Research Article

Existence of Solutions for p-Kirchhoff Problem of Brézis-Nirenberg Type with Singular Terms

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In this paper, we prove the existence of positive solution for a p-Kirchhoff problem of Brézis-Nirenberg type with singular terms, nonlocal term, and the Caffarelli-Kohn-Nirenberg exponent by using variational methods, concentration compactness, and maximum principle.  

1. Introduction and Main Result  

In this paper, we consider the following p-Kirchhoff problem of Brézis-Nirenberg type with singular terms  

\[
\begin{cases}
-\mathcal{H}(u) + \frac{\nabla |u|^{p-2}u}{|x|^\beta} + \mu |u|^{p-2}u + \lambda |u|^{\beta} = 0, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]  

(1)  

where \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^N \), \( 1 < p < N \), \( 0 \in \Omega \), \( \mathcal{H}(u) = a||u||^q + b, a, b, q > 0, 0 \leq \alpha < (N - p)/p \), \( a \leq \beta < \alpha + 1 \), \( 0 \leq \gamma < \rho 

\mu < \bar{\mu} = \left( (N - (\alpha + 1) \rho)/\rho \right) \), \( \lambda > 0 \), \( p^* = pN/(N - p(1 + \beta)) \) is the critical Caffarelli-Kohn-Nirenberg exponent corresponding to the noncompact embedding of \( \mathcal{D}_a(\Omega) \) into \( L_{p^*}(\Omega, |x|^{\beta \gamma}) \), where \( \mathcal{D}_a(\Omega) \) is the closure of \( C_0^\infty(\Omega) \) with respect to the norm  

\[
||u||_p^p = \int_\Omega \frac{|\nabla u|^p}{|x|^\beta} + \mu |u|^p |x|^{p(\alpha+1)} \ dx,
\]  

(2)  

and \( L_{p^*}(\Omega, |x|^{\beta \gamma}) \) denotes the usual weighted \( L_{p^*}(\Omega) \) space with the weight \( |x|^{\beta \gamma} \).  

Kirchhoff-type problems are often referred to as being nonlocal because of the presence of the term \( \mathcal{H}(u) \) which implies that the equation in (1) is no longer a pointwise identity. In the case \( p = 2 \) and \( \alpha = \beta = \gamma = \mu = 0 \), it is analogous to the stationary version of equations that arise in the study of string or membrane vibrations, namely,  

\[
uu - \mathcal{H}(u) \Delta u = g(x, u)
\]  

(3)  

where \( u \) denotes the displacement and \( g(x, u) \) is the external force. Equations of this type were first proposed by Kirchhoff in 1883 [1] to describe the transversal oscillations of a stretched string. These problems serve also to model other physical phenomena as biological systems where \( u \) describes a process which depends on the average of itself (for example, population density). In recent years, Kirchhoff-type problems received much attention, mainly after the famous article of Lions [2]; they have been studied in many papers by using variational methods, see [3–9] and the references therein. The problem (1) without nonlocal term \( (\alpha = 0) \) and without singular terms \( (\alpha = \beta = \gamma = \mu = 0) \) has been treated by Brézis and Nirenberg [10] for \( p = 2 \). Subsequently, an increasing number of researchers have paid attention to...
semilinear or quasilinear elliptic equations with critical exponent of Sobolev or Caffarelli-Kohn-Nirenberg, for example, see [11, 12] and the references therein.

In [7], Naimen generalized the results of [10] to the nonlocal problem (1) with \( N = 3 \) and without singular terms. Kang in [1] generalized the results of [10] to a quasilinear problem with singular terms and without the nonlocal term \((p > 1, \alpha = 0)\) and \((\alpha, \beta, \gamma, \mu) \neq (0, 0, 0, 0))\).

Thus, it is natural for us to consider the quasilinear Brézis-Nirenberg problem in [10] with nonlocal term and singular weights, \((p > 1, \alpha \neq 0)\) and \((\alpha, \beta, \gamma, \mu) \neq (0, 0, 0, 0))\). The competing effect of the nonlocal term with the critical nonlinearity and the lack of compactness of the embedding of \(\mathcal{D}_u(\Omega)\) into \(L^{p^*}(\Omega, |x|^{\beta p^*})\) prevent us from using the variational methods in a standard way. So, motivated by all the works mentioned above, we prove existence results of our problem for large range of \(N\) and under some little, as possible, conditions on \(q\). We show that the existence of solutions depends on the parameter \(\lambda\) and the position of \(q\) with respect to \(p^* - p\). Here, we need more delicate estimates.

To the best of our knowledge, many of the results are new for \(q > 1\), and even in the case \((\alpha, \beta, \gamma, \mu) = (0, 0, 0, 0))\).

Our technique is based on variational methods and concentration compactness argument [2].

