \left(y^{[2]}(\nu) \right)^{1/\gamma_2} H_1(\nu, \tau). \tag{15}$$

So,

$$-y^{\Delta}(\tau) \geq \left(y^{[2]}(v)\right)^{1/\gamma}\left(\frac{H_1(v,\tau)}{p_1(\tau)}\right)^{1/\gamma_1}. \tag{16}$$

Integrating the previous inequality with respect to τ from u to v yields that

$$y(u)-y(v) \geq \left(y^{[2]}(v)\right)^{1/\gamma}\int_u^v\left(\frac{H_1(v,\tau)}{p_1(\tau)}\right)^{1/\gamma_1}\Delta\tau = \left(y^{[2]}(v)\right)^{1/\gamma}H_2^{1/\gamma_1}(u,v). \tag{17}$$

Thus,

$$y^{\gamma}(u) \geq y^{[2]}(v)H_2^{\gamma_2}(u,v). \tag{18}$$

Thus, (12) holds for $u, v \in \xi_0,\infty)_{\mathbb{T}}$. The proof is now complete. □

Lemma 3. *Assume that equation (1) has a solution y such that*

$$y^{[i]}(\xi) > 0, i = 0, 1, 2, \text{ on } [\xi_0,\infty)_{\mathbb{T}}, \tag{19}$$

then for $v \in (u,\infty)_{\mathbb{T}} \subseteq \xi_0,\infty)_{\mathbb{T}}$,

$$\left(\frac{y^{[1]}(v)}{H_1(v,u)}\right)^{\Delta_v} < 0, \tag{20}$$

$$\frac{\phi_{\gamma_2}\left(y^{[1]}(v)\right)}{y^{[2]}(v)} > H_1^{\gamma_2}(v,u), \tag{21}$$

$$\frac{\phi_{\gamma}(y(v))}{y^{[2]}(v)} > H_2^{\gamma_2}(v,u), \tag{22}$$

and

$$\frac{\phi_{\gamma_1}(y(v))}{y^{[1]}(v)} > \frac{H_2(v,u)}{H_1(v,u)}. \tag{23}$$

Proof. Suppose, without losing generality, that $y(g(\xi)) > 0$ on $[\xi_0,\infty)_{\mathbb{T}}$. By dint of (13), $\left(y^{[2]}(\xi)\right)^{\Delta} < 0$ on $[\xi_0,\infty)_{\mathbb{T}}$. Therefore,

$$\begin{aligned} y^{[1]}(v) > y^{[1]}(v) - y^{[1]}(u) &= \int_u^v \phi_{\gamma_2}^{-1}\left(y^{[2]}(\tau)\right)H_0(\tau,u)\Delta\tau \\ &\geq \phi_{\gamma_2}^{-1}\left(y^{[2]}(v)\right)\int_u^v H_0(\tau,u)\Delta\tau \\ &= \phi_{\gamma_2}^{-1}\left(y^{[2]}(v)\right)H_1(v,u). \end{aligned} \tag{24}$$

Thus, (21) holds for $u, v \in \xi_0,\infty)_{\mathbb{T}}$. From (24), we have

$$y^{\Delta}(v) > \phi_{\gamma}^{-1}\left(y^{[2]}(v)\right)\left(\frac{H_1(v,u)}{p_1(v)}\right)^{1/\gamma_1}. \tag{25}$$

Replacing v by s in (25) and integrating with respect to s from u to v gives

$$\begin{aligned} y(v) &\geq \int_u^v \phi_{\gamma}^{-1}\left(y^{[2]}(\tau)\right)\left(\frac{H_1(\tau,u)}{p_1(\tau)}\right)^{1/\gamma_1}\Delta\tau \\ &\geq \phi_{\gamma}^{-1}\left(y^{[2]}(v)\right)\int_u^v\left(\frac{H_1(\tau,u)}{p_1(\tau)}\right)^{1/\gamma_1}\Delta\tau \\ &= \phi_{\gamma}^{-1}\left(y^{[2]}(v)\right)H_2^{1/\gamma_1}(v,u). \end{aligned} \tag{26}$$

Thus, (22) holds for $u, v \in \xi_0,\infty)_{\mathbb{T}}$. By virtue of (24), there is a $v \in (u,\infty)_{\mathbb{T}}$ such that

$$\left(\frac{y^{[1]}(v)}{H_1(v,u)}\right)^{\Delta_v} < 0 \text{for} v \in (u,\infty)_{\mathbb{T}} \subseteq [\xi_0,\infty)_{\mathbb{T}}. \tag{27}$$

Hence, for $v \in (u,\infty)_{\mathbb{T}}$,

$$\begin{aligned} y(v) &> \int_u^v \phi_{\gamma_1}^{-1}\left(\frac{y^{[1]}(\tau)}{H_1(\tau,u)}\right)\left(\frac{H_1(\tau,u)}{p_1(\tau)}\right)^{1/\gamma_1}\Delta\tau \\ &\geq \phi_{\gamma_1}^{-1}\left(\frac{y^{[1]}(v)}{H_1(v,u)}\right)\int_u^v\left(\frac{H_1(\tau,u)}{p_1(\tau)}\right)^{1/\gamma_1}\Delta\tau \\ &= \phi_{\gamma_1}^{-1}\left(\frac{y^{[1]}(v)}{H_1(v,u)}\right)H_2^{1/\gamma_1}(v,u). \end{aligned} \tag{28}$$

Thus, (23) holds for $v \in (u,\infty)_{\mathbb{T}}$. The proof is now complete. □

2.1. Asymptotic Behavior. In this subsection, we debate the asymptotic behavior of the solutions of equation (1) for both cases $\beta \geq \gamma$ and $\beta \leq \gamma$.

