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Research Article

Local and Global Existence and Uniqueness of Solution for Class of Fuzzy Fractional Functional Evolution Equation

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For fuzzy fractional functional evolution equations, the concept of global and local existence and uniqueness will be presented in this work. We employ the contraction principle and successive approximations for global and local existence and uniqueness, respectively, as given $\begin{cases} {}_0^cD_q^H x(\mathfrak{I}) = f(\mathfrak{I}, x_{\mathfrak{I}}) + \int_0^{\mathfrak{I}} g(\mathfrak{I}, s, x_s)ds, \mathfrak{I} \geq \mathfrak{I}_0, \mathfrak{I} \in [0, T], \\ x(\mathfrak{I}) = \psi(\mathfrak{I} - \mathfrak{I}_0) = \psi_0, \in C_\sigma, \mathfrak{I}_0 \geq \mathfrak{I} \geq \mathfrak{I}_0 - \sigma, \\ x'(\mathfrak{I}) = \psi'(\mathfrak{I}) = \psi_1, \end{cases}$ where C_σ denotes the set of fuzzy continuous mapping defined on $[\mathfrak{I}_0 - \sigma, T]$ and $\sigma > 1$. We also use this method to solve fuzzy fractional functional evolution equations with fuzzy population models and distributed delays using fuzzy fractional functional evolution equations. To explain these results, some theorems are given. Finally, certain fuzzy fractional functional evolution equations are illustrated.

1. Introduction

In reality, to show fractional-order demeanor which can change with time and space in case of a large number of physical processes, fractional calculus authorizes the operations of differentiation and integration of fractional-order. The fractional-order can be applied to both imaginary and real numbers. Because of its wide range of applications in disciplines like mechanics, electrical engineering, signal processing, thermal systems, robotics and control, signal processing, and many others, the theory of fuzzy sets continues to attract academics' attention [1–3]. Therefore, it has been noticed that it is the center of increasing interest of researchers during the past few years.

In real-world systems, delays can be recognized everywhere, and there has been widespread interest in the study of delay differential equations for many years. Fractional differential equations are becoming more important in system models in biology, chemistry, physics, and other sciences. There is a large form of evidence about functional differential equations and their methods. On the other hand, we can seldom be certain that dynamic in a system is perfectly modeled using deterministic ordinary differential equations because the knowledge of dynamical systems is either unclear or incomplete. If the model's underlying structure is based on subjective decisions, one way to incorporate these is to use the fuzziness aspect, which contributes to the consideration of fuzzy fractional functional evolution equations. In the context of fuzzy-valued analysis and set-valued differential equations, fuzzy differential equations were first studied as a separate subject. The analysis of fuzzy differential equations can be expressed in a variety of ways. In biology, chemistry, physics, and other sciences, fractional differential equations are becoming more significant in system models. The reader is referred to the monographs [4, 5], and the references therein, as there is a large quantity of literature dealing with delay differential equations and their applications. As a new branch of fuzzy mathematics, the study of fuzzy delay differential equations is growing in popularity. Over the last few years, both theory and applications have been widely discussed. The study of fuzzy delay fractional functional evolution equations has numerous interpretations in the literature.

Puri and Ralescu defined H-differentiability for fuzzy functions using the Hukuhara derivative of multivalued functions. In the context of fuzzy differential equations in a time-dependent manner, Seikkala and Kaleva proposed and investigated this definition. The fuzzy initial value issue has a unique local solution if f is continuous and satisfies the Lipschitz condition with respect to u, as Kaleva established in [6].

$$U'(\mathfrak{I}) = f(\mathfrak{I}, u), u(0) = u_0 \, on(\mathbf{E^m}, \mathscr{D}). \tag{1}$$

He proved that the Peano theorem is invalid in [6], since metric space $(\mathbf{E^m}, \mathscr{D})$ can be locally compact. Peano's existence theorem for FDEs on $(\mathbf{E^m}, \mathscr{D})$ was proven by Nieto [7] if f is bounded and continuous. Buckley and Feuring [8] gave reasonable general formulation to the fuzzy first-order initial value problem. Citations [9, 10] present the existence of theorems for solutions to the fuzzy initial value problem under a wide range of assumptions. This H-differentiability-based approach has the disadvantage of having an increasing length of support for each solution of FDE. As a result, this method is inappropriate for modeling and fails to describe any of the complex properties of ordinary differential equations, that is, stability, periodicity, bifurcation, and other phenomena [11]. This problem is solved using FDE, which can be read as a family of differential inclusions [12]. We do not have a derivative for fuzzy-number-valued equations, which is a key drawback of differential inclusions.

The above-mentioned method for fuzzy-number-valued functions with highly generalized differentiability was recently solved by Bede and Gal [13]. The derivative is maintained in this case, and the support length of the FDE solution may decrease, but the uniqueness is lost. On fuzzy differential equations, there is a lot of literature. In comparison, FFDEs and their implementations were only briefly mentioned in a few articles. Park and his colleagues' [14] approximate solutions of fuzzy functional integral equations were studied. Park et al. [15] examined the presence of almost periodic and asymptotically almost periodic solutions for FFDEs. For nonlinear fuzzy neutral functional differential equations, Balasubramaniam and Muralisankar [16] investigate local uniqueness and existence theorem. Guo et al. [17] developed existence results for fuzzy impulsive functional differential equations using Hüllermeier's level-wise method [13], which they then applied to fuzzy population models. Abbas et al. [18, 19] worked on a partial differential equation. Niazi et al. [20], Iqbal et al. [21], Shafqat et al. [22], Abuasbeh et al. [23], and Alnahdi et al.'s [24] existence and uniqueness of the FFEE were investigated.

Khastan et al. proved the existence of two fuzzy solutions for fuzzy delay differential equations using the concept of generalized differentiability. Hoa et al. established the global existence and uniqueness results for fuzzy delay differential equations using the concept of generalized differentiability. Moreover, the authors have extended and generalized some comparison theorems and stability theorems for fuzzy delay differential equations with the definition of a new Lyapunov-like function. Besides that, some very important extensions of the fuzzy delay dierential equations were introduced. The author considered the FDE with the initial value

$$X'(\mathfrak{I}) = f(\mathfrak{I}, x(\mathfrak{I})), x(\mathfrak{I}_0) = x_0 \in \mathbf{E^d}, \tag{2}$$

where $f : [0,\infty) \times \mathbf{E^d} \longrightarrow \mathbf{E^d}$ and the symbol $'$ denotes the first type of Hukuhara derivative, that is, the classical Hukuhara derivative. O. Kaleva also discussed the properties of differentiable fuzzy mappings and showed that if f is continuous and $f(\mathfrak{I}, x)$ satisfies the Lipschitz condition concerning to x, then, there exists a unique local solution for the fuzzy initial value problem. V. Lupulescu proved several theorems stating the existence, uniqueness, and boundedness of solutions to fuzzy differential equations with the concept of the inner product on the fuzzy space. Guo et al. [25] and Shu et al. [26] studied the fractional differential equation.

In [27], V. Lupulescu considered the fuzzy functional differential equation

$$\begin{aligned} x'(\mathfrak{I}) &= f(\mathfrak{I}, x_{\mathfrak{I}}), \mathfrak{I} \geq \mathfrak{I}_0, \\ x(\mathfrak{I}) &= \phi(\mathfrak{I} - t_0) \in \mathbf{E^d}, \mathfrak{I}_0 \geq \mathfrak{I} \geq \mathfrak{I}_0 - \sigma, \end{aligned} \tag{3}$$

where $f : [0,\infty) \times C_\sigma \longrightarrow \mathbf{E^d}$ and the symbol $'$ denotes the first type Hukuhara derivative called classical Hukuhara derivative. The author studied the local and global existence and uniqueness results by using the method of successive approximations and contraction principle.

We used Caputo derivative to prove the uniqueness and existence of several uniqueness and existence theorems for fuzzy fractional functional differential equations (FFFDEs) under certain conditions, inspired by the above research:

$$\begin{aligned} {}_0^cD_q^H x(\mathfrak{I}) &= f(\mathfrak{I}, x_{\mathfrak{I}}) + \int_0^{\mathfrak{I}} g(\mathfrak{I}, s, x_s) \mathrm{ds}, \mathfrak{I} \geq \mathfrak{I}_0, \mathfrak{I} \in [0, T], \\ x(\mathfrak{I}) &= \psi(\mathfrak{I} - \mathfrak{I}_0) = \psi_0 \in C_\sigma, \mathfrak{I}_0 \geq \mathfrak{I} \geq \mathfrak{I}_0 - \sigma \\ x'(\mathfrak{I}) &= \psi'(\mathfrak{I}) = \psi_1, \end{aligned} \tag{4}$$

where C_σ denotes the set of fuzzy continuous mapping defined on $[\mathfrak{I}_0 - \sigma, T]$ and $\sigma > 1$. $x_{\mathfrak{I}}$ denotes the fuzzy mapping $x(\mathfrak{I}_s)$, $\mathfrak{I}_0 - \sigma \leq s \leq T$; that is, $x_{\mathfrak{I}} \in C_\sigma$. The goal of this study is to use the method of contraction principle and consecutive approximations to show local and global uniqueness and existence theorems for the fuzzy fractional functional differential Equation (4) under certain conditions.

The following is a description of the paper's structure. As a warm-up, we will make some basic observations on fuzzy sets and the differentiability and integrability features of fuzzy functions. In Section 3, we show the local uniqueness and existence theorem for the solution to the initial value problem for FFFDEs using the successive approximation method. Section 4 proves the global uniqueness and existence theorem for the initial value solution. A problem involving fuzzy fractional functional differential equations is solved using contraction theory. Finally, we apply what we have learned about FDEs to two different forms of fuzzy differential equations: FFFDEs with fuzzy population and distributed delays models.

2. Preliminaries

The set of all nonempty, compact convex subsets of $\mathbf{R^m}$ is denoted by $\mathscr{K}_c(\mathbf{R^m})$. The Hausdorff distance between sets $A, B \in \mathscr{K}_c(\mathbf{R^m})$ is defined as

$$d_H(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B}||a-b||, \sup_{b\in B}\inf_{a\in A}||a-b||\right\}. \quad (5)$$

Denote $\{\mathbf{E^m} = x : \mathbf{R^m} \longrightarrow 0, 1| \,; x \text{ satisfies (a)} - \text{(d) below}\}$.