Note that if \(\lambda = 0, b = 1, \text{ and } \Omega = \mathbb{R}^N\), the problem (1) has positive radial ground state solution \(u_\varepsilon(\varepsilon > 0))\). Moreover, for any \(\varepsilon > 0\) the extremal functions

\[
v_\varepsilon(x) = e^{-\left(\frac{|x|^p}{\varepsilon}\right)} u_\varepsilon \left(\frac{|x|}{\varepsilon}\right)
\]

is a minimizer for

\[
S := \inf_{u \in \mathcal{D}_u(\Omega) \setminus \{0\}} \frac{\|u\|^p}{\left(\int_{\mathbb{R}^N} \left(\|u\|^p / |x|^{p\beta p^*}\right) dx\right)^{p/p^*}}
\]

and satisfies

\[
\|v_\varepsilon\|^p = \int_{\mathbb{R}^N} \frac{|v_\varepsilon|^{p^*}}{|x|^{p\beta}} dx = S_{p^*/p}^\varepsilon.
\]

On the other hand, for \(0 \leq \alpha < (N - p)/p, \alpha \leq \beta < \alpha + 1, 0 \leq \gamma < p\), and \(0 \leq \mu < \bar{\mu}\), we defined the first eigenvalue of the problem

\[
-\text{div} \left(\frac{\|u\|^{p-2}}{|x|^{p\beta}} \nabla u\right) - \mu \frac{|u|^{p-2}}{|x|^{p\beta+1}} u = \lambda \frac{|u|^{p-2}}{|x|^{p\beta}}, \quad \text{in } \Omega
\]

\[
u = 0, \quad \text{on } \partial \Omega
\]

as

\[
\lambda_1 := \inf_{u \in \mathcal{D}_u(\Omega) \setminus \{0\}} \frac{\|u\|^p}{\int_\Omega \left(\|u\|^p / |x|^{p\beta}\right) dx},
\]

and it is positive, (see for instance [1]).

The main result is concluded as the following theorem.

**Theorem 1.** Let \(1 < p < N, 0 \leq \alpha < (N - p)/p, \alpha \leq \beta < \alpha + 1, \|

Then, the problem (1) has a positive solution in the following cases:

1. \(q = p^* - p\) and \(0 < a < S_{p^*+p/p}^\varepsilon\)
2. \(q < p^* - p\) and \(a > 0\)

Remark 2. In the case where \(\lambda = 0\) and \(\Omega\) is a star-shaped domain with respect to the origin, we can easily verify that the problem (1) has no nontrivial solution by using a Pohozaev-type identity.

This paper is organized as follows. In Section 2, we study the variational framework and give some preliminary results. In Section 3, we show the existence result and we will prove Theorem 1.

## 2. Variational Framework and Preliminary Results

The starting point of the variational approach to problem (1) is the following Caffarelli-Kohn-Nirenberg inequality in [13] which is also called the Hardy-Sobolev inequality. Assume that \(1 < p < N, 0 \leq \alpha < (N - p)/p, \alpha \leq \beta < \alpha + 1, \text{ and then, the most general Pohozaev-type identity holds}\):

\[
\left(\int_\Omega \frac{|u|^{p^*}}{|x|^{p\beta}} dx\right)^{1/p^*} \leq \tilde{C} \left(\int_\Omega \frac{|\nabla u|^p}{|x|^{p\beta+1}} dx\right)^{p/p^*}
\]

for all \(u \in C_0^\infty(\Omega), \tilde{C}\).

\[
(\tilde{C}) \left(\int_\Omega |u|^p / |x|^{p\beta} dx\right) \leq \tilde{C} \left(\int_\Omega \frac{|\nabla u|^p}{|x|^{p\beta+1}} dx\right)^{p/p^*}
\]

for all \(u \in C_0^\infty(\Omega), \tilde{C}\).

Definition 3. We say that \(u \in \mathcal{D}_u(\Omega) \setminus \{0\}\) is a weak solution of equation (1) if

\[
-\text{div} \left(\frac{\|u\|^{p-2}}{|x|^{p\beta}} \nabla u\right) - \mu \frac{|u|^{p-2}}{|x|^{p\beta+1}} u = \lambda \frac{|u|^{p-2}}{|x|^{p\beta}}, \quad \text{in } \Omega
\]

\[
u = 0, \quad \text{on } \partial \Omega
\]
for any $v \in \mathcal{D}_a(\Omega)$.

Next, we define the energy functional

$$I_\lambda(u) = \frac{a}{q + p} \|u\|^{q+p} + \frac{b}{p} \|u\|^p - \int_{\Omega} \frac{|u|^p}{|x|^{p\alpha}} dx - \lambda \int_{\Omega} \frac{|u|^{p-2} u}{|x|^{p\alpha}} dx$$

(13)

associated to problem (1), for all $u \in \mathcal{D}_a(\Omega)$.

Notice that the functional $I_\lambda$ is well defined in $\mathcal{D}_a(\Omega)$ and belongs to $C^1(\mathcal{D}_a(\Omega), \mathbb{R})$ and a critical point of $I_\lambda$ is a weak solution of problem (1).

**Definition 4.** Let $c \in \mathbb{R}$, a sequence $(u_n) \subset \mathcal{D}_a(\Omega)$ is called a $(PS)_c$ sequence (Palais-Smale sequence at level $c$) if

$$I_\lambda(u_n) \to c \quad \text{and} \quad I_\lambda'(u_n) \to 0 \quad \text{as} \quad n \to +\infty.$$ 

(14)

Let $c \in \mathbb{R}$. We say that $I_\lambda$ satisfies the Palais-Smale condition at level $c$, if any $(PS)_c$ sequence contains a convergent subsequence in $\mathcal{D}_a(\Omega)$.