Theorem 4. *Assume that $l > 0$ and for sufficiently large $T \in [\xi_0,\infty)_{\mathbb{T}}$,*

$$\liminf_{\xi\longrightarrow\infty} H_1^{\gamma_2}(\xi,T)\int_{\xi}^{\infty}\left(\frac{H_2^{\beta}(\psi(\tau),T)}{H_1^{\alpha}(\tau,T)}\right)^{1/\gamma_1}p(\tau)\Delta\tau > \frac{\gamma_2^{\gamma_2}}{l^{\gamma_2|\gamma_2-1|}(\gamma_2+1)^{\gamma_2+1}}. \tag{29}$$

If equation (1) has a nonoscillatory solution $y(\xi)$, then $y^{[i]}(\xi)$ and $i = 0, 1, 2$ converge.

Proof. Suppose, without losing generality, that $y(\xi)$ and $y(g(\xi))$ are eventually positive, by virtue of (1), we deduce that $y^{[i]}(\xi)$ and $i = 1, 2$ are eventually of one sign and by (2), and we can easily see that $y^{[2]}(\xi)$ is eventually positive, see [34], part ($\mathcal{I}$) of the proof of Theorem 4]. Therefore, we consider the following two cases:

$(\mathcal{I}) y^{[1]}(\xi)$ is eventually positive. In this case, there is a $\xi_1 \in [\xi_0,\infty)_{\mathbb{T}}$ such that

$$y^{[i]}(\xi) > 0, i = 0, 1, 2, \text{and} \left(y^{[2]}(\xi)\right)^{\Delta} < 0 \text{ on } [\xi_1,\infty)_{\mathbb{T}}. \quad (30)$$

Consider

$$z(\xi) := \frac{y^{[2]}(\xi)}{\left(y^{[1]}(\xi)\right)^{\gamma_2}}. \quad (31)$$

Hence,

$$\begin{aligned} z^{\Delta}(\xi) &= \frac{\left(y^{[2]}(\xi)\right)^{\Delta}}{\left(y^{[1]}(\xi)\right)^{\gamma_2}} + \left(\frac{1}{\left(y^{[1]}(\xi)\right)^{\gamma_2}}\right)^{\Delta} y^{[2]}(\sigma(\xi)) \\ &= \frac{\left(y^{[2]}(\xi)\right)^{\Delta}}{\left(y^{[1]}(\xi)\right)^{\gamma_2}} - \frac{\left(\left(y^{[1]}(\xi)\right)^{\gamma_2}\right)^{\Delta}}{\left(y^{[1]}(\xi)\right)^{\gamma_2}} \frac{y^{[2]}(\sigma(\xi))}{\left(y^{[1]}(\sigma(\xi))\right)^{\gamma_2}}. \end{aligned} \quad (32)$$

(1) If $0 < \gamma_2 \le 1$, by means of the Pötzsche chain rule and the definitions of $H_0(\xi, \xi_1)$ and $z(\xi)$, we obtain for $\xi \in [\xi_2,\infty)_{\mathbb{T}}$

$$\begin{aligned} z^{\Delta}(\xi) &\le \frac{\left(y^{[2]}(\xi)\right)^{\Delta}}{\left(y^{[1]}(\xi)\right)^{\gamma_2}} - \gamma_2 \frac{\left(y^{[1]}(\sigma(\xi))\right)^{\gamma_2-1} \left[y^{[1]}(\xi)\right]^{\Delta}}{\left(y^{[1]}(\xi)\right)^{\gamma_2}} z(\sigma(\xi)) \\ &= \frac{\left(y^{[2]}(\xi)\right)^{\Delta}}{\left(y^{[1]}(\xi)\right)^{\gamma_2}} - \gamma_2 H_0(\xi, \xi_1) \left(\frac{y^{[1]}(\xi)}{y^{[1]}(\sigma(\xi))}\right)^{1-\gamma_2} \frac{\left[y^{[2]}(\xi)\right]^{1/\gamma_2}}{y^{[1]}(\xi)} z(\sigma(\xi)) \\ &= \frac{\left(y^{[2]}(\xi)\right)^{\Delta}}{\left(y^{[1]}(\xi)\right)^{\gamma_2}} - \gamma_2 H_0(\xi, \xi_1) \left(\frac{y^{[1]}(\xi)}{y^{[1]}(\sigma(\xi))}\right)^{1-\gamma_2} z^{1/\gamma_2}(\xi) z(\sigma(\xi)). \end{aligned} \quad (33)$$

Using the fact that $\left(y^{[1]}(\xi)/H_1(\xi, \xi_1)\right)^{\Delta} < 0$ for $\xi \in [\xi_2,\infty)_{\mathbb{T}} \subseteq (\xi_1,\infty)_{\mathbb{T}}$, we get

$$\left(\frac{y^{[1]}(\xi)}{y^{[1]}(\sigma(\xi))}\right)^{1-\gamma_2} \ge \left(\frac{H_1(\xi, \xi_1)}{H_1(\sigma(\xi), \xi_1)}\right)^{1-\gamma_2}. \quad (34)$$

Therefore, (33) becomes

$$z^{\Delta}(\xi) \le \frac{\left(y^{[2]}(\xi)\right)^{\Delta}}{\left(y^{[1]}(\xi)\right)^{\gamma_2}} - \gamma_2 H_0(\xi, \xi_1) \left(\frac{H_1(\xi, \xi_1)}{H_1(\sigma(\xi), \xi_1)}\right)^{1-\gamma_2} z^{1/\gamma_2}(\xi) z(\sigma(\xi)). \quad (35)$$