In the above equation,

(a) x is normal due to the exists of $\mathbf{R^m}$, $x(u_0) = 1$

(b) x is fuzzy convex, for $\mathbf{R^m}$, $0 \le \lambda \le 1$, $x(\lambda u + (1-\lambda)v) \ge \min\{x(u), x(v)\}$

(c) x is upper semicontinuous function on $\mathbf{R^m}$

(d) $[x]^0 = cl\{s \in \mathbf{R^m}/x(\mathfrak{J}) > 0\}$ is compact

$1 < \beta \le 2$, represent $[x]^\beta = \{u \in \mathbf{R^m}/x(\mathfrak{J}) \ge \beta\}$. Then, from (a) to (b), it shows, β-level set $[x]^\beta \mathfrak{J} \in \mathscr{K}_c(\mathbf{R^m}) \forall 1 \le \beta \le 2$. We define $\tilde{0} \in \mathbf{E^m}$ as $\tilde{0}(u) = 1$ if $u = 0$ and $\tilde{0}(u) = 0$ if $u \ne 0$ for later purposes.

Using Zadeh's extension theorem, we can have scalar multiplication and addition in fuzzy number space $\mathbf{E^m}$ as shown in

$$[x \oplus y]^\beta = [x]^\beta \oplus [y]^\beta, [kx]^\beta = k[x]^\beta, \quad (6)$$

where $x, y \in \mathbf{E^m}, k \in \mathbf{R^m}$ and $1 \le \beta \le 2$.

Define $\mathscr{D} : \mathbf{E^m} \times \mathbf{E^m} \longrightarrow \mathbf{R^+}$ by notation

$$\mathscr{D}(x, y) = \sup_{1\le\beta\le2} d_H\left\{[x]^\beta, [y]^\beta\right\}. \quad (7)$$

where $\mathscr{D}$ is Hausdorff a metric for nonempty compact sets in $\mathbf{R^m}$ and $(\mathbf{E^m}, \mathscr{D})$ is a complete metric space [28].

It is very simple to notice that $\mathscr{D}$ is a metric in $\mathbf{E^m}$. By using the properties of $\mathscr{D}(x, y)$:

(a) $(\mathbf{E^m}, \mathscr{D})$ is a complete metric space

(b) $\mathscr{D}(x \oplus z, y \oplus z) = \mathscr{D}(x, y)$ and $\mathscr{D}(x, y) = \mathscr{D}(x, y) \forall x, y, z \in \mathbf{E^m}$

(c) $\mathscr{D}(\lambda x, \lambda y) = |\lambda|\mathscr{D}(x, y) \forall x, y \in \mathbf{E^m}$ and $\lambda \in \mathbf{R^m}$

(d) $\mathscr{D}(x, y) \le \mathscr{D}(x, z) + \mathscr{D}(z, y)$

If we denote $\|x\|_{\mathfrak{G}} = \mathscr{D}(x, \tilde{0}), x \in \mathbf{E^m}$, then, $\|x\|_{\mathfrak{G}}$ has properties of an usual norm on$\mathbf{E^m}$ [29]:

(i) $\|x\|_{\mathfrak{G}} = 0$ if $x = \tilde{0}$

(ii) $\|\lambda x\|_{\mathfrak{G}} = |\lambda|\|x\|_{\mathfrak{G}} \forall x, y \in \mathbf{E^m}$

(iii) $\|x + y\|_{\mathfrak{G}} \le \|x\|_{\mathfrak{G}} + \|y\|_{\mathfrak{G}} \forall x, y \in \mathbf{E^m}$

(iv) $\mathscr{D}(\beta x, \gamma x) \le |\beta - \gamma| D(x, \tilde{0}), \forall \beta, \gamma \ge 1$ or $\beta, \gamma \le 1, x \in \mathbf{E^m}$

On $\mathbf{E^m}$, we can describe subtraction !, also known as H -difference [30], as follows: $s \ominus v$ has significance if $\omega \in \mathbf{E^m}$, $x = y + z$ exists.

Suppose $a, b \in \mathbf{R^m}, f \in \mathscr{C}(I, \mathbf{E^m})$, if we represent $\|f\| = H(f, \tilde{0})$, then, $\|f\|$ has properties of an usual norm on $\mathbf{E^m}$ [29],

(i) $\|f\| = 0$ if $f = \tilde{0}$

(ii) $\|\lambda f\| = |\lambda|\|f\| \forall f \in \mathbb{C}(I, \mathbf{E^m}), \lambda \in \mathbf{R^m}$

(iii) $\|f \oplus h\| \le \|f\| \oplus \|h\| \forall f, h \in (I, \mathbf{E^m})$

(iv) $H(\beta f, \gamma f) \le |\beta - \gamma| H(f, \tilde{0}), \forall \beta, \gamma \ge 0$ or $\beta, \gamma \le 0$, $f \in \mathbb{C}(I, \mathbf{E^m})$

Definition 1. The mapping $\mathscr{F} : \mathrm{I} \longrightarrow \mathbf{E^m}$ is Hukuhara differentiable at $\mathfrak{J} \in I$ if exists $\mathscr{G}'(\mathfrak{J}) \in \mathbf{E^m}$ similar to the limits:

$$\lim_{h\longrightarrow0^+} \frac{\mathscr{G}(\mathfrak{J}_0 + h) \ominus \mathscr{G}(\mathfrak{J}_0)}{h} \text{and} \lim_{h\longrightarrow0^+} \frac{\mathscr{G}(\mathfrak{J}_0) \ominus \mathscr{G}(\mathfrak{J}_0 - h)}{h}, \quad (8)$$

and is equal and exists to $\mathscr{G}'(\mathfrak{J})$.

We can remember some properties of integrability and measurability for fuzzy set-valued mappings [28].

Definition 2. If $\mathscr{G} : \mathrm{I} \longrightarrow \mathbf{E^m}$ is fuzzy function, that is

$$[\mathscr{G}(\mathfrak{J})]^\beta = \left[\mathscr{G}_1^\beta(\mathfrak{J}), \mathscr{G}_2^\beta(\mathfrak{J})\right], \beta \in [1, 2], \quad (9)$$

and there exists $\mathscr{G}'(\mathfrak{J}_0)$ for some $\mathfrak{J}_0 \in I$, and now

$$\left[\mathscr{G}'(\mathfrak{J}_0)\right]^\beta = \left[\left(\mathscr{G}_1^\beta\right)'(\mathfrak{J}_0), \left(\mathscr{G}_2^\beta\right)'(\mathfrak{J}_0)\right], \beta \in [1, 2]. \quad (10)$$

Definition 3. The mapping $\mathscr{G} : I \in \mathbf{E^m}$ is strongly measurable if for all $\beta \in [1, 2]$, then, the set-valued function $\mathscr{G}_\beta : \mathrm{I} \longrightarrow \mathrm{M_j}\mathbf{R^m}$define by $\mathscr{G}_\beta(\mathfrak{J}) = [\mathscr{G}(\mathfrak{J})]^\beta$ is Lebesgue measurable.

The mapping $\mathscr{G} : I \in \mathbf{E^m}$ is known as integrably bounded if there exists an integrable function j like

$$||x|| \leq J(\mathfrak{I}) \forall x \in \mathscr{G}_0(\mathfrak{I}). \tag{11}$$

Definition 4. Suppose $\mathscr{G} : \mathrm{I} \in \mathbf{E^m}$. Then, the equation defines integral of $\mathscr{G}$ over I, which is expressed by $\int_{\mathrm{I}} \mathscr{G}(\mathfrak{I})\mathrm{dt}$, $[\int_I \mathscr{G}(\mathfrak{I})\mathrm{dt}]^{\beta} = \int_I \mathscr{G}_{\beta}(\mathfrak{I}) d\mathfrak{I} = \{\int_I \mathscr{G}(\mathfrak{I}) d\mathfrak{I}/f : I \longrightarrow \mathbf{R^m}$ is measurable selection for $\mathscr{G}_{\beta}\} \forall \beta \in [1, 2]$.

Also, strongly measurable and integrably bounded mapping $\mathscr{G} : I \longrightarrow \mathbf{E^m}$ is said to be integrable over I if and only if

$$\int_I \mathscr{G}(\mathfrak{I}) d\mathfrak{I} \in \mathbf{E^m}. \tag{12}$$

Proposition 5 (Aumann [31]). *$\mathscr{G}$ is integrable if $\mathscr{G} : I \in \mathbf{E^m}$ is integrably bounded and strongly measurable.*

Proposition 6 (Kaleve [28]). *It is integrable over I if $\mathscr{G} : I \longrightarrow \mathbf{E^m}$ is continuous. Furthermore, function $\mathscr{F}(\mathfrak{I}) = \int_{\mathfrak{I}_0}^{\mathfrak{I}} \mathscr{G}(s)ds$, $\mathfrak{I}_0, \mathfrak{I} \in I$ is differentiable in this case, and $\mathscr{F}'(\mathfrak{I}) = \mathscr{G}(\mathfrak{I})$.*

Proposition 7 (Kaleve [28]). *Suppose $\mathscr{G}, H : I \in \mathbf{E^m}$ be integrable and $\lambda \in \mathbf{R^m}$. Now*

(i) $\int_I (\mathscr{G}(\mathfrak{I}) \oplus H(\mathfrak{I})) d\mathfrak{I} = \int_I \mathscr{G}(\mathfrak{I}) d\mathfrak{I} \oplus \int_I H(\mathfrak{I}) d\mathfrak{I}$

(ii) $\int_I \lambda \mathscr{G}(\mathfrak{I}) d\mathfrak{I} = \lambda \int_I \mathscr{F}(\mathfrak{I}) d\mathfrak{I}$

(iii) $\mathscr{D}(\mathscr{G}, H)$ is integrable

(iv) $\mathscr{D}(\int_I \mathscr{G}(\mathfrak{I}) d\mathfrak{I}, \int_I H(\mathfrak{I}) d\mathfrak{I}) \int_I \mathscr{D}(\mathscr{G}, H)(\mathfrak{I}) d\mathfrak{I}$

(v) $\int_{\mathfrak{I}_0}^{\mathfrak{I}_2} \mathscr{G}(\mathfrak{I}) d\mathfrak{I} = \int_{\mathfrak{I}_0}^{\mathfrak{I}_1} \mathscr{G}(\mathfrak{I}) d\mathfrak{I} + \int_{\mathfrak{I}_1}^{\mathfrak{I}_2} \mathscr{G}(\mathfrak{I}) d\mathfrak{I}$, for $\mathfrak{I}_0$, $\mathfrak{I}_1, \mathfrak{I}_2 \in I$

If I is compact interval of $\mathbf{R^m}$, then represent $\mathbb{C}(I, \mathbf{E^m}) = \{f : I \longrightarrow \mathbf{E^m}; f$ is continuous functions on $I\}$, equipped with metric

$$\mathscr{D}(x, y) = \sup_{\mathfrak{I} \in I} \mathscr{D}(x(\mathfrak{I}), y(\mathfrak{I})). \tag{13}$$

Now, $(\mathbb{C}, H)$ is a complete metric space.