**Lemma 5.** Assume $1 < p < N$, $0 \leq \alpha < (N - p)/p$, $\alpha \leq \beta < \alpha + 1$, $0 \leq \beta < p\alpha$, $b > 0$, $0 < \lambda < b\lambda_1$, and $q \leq p' - p$. Let $c \in \mathbb{R}^+$ and $(u_n) \subset \mathcal{D}_a(\Omega)$ be a $(PS)_c$ sequence for $I_\lambda$. Then,

$$u_n \rightharpoonup u \quad \text{in} \quad \mathcal{D}_a(\Omega)$$

(15)

for some $u \in \mathcal{D}_a(\Omega)$ with $I_\lambda'(u) = 0$.

**Proof.** We have

$$I_\lambda(u_n) \to c,$$

$$I_\lambda'(u_n) \to 0.$$ 

(16)

That is,

$$c + o_n(1) = I_\lambda(u_n),$$

$$o_n(1) \|v\| = \langle I_\lambda'(u_n), v \rangle,$$ 

(17)

for any $v \in \mathcal{D}_a(\Omega)$.

Then, as $n \to +\infty$, it follows that

$$c + o_n(1) - \frac{1}{p} o_n(1) \|u_n\|$$

$$= I_\lambda(u_n) - \frac{1}{p} \langle I_\lambda'(u_n), u_n \rangle$$

$$\geq a p^* - (q + p) \|u_n\|^{q+p} + \left( b - \frac{\lambda}{\lambda_1} \right)^{p^* - p} \|u_n\|^p.$$

(18)

As $\lambda < b\lambda_1$ and $q \leq p^* - p$, we obtain that $(u_n)$ is bounded in $\mathcal{D}_a(\Omega)$. Up to a subsequence if necessary, there exists a function $u \in \mathcal{D}_a(\Omega)$ such that $u_n \to u$ in $\mathcal{D}_a(\Omega)$, $u_n \rightharpoonup u$ in $L^p(\Omega, |x|^{p\alpha})$, $u_n \to u$ in $L^r(\Omega, |x|^{p\alpha})$, for all $r < p^*$ and $u_n \to u$ a.e. on $\Omega$. Then,

$$\langle I_\lambda'(u_n), v \rangle \to 0$$

(19)

and thus $I_\lambda'(u) = 0$. This completes the proof of Lemma 5.

The following lemma is very important for giving the local Palais-Smale condition.

**Lemma 6.** Let $a, b, q > 0, \sigma \geq 1$, and $\bar{y} = ((a/\sigma)S^{q+p})^{1/\sigma - 1}$ for $\sigma > 1$. For $y \geq 0$, we consider the function $f : \mathbb{R}^* \to \mathbb{R}^*$, given by

$$f(y) = S^{-1}y^\sigma - aS^\bar{y} - b.$$ 

(20)

Then,

(1) If $\sigma = 1$ and $0 < a < S^{-q+p}$, then the equation $f(y) = 0$ has a unique positive solution

$$y_1 = \frac{b}{S^{q+p}(S^{-q+p} - a)}$$ 

(21)

and $f(y) \geq 0$ for all $y \geq y_1$.

(2) If $\sigma > 1$, then the equation $f(y) = 0$ has a unique positive solution $y_2 > \bar{y}$ and $f(y) \geq 0$ for all $y \geq y_2$.

**Proof.**

(1) For $\sigma = 1$ and $0 < a < S^{-q+p}$, we have

$$f(y) = S^{q+p}(S^{-q+p} - a)y - b$$ 

(22)

that is, the equation $f(y) = 0$ has a unique positive solution

$$y_1 = \frac{b}{(S^{-q+p} - a)S^{q+p}},$$

(23)

and $f(y) \geq 0$ for all $y \geq y_1$. 

\[\Box\]
For $\sigma > 1$, we have $f'(y) = \sigma S^{-1} y^{\sigma - 1} - a S^{p/p^*}$ and
\begin{align}
 f''(y) = \sigma (\sigma - 1) S^{-1} y^{\sigma - 2} > 0, \forall y > 0. \tag{24}
\end{align}

Then, $f'(y) = 0$ for $y < \bar{y}$ and $f'(y) > 0$ for $y > \bar{y}$.
Hence, $f$ is concave function and
\begin{align}
 \min_{y \geq 0} f(y) = f(\bar{y}) = - (\sigma - 1) S^{-1} \left( \frac{\bar{y}}{\sigma} \right)^{\sigma} < 0. \tag{25}
\end{align}

Moreover, we have $f(\bar{y}) < 0$ and $\lim_{y \to +\infty} f(y) = +\infty$; thus, from (25) and the concavity of $f$, we can conclude that the equation $f(y) = 0$ has a unique positive solution $y_2 > \bar{y}$ and $f(y) \geq 0$ for all $y \geq y_2$.