Now, for any $\varepsilon > 0$, there exists a $\xi_3 \in \xi_2,\infty)_{\mathbb{T}}$ such that

$$\frac{H_1(\xi, \xi_1)}{H_1(\sigma(\xi), \xi_1)} \ge l - \varepsilon \quad \text{and} \quad H_1^{\gamma_2}(\xi, \xi_1) z(\xi) \ge H - \varepsilon \text{ for } \xi \in \xi_3,\infty)_{\mathbb{T}}, \quad (36)$$

where

$$H := \liminf_{\xi \longrightarrow \infty} H_1^{\gamma_2}(\xi, \xi_1) z(\xi), \quad 0 \le H \le 1. \quad (37)$$

Substituting (36) into (35), we get for $\xi \in \xi_3,\infty)_{\mathbb{T}}$,

$$\begin{aligned} z^{\Delta}(\xi) &\le \frac{\left(y^{[2]}(\xi)\right)^{\Delta}}{\left(y^{[1]}(\xi)\right)^{\gamma_2}} - (H-\varepsilon)^{1+1/\gamma_2} \frac{\gamma_2 H_0(\xi, \xi_1)}{H_1^{\gamma_2}(\xi, \xi_1) H_1(\sigma(\xi), \xi_1)} \\ &\le \frac{\left(y^{[2]}(\xi)\right)^{\Delta}}{\left(y^{[1]}(\xi)\right)^{\gamma_2}} - (H-\varepsilon)^{1+1/\gamma_2} (l-\varepsilon)^{1-\gamma_2} \frac{\gamma_2 H_0(\xi, \xi_1)}{H_1(\xi, \xi_1) H_1^{\gamma_2}(\sigma(\xi), \xi_1)} \\ &\le \frac{\left(y^{[2]}(\xi)\right)^{\Delta}}{\left(y^{[1]}(\xi)\right)^{\gamma_2}} - (H-\varepsilon)^{1+1/\gamma_2} (l-\varepsilon)^{1-\gamma_2} \left(\frac{-1}{H_1^{\gamma_2}(\xi, \xi_1)}\right)^{\Delta}. \end{aligned} \quad (38)$$

(2) If $\gamma_2 \ge 1$, by means of the Potzsche chain rule and the definitions of $H_0(\xi, \xi_1)$ and $z(\xi)$, we obtain for $\xi \in [\xi_2,\infty)_{\mathbb{T}}$

$$\begin{aligned} z^{\Delta}(\xi) &\le \frac{\left(y^{[2]}(\xi)\right)^{\Delta}}{\left(y^{[1]}(\xi)\right)^{\gamma_2}} - \gamma_2 \frac{\left(y^{[1]}(\xi)\right)^{\gamma_2-1} \left[y^{[1]}(\xi)\right]^{\Delta}}{\left(y^{[1]}(\xi)\right)^{\gamma_2}} z(\sigma(\xi)) \\ &= \frac{\left(y^{[2]}(\xi)\right)^{\Delta}}{\left(y^{[1]}(\xi)\right)^{\gamma_2}} - \gamma_2 H_0(\xi, \xi_1) \frac{\left[y^{[2]}(\xi)\right]^{1/\gamma_2}}{y^{[1]}(\xi)} z(\sigma(\xi)) \\ &= \frac{\left(y^{[2]}(\xi)\right)^{\Delta}}{\left(y^{[1]}(\xi)\right)^{\gamma_2}} - \gamma_2 H_0(\xi, \xi_1) z^{1/\gamma_2}(\xi) z(\sigma(\xi)). \end{aligned} \quad (39)$$

Substituting (36) into (39), we get for $\xi \in \xi_3,\infty)_{\mathbb{T}}$,

$$\begin{aligned} z^{\Delta}(\xi) &\le \frac{\left(y^{[2]}(\xi)\right)^{\Delta}}{\left(y^{[1]}(\xi)\right)^{\gamma_2}} - (H-\varepsilon)^{1+1/\gamma_2} \frac{\gamma_2 H_0(\xi, \xi_1)}{H_1(\xi, \xi_1) H_1^{\gamma_2}(\sigma(\xi), \xi_1)} \\ &\le \frac{\left(y^{[2]}(\xi)\right)^{\Delta}}{\left(y^{[1]}(\xi)\right)^{\gamma_2}} - (H-\varepsilon)^{1+1/\gamma_2} (l-\varepsilon)^{\gamma_2-1} \frac{\gamma_2 H_0(\xi, \xi_1)}{H_1^{\gamma_2}(\xi, \xi_1) H_1(\sigma(\xi), \xi_1)} \\ &\le \frac{\left(y^{[2]}(\xi)\right)^{\Delta}}{\left(y^{[1]}(\xi)\right)^{\gamma_2}} - (H-\varepsilon)^{1+1/\gamma_2} (l-\varepsilon)^{\gamma_2-1} \left(\frac{-1}{H_1^{\gamma_2}(\xi, \xi_1)}\right)^{\Delta}. \end{aligned} \quad (40)$$

Combining (38) with (40), we conclude that for $\gamma_2 > 0$ and $\xi \in \xi_3,\infty)_{\mathbb{T}}$,

$$z^{\Delta}(\xi) \le \frac{\left(y^{[2]}(\xi)\right)^{\Delta}}{\left(y^{[1]}(\xi)\right)^{\gamma_2}} - (H-\varepsilon)^{1+1/\gamma_2} (l-\varepsilon)^{|\gamma_2-1|} \left(\frac{-1}{H_1^{\gamma_2}(\xi, \xi_1)}\right)^{\Delta}. \quad (41)$$

