Hindawi
Journal of Function Spaces
Volume 2022, Article ID 7474964, 8 pages
https://doi.org/10.1155/2022/7474964

Research Article

A Blow-Up Criterion for 3D Nonhomogeneous Incompressible Magnetohydrodynamic Equations with Vacuum

Shujuan Wang,[1] **Miaoqing Tian**,[1] **and Rijian Su**[2]

[1]*College of Mathematics and Information Science, Zhengzhou University of Light Industry, Zhengzhou 450002, China*
[2]*College of Computer and Communication Engineering, Zhengzhou University of Light Industry, Zhengzhou 450002, China*

Correspondence should be addressed to Rijian Su; rijiansu@126.com

Received 29 March 2022; Accepted 10 May 2022; Published 3 June 2022

Academic Editor: Igor E. Verbitsky

For the strong solutions of the nonhomogeneous incompressible magnetohydrodynamics (MHD) system with vacuum, we establish a blow-up criterion for this system in terms of $\|u\|_{C([0,T];L^3(\mathbb{R}^3))}$. Moreover, the result generalizes previous ones in Giga (1986) and He and Xin (2005) where homogeneous incompressible Navier-Stokes equations and homogeneous incompressible MHD system are considered, respectively, and demonstrates that the velocity field plays a more dominant role in the MHD system.

1. Introduction

Magnetohydrodynamics (MHD) is concerned with the interaction between fluid flow and magnetic field, and the motion of the nonhomogeneous incompressible MHD can be stated as follows (see, e.g., [1–4]):

$$\rho_t + \operatorname{div}(\rho u) = 0, \tag{1}$$

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P = (\nabla \times H) \times H + \mu \Delta u, \tag{2}$$

$$H_t - H \cdot \nabla u + u \cdot \nabla H = \nu \Delta H, \tag{3}$$

$$H_t - H \cdot \nabla u + u \cdot \nabla H = \nu \Delta H, \tag{4}$$

where ρ, $u = (u^1, u^2, u^3) \in \mathbb{R}^3$, P, and $\mathrm{H} = (\mathrm{H}^1, H^2, \mathrm{H}^3) \in \mathbb{R}^3$ represent, respectively, the density, velocity, pressure, and magnetic field. The constants $\mu > 0$ and $\nu > 0$ denote the viscosity of fluid and the relative strengths of advection and diffusion of H. Since the presence of all the physical constants does not create essential mathematical difficulties, for notational simplicity, we will normalize all constants in the system to be one in the sequel.

In recent years, the MHD system has drawn the attention of engineers and applied mathematicians due to its important physical background and mathematical feature. If taking $\rho \equiv$ const, the system (1)-(4) is reduced to the homogeneous incompressible MHD. For this case, Duvaut and Lions [5] constructed a class of global weak solutions, similar to the Leray-Hopf weak solutions to the three-dimensional Navier-Stokes equations. Sermange and Temam [6] first gave a local existence of the strong solution with any given initial data $(u_0, H_0) \in H^m(\mathbb{R}^3)$ $(m \geq 2)$. It should be pointed out that whether this unique local solution can exist globally with general initial data is an outstanding challenging problem in three dimensions. Thus, there are many works to study the regularity criteria for weak or classical solutions, see [7–12]. We also notice that if partial viscosity and resistivity are zero, the global regularity issues have been established in [13].

For the nonhomogeneous case (1)–(4), there are a lot of literature which includes the existence, uniqueness, and regularity of solutions [1, 14–17]. Zhang [18] established local classical solutions of (1)–(4) and showed that as the viscosity μ and resistivity ν went to zero, the solution of (1)–(4) converged to the solution of ideal MHD system (i.e., $\mu = \nu = 0$). Gerbeau [3] and Desjardins and Le Bris [19] considered the global existence of weak solutions of finite energy in the whole space or in the torus. Abidi and Paicu [20] proved the global existence of strong solutions with small initial data in some Besov spaces. Recently, Huang and Wang [21] demonstrated the unique global strong solution with general initial data to (1)–(4) in two dimensions.

In this paper, we are interested in the Cauchy problem of (1)–(4) subject to the following initial conditions:

$$(\rho, u, H)(x, 0) = (\rho_0, u_0, H_0)(x) \text{ and } \rho_0 \le \bar{\rho} \text{ for all } x \in \mathbb{R}^3, \tag{5}$$

and far field conditions:

$$(\rho, u, H) \longrightarrow (0, 0, 0), \text{ as } |x| \longrightarrow \infty, \tag{6}$$

where $\bar{\rho}$ is a given constant.

To state the main results in a precise way, we first introduce some notations and conventions which will be used throughout the paper. For $k \in \mathbb{Z}^+$ and $r > 1$, the standard homogeneous and inhomogeneous Sobolev spaces for scalar/vector functions are denoted by:

$$\begin{cases} L^r = L^r(\mathbb{R}^3), D^{k,r} = \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^3) \,\middle|\, \left\|\nabla^k u\right\|_{L^r} < \infty \right\}, W^{k,r} = L^r \cap D^{k,r}, \\ H^k = W^{k,2}, D^k = D^{k,2}, D^1 = \left\{ u \in L^6 \,|\, \|\nabla \mathrm{u}\|_{L^2} < \infty \right\}, \|u\|_{D^{k,r}} = \left\|\nabla^k u\right\|_{L^r}. \end{cases} \tag{7}$$

.

The strong solutions of the problem (1)–(4) are defined as follows.

Definition 1. A pair of functions (ρ, u, H) is called a strong solution to the problem (1)–(4) in $\mathbb{R}^3 \times (0, T)$, if for some $q_0 \in (3, 6]$,

$$\begin{cases} \rho \in C\left([0, T]; L^1 \cap H^1 \cap W^{1,q_0}\right), \rho_t \in C\left([0, T]; L^2 \cap L^{q_0}\right), \rho \ge 0, \\ u \in C\left([0, T]; D^1 \cap D^2\right) \cap L^2\left(0, T; D^{2,q_0}\right), \sqrt{\rho} u_t \in L^\infty\left(0, T; L^2\right), \\ (u_t, H_t) \in L^2\left(0, T; D^1\right), H \in C\left([0, T]; H^2\right), H_t \in L^\infty\left(0, T; L^2\right), \end{cases} \tag{8}$$

and (ρ, u, H) satisfies (1)–(4) a.e. in $\mathbb{R}^3 \times (0, T)$.