Now, we prove an important lemma which ensures the local compactness of the Palais-Smale sequence for $I_\lambda$.

For $i = 1, 2$, let $y_i$ be defined in Lemma 6 and define
\begin{align}
 y_* = \begin{cases} 
y_1 & \text{if } q = p^* - p \text{ and } 0 < a < S^{\frac{p^* - p}{p}}, \\
y_2 & \text{if } q < p^* - p \text{ and } a > 0.
\end{cases} \tag{26}
\end{align}

and $C_* = A(y_*)$.

\textbf{Lemma 7.} Let $1 < p < N$, $0 < \alpha < (N - p)/p, \alpha \leq \beta \leq \alpha + 1$, $0 \leq y < p_0 > 0$, and $\alpha < b \lambda_1$. Assume that $q = p^* - p$ and $0 < a < S^{-q/p^*}$ or $q < p^* - p$ and $a > 0$. Then, the functional $I_\lambda$ satisfies (PS)$_c$ condition for all $c \in C_*$. \hfill $\Box$

\textbf{Proof.} Let $\{u_n\} \subset D_\lambda(\Omega)$ be a $(PS)_c$ sequence for $I_\lambda$ with $c \in C_*$. By the proof of Lemma 5, we have $\{u_n\}$ is a bounded sequence in $D_\lambda(\Omega)$. Hence, by the concentration-compactness principle due to Lions [2], there exists a subsequence, still denoted by $\{u_n\}$, such that
\begin{align}
 |\nabla u_n|^p - \mu |u_n|^p \over |x|^{p/(p+1)} & \to d\eta \\
 \geq |\nabla u|^p - \mu |u|^p + \sum_{i \in I} \tilde{\delta}_i |\nabla u_i|^p - \mu |u_i|^p + \sum_{i \in I} \tilde{\delta}_i |u_i|^p \to d\theta \tag{28}
\end{align}

and
\begin{align}
 1. \text{ Case } q = p^* - p, b > 0 \text{ and } 0 < a < S^{-q/p^*}.
\end{align}

According to Lemma 6, we have $f(y) = 0$ if $y \geq y_1$ with
\begin{align}
 y_1 = \frac{b}{S^{-q/p^*} - a S^{p/p^*}}, \tag{35}
\end{align}

which implies that
\begin{align}
 S(\theta_{b}) \geq \frac{\tilde{\delta}}{S^{p/p^*}} = B_1. \tag{36}
\end{align}

We claim that $I$ is finite and for any $i \in I$, let $\phi_{\epsilon}(x)$ be a smooth cut-off function centered at $x_i$ such that $0 \leq \phi_{\epsilon}(x) \leq 1$, and
\begin{align}
 \phi_{\epsilon}(x) = \begin{cases} 1 & \text{in } B(x_i, \epsilon), \\
 0 & \text{in } \Omega \setminus B(x_i, 2\epsilon),
\end{cases} \tag{30}
\end{align}

\begin{align}
 |\nabla \phi_{\epsilon}(x)| \leq \frac{2}{\epsilon}. \tag{30}
\end{align}

Since $\{\phi_{\epsilon}(u_n)\}$ is bounded in $W^{1,p}_0(\Omega)$ and $I'_\lambda(u_n) \to 0$ as $n \to \infty$, it holds by Hölder's inequality
\begin{align}
 0 = \lim_{\epsilon \to 0} \lim_{n \to \infty} \left( I'_\lambda(u_n), \phi_{\epsilon}(u_n) \right) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \left( \int_\Omega |\nabla u_n|^p - \mu |u_n|^p \nabla \nabla (\phi_{\epsilon}(u_n)) - \mu |u_n|^p - \mu |u_n|^p \phi_{\epsilon}(u_n) dx \right) \tag{31}
\end{align}

\begin{align}
 \text{Then, } \theta_1 \geq b \theta_1 + a \theta_1^{p/p^*}. \text{ Therefore, by (29), we deduce that}
\end{align}

\begin{align}
 \theta_1 \geq b \theta_1 + a \theta_1^{p/p^*} - b \geq 0. \tag{32}
\end{align}

Assume by contradiction that there exists $i_0 \in I$ such that $\theta_{b} = 0$. Set $y = (\theta_{b})^{p/p^*}$ and $\sigma = p^* - p/q$, then by (32) we get
\begin{align}
 S^{-1} y^p - a S^{p/p^*} - b \geq 0. \tag{33}
\end{align}

It is clear that $\sigma \geq 1$ thanks to $q \leq p^* - p$. So, from (33) and the definition of $f$ in Lemma 6 we get
\begin{align}
 f(y) = S^{-1} y^p - a S^{p/p^*} - b \geq 0. \tag{34}
\end{align}

We will discuss it in two cases:

\textbf{Case 1.} $q = p^* - p, b > 0$ and $0 < a < S^{-q/p^*}$.

Assume by contradiction that there exists $i_0 \in I$ such that $\theta_{b} = 0$. Set $y = (\theta_{b})^{p/p^*}$ and $\sigma = p^* - p/q$, then by (32) we get
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\textbf{Case 1.} $q = p^* - p, b > 0$ and $0 < a < S^{-q/p^*}$.