We call C_{σ} space $C([-\sigma, 0], \mathbf{E^m})$ for positive numbers σ. Represent it as well:

$$\mathscr{D}_{\sigma}(x, y) = \sup_{\mathfrak{I} \in [-\sigma, 0]} \mathscr{D}(x(\mathfrak{I}), y(\mathfrak{I})), \tag{14}$$

metric on space C_{σ}. For a given constant $\rho > 0$, put $B_{\rho} \coloneqq \{\varphi \in C_{\sigma}; \mathscr{D}_{\sigma}(\varphi, 0) \leq \rho\}$.

Suppose $x(.) \in C([-\sigma, \infty), \mathbf{E^m})$. Now, for all $\mathfrak{I} \in [0, \infty)$, denoted by x_1 element of C_{σ} defined by $x_1(s) = x(\mathfrak{I} + s)$, $s \in [-\sigma, 0]$.

Definition 8 (Fuzzy Strongly Continuous Semigroups) [30, 31]. A family $\{\mathrm{T}(\mathfrak{I}), \mathfrak{I} \geq 0\}$ is fuzzy strongly continuous semigroup of operators from $\mathbf{E^m}$ into itself if

(i) $T(0) = k$ identity mapping on $\mathbf{E^m}$

(ii) $T(\mathfrak{I} \oplus m) = T(\mathfrak{I}) T(m) \forall \mathfrak{I}, m \geq 0$

(iii) function $h : [0, \infty[\longrightarrow \mathbf{E^m}$, defined by $h(\mathfrak{I}) = T(\mathfrak{I})x$ at $\mathfrak{I} = 0 \forall x \in \mathbf{E^m}$ is continuous

$$\lim_{\mathfrak{I} \longrightarrow 0^+} T(\mathfrak{I})x = x. \tag{15}$$

(iv) There are two constants $R > 0$ and ω like

$$\mathscr{D}(T(\mathfrak{I})x, T(\mathfrak{I})y) \leq Re^{\omega} D(x, y), \text{for} \mathfrak{I} \geq 0, x, y \in \mathbf{E^m}. \tag{16}$$

Specially, if $\omega = 0$ and $\mathbf{R^m} = 1$, $\{T(\mathfrak{I}), \mathfrak{I} \geq 0\}$ is a contraction fuzzy semigroup.

Lemma 9. *If $\mathscr{G} : [0, \infty) \times C_{\sigma} \longrightarrow \mathbf{E^m}$ is jointly continuous function and $x : [-\sigma, \infty) \longrightarrow \mathbf{E^m}$ is continuous function, now, function $\mathfrak{I} \mapsto \mathscr{G}(\mathfrak{I}, x_{\mathfrak{I}})$: $[0, \infty) \longrightarrow \mathbf{E^m}$ is also continuous.*

Proof. Assume that fixed $(\tau, \varphi) \times C_{\sigma}$ and $\varepsilon > 0$. $\mathscr{G} : [0, \infty) \times C_{\sigma} \longrightarrow \mathbf{E^m}$ are jointly continuous, there exists $\delta_1 > 0$ that is for all $(\mathfrak{I}, \psi) \in [0, \infty) \times C_{\sigma}$ with $|\mathfrak{I} - \tau| + \mathscr{D}_{\sigma}(\varphi, \psi) < \delta_1$, $\mathscr{D}[\mathscr{G}(\mathfrak{I}, \psi), \mathscr{G}(\tau, \varphi)] < \varepsilon$. On the other way, $x : [-\sigma, \infty) \longrightarrow \mathbf{E^m}$ is continuous; now, it is uniformly continuous on compact interval $I_1 = [\max\{-\sigma, \tau - \sigma - \delta_1\}, \tau + \delta_1]$. There exists $\delta_2 > 0$; for all $\mathfrak{I}_1, \mathfrak{I}_2 \in I_1$ with $|\mathfrak{I}_1 - \mathfrak{I}_2| < \delta_2$, we have $\mathscr{D}[x(\mathfrak{I}_1), x(\mathfrak{I}_2)] < \delta_1/2$. After, for all $s \in [-\sigma, 0]$, $\tau + s \in I_1$, and $\mathfrak{I} + \mathrm{s} \in I_1$ if $|\mathfrak{I} - \tau| < \delta_1/2$, now, $|(\mathfrak{I} + \mathrm{s}) - (\tau + \mathrm{s})| < \delta_2$, and it shows that

$$\begin{aligned} \mathscr{D}_{\sigma}(x_{\mathfrak{I}}, x_{\tau}) &= \sup_{-\sigma \leq s \leq 0} \mathscr{D}[x_{\mathfrak{I}}(\mathrm{s}), x_{\tau}(\mathrm{s})] \\ &= \sup_{-\sigma \leq s \leq 0} \mathscr{D}[x(\mathfrak{I} + \mathrm{s}), x(\tau + \mathrm{s})] \leq \delta_1/2. \end{aligned} \tag{17}$$

Therefore, $|\mathfrak{I} - \tau| + \mathscr{D}_{\sigma}(x_{\mathfrak{I}}, x_{\tau}) < \delta_1$, since $\mathscr{G}$ is jointly continuous, $\mathscr{D}[\mathscr{G}(\mathfrak{I}, x_{\mathfrak{I}}), \mathscr{G}(\tau, x_{\tau})] < \varepsilon$. This implies that function $\mathfrak{I} \mapsto \mathscr{G}(\mathfrak{I}, x_{\mathfrak{I}})$: $[0, \infty) \longrightarrow \mathbf{E^m}$ is continuous. □

Remark 10. If $\mathscr{G} : [0, \infty) \times C_{\sigma} \longrightarrow \mathbf{E^m}$ is jointly continuous function and $x : [-\sigma, \infty) \longrightarrow \mathbf{E^m}$ is continuous function, then, function $\mathfrak{I} \mapsto \mathscr{G}(\mathfrak{I}, x_1)$: $[0, \infty) \longrightarrow \mathbf{E^m}$ on each compact interval $[\tau, T]$ is integrable. Furthermore, function $\mathscr{F}(\mathfrak{I}) = \int_{\tau}^{\mathfrak{I}} \mathscr{G}(s, x_s)\mathrm{ds}$, $\mathfrak{I} \in [\tau, T]$ is differentiable in this case, and $\mathscr{F}'(\mathfrak{I}) = \mathscr{G}(\mathfrak{I}, x_{\mathfrak{I}})$.

Remark 11. If $\mathscr{G} : [0, \infty) \times C_{\sigma} \longrightarrow \mathbf{E^m}$ is jointly continuous function and $x : [-\sigma, \infty) \longrightarrow \mathbf{E^m}$ is continuous function, then, function $\mathfrak{I} \mapsto \mathscr{G}(\mathfrak{I}, x_1)$: $[0, \infty) \longrightarrow \mathbf{E^m}$ on each compact interval $[\tau, T]$ is bounded. On each compact interval $[0, T]$, function $\mathfrak{I} \mapsto \mathscr{G}(\mathfrak{I}, 0): [0,\infty) \longrightarrow \mathbf{E^m}$ is also bounded.

Definition 12. We say that $\mathscr{G} : [0,\infty) \times C_\sigma \longrightarrow \mathbf{E^m}$ is locally Lipschitz if $a, b \in [0,\infty)$ and $\rho > 0$, and there exists $L > 0$,

$$\mathscr{D}[\mathscr{G}(\mathfrak{I}, \varphi), \mathscr{G}(\mathfrak{I}, \psi) \le L\mathscr{D}_\sigma(\varphi, \psi), a \le \mathfrak{I} \le b, \varphi, \psi \in B_\rho. \tag{18}$$

Lemma 13. *Assume that $\mathscr{G} : [0,\infty) \times C_\sigma \longrightarrow \mathbf{E^m}$ is locally Lipschitz and continuous. Now, for all compact interval $J \subset [0,\infty)$ and $\rho > 0$, there exists $\mathscr{K} > 0$,*

$$\mathscr{D}\left[\mathscr{G}(\mathfrak{I}, \varphi), \tilde{0}\right] \le \mathscr{K}, \mathfrak{I} \in J, \varphi \in B_\rho. \tag{19}$$

Proof. $\mathfrak{I} \in J$, then

$$\begin{aligned}\mathscr{D}\left[\mathscr{G}(\mathfrak{I}, \varphi), \tilde{0}\right] &\le \mathscr{D}[\mathscr{G}(\mathfrak{I}, \varphi), \mathscr{G}(\mathfrak{I}, 0)] + \mathscr{D}\left[\mathscr{G}(\mathfrak{I}, 0), \tilde{0}\right] \\ &\le L\mathscr{D}_\sigma(\varphi, 0) + \mathscr{D}\left[\mathscr{G}(\mathfrak{I}, 0), \tilde{0}\right] \le \rho L + \eta,\end{aligned} \tag{20}$$

where $\eta \coloneqq \sup_{\mathfrak{I}\in J} D[\mathscr{G}(\mathfrak{I}, 0), \tilde{0}]$. □

Definition 14 (see [32]). The RL fractional derivative is defined as

$${}_a\mathscr{D}^p_{\mathfrak{I}} f(\mathfrak{I}) = \left(\frac{d}{d\mathfrak{I}}\right)^{n+1} \int_a^{\mathfrak{I}} (\mathfrak{I} - \tau)^{n-p} f(\tau) d\tau, n \le p \le n+1. \tag{21}$$

Definition 15 (see [32]). The Caputo fractional derivatives ${}^c_a\mathscr{D}^\alpha_{\mathfrak{I}} f(\mathfrak{I})$ of order $\alpha \in \mathbf{E}^+$ are defined by

$${}^c_a\mathscr{D}^\alpha_{\mathfrak{I}} f(\mathfrak{I}) = {}_a\mathscr{D}^\alpha_{\mathfrak{I}} \left(f(\mathfrak{I}) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (\mathfrak{I} - a)^k \right), \tag{22}$$

respectively, where $n = [\alpha] + 1$ for $\alpha \notin N_0$; $n = \alpha$ for $\alpha \in N_0$.

We investigate the Caputo fractional derivative of order $1 < \alpha \le 2$ in this study; e.g.,

$${}^c_a\mathscr{D}^{3/2}_{\mathfrak{I}} f(\mathfrak{I}) = {}_a D^{3/2}_{\mathfrak{I}} \left(f(\mathfrak{I}) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (\mathfrak{I} - a)^k \right). \tag{23}$$

Definition 16 (see [33]). The Wright function ψ_α is defined by

$$\begin{aligned}\psi_\alpha(\theta) &= \sum_{n=0}^{\infty} \frac{(-\theta)^n}{n!\Gamma(-\alpha n + 1 - \alpha)} \\ &= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-\theta)^n}{(n-1)!} \Gamma(n\alpha) \sin(n\pi\alpha),\end{aligned} \tag{24}$$

where $\theta \in \mathbb{C}$ with $0 < \alpha < 1$.