Before stating the main result of this paper, we first state a local existence of strong solutions to (1)–(4). The following local well-posedness theorem of strong solutions was given in [16].

Proposition 2. *Assume that for some $q \in (3, 6]$ and the initial data (ρ_0, u_0, H_0) satisfying*

$$\begin{aligned} \rho_0 \ge 0, \rho_0 \in L^1 \cap W^{1,q}, \operatorname{div} u_0 = 0, u_0 \\ \in D^1 \cap D^2, \operatorname{div} H_0 = 0, H_0 \in H^2, \end{aligned} \tag{9}$$

$$-\triangle u_0 + \nabla P_0 - (\nabla \times H_0) \times H_0 = \rho_0^{1/2} g, \tag{10}$$

for some $g \in L^2$. Then, there exist a time $T_* > 0$ and a unique strong solution (ρ, u, H) to (1)–(4) together with (5)–(6) in $\mathbb{R}^3 \times (0, T_*)$, such that

$$\begin{cases} \rho \ge 0, \rho \in C([0, T_*]; W^{1,q}), \rho_t \in C([0, T_*]; L^q), \\ u \in C\left([0, T_*]; D^1 \cap D^2\right) \cap L^2\left(0, T_*; D^{2,q}\right), \sqrt{\rho} u_t \in L^\infty\left(0, T_*; L^2\right), \\ (u_t, H_t) \in L^2\left(0, T_*; D^1\right), H \in C\left([0, T_*]; H^2\right) \cap L^2\left(0, T_*; W^{2,q}\right), \\ P \in C\left([0, T_*]; H^1\right) \cap L^2\left(0, T_*; W^{1,q}\right). \end{cases} \tag{11}$$

Although significant progress has been made in the study of multidimensional nonhomogeneous incompressible MHD system, many physically important and mathematically fundamental problems are still open due to the lack of smoothing mechanism and the strong nonlinearity. Similar to that for the three-dimensional incompressible Navier-Stokes equations, whether the unique local strong solution obtained in Proposition 2 can exist globally is an outstanding challenging open problem. If the answer is negative, then it simultaneously raises the interesting questions of the mechanism of blowup and the structure of possible singularities.

In the recent paper [7], He and Xin proved a blow-up criterion to nonhomogeneous incompressible magnetohydrodynamic equations; that is, if $u \in C([0, T]; L^3)$ is bounded above, then the local strong solution, in fact, is a global one. This criterion is analogous to the criterion on the weak solutions to the 3D incompressible Navier-Stokes equations (see [22]). Motivated by these works on the blow-up criterion of local strong solutions to the Navier-Stokes equation and homogeneous incompressible MHD system, we will generalize this result in [7, 22] to the 3D nonhomogeneous incompressible MHD system (1)–(4). Our main result of this paper is stated as follows.

Theorem 3. *Suppose that the assumptions in Proposition 2 are satisfied. Let (ρ, u, H) be a strong solution to (1)–(4) with regularity (10). If $T^* < \infty$ is the maximal time of existence, then*

$$\lim_{T \longrightarrow T^*} \|u\|_{C\left([0,T]; L^3(\mathbb{R}^3)\right)} = \infty. \tag{12}$$

Remark 4. The proof of this theorem together with the result in [23], we can extend this result from incompressible magnetohydrodynamic equations to the compressible case, which is our work in the future.

To prove Theorem 3, the main two steps are to estimate the L^p-norm of the magnetic field H and L^2-norm of the gradient of the velocity. To do this, the key observation of the present paper lies in the following simple fact:

Proposition 5. *For $U \in C([0, T]; L^3)$, there exist $U^1 \in C([0, T]; L^3)$ and $U^2 \in L^\infty(0, T; L^\infty)$ such that for any $\delta > 0$,*

$$U = U^1 + U^2, \left\|U^1\right\|_{C\left([0,T]; L^3\right)} \le \delta, \left\|U^2\right\|_{L^\infty(0,T; L^\infty)} \le C(\delta, M), \tag{13}$$

where $M \triangleq \|u\|_{C([0,T];L^3)}$ *and* $C(\delta, M)$ *is a positive constant depending only on* δ, M.

Thus, by choosing $\delta > 0$ suitably small, we then succeed in obtaining the estimates on $\|\nabla u\|_{L^2}$ by utilizing the preliminary estimates of the vorticity $\omega = \nabla \times u$ (see Lemma 7) to control the L^p-norm of ∇u in the proof of Lemma 10. With the estimate of $\|\nabla u\|_{L^2}$ at hand, we can give the higher-order estimate of (ρ, u, H) and thus finish the proof of Theorem 3.

2. Auxiliary Lemmas

We state the well-known Gagliardo-Nirenberg inequality (see, for instance, [24]).

Lemma 6. *Assume that* $f \in H^1$ *and* $g \in H^2$ *with* $q > 1$ *and* $r > 3$. *Then, for any* $p \in [2,6]$, *there exists a positive constant* C, *depending only on p, q, and r, such that*

$$\begin{aligned} \|f\|_{L^p} &\le C\|f\|_{L^2}^{(6-p)/(2p)} \|\nabla f\|_{L^2}^{(3p-6)/2p}, \\ \|g\|_{L^\infty} &\le C\|g\|_{L^2}^{1/4} \|\nabla^2 g\|_{L^2}^{3/4}. \end{aligned} \tag{14}$$

To complete some estimates in Section 3, we need the following L^p-estimate for vorticity $\omega = \nabla \times u$. In fact, we deduce ω satisfy the following elliptic system by the momentum equation (2)

$$\Delta\omega = \nabla \times (\rho u_t + \rho u \cdot \nabla u - H \cdot \nabla H), \tag{15}$$

due to $(\nabla \times H) \times H = H \cdot \nabla H - \nabla |H|^2/2$. By virtue of the standard L^p-estimate of the elliptic system, we have the following.