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\begin{align}
 S^{-1} y^p - a S^{p/p^*} - b \geq 0. \tag{33}
\end{align}

It is clear that $\sigma \geq 1$ thanks to $q \leq p^* - p$. So, from (33) and the definition of $f$ in Lemma 6 we get
\begin{align}
 f(y) = S^{-1} y^p - a S^{p/p^*} - b \geq 0. \tag{34}
\end{align}

We will discuss it in two cases:
Case 2. $q < p^* - p, b > 0$, and $a > 0$. In this case, from Lemma 6, we get $f(y_2) = 0$ and $f(y) \geq 0$ if $y \geq y_2$ with

$$y_2 > \left( \frac{aq}{p^* - p} \right)^{\frac{1}{q(p^*)}} S(y),$$

(37)

which implies that

$$\mathcal{S}(\theta_0, \frac{p^*}{p}) \geq S y_2^\frac{p^*}{p} = B_2.$$  

(38)

Hence, using (29), we deduce $\eta_i \geq S \theta_i^{p^*} \geq$

$$\begin{cases} 
B_1 & \text{if } q = p^* - p \text{ and } 0 < a < S^{q/p} \\
B_2 & \text{if } q < p^* - p \text{ and } a > 0.
\end{cases}$$

By Young inequality we have

$$c = \lim_{n \to -\infty} I_i(u_n) - \frac{1}{q + p} \left( I_i(u_n) \right) = \lim_{n \to -\infty} \frac{q}{(q + p)\lambda} b \|u_n\|^p + \frac{p^* - (q + p)}{(q + p)\lambda} \int_{\Omega} \frac{|u_n|^p}{|x|^p} dx - \lambda \frac{q}{(q + p)\lambda} \int_{\Omega} \frac{|u_n|^p}{|x|^p} dx 
\geq \frac{q}{(q + p)\lambda} b (\|u\|^p + \eta_i) + \frac{p^* - (q + p)}{(q + p)\lambda} \left( \int_{\Omega} \frac{|u|^p}{|x|^p} dx + \eta_i \right) 
\geq \frac{q}{(q + p)\lambda} \eta_i + \frac{p^* - (q + p)}{(q + p)\lambda} \theta_i, 
$$

(39)

we observe that $(q/(q+p))(b-\lambda/\lambda_1) > 0, p^* - q - p \geq 0$; thus, for $j \in \{1, 2\}$ we get

$$c \geq \frac{q}{(q + p)\lambda} b \eta_i + \frac{p^* - (q + p)}{(q + p)\lambda} \theta_i 
\geq (\frac{1}{p} - \frac{1}{q + p}) b B_j + \frac{p^* - (q + p)}{(q + p)\lambda} S_j^\frac{p^*}{p} 
\geq (\frac{1}{p} - \frac{1}{q + p}) b B_j + \frac{p^* - (q + p)}{(q + p)\lambda} S_j^\frac{p^*}{p} 
+ \frac{p^* - (q + p)}{(q + p)\lambda} a b \eta_i + \frac{p^* - (q + p)}{(q + p)\lambda} S_j^\frac{p^*}{p} 
+ \frac{1}{p^*} b B_j - \frac{1}{p^*} b B_j 
= p^* - (q + p) \frac{p^*}{p} b B_j + p^* - (q + p) \frac{p^*}{p} b B_j + \frac{1}{p^*} b B_j 
\geq p^* - (q + p) \frac{p^*}{p} b B_j + \frac{1}{p^*} b B_j,$$

(40)

since $f(y_j) = 0$ for $j \in \{1, 2\}$ and $C_s$ defined in Lemma 7. Contraadiction with $c < C_s$. Then, $I$ is empty, which implies that

$$\int_{\Omega} |u_n|^p dx \to \int_{\Omega} |u|^p dx.$$  

(41)

Now, set $l = \lim |u_n|$ as $n \to +\infty$; then, we have

$$\left( I_i(u_n), u_n \right) = (a\|u_n\| + b\|u_n\|^q - \int_{\Omega} |u_n|^p dx 
- \lambda \int_{\Omega} \frac{|u_n|^p}{|x|^p} dx 
= \lambda \int_{\Omega} \frac{|u_n|^p}{|x|^p} dx 
\right) 
= (a\|u\| + b\|u\|^q - \int_{\Omega} |u|^p dx 
- \lambda \int_{\Omega} \frac{|u|^p}{|x|^p} dx 
= \lambda \int_{\Omega} \frac{|u|^p}{|x|^p} dx 
)$$

(42)

for any $v \in \mathcal{D}_0(\Omega)$. Let $n \to +\infty$, and then, from (42) and (43), we deduce that

$$\int_{\Omega} |u|^p dx \to \int_{\Omega} |u|^p dx = 0.$$  

(44)

Taking the test function $v = u$ in (45), we get

$$\int_{\Omega} |u|^p dx \to \int_{\Omega} |u|^p dx = 0.$$  

(46)

Therefore, the equalities (44) and (45) imply that $\|u\| = l$. Consequently, $\{u_n\}$ converges strongly in $\mathcal{D}_0(\Omega)$, which is the desired result.
3. Proof of the Main Result

Let $R$ be a positive constant and set $\varphi \in C_0^\infty(\Omega)$ such that $0 \leq \varphi(x) \leq 1$ for $|x| \leq R$ and $\varphi(x) \equiv 1$ for $|x| \leq R/2$ and $B_R(0) \subset \Omega$. Set $z_\varepsilon(x) = \varphi(x)u_\varepsilon(x)$.