Lemma 17 (see [33]). *Let $\{C(\mathfrak{I})\}\mathfrak{I} \in$ **Rm** be a strongly continuous cosine family in X satisfying $\|C(\mathfrak{I})\|_{L_b(X)} \le Me^{\omega|\mathfrak{I}|}$, $\mathfrak{I} \in \mathbf{R^m}$, and let A be the infinitesimal generator of $\{C(\mathfrak{I})\}$ $\mathfrak{I} \in$ **Rm**. Then, for* Re $\lambda > \omega, \lambda^2 \in \rho(A)$

$$\begin{aligned}\lambda R\left(\lambda^2 ; A\right)x &= \int_0^\infty e^{-\lambda\mathfrak{I}} C(\mathfrak{I}) x d\mathfrak{I}, R\left(\lambda^2 ; A\right)x \\ &= \int_0^\infty e^{-\lambda\mathfrak{I}} S(\mathfrak{I}) x d\mathfrak{I}, \textit{for } x \in X.\end{aligned} \tag{25}$$

Lemma 18. *For $x(\mathfrak{I}) = \psi_0$, if $u_{\mathfrak{I}}$ is the solution of Equation (4), then, the solution $u_{\mathfrak{I}}$ is given by*

$$\begin{aligned}x_{\mathfrak{I}} = C_q(\mathfrak{I})\psi_0 + \mathscr{K}_q(\mathfrak{I})\psi_1 + \int_0^{\mathfrak{I}} (\mathfrak{I} - s)^{q-1} P_q(\mathfrak{I} - s) \\ \cdot \left[f(s, x_s) + \int_0^{\mathfrak{I}} g(\mathfrak{I}, s, x_s) ds \right] ds, \mathfrak{I} \in [0, T],\end{aligned} \tag{26}$$

such that

$$\begin{aligned}C_q(\mathfrak{I}) = \int_0^\infty M_q C(\mathfrak{I}^q \zeta) d\zeta, \mathscr{K}_q(\mathfrak{I}) = \int_0^{\mathfrak{I}} C_q(s) ds, P_q(\mathfrak{I}) \\ = \int_0^\infty q\zeta M_q C(\mathfrak{I}^q \zeta) d\zeta,\end{aligned} \tag{27}$$

where $C_q(\mathfrak{I})$ and $\mathscr{K}_q(\mathfrak{I})$ are continuous with $C(0) = I$ and $\mathscr{K}(0) = I$, $|C_q(\mathfrak{I})| \le c, c > 1$ and $|\mathscr{K}_q(\mathfrak{I})| \le c, c > 1, \forall \mathfrak{I} \in [0, T]$.

3. Local Uniqueness and Existence

For $\mathscr{G} : [0,\infty) \times C_\sigma \longrightarrow \mathbf{E^m}$, we assume the fuzzy Caputo functional equation:

$$\begin{gathered}{}^C_0 D^q_H x(\mathfrak{I}) = f(\mathfrak{I}, x_{\mathfrak{I}}) + \int_0^{\mathfrak{I}} g(\mathfrak{I}, s, x_s) ds, \mathfrak{I} \ge \mathfrak{I}_0, \mathfrak{I} \in [0, T], \\ x(\mathfrak{I}) = \psi(\mathfrak{I} - \mathfrak{I}_0) = \psi_0 \in C_\sigma, \mathfrak{I}_0 \ge \mathfrak{I} \ge \mathfrak{I}_0 - \sigma, \\ u'(\mathfrak{I}) = \psi'(\mathfrak{I}) = \psi_1.\end{gathered} \tag{28}$$

According to the solution of FFFDE (4) on interval $[\mathfrak{I}_0, b]$, we mean continuous function $x : [\mathfrak{I}_0 - \sigma, b) \longrightarrow \mathbf{E^m}$; that is, $x(\mathfrak{I}) = \varphi(\mathfrak{I} - \mathfrak{I}_0)$ for $\mathfrak{I} \in [\mathfrak{I}_0 - \sigma, b]$ for $\mathfrak{I} \in [0, T]$ and x is differentiable on $(\mathfrak{I}_0, b]$ and ${}^c_0\mathscr{D}^q_H x(\mathfrak{I}) = f(\mathfrak{I}, x_{\mathfrak{I}}) + \int_0^{\mathfrak{I}} g(\mathfrak{I}, s, x_s) ds, \mathfrak{I} \in [0, T]$.

Theorem 19. *Suppose set $\mathscr{G} : [0,\infty) \times C_\sigma \longrightarrow \mathbf{E^m}$ is locally Lipschitz and continuous. Now, for all $(\mathfrak{I}_0, \varphi) \in [0,\infty) \times C_\sigma$, there exists $\mathfrak{I} > \mathfrak{I}_0$; that is, FFFDE (4) has unique solution $x : [\mathfrak{I}_0 - \sigma, \mathfrak{I}] \longrightarrow \mathbf{E^m}$.*

Proof. Any positive number will satisfy as $\rho > 0$. Then, there exists $L > 0$; that is, $\mathscr{G}$ is Lipschitz locally.

$$\mathscr{D}[\mathscr{G}(\mathfrak{I}, \varphi), \mathscr{G}(\mathfrak{I}, \psi)] \le L\mathscr{D}_\sigma(\varphi, \psi), \mathfrak{I}_0 \le \mathfrak{I} \le h, \varphi, \psi \in B_{2\rho}, \tag{29}$$

for some $h > \mathfrak{I}_0$. According to Lemma 13, there exists $\mathscr{K} > 0$, $\mathscr{D}[\mathscr{F}(\mathfrak{I}, \varphi), \tilde{0}] \le \mathscr{K}$ for $(\mathfrak{I}, \varphi) \in [\mathfrak{I}_0, h] \times B_{2\rho}$. Suppose $T \coloneqq \min\{h, \rho/\mathscr{K}\}$. We assume set $\mathbf{E^m}$ of all functions $x \in C([\mathfrak{I}_0 - \sigma, T], \mathbf{E^m})$; then, $x(\mathfrak{I}) = \varphi(\mathfrak{I} - \mathfrak{I}_0)$ on $[\mathfrak{I}_0 - \sigma, \mathfrak{I}_0]$ and $\mathscr{D}[x(\mathfrak{I}), \tilde{0}] \le 2\rho$ on $[\mathfrak{I}_0, T]$. If $y \in \mathbf{E^m}$, we define continuous function $\omega : [\mathfrak{I}_0 - \sigma, T] \longrightarrow \mathbf{E^m}$ by

$$w(\mathfrak{I}) \coloneqq \begin{cases} f(\mathfrak{I}, x_{\mathfrak{I}}) + \int_0^{\mathfrak{I}} g(\mathfrak{I}, \mathrm{s}, x_{\mathrm{s}})\mathrm{ds}, \mathfrak{I} \ge \mathfrak{I}_0, \mathrm{t}_0 \ge \mathfrak{I} \ge \mathfrak{I}_0 - \sigma, \\ \varphi(0) + C_q(\mathfrak{I})\psi_0 + \mathscr{K}_q(\mathfrak{I})\psi_1 + \int_0^{\mathfrak{I}} (\mathfrak{I} - \mathrm{s})^{q-1} P_q(\mathfrak{I} - \mathrm{s}) \left[f(\mathrm{s}, y_{\mathrm{s}}) + \int_0^{\mathfrak{I}} g(\mathfrak{I}, \mathrm{s}, y_{\mathrm{s}}) ds \right] \mathrm{ds} \in [0, T]. \end{cases} \tag{30}$$

Now, for $\mathfrak{I} \in [0, T]$

$$\begin{aligned} \mathscr{D}\left[w(\mathfrak{I}), \tilde{0}\right] \ge{}& \mathscr{D}\left[\left[\int_0^{\mathfrak{I}} f(\mathrm{s}, y_{\mathrm{s}}), \tilde{0}\right)\mathrm{ds}\right] \\ &+ \int_0^{\mathfrak{I}} \mathscr{D}\left[\left[\int_0^{\mathfrak{I}} \left(g(\mathfrak{I}, \mathrm{s}, y_{\mathrm{s}}), \tilde{0}\right)\mathrm{ds}\right]\mathrm{ds} \ge 2\rho T, \end{aligned} \tag{31}$$

and so $\omega \in \mathbf{E^m}$. We will use method of successive approximations to solve (4) by constructing series of continuous functions. $x^m : [\mathfrak{I}_0 - \sigma, T] \longrightarrow \mathbf{E^m}$ beginning with initial continuous function

$$x^0(\mathfrak{I}) \coloneqq \begin{cases} f(\mathfrak{I}, x_{\mathfrak{I}}) + \int_0^{\mathfrak{I}} g(\mathfrak{I}, \mathrm{s}, x_{\mathrm{s}})\mathrm{ds}, \mathfrak{I} \ge \mathfrak{I}_0, \mathfrak{I}_0 \ge \mathfrak{I} \ge \mathfrak{I}_0 - \sigma, \\ \varphi(0) + C_q(\mathfrak{I})\psi_0 + \mathscr{K}_q(\mathfrak{I})\psi_1 \mathfrak{I} \in [0, T]. \end{cases} \tag{32}$$

Clearly, $\mathscr{D}[{}_0^c\mathscr{D}_H^q x^0(\mathfrak{I}), \tilde{0}] \le \rho$ on $[0, T]$. Further, define

$$x^{m+1}(\mathfrak{I}) \coloneqq \begin{cases} f(\mathfrak{I}, x_{\mathfrak{I}}) + \int_0^{\mathfrak{I}} g(\mathfrak{I}, \mathrm{s}, x_{\mathrm{s}})ds, \mathfrak{I} \ge \mathfrak{I}_0, \mathfrak{I}_0 \ge \mathfrak{I} \ge \mathfrak{I}_0 - \sigma, \\ \varphi(0) + C_q(\mathfrak{I})\psi_0 + \mathscr{K}_q(\mathfrak{I})\psi_1 + \int_0^{\mathfrak{I}} (\mathfrak{I} - \mathrm{s})^{q-1} P_q(\mathfrak{I} - \mathrm{s}) \left[f(\mathrm{s}, x_{\mathrm{s}}^m) + \int_0^{\mathfrak{I}} g(\mathfrak{I}, \mathrm{s}, x_{\mathrm{s}}^m) ds \right] ds, \mathfrak{I} \in [0, T], \end{cases} \tag{33}$$

if $= 0, 1, \cdots$. Then, for $\mathfrak{I} \in [0, T]$, now

$$\begin{aligned} \mathscr{D}\left[x^1(\mathfrak{I}), u^0(\mathfrak{I})\right] \le{}& \mathscr{D}\left(\int_0^{\mathfrak{I}} (\mathfrak{I} - \mathrm{s})^{q-1} P_q(\mathfrak{I} - \mathrm{s}) \right. \\ &\left. \cdot \left[f\left(\mathrm{s}, x_{\mathrm{s}}^0\right) + \int_0^{\mathfrak{I}} g\left(\mathfrak{I}, \mathrm{s}, x_{\mathrm{s}}^0\right) ds \right] \mathrm{ds}, \tilde{0} \right) \\ \le{}& \mathscr{K}(T - 0). \end{aligned} \tag{34}$$