Lemma 7. *Let* (ρ, u, H) *be a smooth solution of (1) and (3); if* $0 \le \rho \le \tilde{\rho}$, *then there exists a generic positive constant depending only on* ρ^- *such that*

$$\begin{aligned} \|\nabla\omega\|_{L^2} &\le C(\|\rho u_t\|_{L^2} + \|\rho u \nabla u\|_{L^2} + \|H \nabla H\|_{L^2}), \\ \|\nabla\omega\|_{L^6} &\le C(\|\rho u_t\|_{L^6} + \|\rho u \nabla u\|_{L^6} + \|H \nabla H\|_{L^6}), \end{aligned} \tag{16}$$

where $\tilde{\rho}$ *is a given constant.*

Proof. Using Lemma 6, one deduces from (15) and the standard L^p-estimate of the elliptic system that

$$\begin{aligned} \|\nabla\omega\|_{L^2} &\le C(\|\rho u_t\|_{L^2} + \|\rho u \cdot \nabla u\|_{L^2} + \|H \cdot \nabla H\|_{L^2}), \\ &\le C\left(\left\|\rho^{1/2} u_t\right\|_{L^2} + \|u \nabla u\|_{L^2} + \|H \nabla H\|_{L^2}\right), \\ \|\nabla\omega\|_{L^6} &\le C\left(\left\|\rho\, \dot{u}\right\|_{L^6} + \|H \cdot \nabla H\|_{L^6}\right) \\ &\le C\left(\left\|\nabla \dot{u}\right\|_{L^2} + \|H \nabla H\|_{L^6}\right), \end{aligned} \tag{17}$$

which immediately finish the proof of Lemma 7. □

3. A Priori Estimates

Let (ρ, u, H) be strong solutions to the problem (1)–(4) as described in Proposition 2. We will prove Theorem 3 by a contradiction argument. To this end, we suppose that for any $T < T^* < \infty$

$$\|u\|_{C([0,T];L^3)} \le M < \infty. \tag{18}$$

Then, we will deduce a contradiction to the maximality of T^*.

Throughout this paper, we will denote by C the various generic positive constants, which may depend on the initial data, M and T. Special dependence will be pointed out explicitly in this paper if necessary.

First of all, by the method of characteristics, it is easy to see that

$$0 \le \rho(x, t) \le \sup_{x \in \mathbb{R}^2} \rho_0(x) \le \bar{\rho} \text{ for all } x \in \mathbb{R}^3, t \in (0, T). \tag{19}$$

Next, we give the standard energy estimate as follows.

Lemma 8. *Let* (ρ, u, H) *be a smooth solution of (1)–(4) on* $\mathbb{R}^3 \times (0, T]$. *Then, there exist a constant* C *such that*

$$\sup_{0 \le t \le T} \int \left(\frac{1}{2}\rho|u|^2 + \frac{1}{2}|H|^2\right) dx + \int_0^T \left(\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2\right) dt \le C. \tag{20}$$

Proof. Multiplying (2) and (3) by u and H, respectively, integrating by parts, and adding them together, one immediately gets (20). □

Under the assumption (18), we can improve the integrability of magnetic field H which will be frequently used in the sequel.

Lemma 9. *Under the assumption (18), for any* $T < T^*$, *it holds that*

$$\|H\|_{L^\infty(0,T;L^q)} \le C, \forall q \in [2,\infty). \tag{21}$$

Proof. Multiplying (3) by $q|H|^{q-2}H$ and integrating the resulting equations over $\mathbb{R}^3$ lead to

$$\frac{d}{dt}\int |H|^q dx = q\int (\Delta H + H \cdot \nabla u - u \cdot \nabla H) \cdot |H|^{q-2} H dx = \sum_{i=1}^3 I_i. \tag{22}$$

For the terms on the right-hand side of the equation above, we get by integrations by parts

$$I_1 = q\int \Delta H \cdot |H|^{q-2} H dx = -q\int |H|^{q-2}|\nabla H|^2 dx - \frac{q(q-2)}{2}\int |H|^{q-4}\left|\nabla |H|^2\right|^2 dx,$$
$$I_2 = q\int H\cdot\nabla u\cdot |H|^{q-2}Hdx = -\int\left(q|H|^{q-2}H \cdot\nabla H\cdot u + \frac{q(q-2)}{2}|H|^{q-4}\left(H\cdot\nabla|H|^2\right)(u\cdot H)\right)dx,$$
$$I_3 = -q\int u\cdot\nabla H\cdot|H|^{q-2}Hdx = -\frac{1}{2}\int u\cdot\nabla|H|^q dx = 0. \tag{23}$$

Substituting $I_1 - I_3$ into (21) and using Young inequality lead to

$$\begin{aligned}&\frac{d}{d}\int |H|^q dx + \int\left(q|H|^{q-2}|\nabla H|^2 + \frac{q(q-2)}{2}|H|^{q-4}\left|\nabla|H|^2\right|^2\right)dx\\ &= -\int\left(q|H|^{q-2}H\cdot\nabla H\cdot u + \frac{q(q-2)}{2}|H|^{q-4}\left(H\cdot\nabla|H|^2\right)(u\cdot H)\right)dx\\ &\le \frac{1}{2}\left(\int\left(q|H|^{q-2}|\nabla H|^2 + \frac{q(q-2)}{2}|H|^{q-4}\left|\nabla|H|^2\right|^2\right)dx\right)\\ &\quad + C\int |u|^2|H|^q dx,\end{aligned} \tag{24}$$

which immediately implies that

$$\frac{d}{dt}\int |H|^q dx + \int\left|\nabla|H|^{q/2}\right|^2 dx \le C\int |u|^2\left||H|^{q/2}\right|^2 dx. \tag{25}$$

Due to the fact that $u\in C([0,T];L^3)$, we can decompose u into the following two parts:

$$u \triangleq U^1 + U^2, \tag{26}$$

with

$$\left\|U^1\right\|_{C([0,T];L^3)} \le \delta,\ \left\|U^2\right\|_{L^\infty(0,T;L^\infty)} \le C(\delta, M_0), \tag{27}$$

for $M_0 \triangleq \|u\|_{C([0,T];L^3)}$ and any $\delta\in(0,1)$.