From (1), we see that

$$z^{\Delta}(\xi) \le -\left(\frac{y^{\gamma_1}(g(\xi))}{y^{[1]}(\xi)}\right)^{\beta/\gamma_1}\left(y^{[1]}(\xi)\right)^{\beta/\gamma_1-\gamma_2} p(\xi) - (H-\varepsilon)^{1+1/\gamma_2}(l-\varepsilon)^{|\gamma_2-1|}\left(\frac{-1}{H_1^{\gamma_2}(\xi,\xi_1)}\right)^{\Delta}. \tag{42}$$

If $\beta \ge \gamma$, by the fact that $y^{[2]}(\xi) > 0$ for $\xi \in [\xi_3,\infty)_{\mathbb{T}}$, then

$$\left(y^{[1]}(\xi)\right)^{\beta/\gamma_1-\gamma_2} \ge \left(y^{[1]}(\xi_0)\right)^{\beta/\gamma_1-\gamma_2} =: k_1 > 0 \quad \text{for} \xi \ge \xi_3, \tag{43}$$

whereas if $\beta \le \gamma$, by the fact that $(y^{[1]}(\xi)/H_1(\xi,\xi_1))^{\Delta} < 0$ for $\xi \in [\xi_3,\infty)_{\mathbb{T}}$, then

$$\left(\frac{y^{[1]}(\xi)}{H_1(\xi,\xi_1)}\right)^{\beta/\gamma_1-\gamma_2} \ge \left(\frac{y^{[1]}(\xi_1)}{H_1(\xi_2,\xi_1)}\right)^{\beta/\gamma_1-\gamma_2} =: k_2 > 0 \quad \text{for} \xi \ge \xi_3. \tag{44}$$

Now, consider the case when $g(\xi) \le \xi$. From (23) and using the fact that $(y^{[1]}(\xi)/H_1(\xi,\xi_1))^{\Delta} < 0$, we deduce that

$$y^{\gamma_1}(g(\xi)) \ge \frac{y^{[1]}(g(\xi))}{H_1(g(\xi),\xi_1)} H_2(g(\xi),\xi_1) \ge \frac{y^{[1]}(\xi)}{H_1(\xi,\xi_1)} H_2(g(\xi),\xi_1). \tag{45}$$

Next consider the case when $g(\xi) \ge \xi$. From (23) and using the fact that $y^{[1]}(\xi) > 0$, we conclude that

$$y^{\gamma_1}(g(\xi)) \ge y^{\gamma_1}(\xi) \ge \frac{y^{[1]}(\xi)}{H_1(\xi,\xi_1)} H_2(\xi,\xi_1). \tag{46}$$

By combining (45) with (46), we have

$$\frac{y^{\gamma_1}(g(\xi))}{y^{[1]}(\xi)} \ge \frac{H_2(\psi(\xi),\xi_1)}{H_1(\xi,\xi_1)} \text{ for } \xi \in \xi_3,\infty)_{\mathbb{T}}. \tag{47}$$

From (43), (44), and (47), we get for $\xi \in \xi_3,\infty)_{\mathbb{T}}$,

$$\left(\frac{y^{\gamma_1}(g(\xi))}{y^{[1]}(\xi)}\right)^{\beta/\gamma_1}\left(y^{[1]}(\xi)\right)^{\beta/\gamma_1-\gamma_2} \ge k\left(\frac{H_2^{\beta}(\psi(\xi),\xi_1)}{H_1^{\alpha}(\xi,\xi_1)}\right)^{1/\gamma_1}, \tag{48}$$

where $k := \min\{k_1, k_2\}$. Substituting (47) into (42), we obtain for $\xi \in \xi_3,\infty)_{\mathbb{T}}$,

$$z^{\Delta}(\xi) \le -k\left(\frac{H_2^{\beta}(\psi(\xi),\xi_1)}{H_1^{\alpha}(\xi,\xi_1)}\right)^{1/\gamma_1} p(\xi) - (H-\varepsilon)^{1+}1/\gamma_2(l-\varepsilon)^{|\gamma_2-1|}\left(\frac{-1}{H_1^{\gamma_2}(\xi,\xi_1)}\right)^{\Delta}. \tag{49}$$

Integrating (49) from ξ to v, we have

$$z(v) - z(\xi) \le k\int_{\xi}^{v}\left(\frac{H_2^{\beta}(\psi(\tau),\xi_1)}{H_1^{\alpha}(\tau,\xi_1)}\right)^{1/\gamma_1} p(\tau)\Delta\tau - (H-\varepsilon)^{1+1/\gamma_2}(l-\varepsilon)^{|\gamma_2-1|}\left(\frac{1}{H_1^{\gamma_2}(\xi,\xi_1)} - \frac{1}{H_1^{\gamma_2}(v,\xi_1)}\right). \tag{50}$$

Taking into consideration that $z > 0$ and passing to the limit as $v \longrightarrow \infty$, we conclude that

$$k\int_{\xi}^{\infty}\left(\frac{H_2^{\beta}(\psi(\tau),\xi_1)}{H_1^{\alpha}(\tau,\xi_1)}\right)^{1/\gamma_1} p(\tau)\Delta\tau \le z(\xi) - (H-\varepsilon)^{1+1/\gamma_2}(l-\varepsilon)^{|\gamma_2-1|}\left(\frac{1}{H_1^{\gamma_2}(\xi,\xi_1)}\right). \tag{51}$$