We have the well-known estimates as $\varepsilon \to 0$:

$$
\begin{align*}
\left\|z_\varepsilon\right\|_p &= \sqrt{2} + O\left(\varepsilon^{p+\gamma(p+\alpha+1)-N}\right) \\
\int_{|x|^{p'-\varepsilon}} \left\|z_\varepsilon\right\|_{p'}^p dx &= \sqrt{2} + O\left(\varepsilon^{p+\gamma(p+\alpha+1)-N}\right),
\end{align*}
$$

where $\xi_\mu$ and $\omega_\mu$ are zeroes of the function

$$
f(t) = (p-1)t^p - (N - p(a + 1))t^{p-1} + \mu, t \geq 0, 0 \leq \mu < \mu_0,
$$

that satisfy

$$
0 \leq \xi_\mu < t_\star < \omega_\mu < \frac{N - p(a + 1)}{p - 1}.
$$

(49)

(see [1]).

On the other hand, the function $f(t)$ has the unique minimal point

$$
t_\star = \frac{N - p(a + 1)}{p - 1}
$$

and is increasing on $[t_\star, +\infty]$. Thus, if $\gamma \geq N - p\omega_\mu$, i.e., $\omega_\mu \geq N - \gamma/p$ we have

$$
0 = f(\omega_\mu) \geq f\left(\frac{N - \gamma}{p}\right).
$$

(51)

Therefore, the equalities (48) and (51) imply that

$$
0 \leq \mu < \tilde{\mu} = \frac{N - p^2(a + 1) - \gamma(1-p)}{p} \left(\frac{N - \gamma}{p}\right)^{p-1}.
$$

(52)

Consequently,

$$
N - p^2(a + 1) - \gamma(1-p) \geq 0, \quad \frac{\gamma(p+\alpha+1)-N}{p-1} \leq \gamma < p \text{ and } a < \frac{N-p}{p^2}.
$$

(53)

Next, we show that $\tilde{\mu} \leq \mu$. For any $\gamma \geq 0$, let

$$
\tilde{\mu} = \Psi(\gamma) = \frac{N - p^2(a + 1) - \gamma(1-p)}{p} \left(\frac{N - \gamma}{p}\right)^{p-1}.
$$

(54)

We can show easily that since $\Psi$ is increasing on $[0, p(a + 1)]$ and decreasing on $[p(a + 1), +\infty)$ and $\Psi(p(a + 1)) = \mu$. So, $\mu \leq \mu_0$.

Lemma 8. Let $1 < p < N$, $0 \leq \alpha < (N - p)/p^2, \alpha \leq \beta < a + 1$, $0 < \gamma < p,b > 0, 0 < \lambda < \lambda_1$, and $0 \leq \mu < \mu_0$. Assume that $q = p^* - p$ and $0 < a < S^\gamma \alpha p^p$, or $q < p^* - p$ and $a > 0$. Then, $\sup_{t \geq 0} I_{\lambda}(t) < C_\star$.

Proof. We define the following functions

$$
g(t) = I_{\lambda}(t) = \frac{a}{q + p} t^{p+\gamma} \left|z_\varepsilon\right|^{p+\gamma} + \frac{b}{p} \left|z_\varepsilon\right|^p
$$

$$
= \frac{1}{p^*} \int_{\Omega} |z_\varepsilon|^{p+\gamma} dx - \frac{\lambda}{p^*} \int_{\Omega} \left|z_\varepsilon\right|^p dx
$$

$$
= \frac{a}{q + p} t^{p+\gamma} \left|z_\varepsilon\right|^{p+\gamma} + \frac{b}{p} \left|z_\varepsilon\right|^p - \frac{1}{p^*} \int_{\Omega} \left|z_\varepsilon\right|^{p+\gamma} dx
$$

$$
= \frac{1}{p^*} \left(\int_{\Omega} \left|z_\varepsilon\right|^{p+\gamma} dx - S^\gamma \alpha p^p \left|z_\varepsilon\right|^p \right) \left|z_\varepsilon\right|^{p+\gamma} - \frac{\lambda}{p^*} \int_{\Omega} \left|z_\varepsilon\right|^p dx,
$$

$$
h(t) = -\frac{1}{p^*} \int_{\Omega} \left|z_\varepsilon\right|^{p+\gamma} dx + \frac{a}{q + p} t^{p+\gamma} \left|z_\varepsilon\right|^{p+\gamma} + \frac{b}{p} \left|z_\varepsilon\right|^p.
$$

(55)