From (25)–(27) and using Hölder inequality and imbedding inequality, we have

$$\begin{aligned}&\frac{d}{dt}\int |H|^q dx + \int\left|\nabla|H|^{q/2}\right|^2 dx\\ &\le C\left\|U^1\right\|_{L^3}^2\left\||H|^{q/2}\right\|_{L^6}^2 + \left\|U^2\right\|_{L^\infty((0,T)\times(\mathbb{R}^3))}^2\int |H|^q dx\\ &\le C\delta\left\|\nabla|H|^{q/2}\right\|_{L^2}^2 + C\int |H|^q dx.\end{aligned} \tag{28}$$

This, together with taking δ suitably small and applying Gronwall's inequality, immediately leads to the desired estimate (21). □

Under the assumption (18) and Lemmas 8 and 9, we prove the following crucial estimate concerning the estimates of the gradients of u and H.

Lemma 10. *Under the assumption (18), for any $T < T^*$, it holds that*

$$\sup_{0\le t\le T}\left(\|\nabla u\|_{L^2}^2 + \|H\|_{H^1}^2\right) + \int_0^T\left(\left\|\rho^{1/2}u_t\right\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \|\nabla u\|_{H^1}^2 + \|H\|_{H^2}^2\right)dt \le C. \tag{29}$$

Proof. Multiplying (2) by u_t in L^2 and integrating the resulting equations by parts, we obtain after summing them up that

$$\frac{1}{2}\frac{d}{dt}\int |\nabla u|^2 dx + \int \rho|u_t|^2 dx = -\int \rho u\cdot\nabla u\cdot u_t dx - \int H\cdot\nabla u_t\cdot H dx, \tag{30}$$

where we use the fact div $u_t = 0$.

In addition, it follows from (3) that

$$\frac{d}{dt}\|\nabla H\|_{L^2}^2 + \left(\|H_t\|_{L^2}^2 + \|\Delta H\|_{L^2}\right) = \int |H_t - \Delta H|^2 dx = \int |H\cdot\nabla u - u\cdot\nabla H|^2 dx. \tag{31}$$

Putting (30) and (31) together leads to

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\|\nabla u\|^2 + \|\nabla H\|_{L^2}^2\right) + \left(\left\|\rho^{1/2}u_t\right\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \|\Delta H\|_{L^2}\right)\\ &= -\int \rho u\cdot\nabla u\cdot u_t dx - \int H\cdot\nabla u_t\cdot H dx\\ &\quad + \int |H\cdot\nabla u - u\cdot\nabla H|^2 dx = \sum_{i=1}^3 J_i.\end{aligned} \tag{32}$$

We estimate the three terms on the right-hand side of (32) term by term. Following from Young inequality, one has that

$$\begin{aligned}J_1 &\le \frac{1}{4}\left\|\rho^{1/2}u_t\right\|_{L^2}^2 + C\|u\nabla u\|_{L^2}^2,\\ J_2 &= \int H\cdot\nabla u_t\cdot H dx = \frac{d}{dt}\int H\cdot\nabla u\cdot H dx - \int H_t\cdot\nabla u\cdot H dx - \int H\cdot\nabla u\cdot H_t dx\\ &\le \frac{d}{dt}\int H\cdot\nabla u\cdot H dx + \frac{1}{4}\|H_t\|_{L^2}^2 + C\|H\nabla u\|_{L^2}^2,\\ J_3 &= \int |H\cdot\nabla u - u\cdot\nabla H|^2 dx \le C\left(\|u\nabla H\|_{L^2}^2 + \|H\nabla u\|_{L^2}^2\right).\end{aligned} \tag{33}$$

Putting $J_1 - J_3$ into (31), we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\|\nabla u\|^2 + \|\nabla H\|_{L^2}^2\right) + \frac{1}{2}\left(\left\|\rho^{1/2}u_t\right\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \|\Delta H\|_{L^2}\right) \leq -\frac{d}{dt}\int H\cdot\nabla u\cdot H dx + C_1\left(\|u\nabla H\|_{L^2}^2 + \|u\nabla u\|_{L^2}^2 + \|H\nabla u\|_{L^2}^2\right), \tag{34}$$

where C_1 is a positive constant depending only on the initial data.

By virtue of (21) and (27), we can deal with the second term on the right-hand side of (34) as follows:

$$\begin{aligned} \|u\nabla H\|_{L^2}^2 &\leq \int \left|U^1\right|^2|\nabla H|^2 dx + \int \left|U^2\right|^2|\nabla H|^2 dx \\ &\leq \left\|U^1\right\|_{L^3}^2\|\nabla H\|_{L^6}^2 + \left\|U^2\right\|_{L^\infty}^2\|\nabla H\|_{L^2}^2 \\ &\leq \delta^2\|\nabla H\|_{L^6}^2 + C(\delta, M)\|\nabla H\|_{L^2}^2, \\ \|u\nabla u\|_{L^2}^2 &\leq \int \left|U^1\right|^2|\nabla u|^2 dx + \int \left|U^2\right|^2|\nabla u|^2 dx \\ &\leq \left\|U^1\right\|_{L^3}^2\|\nabla u\|_{L^6}^2 + \left\|U^2\right\|_{L^\infty}^2\|\nabla u\|_{L^2}^2 \\ &\leq \delta^2\|\nabla u\|_{L^6}^2 + C(\delta, M)\|\nabla u\|_{L^2}^2, \\ \|H\nabla u\|_{L^2}^2 &\leq C\|H\|_{L^6}^2\|\nabla u\|_{L^3}^2 \leq C\|\nabla u\|_{L^2}\|\nabla u\|_{L^6} \\ &\leq \delta^2\|\nabla u\|_{L^6}^2 + C(\delta)\|\nabla u\|_{L^2}^2. \end{aligned} \tag{35}$$