Multiplying both sides of (51) by $H_1^{\gamma_2}(\xi,\xi_1)$, we get for $\xi \in \xi_3,\infty)_{\mathbb{T}}$,

$$kH_1^{\gamma_2}(\xi,\xi_1)\int_{\xi}^{\infty}\left(\frac{H_2^{\beta}(\psi(\tau),\xi_1)}{H_1^{\alpha}(\tau,\xi_1)}\right)^{1/\gamma_1} p(\tau)\Delta\tau \le H_1^{\gamma_2}(\xi,\xi_1)z(\xi) - (H-\varepsilon)^{1+1/\gamma_2}(l-\varepsilon)^{|\gamma_2-1|}. \tag{52}$$

Taking the liminf of both sides of the last inequality (52) as $\xi \longrightarrow \infty$, we obtain

$$k\liminf_{\xi\longrightarrow\infty} H_1^{\gamma_2}(\xi,\xi_1)\int_{\xi}^{\infty}\left(\frac{H_2^{\beta}(\psi(\tau),\xi_1)}{H_1^{\alpha}(\tau,\xi_1)}\right)^{1/\gamma_1} p(\tau)\Delta\tau \le H - (H-\varepsilon)^{1+1/\gamma_2}(l-\varepsilon)^{|\gamma_2-1|}. \tag{53}$$

By dint of the fact that $k, \varepsilon > 0$ is arbitrary, we deduce that

$$\liminf_{\xi\longrightarrow\infty} H_1^{\gamma_2}(\xi,\xi_1)\int_{\xi}^{\infty}\left(\frac{H_2^{\beta}(\psi(\tau),\xi_1)}{H_1^{\alpha}(\tau,\xi_1)}\right)^{1/\gamma_1} p(\tau)\Delta\tau \le H - H^{1+1/\gamma_2} l^{|\gamma_2-1|}. \tag{54}$$

Setting

$$X := H l^{\gamma_2|\gamma_2-1|/1+\gamma_2} \quad \text{and} \quad Y := \left(\frac{\gamma_2}{1+\gamma_2}\right)^{\gamma_2} l^{-\gamma_2^2|\gamma_2-1|/\gamma_2+1}, \tag{55}$$

and $\lambda := \gamma_2 + 1/\gamma_2 > 1$. Then, using inequality (see [35])

$$\lambda X Y^{\lambda-1} - X^{\lambda} \le (\lambda-1) Y^{\lambda}, \tag{56}$$

we achieve

$$H - H^{1+1/\gamma_2} l^{|\gamma_2-1|} \le \frac{\gamma_2^{\gamma_2}}{l^{\gamma_2|\gamma_2-1|}(\gamma_2+1)^{\gamma_2+1}}. \tag{57}$$

Thus, (54) becomes

$$\liminf_{\xi\longrightarrow\infty} H_1^{\gamma_2}(\xi,\xi_1)\int_{\xi}^{\infty}\left(\frac{H_2^{\beta}(\psi(\tau),\xi_1)}{H_1^{\alpha}(\tau,\xi_1)}\right)^{1/\gamma_1} p(\tau)\Delta\tau \le \frac{\gamma_2^{\gamma_2}}{l^{\gamma_2|\gamma_2-1|}(\gamma_2+1)^{\gamma_2+1}}, \tag{58}$$

as a result of which there is a contradiction with (29).

$(\mathscr{I}\mathscr{I})y^{[1]}(\xi)$ is eventually positive. In this case, there is a $\xi_1\in[\xi_0,\infty)_{\mathbb{T}}$ such that

$$(-1)^i y^{[i]}(\xi)>0,\ i=0,1,2, \text{and} \left(y^{[2]}(\xi)\right)^{\Delta}<0 \text{ on } [\xi_1,\infty)_{\mathbb{T}}. \tag{59}$$

By virtue of (59), it is easy to see that $y^{[i]}(\xi)$ and $i=0,1,2$ converge. This completes the proof. □

Remark 5. [1] The deduction of Theorem 4 keeps intact if assumption (29) is replaced by

$$\int_{\xi_0}^{\infty}\left(\frac{H_2^{\beta}(\psi(\tau),T)}{H_1^{\alpha}(\tau,T)}\right)^{1/\gamma_1} p(\tau)\Delta\tau=\infty. \tag{60}$$

2) If either

$$\int_{\xi_0}^{\infty} p(\tau)\Delta\tau=\infty;\int_{\xi_0}^{\infty}\left(\frac{1}{p_2(\tau)}\int_{\tau}^{\infty}p(\omega)\Delta\omega\right)^{1/\gamma_2}\Delta\tau=\infty \tag{61}$$

or

$$\int_{\xi_0}^{\infty}\left[\frac{1}{p_1(\tau)}\int_{\tau}^{\infty}\left(\frac{1}{p_2(\omega)}\int_{\omega}^{\infty}p(s)\Delta s\right)^{1/\gamma_2}\Delta\omega\right]^{1/\gamma_1}\Delta\tau=\infty, \tag{62}$$

then nonoscillatory solutions of the investigated equation (1) are convergent to zero, see [35], [Theorem 4].

2.2. Oscillation Criteria. In this subsection, we establish oscillation criteria of the solutions of equation (1) when $\beta=\gamma$ and $g(\xi)\le\xi$ on $[\xi_0,\infty)_{\mathbb{T}}$. This solves an open problem posed in [1].