By Equations (29) and (33), we find

$$\begin{aligned} \mathscr{D}\left[x^{m+1}(\mathfrak{I}), x^m(\mathfrak{I})\right] \le{}& \mathscr{D}\left(\int_0^{\mathfrak{I}} (\mathfrak{I} - \mathrm{s})^{q-1} P_q(\mathfrak{I} - \mathrm{s}) \right. \\ & \cdot \left[f\left(\mathrm{s}, x_{\mathrm{s}}^m\right) + \int_0^{\mathfrak{I}} g\left(\mathfrak{I}, \mathrm{s}, x_{\mathrm{s}}^m\right) ds \right] \mathrm{ds}, \\ & \int_0^{\mathfrak{I}} (\mathfrak{I} - \mathrm{s})^{q-1} P_q(\mathfrak{I} - \mathrm{s}) \\ & \left. \cdot \left[f\left(\mathrm{s}, x_{\mathrm{s}}^{m-1}\right) + \int_0^{\mathfrak{I}} g\left(\mathfrak{I}, \mathrm{s}, x_{\mathrm{s}}^{m-1}\right) ds \right] \mathrm{ds} \right) \\ \le{}& \int_0^{\mathfrak{I}} L^2 \mathscr{D}_\sigma \left(\left[x_{\mathrm{s}}^m + \int_0^{\mathfrak{I}} g\left(\mathfrak{I}, \mathrm{s}, x_{\mathrm{s}}^m\right) \right], \right. \\ & \left. \left[x_{\mathrm{s}}^{m-1} + \int_0^{\mathfrak{I}} g\left(\mathfrak{I}, \mathrm{s}, x_{\mathrm{s}}^{m-1}\right) ds \right] \right) \mathrm{ds} \\ \le{}& \int_0^{\mathfrak{I}} L^2 \sup_{\theta \in [s-T, s]} D \left(\left[x^m(\theta) + \int_0^{\mathfrak{I}} g\left(\mathfrak{I}, \mathrm{s}, x^m(\theta)\right) ds \right], \right. \\ & \left. \left[x^{m-1}(\theta) + \int_0^{\mathfrak{I}} g\left(\mathfrak{I}, \mathrm{s}, x^{m-1}(\theta)\right) ds \right] \right) \mathrm{ds}, \mathfrak{I} \in [0, T]. \end{aligned} \tag{35}$$

In particular,

$$\mathscr{D}\left[x^2(\mathfrak{I}), x^1(\mathfrak{I})\right] = \frac{\mathscr{K}}{L^2} \frac{\left[L^2(\mathfrak{I} - T)\right]^2}{2!}, \mathfrak{I} \in [0, T]. \tag{36}$$

If we suppose

$$\mathscr{D}\left[x^m(\mathfrak{I}), x^{m-1}(\mathfrak{I})\right] \le \frac{\mathscr{K}}{L^2} \frac{\left[L^2(\mathfrak{I} - T)\right]^m}{m!}, \mathfrak{I} \in [0, T], \tag{37}$$

now

$$\mathscr{D}\left[x^{m+1}(\mathfrak{I}), x^m(\mathfrak{I})\right] = \frac{\mathscr{K}}{L^2} \frac{\left[L^2(\mathfrak{I} - T)\right]^{m+1}}{(m+1)!}, \mathfrak{I} \in [0, T]. \tag{38}$$

(37) holds for any $m \ge 2$, according to mathematical induction. As a result, the sequence $\sum_{m=2}^{\infty} \mathscr{D}[x^m(\mathfrak{I}), x^{m-1}(\mathfrak{I})]$ is a sequence $\{x^m\}_{m \ge 0}$ that is uniformly convergent on $[0, T]$. As a result, there is a continuous function $x : [0, T] \longrightarrow \mathbf{E^m}$,

which is $\sup_{0\le\mathfrak{T}\le T} \mathscr{D}[x^m(\mathfrak{T}), x(\mathfrak{T})] \longrightarrow 0$ as $m \longrightarrow \infty$. Since then

$$\begin{aligned}&\mathscr{D}\left(\left[f(\mathrm{s}, x_{\mathrm{s}}^m)+\int_0^{\mathfrak{T}} g(\mathfrak{T}, \mathrm{s}, x_{\mathrm{s}}^m)ds\right], \left[f\left(\mathrm{s}, x_{\mathrm{s}}^{m-1}\right)+\int_0^{\mathfrak{T}} g\left(\mathfrak{T}, \mathrm{s}, x_{\mathrm{s}}^{m-1}\right)ds\right]\right)\\ &\le L^2\mathscr{D}_\sigma\left(\left[x_{\mathrm{s}}^m+\int_0^{\mathfrak{T}} g(\mathfrak{T}, \mathrm{s}, x_{\mathrm{s}}^m)ds\right], \left[x_{\mathrm{s}}+\int_0^{\mathfrak{T}} g(\mathfrak{T}, \mathrm{s}, x_{\mathrm{s}})ds\right]\right)\mathrm{ds}\\ &\le \sup_{0\le\mathfrak{T}\le T} \mathscr{D}[x^m(\mathfrak{T}), x(\mathfrak{T})].\end{aligned} \tag{39}$$

We have deduced

$$\begin{aligned}&\mathscr{D}\left(\left[f(\mathrm{s}, x_{\mathrm{s}}^m)+\int_0^{\mathfrak{T}} g(\mathfrak{T}, \mathrm{s}, x_{\mathrm{s}}^m)ds\right]\mathrm{ds},\right.\\ &\left.\left[f\left(\mathrm{s}, x_{\mathrm{s}}^{m-1}\right)+\int_0^{\mathfrak{T}} g\left(\mathfrak{T}, \mathrm{s}, x_{\mathrm{s}}^{m-1}\right)ds\right]\mathrm{ds}\right)\longrightarrow 0,\end{aligned} \tag{40}$$

uniformly on $[0, T]$ as $m \longrightarrow \infty$. Therefore,

$$\begin{aligned}&\mathscr{D}\left(\int_0^{\mathfrak{T}}\left[f(\mathrm{s}, x_{\mathrm{s}}^m)+\int_0^{\mathfrak{T}} g(\mathfrak{T}, \mathrm{s}, x_{\mathrm{s}}^m)ds\right]\mathrm{ds},\right.\\ &\left.\int_0^{\mathfrak{T}}\left[f\left(s, x_{\mathrm{s}}^{m-1}\right)+\int_0^{\mathfrak{T}} g\left(t, s, x_{\mathrm{s}}^{m-1}\right)ds\right]\mathrm{ds}\right)\\ &\le \int_0^{\mathfrak{T}}\mathscr{D}\left(\left[f(\mathrm{s}, x_{\mathrm{s}}^m)+\int_0^{\mathfrak{T}} g(\mathfrak{T}, \mathrm{s}, x_{\mathrm{s}}^m)ds\right]\mathrm{ds},\right.\\ &\left.\left[f\left(\mathrm{s}, x_{\mathrm{s}}^{m-1}\right)+\int_0^{\mathfrak{T}} g\left(\mathfrak{T}, \mathrm{s}, x_{\mathrm{s}}^{m-1}\right)ds\right]\mathrm{ds}\right)\mathrm{ds}.\end{aligned} \tag{41}$$

It follows that

$$\begin{aligned}&\lim_{m\longrightarrow\infty}\int_0^{\mathfrak{T}}\left[f(\mathrm{s}, x_{\mathrm{s}}^m)+\int_0^{\mathfrak{T}} g(\mathfrak{T}, \mathrm{s}, x_{\mathrm{s}}^m)ds\right]\mathrm{ds}\\ &=\int_0^{\mathfrak{T}}\left[f\left(\mathrm{s}, x_{\mathrm{s}}^{m-1}\right)+\int_0^{\mathfrak{T}} g\left(\mathfrak{T}, \mathrm{s}, x_{\mathrm{s}}^{m-1}\right)ds\right]\mathrm{ds}, \mathfrak{T}\in[0, T].\end{aligned} \tag{42}$$

Extending x to $[\mathfrak{T}_0-\sigma, \mathfrak{T}_0]$ in usual way by $x(\mathfrak{T}) = f(\mathfrak{T}, x_{\mathfrak{T}}) + \int_0^{\mathfrak{T}} g(\mathfrak{T}, \mathrm{s}, x_{\mathrm{s}})ds$ for $\mathfrak{T}\in[0, T]$, then, by (33), we obtain that

$$x(\mathfrak{T}) = \begin{cases} f(\mathfrak{T}, x_{\mathfrak{T}})+\int_0^{\mathfrak{T}} g(\mathfrak{T}, \mathrm{s}, x_{\mathrm{s}})\mathrm{ds}, \mathfrak{T}\ge\mathfrak{T}_0, \mathfrak{T}_0\ge\mathfrak{T}\ge\mathfrak{T}_0-\sigma,\\ \varphi(0)+C_q(\mathfrak{T})\psi_0+\mathscr{K}_q(\mathfrak{T})\psi_1+\int_0^{\mathfrak{T}}(\mathfrak{T}-\mathrm{s})^{q-1}P_q(\mathfrak{T}-\mathrm{s})\left[f(\mathrm{s}, x_{\mathrm{s}})+\int_0^{\mathfrak{T}} g(\mathfrak{T}, \mathrm{s}, x_{\mathrm{s}})\right]\mathrm{ds},\in[0, T],\end{cases} \tag{43}$$

and x is the solution for Equation (4). To prove uniqueness, suppose $y : [\mathfrak{T}_0-\sigma, T] \longrightarrow \mathbf{E^m}$ be second solution for (4). For all $\mathfrak{T}\in[0, T]$,

$$\begin{aligned}\mathscr{D}[x(\mathfrak{T}), y(\mathfrak{T})] &= \mathscr{D}\left(\int_0^{\mathfrak{T}}(\mathfrak{T}-\mathrm{s})^{q-1}P_q(\mathfrak{T}-\mathrm{s})[f(\mathrm{s}, x_{\mathrm{s}})\right.\\ &\quad+\left.\int_0^{\mathfrak{T}} g(\mathfrak{T}, \mathrm{s}, x_{\mathrm{s}})ds\right]\mathrm{ds}, \int_0^{\mathfrak{T}}(\mathfrak{T}-\mathrm{s})^{q-1}P_q(\mathfrak{T}-\mathrm{s})\\ &\quad\left.\cdot\left[f(\mathrm{s}, y_{\mathrm{s}})+\int_0^{\mathfrak{T}} g(\mathfrak{T}, \mathrm{s}, y_{\mathrm{s}})ds\right]\mathrm{ds}\right)\le\mathscr{D}_\sigma(x_{\mathrm{s}}, y_{\mathrm{s}})\\ &\le L^2\int_0^{\mathfrak{T}}\sup_{\theta\in[\mathrm{s}-\sigma,\mathrm{s}]}\mathscr{D}[x(\theta), y(\theta)]\mathrm{ds}.\end{aligned} \tag{44}$$