Thus,

$$\|u\nabla H\|_{L^2}^2 + \|u\nabla u\|_{L^2}^2 + \|H\nabla u\|_{L^2}^2 \leq \delta^2\left(\|\nabla u\|_{L^6}^2 + \|\nabla H\|_{L^6}^2\right) + C(\delta)\left(\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2\right). \tag{36}$$

Next, we turn to estimate $\|\nabla u\|_{L^6}$ and $\|\nabla H\|_{L^6}$. Note that, from (3), we obtain an elliptic system as follows:

$$-\Delta H = -H_t - u\cdot\nabla H + H\cdot\nabla u. \tag{37}$$

Applying standard L^p-estimate to elliptic systems (15) and (37), we obtain that

$$\begin{aligned} \|\nabla u\|_{L^6} &\leq C(\|\operatorname{div} u\|_{L^6} + \|\nabla\times u\|_{L^6}) \leq C\|\omega\|_{L^6} \\ &\leq C\|\nabla\omega\|_{L^2} \leq C\left(\left\|\rho^{1/2}u_t\right\|_{L^2} + \|u\nabla u\|_{L^2} + \|H\nabla H\|_{L^2}\right) \\ &\leq C\left(\left\|\rho^{1/2}u_t\right\|_{L^2} + \|u\nabla u\|_{L^2} + \|H\|_{L^3}\|\nabla H\|_{L^6}\right) \\ &\leq C\left(\left\|\rho^{1/2}u_t\right\|_{L^2} + \|u\nabla u\|_{L^2} + \|\nabla H\|_{L^6}\right), \\ \|\nabla H\|_{L^6} &\leq \left\|\nabla^2 H\right\|_{L^2} \leq C(\|H_t\|_{L^2} + \|u\nabla H\|_{L^2} + \|H\nabla u\|_{L^2}), \end{aligned} \tag{38}$$

which imply that

$$\|\nabla u\|_{L^6} + \|\nabla \mathrm{H}\|_{L^6} \leq C_2\left(\left\|\rho^{1/2}u_t\right\|_{L^2} + \|H_t\|_{L^2} + \|u\nabla u\|_{L^2} + \|u\nabla \mathrm{H}\|_{L^2} + \|\mathrm{H}\nabla u\|_{L^2}\right) \tag{39}$$

Putting (39) into (36), we have by choosing $\delta > 0$ sufficiently small that

$$\begin{aligned} &\|u\nabla H\|_{L^2}^2 + \|u\nabla u\|_{L^2}^2 + \|H\nabla u\|_{L^2}^2 \\ &\quad\leq \frac{1}{4C_1}\left(\left\|\rho^{1/2}u_t\right\|_{L^2} + \|H_t\|_{L^2}\right) \\ &\qquad + C(\delta)\left(\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2\right). \end{aligned} \tag{40}$$

where the C_1 is given in (34).

Substituting (40) into (34) leads to

$$\frac{d}{dt}\left(\frac{1}{2}\|\nabla u\|^2 + \|\nabla H\|_{L^2}^2\right) + \frac{1}{4}\left(\left\|\rho^{1/2}u_t\right\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \|\Delta H\|_{L^2}\right) \leq -\frac{d}{dt}\int u\cdot H\cdot\nabla H dx + C\left(\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2\right). \tag{41}$$

By (21) and the Young inequality, we easily see that

$$\left|\int H\cdot\nabla u\cdot H dx\right| \leq \frac{1}{4}\|\nabla u\|_{L^2}^2 + C. \tag{42}$$

Taking this into account, we then conclude from (18)–(20), (41), and Gronwall's inequality for any $0 \leq T < T^*$ that holds

$$\sup_{0\leq t\leq T}\left(\|\nabla u\|_{L^2}^2 + \|H\|_{H^1}^2\right) + \int_0^T\left(\left\|\rho^{1/2}u_t\right\|_{L^2}^2 + \|H_t\|_{L^2}^2\right)dt \leq C. \tag{43}$$

Applying the standard elliptic L^2-estimates to (37) leads to

$$\begin{aligned} \|H\|_{H^2} &\leq \|H_t\|_{L^2} + \|u\cdot\nabla H\|_{L^2} + \|H\cdot\nabla u\|_{L^2} \\ &\leq \|H_t\|_{L^2} + C\|u\|_{L^6}\|\nabla H\|_{L^3} + C\|H\|_{L^6}\|\nabla u\|_{L^3} \\ &\leq \|H_t\|_{L^2} + C\|\nabla u\|_{L^2}\|\nabla H\|_{L^2}^{1/2}\left\|\nabla^2 H\right\|_{L^2}^{1/2} \\ &\quad + C\|\nabla u\|_{L^2}^{1/2}\left\|\nabla^2 u\right\|_{L^2}^{1/2} \\ &\leq \|H_t\|_{L^2} + \frac{1}{4}\left(\left\|\nabla^2 u\right\|_{L^2} + \left\|\nabla^2 H\right\|_{L^2}\right) + C, \end{aligned} \tag{44}$$

where we use (21), (43), and Gagliardo-Nirenberg inequality.

On the other hand, since (u, P) is a solution of the stationary Stokes equations

$$-\Delta u + \nabla P = F \text{ and } \operatorname{div} u = 0 \text{ in } \mathbb{R}^3, \tag{45}$$

where $F = -\rho u_t - \rho u \cdot \nabla u - (1/2)\nabla |H|^2 + H \cdot \nabla H$. It follows from the classical regularity theory that

$$\begin{aligned} \|\nabla u\|_{H^1} &\le C(\|\rho u_t\|_{L^2} + \|\rho u \cdot \nabla u\|_{L^2} + \|H \cdot \nabla H\|_{L^2}) \\ &\le C\left\|\rho^{1/2} u_t\right\|_{L^2} + C\|u\|_{L^6}\|\nabla u\|_{L^3} + C\|H\|_{L^6}\|\nabla H\|_{L^3} \\ &\le C\left\|\rho^{1/2} u_t\right\|_{L^2} + \frac{1}{4}\|\nabla u\|_{H^1} + \frac{1}{4}\|H\|_{H^2} + C. \end{aligned} \quad (46)$$

Adding (44) to (46), we obtain

$$\begin{aligned} &\int_0^T \|\nabla u\|_{H^1}^2 + \|H\|_{H^2}^2 ds \\ &\quad \le C\int_0^T \left(\left\|\rho^{1/2} u_t\right\|_{L^2}^2 + \|H_t\|_{L^2}^2 + 1\right) ds \le C. \end{aligned} \quad (47)$$

This, together with (43), immediately implies (27). This lemma is completed. □

The following lemma is concerned with the L^2-estimate of $\rho^{1/2} u_t$ and H_t.