Note that $\lim_{t \to +\infty} g(t) = -\infty$ and $g(t) > 0$ when $t$ is close to 0, so that $\sup_{t \geq 0} g(t)$ is attained for some $T_\varepsilon > 0$. Furthermore, from $g'(T_\varepsilon) = 0$ it follows that

$$
-T_\varepsilon^{p+\gamma} \int_{\Omega} \left|z_\varepsilon\right|^{p+\gamma} dx + a T_\varepsilon^q \left|z_\varepsilon\right|^{q+\gamma} + b \left|z_\varepsilon\right|^p - \lambda \int_{\Omega} \left|z_\varepsilon\right|^p dx = 0.
$$

(56)

Therefore,

$$
T_\varepsilon^{p+\gamma} \int_{|x|^{p+\gamma}} \left|z_\varepsilon\right|^{p+\gamma} dx = a T_\varepsilon^q \left|z_\varepsilon\right|^{q+\gamma} + b \left|z_\varepsilon\right|^p - \lambda \int_{\Omega} \left|z_\varepsilon\right|^p dx
$$

$$
\geq \left(b - \frac{\lambda}{\lambda_1}\right) \left|z_\varepsilon\right|^p.
$$

(57)

Choose $\varepsilon$ small enough so that by (47) we have $T_\varepsilon \geq t_0$ for some $t_0 > 0$. 

□

Besides, it holds

$$
-T_\varepsilon^{p+\gamma} \int_{|x|^{p+\gamma}} \left|z_\varepsilon\right|^{p+\gamma} dx = a \left|z_\varepsilon\right|^{q+\gamma} + \frac{b}{T_\varepsilon^q} \left|z_\varepsilon\right|^p - \lambda \frac{1}{T_\varepsilon} \int_{\Omega} \left|z_\varepsilon\right|^p dx
$$

$$
\leq a \left|z_\varepsilon\right|^{q+\gamma} + \frac{b}{T_\varepsilon^q} \left|z_\varepsilon\right|^p.
$$

(58)
For $q < p^* - p, a > 0, b > 0$ we have by (47)

$$T_p^{(q, p)} \leq \frac{\|z_p^p\|}{t^p_0} \leq a \|z_1^p\|^p + b \|z_2^p\|^p \int_0^t \frac{1}{t^p_0},$$

(59)

Then, for $\epsilon$ small enough, the above estimates yield $T_\epsilon < t_0'$ for some $t_0' > 0$ (independently of $\epsilon$).

For $q = p^* - p, 0 < a < S^{-q\rho}\mu^p, b > 0$ and for $\epsilon$ small enough we have by (56),

$$T_\epsilon^p = \frac{(b \|z_1^p\|^p - \lambda \int_\Omega (|z_1^p|^p/|x|^p) dx)}{(\int_\Omega (|z_1^p|^p/|x|^p) dx - a \|z_1^p\|^p)},$$

(60)

which implies that $T_\epsilon$ is bounded above for all $\epsilon > 0$, that is, there exists a positive real number $t_0' > 0$ (independently of $\epsilon$).

Now, we estimate $g(T_\epsilon)$. It follows from $h'(t) = 0$

$$-\left[ S^{-p} \|z_1^p\|^p - \lambda \int_\Omega (|z_1^p|^p/|x|^p) dx \right] = 0.$$

(61)

Set $y = tS^{-q\rho}\|z_1^p\|^p$ and $\sigma = p^* - p/q > 1$. Then, by (61) the definition of $f$ we get

$$-\left[ S^{-1} \|y\|^p - a \|y\|^p - b \right] = 0,$$

(62)

which implies from (26) and the proof of Lemma 6 that $f(y, 0) = 0$. Therefore, $h'(t_*) = 0$, where $t_* = S^{-q\rho}\|z_1^p\|^p$. As $f(y)$ is concave then $h'(t)$ is convex and so,

$$\max_{t \geq 0} h(t) = h(t_*) = \frac{1}{p^*} S^{-p} \|z_1^p\|^p + \frac{a}{q + p} \|z_1^p\|^p \|z_1^p\|^p,$$

(63)

By $h'(t_*) = 0$, we have

$$S^{-p} \|z_1^p\|^p = a \|z_1^p\|^p + b \|z_1^p\|^p.$$

(64)

So, from (64) we deduce that

$$\max_{t \geq 0} h(t) = \frac{1}{p^*} \left( a \|z_1^p\|^p + b \|z_1^p\|^p \right)$$

$$+ \frac{a}{q + p} \|z_1^p\|^p + b \|z_1^p\|^p = a \left( \frac{1}{q + p} \right) \frac{1}{p^*} \|z_1^p\|^p + b \left( \frac{1}{p^*} \right) \|z_1^p\|^p$$

$$= a \left( \frac{1}{q + p} \right) S^{-p} \|y_1^p\|^p + b \|z_1^p\|^p.$$

(65)