Theorem 6. *Assume that (29) and either*

$$\limsup_{\xi\longrightarrow\infty}\int_{g(\xi)}^{\xi} H_2^{\gamma_2}(g(\tau),g(\xi))p(\tau)\Delta\tau>1, \tag{63}$$

or

$$\limsup_{\xi\longrightarrow\infty}\int_{g(\xi)}^{\xi}\left(\frac{1}{p_1(\tau)}\int_{\tau}^{\xi}\left(\frac{1}{p_2(\omega)}\int_{\omega}^{\xi}p(s)\Delta s\right)^{1/\gamma_2}\Delta\omega\right)^{1/\gamma_1}\Delta\tau>1, \tag{64}$$

hold. Then, all solutions to equation (1) are oscillatory.

Proof. Suppose, without losing generality, that $y(\xi)$ and $y(g(\xi))$ are eventually positive, by virtue of (1), we deduce that $y^{[i]}(\xi)$ and $i=1,2$ are eventually of one sign and by (2), we can easily see that $y^{[2]}(\xi)$ is eventually positive. Therefore, we consider the following two cases:

$(\mathscr{I})y^{[1]}(\xi)$ is eventually positive. The same proof as in part $(\mathscr{I})$ of the proof of Theorem 4 hence is omitted.

$(\mathscr{I}\mathscr{I})y^{[1]}(\xi)$ is eventually positive. In this case, there is a $\xi_1\in[\xi_0,\infty)_{\mathbb{T}}$ such that (59) for $\xi\in[\xi_1,\infty)_{\mathbb{T}}$.

Let (63) hold. Integrating (1) form $g(\xi)$ to ξ gives

$$y^{[2]}(g(\xi))\ge y^{[2]}(g(\xi))-y^{[2]}(\xi)=\int_{g(\xi)}^{\xi}p(\tau)\phi_{\gamma}(y(g(\tau)))\Delta\tau. \tag{65}$$

By using Lemma 2 and setting $u=g(\tau)$ and $v=g(\xi)$, we obtain

$$\phi_{\gamma}(y(g(\tau)))\ge H_2^{\gamma_2}(g(\tau),g(\xi))y^{[2]}(g(\xi)). \tag{66}$$

Substituting (66) into (65) yields

$$y^{[2]}(g(\xi))\ge y^{[2]}(g(\xi))\int_{g(\xi)}^{\xi}H_2^{\gamma_2}(g(\tau),g(\xi))p(\tau)\Delta\tau. \tag{67}$$

We get a contradiction with (63) by taking the lim sup on both sides of the last inequality as $\xi\longrightarrow\infty$.

Let (64) hold. Integrating (1) form τ to ξ, we get

$$y^{[2]}(\omega)\ge y^{[2]}(\omega)-y^{[2]}(\xi)=\int_{\omega}^{\xi}p(s)y^{\gamma}(g(s))\Delta s\ge y^{\gamma}(g(\xi))\int_{\omega}^{\xi}p(s)\Delta s. \tag{68}$$

Subsequently,

$$\left(y^{[1]}(\omega)\right)^{\Delta}\ge y^{\gamma_1}(g(\xi))\left(\frac{1}{p_2(\omega)}\int_{\omega}^{\xi}p(s)\Delta s\right)^{1/\gamma_2}. \tag{69}$$

Integrating the last inequality form τ to ξ, we conclude that

$$-y^{[1]}(\tau)\ge y^{[1]}(\xi)-y^{[1]}(\tau)\ge y^{\gamma_1}(g(\xi))\int_{\tau}^{\xi}\left(\frac{1}{p_2(\omega)}\int_{\omega}^{\xi}p(s)\Delta s\right)^{1/\gamma_2}\Delta\omega. \tag{70}$$

As a result,

$$-y^{\Delta}(\tau)\ge y(g(\xi))\left(\frac{1}{p_1(\tau)}\int_{\tau}^{\xi}\left(\frac{1}{p_2(\omega)}\int_{\omega}^{\xi}p(s)\Delta s\right)^{1/\gamma_2}\Delta\omega\right)^{1/\gamma_1}. \tag{71}$$

Integrating again the last inequality form $g(\xi)$ to ξ, we achieve

$$y(g(\xi)) \geq y(g(\xi)) \int_{g(\xi)}^{\xi} \left(\frac{1}{p_1(\tau)} \int_{\tau}^{\xi} \left(\frac{1}{p_2(\omega)} \int_{\omega}^{\xi} p(s) \Delta s \right)^{1/\gamma_2} \Delta\omega \right)^{1/\gamma_1} \Delta\tau. \tag{72}$$

We get a contradiction with (64) by taking the lim sup on both sides of the last inequality as $\xi \longrightarrow \infty$. The proof is complete. □

Example 7. Consider the third-order delay dynamic equation

$$\left\{ \frac{1}{4\xi^2} \phi_2 \left(\left[\frac{1}{4^{10}} \left(\frac{3}{\xi} \right)^3 \phi_3 \left(y^{\Delta}(\xi) \right) \right]^{\Delta} \right) \right\}^{\Delta} + \frac{1}{\xi^{17}} \phi_6 \left(y \left(\frac{\xi}{2} \right) \right) = 0, \quad \xi \in [1, \infty). \tag{73}$$

It is obvious that condition (2) is fulfilled. Now,

$$\begin{aligned} &\limsup_{\xi \longrightarrow \infty} \int_{g(\xi)}^{\xi} H_2^{\gamma_2}(g(\tau), g(\xi)) p(\tau) \Delta\tau = \limsup_{\xi \longrightarrow \infty} \int_{\xi/2}^{\xi} \frac{4^{11}}{\tau^{17}} \left(\frac{\tau^2}{4} - \frac{\xi^2}{4} \right)^8 d\tau = \frac{664548}{35} + 64 \ln(2), \\ &\liminf_{\xi \longrightarrow \infty} H_1^{\gamma_2}(\xi, \xi_0) \int_{\xi}^{\infty} \left(\frac{H_2^{\beta}(\psi(\tau), T)}{H_1^{\alpha}(\tau, T)} \right)^{1/\gamma_1} p(\tau) \Delta\tau = \liminf_{\xi \longrightarrow \infty} \left(\xi^2 - 1 \right)^2 \int_{\xi}^{\infty} \left(\frac{2(4)^5 \left(\tau^2/4 - 1 \right)^4}{\tau^2 - 1} \right)^2 \frac{1}{\tau^{17}} d\tau = 16. \end{aligned} \tag{74}$$

Therefore, the conditions (63) and (29) are satisfied. Then, all solutions to equation (73) are oscillatory, according to Theorem 6.