Lemma 11. *Under the assumption (22), it holds for any $t \in (0, T]$ such that*

$$\sup_{0\le t\le T}\left(\left\|\rho^{1/2} u_t\right\|_{L^2}^2 + \|H_t\|_{L^2}^2\right) + \int_0^T \|\nabla u_t\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2 dt \le C. \quad (48)$$

Proof. Differentiating the momentum equations (2) with respect to t yields

$$\begin{aligned} &\rho u_{tt} + \rho u \cdot \nabla u_t + \rho u_t \cdot \nabla u + \rho_t(u_t + u \cdot \nabla u) + \nabla P_t \\ &\quad = \Delta u_t + \left(H \cdot \nabla H - \frac{1}{2}\nabla |H|^2\right)_t. \end{aligned} \quad (49)$$

Multiplying the equation above with u_t and integrating by parts, one gets

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\|\rho^{1/2} u_t\right\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 \\ &\quad = -\int \rho_t |u_t|^2 dx - \int \rho(u_t \cdot \nabla u) \cdot u_t dx - \int \rho_t (u \cdot \nabla u) \\ &\qquad \cdot u_t dx + \int (H_t \cdot \nabla H + H \cdot \nabla H_t) \cdot u_t dx, \end{aligned} \quad (50)$$

due to div $u_t = 0$.

Differentiating (3) with respect to t and multiplying the resulting equation by H_t, we obtain after integrating by parts that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|H_t\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2 &= -\int u_t \cdot \nabla H \cdot H_t dx + \int H_t \\ &\quad \cdot \nabla u \cdot H_t + H \cdot \nabla u_t \cdot H_t dx, \end{aligned} \quad (51)$$

where we have used div $H_t = 0$ and div $u = 0$.

Putting (50) and (51) together leads to

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|H_t\|_{L^2}^2 + \left\|\rho^{1/2} u_t\right\|_{L^2}^2\right) + \|\nabla u_t\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2 \\ &\quad = -\int \rho_t |u_t|^2 dx - \int \rho(u_t \cdot \nabla u) \cdot u_t dx - \int \rho_t (u \cdot \nabla u) \cdot u_t dx \\ &\qquad + \int H_t \cdot \nabla H \cdot u_t - u_t \cdot \nabla H \cdot H_t dx + \int H_t \cdot \nabla u \cdot H_t dx \\ &\qquad + \int H \cdot \nabla u_t \cdot H_t + H \cdot \nabla H_t \cdot u_t dx = \sum_{i=1}^{6} R_i. \end{aligned} \quad (52)$$

We now estimate each term on the right-hand side of (52) by using the previous estimates.

First, by virtue of (1), we obtain

$$\begin{aligned} |R_1| &= \left|\int \rho u \cdot \nabla |u_t|^2 dx\right| \\ &\le C\|\nabla u_t\|_{L^2}\|u\|_{L^6}\|\rho u_t\|_{L^3} \\ &\le C\|\nabla u_t\|_{L^2}\|\rho u_t\|_{L^2}^{1/2}\|\rho u_t\|_{L^6}^{1/2} \\ &\le C\|\nabla u_t\|_{L^2}^{3/2}\|\rho u_t\|_{L^2}^{1/2} \\ &\le \varepsilon\|\nabla u_t\|_{L^2}^2 + C(\varepsilon)\left\|\rho^{1/2} u_t\right\|_{L^2}^2. \end{aligned} \quad (53)$$

Similarly, the estimate of R_2 is given as follows

$$\begin{aligned} |R|_2 &\le C\int \rho |u_t|^2 |\nabla u| dx \le C\|\rho u_t\|_{L^2}\|u_t\|_{L^6}\|\nabla u\|_{L^3} \\ &\le C\left\|\rho^{1/2} u_t\right\|_{L^2}\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^2}^{1/2}\left\|\nabla^2 u\right\|_{L^2}^{1/2} \\ &\le \varepsilon\|\nabla u_t\|_{L^2}^2 + C(\varepsilon)\left\|\rho^{1/2} u_t\right\|_{L^2}^4 + C(\varepsilon)\left\|\nabla^2 u\right\|_{L^2}^2. \end{aligned} \quad (54)$$

For R_3, we have that

$$\begin{aligned} |R_3| &= \left|\int \rho u \cdot \nabla(u \cdot \nabla u \cdot u_t) dx\right| \\ &\le C\int |u||\nabla u|^2|u_t| + |u|^2\left|\nabla^2 u\right||u_t| \\ &\quad + |u|^2|\nabla u||\nabla u_t| dx \triangleq \sum_{i=1}^{3} J_i. \end{aligned} \quad (55)$$