If we assume $\xi(\mathrm{s}) \coloneqq \sup_{r\in[\mathrm{s}-\sigma,\mathrm{s}]}\mathscr{D}[x(r), y(r)]$, $\mathrm{s}\in[0, T]$, now

$$\xi(\mathfrak{T}) \le L^2\int_0^{\mathfrak{T}}\xi(s)\mathrm{ds}, \tag{45}$$

and by Gronwall's lemma, we obtained $\xi(\mathfrak{T}) = 0$ on $[0, T]$. This establishes uniqueness solutions for (4). □

Remark 20. The contraction principle can be used to prove local uniqueness and existence theorem for initial value problems (28). Suppose $P : \mathbf{E^m} \longrightarrow \mathbf{E^m}$ be defined as

$$(Px)(\mathfrak{T}) = \begin{cases} f(\mathfrak{T}, x_{\mathfrak{T}})+\int_0^{\mathfrak{T}} g(\mathfrak{T}, \mathrm{s}, x_{\mathrm{s}})ds, \mathfrak{T}\ge\mathfrak{T}_0, \mathfrak{T}_0\ge\mathfrak{T}\ge\mathfrak{T}_0-\sigma,\\ \varphi(0)+C_q(\mathfrak{T})\psi_0+\mathscr{K}_q(\mathfrak{T})\psi_1+\int_0^{\mathfrak{T}}(\mathfrak{T}-\mathrm{s})^{q-1}P_q(\mathfrak{T}-\mathrm{s})\left[f(\mathrm{s}, x_{\mathrm{s}})+\int_0^{\mathfrak{T}} g(\mathfrak{T}, \mathrm{s}, x_{\mathrm{s}})ds\right]ds, \mathfrak{T}\in[0, T],\end{cases} \tag{46}$$

For $\mathfrak{T}\in[0, T]$,

$$\begin{aligned}\mathscr{D}[(Px)(\mathfrak{T}), (Py)(\mathfrak{T})] &= \mathscr{D}\left(\int_0^{\mathfrak{T}}(\mathfrak{T}-\mathrm{s})^{q-1}P_q(\mathfrak{T}-\mathrm{s})[f(\mathrm{s}, x_{\mathrm{s}})\right.\\ &\quad+\left.\int_0^{\mathfrak{T}} g(\mathfrak{T}, \mathrm{s}, x_{\mathrm{s}})ds\right]\mathrm{ds}, \int_0^{\mathfrak{T}}(\mathfrak{T}-\mathrm{s})^{q-1}\\ &\quad\left.\cdot P_q(\mathfrak{T}-\mathrm{s})\left[f(\mathrm{s}, y_{\mathrm{s}})+\int_0^{\mathfrak{T}} g(\mathfrak{T}, \mathrm{s}, y_{\mathrm{s}})ds\right]\mathrm{ds}\right)\\ &\le L^2\mathscr{D}_\sigma(x_{\mathrm{s}}, y_{\mathrm{s}})\le L^2\int_0^{\mathfrak{T}}\sup_{\theta\in[\mathrm{s}-\sigma,\mathrm{s}]}\mathscr{D}[x(\theta), y(\theta)]\mathrm{ds}\\ &\le L^2\mathfrak{T}\mathscr{D}(x, y).\end{aligned} \tag{47}$$

Hence

$$\mathscr{D}(Px, Py) \le L^2\mathfrak{T}\mathscr{D}(x, y)\forall x, y\in\mathbf{E^m}, \tag{48}$$

where $P : \mathbf{E^m} \longrightarrow \mathbf{E^m}$ is contraction only if $L\mathfrak{T} < 2$. However, if we deal with successive approximations indirectly (33), we can show that iterations converge, and initial value problem (28) has unique solution on interval $[0, T]$ under merely the assumption $\mathscr{K}T < \rho$, without constraint $L\mathfrak{T} < 2$. The discrepancy is resolved by noting that all functions $x\in C([\mathfrak{T}_0-\sigma, T], \mathbf{E^m})$, $x(\mathfrak{T}) = \varphi(\mathfrak{T}-\mathfrak{T}_0)$ on $[0, T]$ have several equivalent metrics on space $\mathbf{E^m}$. In fact, metric

$$\mathscr{D}_\sigma(x, y) = \sup_{\mathfrak{I}_0-\sigma\le\mathfrak{I}\le T} \mathscr{D}[x(\mathfrak{I}), y(\mathfrak{I})]e^{-a\mathfrak{I}}, a > 0, \tag{49}$$

is equivalent to metric $\mathscr{D}(x, y)$. Then

$$\mathscr{D}(x, y)e^{-aT} \le \mathscr{D}_\sigma(x, y) \le \mathscr{D}(x, y)\forall x, y \in \mathbf{E^m}. \tag{50}$$

Using metric (49) and function $\mathscr{G} : [0,\infty) \times C_\sigma \longrightarrow \mathbf{E^m}$ is continuous and satisfies the global Lipschitz condition:

$$\mathscr{D}[\mathscr{G}(\mathfrak{I}, \varphi), \mathscr{G}(\mathfrak{I}, \psi)] \le L^2\mathscr{D}_\sigma(\varphi, \psi), 0 \le \mathfrak{I} \le T, \varphi, \psi \in C_\sigma. \tag{51}$$

In [34], uniqueness and existence of solution for (28) on interval $[0, T]$ were illustrated.

Theorem 21. *Let function $\mathscr{G} : [0,\infty) \times C_\sigma \longrightarrow \mathbf{E^m}$ be locally Lipschitz and continuous. If $(\mathfrak{I}_0, \varphi), (\mathfrak{I}_0, \psi) \in [0,\infty) \times C_\sigma$ and $x(\varphi)$: $[\mathfrak{I}_0 - \sigma, \omega_1) \longrightarrow \mathbf{E^m}$ and $x(\psi)$: $[\mathfrak{I}_0 - \sigma, \omega_2) \longrightarrow \mathbf{E^m}$ are unique solutions of (28) with $x(\mathfrak{I}) = f(\mathfrak{I}, x_\mathfrak{I}) + \int_0^\mathfrak{I} g(\mathfrak{I}, s, x_s)ds$ on $[\mathfrak{I}_0 - \sigma, \mathfrak{I}_0]$, now*

$$\mathscr{D}[x(\varphi)(\mathfrak{I}), x(\psi)(\mathfrak{I})] \le \mathscr{D}_\sigma(\varphi, \psi)e^{L^2(\mathfrak{I}-0)}\forall\mathfrak{I} \in [\mathfrak{I}_0, \omega), \tag{52}$$

where $\omega = \min\{\omega_1, \omega_2\}$.

Proof. On $[\mathfrak{I}_0, \omega)$ solution, $x(\varphi)$ satisfies relation

$$x(\mathfrak{I}) = \begin{cases} f(\mathfrak{I}, x_\mathfrak{I}) + \int_0^\mathfrak{I} g(\mathfrak{I}, s, x_s)ds, \mathfrak{I} \ge \mathfrak{I}_0, \mathfrak{I}_0 \ge \mathfrak{I} \ge \mathfrak{I}_0 - \sigma, \\ \varphi(0) + C_q(\mathfrak{I})\psi_0 + \mathscr{K}_q(\mathfrak{I})\psi_1 + \int_0^\mathfrak{I} (\mathfrak{I} - s)^{q-1}P_q(\mathfrak{I} - s)\left[f(s, x_s) + \int_0^\mathfrak{I} g(\mathfrak{I}, s, x_s)ds\right]ds, \mathfrak{I} \in [0, \omega], \end{cases} \tag{53}$$

and $x(\psi)$ satisfies the same relation as φ, but with ψ instead of φ. Then, for $\mathfrak{I} \in [\mathfrak{I}_0, \omega)$,

$$\begin{aligned} \mathscr{D}[x(\varphi)(\mathfrak{I}), x(\psi)(\mathfrak{I})] &\le \mathscr{D}[\varphi(0), \psi(0)] + \mathscr{D}\left[C_q(\mathfrak{I})\varphi_0, C_q(\mathfrak{I})\psi_0\right] \\ &+ \mathscr{D}\left[\mathscr{K}_q(\mathfrak{I})\varphi_1, \mathscr{K}_q(\mathfrak{I})\psi_1\right] + \int_0^\mathfrak{I} \mathscr{D}\left[(\mathfrak{I} - s)^{q-1}P_q(\mathfrak{I} - s)\right. \\ &\cdot \left[f(s, x_s) + \int_0^\mathfrak{I} g(\mathfrak{I}, s, x_s)ds\right] ds, (\mathfrak{I} - s)^{q-1}P_q(\mathfrak{I} - s) \\ &\cdot \left.\left[f(s, y_s) + \int_0^\mathfrak{I} g(\mathfrak{I}, s, y_s)ds\right] ds\right] \le \mathscr{D}_\sigma(\varphi, \psi) \\ &+ \mathscr{D}_\sigma(\varphi_0, \psi_0) + \mathscr{D}_\sigma(\varphi_1, \psi_1) \\ &+ L^2\int_0^\mathfrak{I} \sup_{r\in[\mathfrak{I}_0-\sigma,s]} \mathscr{D}_\sigma[x(\varphi)(r), x(\psi)(r)]ds. \end{aligned} \tag{54}$$

If suppose $\omega(s) = \sup_{r\in[\mathfrak{I}_0-\sigma,s]} \mathscr{D}_\sigma[x(\varphi)(r), x(\psi)(r)], \mathfrak{I}_0 \le s \le \mathfrak{I}$, then

$$\omega(\mathfrak{I}) \le \mathscr{D}_\sigma(\varphi, \psi)e^{L\mathfrak{I}}, \mathfrak{I}_0 \le \mathfrak{I} < \omega, \tag{55}$$

implying that (52) holds. □

4. Global Existence and Uniqueness

For a given constant $a > 0$, consider set $\mathbf{E^m_a}$ of all functions $x \in C([\mathfrak{I}_0 - \sigma,\infty), \mathbf{E^m})$; that is, $x(\mathfrak{I}) = f(\mathfrak{I}, x_\mathfrak{I}) + \int_0^\mathfrak{I} g(\mathfrak{I}, s, x_s)ds$ on $[\mathfrak{I}_0 - \sigma, \mathfrak{I}_0]$ and $\sup_{\mathfrak{I}\ge\mathfrak{I}_0-\sigma} \mathscr{D}[x(\mathfrak{I}), \tilde{0}]e^{-a\mathfrak{I}} < \infty$. On $\mathbf{E^m_a}$, define the following metric:

$$\mathscr{D}_\sigma(x, y) = \sup_{\mathfrak{I}\mathfrak{I}_0-\sigma} \mathscr{D}[x(\mathfrak{I}), y(\mathfrak{I})]e^{-a\mathfrak{I}}. \tag{56}$$