Consequently, by (47)

$$\sup_{t \geq 0} I_1(t) \leq \sup_{t \geq 0} I_1(t) + \frac{1}{p^*} \left( S^{-p} \|z_1^p\|^p - \frac{1}{p^*} \|z_1^p\|^p \right) \|z_1^p\|^p.$$
On the other hand, using (47) and taking \( \varepsilon_1 > 0 \) small enough, we get
\[
I_\lambda(t z_\varepsilon) \leq \frac{a}{q + p} \| z_\varepsilon \|^{q + p} + \frac{b}{p} \| z_\varepsilon \|^{q + p} - \frac{p^\ast}{p^\ast} \int_\Omega \frac{|\nabla z_\varepsilon|^p}{|x|^{p(\alpha + 1) - N}} dx \\
\leq \frac{1}{q + p} \left( a \frac{\varepsilon^2}{2} - 1 \right) \| z_\varepsilon \|^{q + p} + \frac{b}{p} \frac{\varepsilon^{\alpha + 1}}{2} \| z_\varepsilon \|^{q + p} + O \left( \varepsilon^{p^\ast - (p + 1) + \alpha} \right)
\]
(72)
for all \( \varepsilon \in (0, \varepsilon_1) \). Then, as \( 0 < a < S^{q + p} / p^\ast \), it follows from the above inequality, \( I_\lambda(t z_\varepsilon) \to -\infty \) as \( t \to +\infty \). Thus, choosing \( t_2 > 0 \) sufficiently large such that \( \| t_2 z_\varepsilon \| > \rho_2 \) and \( I_\lambda(t_2 z_\varepsilon) < 0 \).

Set
\[
 c = \inf_{y \in G(t \gamma, \varepsilon)} \max_{t \in [0,1]} I_\lambda(y(t)),
\]
(73)
where
\[
G = \{ y \in C([0,1], \mathcal{D}_a(\Omega)), y(0) = 0, y(1) = t_\ast z_\varepsilon \},
\]
(74)
\[
 t_\ast = \begin{cases} 
 t_2 & \text{if } q < p^\ast - p \text{ and } a > 0 \\
 t_2 & \text{if } q = p^\ast - p \text{ and } 0 < a < S^{q + p} / p^\ast.
\end{cases}
\]

By the Mountain Pass Theorem, there exists a Palais-Smale sequence \( \{ u_n \} \) at level \( c \). Using Lemma 5, we have that \( \{ u_n \} \) has a subsequence, still denoted by \( \{ u_n \} \), such that \( u_n \rightharpoonup u \) in \( \mathcal{D}_a(\Omega) \) as \( n \to +\infty \). Hence, from Lemmas 7 and 8, we have \( u_n \rightharpoonup u \) in \( \mathcal{D}_a(\Omega) \) as \( n \to +\infty \). Hence, \( I_\lambda(u_n) \to 0 \) and \( I_\lambda(u) = c > 0 \). So, as \( c > 0 = I_\lambda(0) \), we can conclude that \( u \) is a nonzero solution of (1) with positive energy. Now, we show that \( u > 0 \). Because
\[
0 = \left( I_\lambda'(u), u^- \right) + \int_\Omega \frac{\left| \nabla u \right|^p}{|x|^{p(\alpha + 1) - N}} \sqrt{u} \nabla u^- - \mu \left| u \right|^{p - 2} u^- u^- dx \\
- \int_\Omega \left| \nabla u \right|^{p - 2} \left| \nabla u \right| \sqrt{u} \nabla u^- dx + \lambda \int_\Omega \left| \nabla u \right|^{p - 2} \left| \nabla u \right| \sqrt{u} \nabla u^- dx \\
\geq (a) \| u^- \|^p + b \left( \int_\Omega \frac{\left| \nabla u \right|^p}{|x|^{p(\alpha + 1) - N}} dx \right) \sqrt{u} \nabla u^- dx + \int_\Omega \left| \nabla u \right|^{p - 2} \left| \nabla u \right| \sqrt{u} \nabla u^- dx \\
+ \lambda \int_\Omega \left| \nabla u \right|^{p - 2} \left| \nabla u \right| \sqrt{u} \nabla u^- dx \geq b \| u^- \|^p,
\]
(75)
which implies that \( u^- = 0 \). By the strong maximum principle one has \( u > 0 \). This completes the proof of Theorem 1.

4. Conclusion 1

In our work, we have searched the critical points as the minimizers of the energy functional associated to the problem. Our results and setting were more general and delicate, it difficult to obtain the solution in the degenerate case. Our technique was based on variational methods and concentration compactness argument and we needed to estimate the energy levels. We have shown the existence result for our problem (1) if \( 1 < p < N, \ 0 \leq \alpha < (N - p)p^\ast, \alpha \leq \beta < \alpha + 1, \ [p^\ast(\alpha + 1) - N] / (p - 1) \leq \gamma < p, b > 0, 0 < \lambda < b \lambda_1 \), and \( 0 < \mu < \mu_\ast \) with
\[
\mu_\ast = \frac{N - (p^\ast(\alpha + 1) + \gamma(1 - p))}{p},
\]
and the problem (1) has a positive solution in the following cases: (1) \( q = p^\ast - p \) and \( 0 < a < S^{q + p(p - 1)} / p^\ast \), (2) \( q < p^\ast - p \) and \( a > 0 \).

Data Availability

The [DATA TYPE] data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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