From Lemma 8 to 9, we can reduce that

$$
\begin{aligned}
|J_1| &\le C\int |u||\nabla u|^2|u_t|dx \\
&\le C\|\nabla u\|_{L^3}^2\|u\|_{L^6}\|u_t\|_{L^6} \\
&\le C\|\nabla u\|_{L^2}^2\left\|\nabla^2 u\right\|_{L^2}\|\nabla u_t\|_{L^2} \\
&\le \varepsilon\|\nabla u_t\|_{L^2}^2 + C(\varepsilon)\left\|\nabla^2 u\right\|_{L^2}^2, \\
|J_2| &\le C\left\|\nabla^2 u\right\|_{L^2}\|u\|_{L^6}^2\|u_t\|_{L^6} \\
&\le C\left\|\nabla^2 u\right\|_{L^2}\|\nabla u\|_{L^2}^2\|\nabla u_t\|_{L^2} \\
&\le \varepsilon\|\nabla u_t\|_{L^2}^2 + C(\varepsilon)\left\|\nabla^2 u\right\|_{L^2}^2, \\
|J_3| &\le C\|\nabla u\|_{L^6}\|u\|_{L^6}^2\|\nabla u_t\|_{L^2} \\
&\le C\left\|\nabla^2 u\right\|_{L^2}\|\nabla u\|_{L^2}^2\|\nabla u_t\|_{L^2} \\
&\le \varepsilon\|\nabla u_t\|_{L^2}^2 + C(\varepsilon)\left\|\nabla^2 u\right\|_{L^2}^2.
\end{aligned} \tag{56}
$$

From the $J_1 - J_3$, we get estimate of R_3

$$
|R_3| \le C\left\|\nabla^2 u\right\|_{L^2}^2 + 3\varepsilon\|\nabla u_t\|_{L^2}^2. \tag{57}
$$

Similarly, we have

$$
\begin{aligned}
|R_4| &\le C\int |H_t||\nabla u||u_t|dx \le C\|\nabla u\|_{L^3}\|H_t\|_{L^2}\|u_t\|_{L^6} \\
&\le C\|\nabla u\|_{L^2}^{1/2}\left\|\nabla^2 u\right\|_{L^2}^{1/2}\|H_t\|_{L^2}\|\nabla u_t\|_{L^2} \\
&\le \varepsilon\|\nabla u_t\|_{L^2}^2 + C(\varepsilon)\left\|\nabla^2 u\right\|_{L^2}\|H_t\|_{L^2}^2 \\
&\le \varepsilon\|\nabla u_t\|_{L^2}^2 + C(\varepsilon)\|H_t\|_{L^2}^4 + C(\varepsilon)\left\|\nabla^2 u\right\|_{L^2}^2, \\
|R_5| &\le C\int |H_t|^2|\nabla u|dx \le C\|H_t\|_{L^4}^2\|\nabla u\|_{L^2} \\
&\le C\|H_t\|_{L^2}^{1/2}\|\nabla H_t\|_{L^2}^{3/2} \le \varepsilon\|\nabla H_t\|_{L^2}^2 + C(\varepsilon)\|H_t\|_{L^2}^2.
\end{aligned} \tag{58}
$$

It is easy to prove that $R_6 = 0$. Thus, taking ε suitable small and substituting the estimates of $R_1 - R_6$ into (52) lead to

$$
\begin{aligned}
&\frac{d}{dt}\left(\|H_t\|_{L^2}^2 + \left\|\rho^{1/2}u_t\right\|_{L^2}^2\right) + \left(\|\nabla u_t\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2\right) \\
&\quad\le C\left\|\nabla^2 u\right\|_{L^2}^2 + C\left(1 + \|H_t\|_{L^2}^2\right)\|H_t\|_{L^2}^2 \\
&\qquad + C\left(1 + \left\|\rho^{1/2}u_t\right\|_{L^2}^2\right)\left\|\rho^{1/2}u_t\right\|_{L^2}^2,
\end{aligned} \tag{59}
$$

which, together with Gronwall's inequality, immediately leads to the desired estimate (50) since (27) implies $\nabla^2 u \in L^2(0,T)$, $1 + \|H_t\|_{L^2}^2 \in L^1(0,T)$, and $1 + \|\rho^{1/2}u_t\|_{L^2}^2 \in L^1(0,T)$. □

Lemmas 12 and 13 deal with the higher-order estimates of the solutions which are needed to guarantee the extension of a local strong solution to a global one.

Lemma 12. *Under the assumption (22), it holds for any $t \in (0,T]$ such that*

$$
\sup_{0\le t\le T}\left(\|\nabla u\|_{H^1} + \|H\|_{H^2}\right) + \int_0^T \|u\|_{W^{2,6}}^2 + \|H\|_{W^{2,6}}^2 dx \le C(T). \tag{60}
$$

Proof. Applying classical regularity theory to (45) again, we have

$$
\begin{aligned}
\sup_{0\le t\le T}\|\nabla u\|_{W^{1,6}} &\le C(\|\rho u_t\|_{L^6} + \|u\|_{L^\infty}\|\nabla u\|_{L^6} \\
&\quad + \|H\|_{L^\infty}\|\nabla H\|_{L^6} + \|\nabla u\|_{L^6} + 1) \\
&\le C\left(\|\nabla u_t\|_{L^2} + \|\nabla u\|_{H^1}^2 + \|H\|_{H^2}^2\right).
\end{aligned} \tag{61}
$$

Integrating the inequality above over $(0,t)$ and by (27) lead to

$$
\int_0^t \|u\|_{W^{2,6}}^2 ds \le C\int_0^t \|\nabla u_t\|_{L^2} + \|\nabla u\|_{H^1}^2 + \|H\|_{H^2}^2 ds \le C. \tag{62}
$$

Similar proof leads to the same conclusion for H

$$
\int_0^t \|H\|_{W^{2,6}}^2 \mathrm{d}s \le C. \tag{63}
$$

By virtue of (44), (46), and (48), it is easily to prove that

$$
\sup_{0\le t\le T}\left(\|\nabla u\|_{H^1} + \|H\|_{H^2}\right) \le C. \tag{64}
$$

Thus, Lemma 11 is proved. □

Finally, the following lemma gives bounds of the first spatial derivatives of the density ρ.