Lemma 22. *$(\mathbf{E^m_a}, \mathscr{D}_a)$ is complete metric space.*

Proof. Suppose $\{x_m\}_{m\ge2}$ be Cauchy sequence in $\mathbf{E^m_a}$. Now, for each $\varepsilon > 0$, there exists $m_\varepsilon \in \mathbb{N}\forall m, p \ge m_\varepsilon$, and we obtain $\mathscr{D}_\sigma(x_m, y_p) < \varepsilon$. Hence

$$\mathscr{D}\left[x_m(\mathfrak{I}), y_p(\mathfrak{I})\right] \le \mathscr{D}_\sigma\left(x_m, y_p\right)e^{a\mathfrak{I}} \le \varepsilon e^\mathfrak{I}, \tag{57}$$

and so

$$\mathscr{D}\left[x_m(\mathfrak{I}), x_p(\mathfrak{I})\right] \le \varepsilon e^{a\mathfrak{I}}\forall m, p \ge m_\varepsilon \text{ and } \mathfrak{I} \ge \mathfrak{I}_0 - \sigma. \tag{58}$$

For each $\mathfrak{I} \ge \mathfrak{I}_0 - \sigma$, $\{x_m(\mathfrak{I})\}_{m\ge2}$ is Cauchy sequence in $\mathbf{E^m}$. $(\mathbf{E^m}, \mathscr{D})$ is a complete metric space, and there exists $x(\mathfrak{I}) = \lim_{m\longrightarrow\infty} x_m(\mathfrak{I})$ for $\mathfrak{I} \ge \mathfrak{I}_0 - \sigma$. Now, $x \in \mathbf{E}_\sigma$. Evidently, $x(\mathfrak{I}) = f(\mathfrak{I}, x_\mathfrak{I}) + \int_0^\mathfrak{I} g(\mathfrak{I}, s, x_s)ds$ on $[\mathfrak{I}_0 - \sigma, \mathfrak{I}_0]$. From (58), we get $\lim_{p\longrightarrow\infty} \mathscr{D}[x_m(\mathfrak{I}), x(\mathfrak{I})] \le \varepsilon e^{a\mathfrak{I}}, \forall m \ge m_\varepsilon$ and $\mathfrak{I} \ge \mathfrak{I}_0$. Now, x is continuous function on $[\mathfrak{I}_0, \infty)$. Suppose $\varepsilon > 0$ and $s \ge \mathfrak{I}_0$. Then, there exists $m = m'_\varepsilon \in \mathbb{N}$, $\mathscr{D}[x_m(\mathfrak{I}), x(\mathfrak{I})](\varepsilon/6)e^{a(\mathfrak{I}-s)}, \forall\mathfrak{I} \ge \mathfrak{I}_0$. Since x_m is continuous function, now, there exists $\delta^1_\varepsilon > 1$, $\mathscr{D}[x_m(\mathfrak{I}), x_m(s)] \le (\varepsilon/3)$ for $\mathfrak{I} \ge \mathfrak{I}_0$. Since$x_m$ is continuous function, then, there exists $\delta^1_\varepsilon > 1$ that is $\mathscr{D}[x_m(\mathfrak{I}), x_m(s)] \le (\varepsilon/3)$ for $\mathfrak{I} \ge \mathfrak{I}_0$ with $|\mathfrak{I} - s| \le \delta^1_\varepsilon$. There exists $\delta^2_\varepsilon > 1$; that is, $e^{a(\mathfrak{I}-s)} \le 2$ for $\mathfrak{I} \ge \mathfrak{I}_0$ with $|\mathfrak{I} - s| \le \delta^2_\varepsilon$. Assume $\delta_\varepsilon = \min\{\delta^1_\varepsilon, \delta^2_\varepsilon\}$. Now, for every $\mathfrak{I} \ge \mathfrak{I}_0$ with $|\mathfrak{I} - s| \le \delta_\varepsilon$,

$$\begin{aligned} \mathscr{D}[x(\mathfrak{I}), x(s)] &\le \mathscr{D}[x(\mathfrak{I}), x_m(\mathfrak{I})] + \mathscr{D}[x_m(\mathfrak{I}), x_m(s)] \\ &+ \mathscr{D}[x_m(s), x(s)] \le \left(\frac{\varepsilon}{6}\right)e^{a(\mathfrak{I}-s)} + \frac{\varepsilon}{3} + \frac{\varepsilon}{6} \le \varepsilon, \end{aligned} \tag{59}$$

where x is continuous function on $[\mathfrak{I}_0, \infty)$. Now

$$\sup_{\mathfrak{I}\ge\mathfrak{I}_0-\sigma} \mathscr{D}\left[x(\mathfrak{I}), \tilde{0}\right]e^{-a\mathfrak{I}} < \infty. \tag{60}$$

Since

$$\mathscr{D}\left[x(\mathfrak{I}),\tilde{0}\right] \leq \mathscr{D}[x(\mathfrak{I}),x_m(\mathfrak{I})] + \mathscr{D}\left[x_m(\mathfrak{I}),\tilde{0}\right] \forall \mathfrak{I} \geq \mathfrak{I}_0 - \sigma \text{ and } m \geq 1. \tag{61}$$

Now

$$\begin{aligned}\sup_{\mathfrak{I}\geq\mathfrak{I}_0-\sigma} \mathscr{D}\left[x(\mathfrak{I}),\tilde{0}\right]e^{-a\mathfrak{I}} &\leq \sup_{\mathfrak{I}\geq\mathfrak{I}_0-\sigma} \mathscr{D}[x(\mathfrak{I}),x_m(\mathfrak{I})]e^{-a\mathfrak{I}} + \sup_{\mathfrak{I}\geq\mathfrak{I}_0-\sigma} \mathscr{D}\left[x_m(\mathfrak{I}),\tilde{0}\right]e^{-a\mathfrak{I}} \\ &= \mathscr{D}_\sigma(x,x_m) + \sup_{\mathfrak{I}\geq\mathfrak{I}_0-\sigma} \mathscr{D}\left[x_m(\mathfrak{I}),\tilde{0}\right]e^{-a\mathfrak{I}},\end{aligned} \tag{62}$$

$\lim_{m\longrightarrow\infty} \mathscr{D}_\sigma(x,x_m) = 1$ and $x_m \in \mathbf{E}_\mathbf{a}^\mathbf{m} \forall m \geq 2$, we get

$$\sup_{\mathfrak{I}\geq\mathfrak{I}_0-\sigma} \mathscr{D}\left[x(\mathfrak{I}),\tilde{0}\right]e^{-a\mathfrak{I}} < \infty. \tag{63}$$

Moreover, $x \in \mathbf{E}_\mathbf{a}^\mathbf{m}$. So, $(\mathbf{E}_\mathbf{a}^\mathbf{m}, \mathscr{D}_a)$ is complete metric space.

The fuzzy differential Equation (28) is then considered under the following conditions:

$(\mathbf{J_1})$ There exist $L > 0$; that is

$$\mathscr{D}[\mathscr{G}(\mathfrak{I},\varphi),\mathscr{G}(\mathfrak{I},\psi)] < L^2 D_\sigma(\varphi,\psi) \forall \varphi,\psi \in C_\sigma \text{ and } \mathfrak{I} \geq 1. \tag{64}$$

$(\mathbf{J_2})\mathscr{G} : [0,\infty) \times C_\sigma \longrightarrow \mathbf{E}^\mathbf{m}$ is jointly continuous.

$(\mathbf{J_3})$ There exists $M > 0$ and $b > 0$,

$$\mathscr{D}\left[\mathscr{G}(\mathfrak{I},0),\tilde{0}\right] \leq Me^{b\mathfrak{I}} \forall \mathfrak{I} \geq 1. \tag{65}$$

Suppose $P : C([-\sigma,\infty),\mathbf{E}^\mathbf{m}) \longrightarrow C([-\sigma,\infty),\mathbf{E}^\mathbf{m})$, defined as

$$(Px)(\mathfrak{I}) = \begin{cases} f(\mathfrak{I},x_\mathfrak{I}) + \int_0^\mathfrak{I} g(\mathfrak{I},\mathrm{s},x_\mathrm{s})ds, \mathfrak{I} \geq \mathfrak{I}_0, \mathfrak{I}_0 \geq \mathfrak{I} \geq \mathfrak{I}_0 - \sigma, \\ \mathscr{D}_\sigma(\varphi,\psi) + C_q(\mathfrak{I})\psi_0 + \mathscr{K}_q(\mathfrak{I})\psi_1 + \int_0^\mathfrak{I} (\mathfrak{I}-\mathrm{s})^{q-1}P_q(\mathfrak{I}-\mathrm{s})\left[f(\mathrm{s},x_\mathrm{s}) + \int_0^\mathfrak{I} g(\mathfrak{I},\mathrm{s},x_\mathrm{s})\right]ds, \mathfrak{I} \in [0,T]. \end{cases} \tag{66}$$

□

Lemma 23. *If $\mathscr{G} : [0,\infty) \times C_\sigma \longrightarrow \mathbf{E}^\mathbf{m}$ satisfies assumptions $(\mathbf{J}_1) - (\mathbf{J}_2)$ and $a > b$, then, $P(\mathbf{E}_\mathbf{a}^\mathbf{m}) \subset \mathbf{E}_\mathbf{a}^\mathbf{m}$.*