3. Conclusions

(1) The important point to note here is that the proposed results in Theorem 6 are new for third-order dynamic equation (1) and also, these results, in a special case, have answered an open problem stated by [1] (Remark 3.3), which is concerned with studying the sufficient conditions that guarantee that all solutions of third-order delay dynamic equations oscillate

(2) In comparison to the results in the cited papers [14–16, 18], Hille-type criteria for dynamic equation (1) in the cases $\beta \geq \gamma, \beta \leq \gamma, g(\xi) \leq \xi$, and $g(t) \geq \xi$ have been developed, and the results in this study are a significant improvement; for more details, see ([19], in Discussions and Conclusions). Moreover, our results improve and expand upon those described in [19], see the following details

(i) If $\beta = \gamma$ and $0 < \gamma_2 \leq 1$, then condition (29) becomes

$$\liminf_{\xi \longrightarrow \infty} H_1^{\gamma_2}(\xi, T) \int_{\xi}^{\infty} \left(\frac{H_2(\psi(\tau), T)}{H_1(\tau, T)} \right)^{\gamma_2} p(\tau) \Delta\tau > \frac{\gamma_2^{\gamma_2}}{l^{\gamma_2(1-\gamma_2)} (\gamma_2 + 1)^{\gamma_2 + 1}}. \tag{75}$$

By dint of

$$H_1^{\gamma_2}(\xi, T) \int_{\xi}^{\infty} \left(\frac{H_2(\psi(\tau), T)}{H_1(\tau, T)} \right)^{\gamma_2} p(\tau) \Delta\tau \geq H_1^{\gamma_2}(\xi, T) \int_{\xi}^{\infty} \left(\frac{H_2(\psi(\tau), T)}{H_1(\sigma(\tau), T)} \right)^{\gamma_2} p(\tau) \Delta\tau, \tag{76}$$

Hille-type criterion (75) improves (5);

(ii) Condition (29) reduces to (6) in the case where $\beta = \gamma$ and $\gamma_2 \geq 1$

(3) The asymptotic behavior of solutions is viable to dynamic equation (1) for both $g(\xi) \leq \xi$ and $g(\xi) \geq \xi$, whereas the oscillation conditions are viable to dynamic equation (1) for $g(\xi) \leq \xi$. As a result, oscillations can be ensured by a delay in equations

(4) The results presented here are for equation (1) on an unbounded above arbitrary time scale; therefore, they are applicable to different of time scales

(5) It would be interesting to define Hille-type criteria for the third-order dynamic equation (1) under noncanonical assumptions

$$\int_{\xi_0}^{\infty} \frac{\Delta\tau}{p_i^{1/\gamma_i}(\tau)} < \infty, i = 1, 2. \tag{77}$$

Data Availability

The numerical data used to support the findings of this study are included in the article.

Conflicts of Interest

The authors declare that they have no competing interests. There are no any nonfinancial competing interests (political, personal, religious, ideological, academic, intellectual, commercial, or any other) to declare in relation to this manuscript.

Authors' Contributions

Hassan oversaw the study and help inspection. All the authors carried out the main results of this article and drafted the manuscript and read and approved the final manuscript.

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References

[1] S. Hilger, "Analysis on measure chains — a unified approach to continuous and discrete calculus," *Results in Mathematics*, vol. 18, no. 1-2, pp. 18–56, 1990.

[2] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.

[3] R. P. Agarwal, M. Bohner, D. O'Regan, and A. Peterson, "Dynamic equations on time scales: a survey," *Journal of Computational and Applied Mathematics*, vol. 141, no. 1-2, pp. 1–26, 2002.

[4] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

[5] E. Hille, "Non-oscillation theorems," *Transactions of the American Mathematical Society*, vol. 64, no. 2, pp. 234–252, 1948.

[6] J. S. Wong, "Second order oscillation with retarded arguments," in *Ordinary differential equations*, pp. 581–596, Academic press, New York and London, 1972.

[7] L. Erbe, "Oscillation criteria for second order nonlinear delay equations," *Canadian Mathematical Bulletin*, vol. 16, no. 1, pp. 49–56, 1973.

[8] T. S. Hassan, R. A. El-Nabulsi, and A. Abdel Menaem, "Amended criteria of oscillation for nonlinear functional dynamic equations of second-order," *Mathematics*, vol. 9, no. 11, p. 1191, 2021.

[9] T. S. Hassan, Y. Sun, and A. Abdel Menaem, "Improved oscillation results for functional nonlinear dynamic equations of second order," *Mathematics*, vol. 8, no. 11, p. 1897, 2020.

[10] M. Bohner, T. S. Hassan, and T. Li, "Fite-Hille-Wintner-type oscillation criteria for second-order half-linear dynamic equations with deviating arguments," *Indagationes Mathematicae*, vol. 29, no. 2, pp. 548–560, 2018.