Lemma 13. *Under the assumption (18), it holds for any $t \in (0,T]$ such that*

$$
\sup_{0\le t\le T}\|\rho\|_{W^{1,q}} \le C(T), \text{ for } q \in [2,6]. \tag{65}
$$

Proof. Differentiating (1) with respect to x_i $(i = 1,2,3)$, multiplying it by $|\partial_i\rho|^{p-2}\partial_i\rho$ with $p \ge 2$, and integrating the resulting equation by parts, we obtain after summing over i from 1 to 3 that

$$
\frac{d}{dt}\|\nabla\rho\|_{L^p}^p \le C\int |\nabla u||\nabla\rho|^p dx \le C\|\nabla u\|_{W^{1,6}}\|\nabla\rho\|_{L^p}^p, \tag{66}
$$

which, together with Gronwall's inequality, leads to

$$
\|\nabla\rho\|_{L^p}^p \le C\|\nabla\rho_0\|_{L^p}^p \exp\left(C\int_0^T \|u\|_{W^{2,6}} dt\right) \le C. \tag{67}
$$

This finishes the proof of Lemma 13. □

With all the a priori estimates in Section 3 at hand, we are ready to prove the main result of this paper.

Basing on Lemmas 8–13 and using the local existence theorem (cf. Proposition 2), one can easily extend the strong solutions of (ρ, u, H) beyond $t > T^*$ by the standard method. This leads to a contradiction of the assumption on T^*. The proof of Theorem 3 is therefore complete.

Data Availability

All data, models, and code generated or used during the study appear in the submitted article.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors have contributed their parts equally and have also read and approved the final manuscript.

Acknowledgments

This research was partly supported by the National Natural Science Foundation of China (Grant Nos. 61773018, 11971446, 11701525, and 12101569), the Natural Science Foundation of Henan (212300410301), and the China Scholarship Council (Grant No. 201709440003).

References

[1] H. Cabannes, *Theoretical Magnetofluiddynamics*, Academic Press, New York, 1970.

[2] P. A. Davidson, *An Introduction to Magenetohydrodynamics*, Cambridge University Press, Cambridge, 2010.

[3] J. F. Gerbeau, "Existence of solution for a density-dependent magneto hydrodynamic equation," *Advances in Differential Equations*, vol. 2, p. 427, 1997.

[4] L. C. Woods, *Priciples of Magnetoplasma Dynamics*, Oxford University Press, Oxford, 1987.

[5] G. Duvaut and J. L. Lions, "Inéquations en thermoélasticité et magnétohydrodynamique," *Archive for Rational Mechanics and Analysis*, vol. 46, no. 4, pp. 241–279, 1972.

[6] M. Sermange and R. Temam, "Some mathematical questions related to the MHD equations," *Communications on Pure and Applied Mathematics*, vol. 36, no. 5, pp. 635–664, 1983.

[7] C. He and Z. Xin, "On the regularity of weak solutions to the magnetohydrodynamic equations," *Journal of Differential Equations*, vol. 213, no. 2, pp. 235–254, 2005.

[8] C. He and Y. Wang, "On the regularity criteria for weak solutions to the magnetohydrodynamic equations," *Journal of Differential Equations*, vol. 238, no. 1, pp. 1–17, 2007.

[9] X. J. Jia and Y. Zhou, "A new regularity criterion for the 3D incompressible MHD equations in terms of one component of the gradient of pressure," *Journal of Mathematical Analysis and Applications*, vol. 396, no. 1, pp. 345–350, 2012.

[10] Y. Zhou, "Remarks on regularities for the 3D MHD equations," *Discrete & Continuous Dynamical Systems*, vol. 12, no. 5, pp. 881–886, 2005.

[11] L. Ni, Z. Guo, and Y. Zhou, "Some new regularity criteria for the 3D MHD equations," *Journal of Mathematical Analysis and Applications*, vol. 396, no. 1, pp. 108–118, 2012.

[12] X. Zhong, "A blow-up criterion for strong solutions to three-dimensional compressible magnetohydrodynamic equations," 2016, https://arxiv.org/abs/1603.08516.

[13] C. Cao and J. Wu, "Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion," *Advances in Mathematics*, vol. 226, no. 2, pp. 1803–1822, 2011.

[14] L. Li, W. Wang, and L. Liu, "A new regularity criterion for the three-dimensional incompressible magnetohydrodynamic equations in the Besov spaces," *Journal of Function Spaces*, vol. 2021, 7 pages, 2021.

[15] C. Tian, "A regularity criterion for the 3D incompressible magnetohydrodynamics equations in the multiplier spaces," *Journal of Function Spaces*, vol. 2017, 5 pages, 2017.

[16] Q. Chen, Z. Tan, and Y. J. Wang, "Strong solutions to the incompressible magnetohydrodynamic equations," *Mathematical methods in the applied sciences*, vol. 34, no. 1, pp. 94–107, 2011.

[17] S. Kawashima, *Systems of a Hyperbolic-Parabolic Composite Type, with Applications Fo the Equations of Magnetohydrodynamics*, Ph. D, Thesis, Kyoto University, 1983.

[18] J. Zhang, "The inviscid and non-resistive limit in the Cauchy problem for 3-D nonhomogeneous incompressible magnetohydrodynamics," *Acta Mathematica Scientia*, vol. 31, no. 3, pp. 882–896, 2011.

[19] B. Desjardins and C. Le Bris, "Remarks on a nonhomogeneous model of magnetohydrodynamics," *Differential and Integral Equations*, vol. 11, pp. 377–394, 1998.

[20] H. Abidi and M. Paicu, "Global existence for the magnetohydrodynamic system in critical spaces," *Proceedings of the Royal Society of Edinburgh*, vol. 138, no. 3, pp. 447–476, 2008.

[21] X. D. Huang and Y. Wang, "Global strong solution to the 2D nonhomogeneous incompressible MHD system," *Journal of Differential Equations*, vol. 254, no. 2, pp. 511–527, 2013.

[22] Y. Giga, "Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system," *Journal of Differential Equations*, vol. 62, no. 2, pp. 186–212, 1986.

[23] S. Liu and J. Zhang, "A note on the blow-up criterion for the compressible isentropic Navier-Stokes equations with vacuum," *Applied Mathematics Letters*, vol. 26, no. 5, pp. 584–588, 2013.

[24] O. A. Ladyzenskaja, V. A. Solonnikov, and N. N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, American Mathematical Society, Providence, RI, 1968.