[11] L. Erbe, T. S. Hassan, A. Peterson, and S. H. Saker, "Oscillation criteria for half-linear delay dynamic equations on time scales," *Nonlinear Dynamics and Systems Theory*, vol. 9, pp. 51–68, 2009.

[12] L. Erbe, T. S. Hassan, A. Peterson, and S. H. Saker, "Oscillation criteria for sublinear half-linear delay dynamic equations on time scales," *International Journal of Difference Equations*, vol. 3, pp. 227–245, 2008.

[13] B. Karpuz, "Hille-Nehari theorems for dynamic equations with a time scale independent critical constant," *Applied Mathematics and Computation*, vol. 346, pp. 336–351, 2019.

[14] R. P. Agarwal, M. Bohner, T. Li, and C. Zhang, "Hille and Nehari type criteria for third-order delay dynamic equations," *Journal of Difference Equations and Applications*, vol. 19, no. 10, pp. 1563–1579, 2013.

[15] L. Erbe, A. Peterson, and S. H. Saker, "Hille and Nehari type criteria for third-order dynamic equations," *Journal of Mathematical Analysis and Applications*, vol. 329, no. 1, pp. 112–131, 2007.

[16] S. H. Saker, "Oscillation of third-order functional dynamic equations on time scales," *SCIENCE CHINA Mathematics*, vol. 54, no. 12, pp. 2597–2614, 2011.

[17] T. S. Hassan, R. P. Agarwal, and W. W. Mohammed, "Oscillation criteria for third-order functional half-linear dynamic equations," *Adv. Difference Equ.*, vol. 2017, article 111, 2017.

[18] T. S. Hassan, A. O. Almatroud, M. M. Al-Sawalha, and I. Odinaev, "Asymptotics and Hille-type results for dynamic equations of third order with deviating arguments," *Symmetry*, vol. 2021, no. 13, pp. 1–13, 2021.

[19] T. S. Hassan, R. A. Ramadan, Z. Alsheekhhussain, A. Y. Khedr, A. Abdel Menaem, and I. Odinaev, "Improved Hille oscillation criteria for nonlinear functional dynamic equations of third-order," *Mathematics*, vol. 10, no. 7, p. 1078, 2022.

[20] B. Baculikova, "Oscillation of second-order nonlinear noncanonical differential equations with deviating argument," *Applied Mathematics Letters*, vol. 91, pp. 68–75, 2019.

[21] G. E. Chatzarakis, J. Džurina, and I. Jadlovská, "Oscillatory and asymptotic properties of third-order quasilinear delay differential equations," *Journal of Inequalities and Applications*, vol. 2019, no. 1, 2019.

[22] L. Erbe, T. S. Hassan, and A. Peterson, "Oscillation of third order nonlinear functional dynamic equations on time scales," *Differential Equations and Dynamical Systems*, vol. 18, no. 1-2, pp. 199–227, 2010.

[23] O. Moaaz, R. A. El-Nabulsi, W. Muhsin, and O. Bazighifan, "Improved oscillation criteria for 2nd-order neutral differential equations with distributed deviating arguments," *Mathematics*, vol. 8, no. 5, p. 849, 2020.

[24] T. S. Hassan, "Oscillation of third order nonlinear delay dynamic equations on time scales," *Mathematical and Computer Modelling*, vol. 49, no. 7-8, pp. 1573–1586, 2009.

[25] Z. Han, T. Li, S. Sun, and M. Zhang, "Oscillation behavior of solutions of third-order nonlinear delay dynamic equations on time scales," *Korean Mathematical Society*, vol. 26, no. 3, pp. 499–513, 2011.

[26] T. Li, Z. Han, S. Sun, and Y. Zhao, "Oscillation results for third-order nonlinear delay dynamic equations on time scales," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 34, pp. 639–648, 2011.

[27] T. Li, Z. Han, Y. Sun, and Y. Zhao, "Asymptotic behavior of solutions for third-order half-linear delay dynamic equations on time scales," *Journal of Applied Mathematics and Computing*, vol. 36, no. 1-2, pp. 333–346, 2011.

[28] G. Hovhannisy, "On oscillations of solutions of third-order dynamic equation," *Abstract and Applied Analysis*, vol. 2012, Article ID 715981, 2012.

[29] M. T. Senel, "Behavior of solutions of a third-order dynamic equation on time scales," *Journal of Inequalities and Applications*, vol. 2013, no. 1, 2013.

[30] Y. Sun, Z. Han, Y. Sun, and Y. Pan, "Oscillation theorems for certain third-order nonlinear delay dynamic equations on time scales," *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 75, pp. 1–14, 2011.

[31] Y. Wang and Z. Xu, "Asymptotic properties of solutions of certain third-order dynamic equations," *Journal of Computational and Applied Mathematics*, vol. 236, no. 9, pp. 2354–2366, 2012.

[32] Z. Yu and Q. Wang, "Asymptotic behavior of solutions of third-order nonlinear dynamic equations on time scales,"

Journal of Computational and Applied Mathematics, vol. 225, no. 2, pp. 531–540, 2009.

[33] T. S. Hassan and B. M. El-Matary, "Oscillation criteria for third order nonlinear neutral differential equation," *PLOMS Mathematics*, vol. 1, p. 12, 2021.

[34] T. S. Hassan and Q. Kong, "Asymptotic behavior of third order functional dynamic equations with γ-Laplacian and nonlinearities given by Riemann-Stieltjes integrals," *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 2014, no. 40, pp. 1–21, 2014.

[35] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, second ed. edition, 1988.