Proof. Suppose $x \in \mathbf{E}_\mathbf{a}^\mathbf{m}$. For each $\mathfrak{I} \geq \mathfrak{I}_0$,

$$\begin{aligned}\mathscr{D}\left[(Px)(\mathfrak{I}),\tilde{0}\right] &= \mathscr{D}\left[\varphi(0) + C_q(\mathfrak{I})\psi_0 + \mathscr{K}_q(\mathfrak{I})\psi_1 + \int_0^\mathfrak{I} (\mathfrak{I}-\mathrm{s})^{q-1} \cdot P_q(\mathfrak{I}-\mathrm{s})\left[f(\mathrm{s},x_\mathrm{s}) + \int_0^\mathfrak{I} g(\mathfrak{I},\mathrm{s},x_\mathrm{s})\mathrm{ds}\right]\mathrm{ds},\tilde{0}\right] \\ &\leq \mathscr{D}\left[\varphi(0),\tilde{0}\right] + \mathscr{D}\left[C_q(\mathfrak{I})\psi_0,\tilde{0}\right] + \mathscr{D}\left[\mathscr{K}_q(\mathfrak{I})\psi_1,\tilde{0}\right] \\ &\quad + \mathscr{D}\left[\left[\int_0^\mathfrak{I} (\mathfrak{I}-\mathrm{s})^{q-1}P_q(\mathfrak{I}-\mathrm{s})[f(\mathrm{s},x_\mathrm{s}) + \int_0^\mathfrak{I} g(\mathfrak{I},\mathrm{s},x_\mathrm{s})\mathrm{ds}\right]\mathrm{ds},\tilde{0}\right] \leq \mathscr{D}\left[\varphi(0),\tilde{0}\right] \\ &\quad + \mathscr{D}\left[C_q(\mathfrak{I})\psi_0,\tilde{0}\right] + \mathscr{D}\left[\mathscr{K}_q(\mathfrak{I})\psi_1,\tilde{0}\right] + \int_0^\mathfrak{I}\left(L^2 D_\sigma\left(x_\mathrm{s},\tilde{0}\right) + Me^{\mathrm{bs}}\right)\mathrm{ds} \leq \mathscr{D}\left[\varphi(0),\tilde{0}\right] \\ &\quad + \mathscr{D}\left[C_q(\mathfrak{I})\psi_0,\tilde{0}\right] + \mathscr{D}\left[\mathscr{K}_q(\mathfrak{I})\psi_1,\tilde{0}\right] + L^2\int_0^\mathfrak{I}\left(D_\sigma\left(x_\mathrm{s},\tilde{0}\right)\right)\mathrm{ds} + \frac{M}{b}e^{b\mathfrak{I}}.\end{aligned} \tag{67}$$

Since $x \in \mathbf{E}_\mathbf{a}^\mathbf{m}$, there exists $\rho > 1$, $\mathscr{D}[x(\mathfrak{I}),\tilde{0}] \leq \rho e^{a\mathfrak{I}} \forall \mathfrak{I} \geq \mathfrak{I}_0 - \sigma$,

$$\sup_{\theta\in[-\sigma,0]} D\left[x(\mathfrak{I}+0),\tilde{0}\right] \leq \mathscr{D}\left[\varphi(0),\tilde{0}\right] \leq \rho e^{a\mathfrak{I}} \forall \mathfrak{I} \geq \mathfrak{I}_0,$$

$$\begin{aligned}\mathscr{D}\left[(Px)(\mathfrak{I}),\tilde{0}\right] &\leq \mathscr{D}\left[\varphi(0),\tilde{0}\right] + \mathscr{D}\left[C_q(\mathfrak{I})\psi_0,\tilde{0}\right] + \mathscr{D}\left[\mathscr{K}_q(\mathfrak{I})\psi_1,\tilde{0}\right] + L^2\int_0^\mathfrak{I} \sup_{\theta\in[-\sigma,0]} \mathscr{D}\left[x(\mathfrak{I}+0),\tilde{0}\right]ds + \frac{M}{b}e^{b\mathfrak{I}} \\ &\leq \mathscr{D}\left[\varphi(0),\tilde{0}\right] + \mathscr{D}\left[C_q(\mathfrak{I})\psi_0,\tilde{0}\right] + \mathscr{D}\left[\mathscr{K}_q(\mathfrak{I})\psi_1,\tilde{0}\right] + \frac{\rho L^2}{a}e^{a\mathfrak{I}_0} + \frac{M}{b}e^{b\mathfrak{I}}.\end{aligned} \tag{68}$$

Thus

$$\begin{aligned}\sup_{t\geq\mathfrak{I}_0} \mathscr{D}\left[(Px)(\mathfrak{I}),\tilde{0}\right]e^{-a\mathfrak{I}} &\leq \sup_{\mathfrak{I}\geq\mathfrak{I}_0}\left(\mathscr{D}\left[\varphi(0),\tilde{0}\right] + \mathscr{D}\left[C_q(\mathfrak{I})\psi_0,\tilde{0}\right] + \mathscr{D}\left[\mathscr{K}_q(\mathfrak{I})\psi_1,\tilde{0}\right] + \frac{\rho L^2}{a}e^{a\mathfrak{I}_0} + \frac{M}{b}e^{b\mathfrak{I}}\right)e^{-a\mathfrak{I}} \\ &\leq \mathscr{D}\left[\varphi(0),\tilde{0}\right] + \mathscr{D}\left[C_q(\mathfrak{I})\psi_0,\tilde{0}\right] + \mathscr{D}\left[\mathscr{K}_q(\mathfrak{I})\psi_1,\tilde{0}\right] + \frac{1}{b}\left(\rho L^2 + M\right).\end{aligned} \tag{69}$$

Let

$$3\mathscr{K} = \sup_{\theta\in[\mathfrak{J}_0-\sigma,\mathfrak{J}_0]} \mathscr{D}\left[\varphi(0),\tilde{0}\right] + \mathscr{D}\left[C_q(\mathfrak{J})\psi_0,\tilde{0}\right] + \mathscr{D}\left[\mathscr{K}_q(\mathfrak{J})\psi_1,\tilde{0}\right]. \tag{70}$$

Now

$$\sup_{\mathfrak{J}\geq\mathfrak{J}_0} \mathscr{D}\left[(Px)(\mathfrak{J}),\tilde{0}\right]e^{-a\mathfrak{J}} \leq 3\mathscr{K} + \frac{1}{b}\left(\rho L^2 + M\right) < \infty, \tag{71}$$

and $Px \in \mathbf{E}_\mathbf{a}^\mathbf{m}$. □

Lemma 24. *If $\mathscr{F} : [0,\infty)\times C_\sigma \longrightarrow \mathbf{E}^\mathbf{m}$ satisfies $(\mathbf{J}_1)-(\mathbf{J}_3)$ and $L < a$, then, P is contraction on $\mathbf{E}_\mathbf{a}^\mathbf{m}$.*

Proof. Suppose $x, y \in \mathbf{E}_\mathbf{a}^\mathbf{m}$. Now, for each $\mathfrak{J} \geq \mathfrak{J}_0$

$$\begin{aligned}\mathscr{D}[(Px)(\mathfrak{J}),(Py)(\mathfrak{J})] &= \mathscr{D}\left[\int_0^{\mathfrak{J}} (\mathfrak{J}-\mathrm{s})^{q-1}P_q(\mathfrak{J}-\mathrm{s})\right.\\ &\quad\cdot\left[f(\mathrm{s},x_\mathrm{s}) + \int_0^{\mathfrak{J}} g(\mathfrak{J},\mathrm{s},x_\mathrm{s})\mathrm{ds}\right]\mathrm{ds}, \int_0^{\mathfrak{J}}(\mathfrak{J}-\mathrm{s})^{q-1}P_q(\mathfrak{J}-\mathrm{s})\\ &\quad\left.\cdot\left[f(\mathrm{s},y_\mathrm{s}) + \int_0^{\mathfrak{J}} g(\mathfrak{J},\mathrm{s},y_\mathrm{s})\mathrm{ds}\right]\mathrm{ds}\right] \leq \int_0^{\mathfrak{J}} L^2\mathscr{D}_\sigma(x_\mathrm{s},y_\mathrm{s})\mathrm{ds}\\ &= L^2\int_0^{\mathfrak{J}} \sup_{r\in[-\sigma,0]} \mathscr{D}[x(r+\mathrm{s}),y(r+\mathrm{s})]ds\\ &= L^2\int_0^{\mathfrak{J}} \sup_{\theta\in[\mathrm{s}-\sigma,\mathrm{s}]} \mathscr{D}[x(\theta),y(\theta)]\mathrm{ds}.\end{aligned} \tag{72}$$

From (29), $\mathfrak{D}[x(\mathfrak{J}),y(\mathfrak{J})] \leq \mathscr{D}_\sigma(x,y)e^{a\mathfrak{J}} \forall\mathfrak{J} \geq \mathfrak{J}_0 - \sigma$. So

$$\sup_{r\in[-\sigma,0]} \mathscr{D}[x(r),y(r)] \leq \mathscr{D}_\sigma(x,y)e^{\mathfrak{J}}\forall\mathfrak{J} \geq \mathfrak{J}_0. \tag{73}$$

For every $\mathfrak{J} \geq \mathfrak{J}_0$,

$$\begin{aligned}\mathscr{D}[(Px)(\mathfrak{J}),(Py)(\mathfrak{J})] &\leq L^2 \sup_{r\in[-\sigma,0]} \mathscr{D}[x(r),y(r)]ds\\ &= \frac{L^2}{a}\mathscr{D}_a(x,y)e^{a\mathfrak{J}_0}\left[e^{a(\mathfrak{J}-\mathfrak{J}_0)} - 1\right],\end{aligned} \tag{74}$$

and so

$$\begin{aligned}\mathscr{D}_\sigma(Px,Py) &= \sup_{\mathfrak{J}\geq\mathfrak{J}_0-\sigma} \mathscr{D}[(Px)(\mathfrak{J}),(Py)(\mathfrak{J})]e^{-a\mathfrak{J}}\\ &\leq \frac{L^2}{a}\mathscr{D}_a(x,y) \leq \mathscr{D}_a(x,y).\end{aligned} \tag{75}$$

Hence

$$\frac{L^2}{a} < 1. \tag{76}$$

Therefore, P is contraction on $\mathbf{E}_\mathbf{a}^\mathbf{m}$. □

Theorem 25. *Let function $\mathscr{G} : [0,\infty)\times C_\sigma \longrightarrow \mathbf{E}^\mathbf{m}$ satisfies assumptions $(\mathbf{J}_1)-(\mathbf{J}_3)$. Then, for each $(\mathfrak{J}_0,\varphi) \in C_\sigma$, FFDE (28) has unique solution on $[\mathfrak{J}_0,\infty)$.*

Proof. Assume

$$a > \max\left\{b, L^2\right\}. \tag{77}$$

We can deduce that the operator $P : \mathbf{E}_\mathbf{a} \longrightarrow \mathbf{E}_\mathbf{a}$ is contraction using Lemmas 23 and 24. As a result, there is only one $x \in \mathbf{E}_\mathbf{a}^\mathbf{m}$, which is $Px = x$. x is continuous function,

$$x(\mathfrak{J}) = f(\mathfrak{J},x_\mathfrak{J}) + \int_0^{\mathfrak{J}} g(\mathfrak{J},\mathrm{s},x_\mathrm{s})\mathrm{ds}, \tag{78}$$

on $[\mathfrak{J}_0 - \sigma, T]$. Moreover,

$$\begin{aligned}x(\mathfrak{J}) &= C_q(\mathfrak{J})\psi_0 + \mathscr{K}_q(\mathfrak{J})\psi_1 + \int_0^{\mathfrak{J}}(\mathfrak{J}-\mathrm{s})^{q-1}P_q(\mathfrak{J}-\mathrm{s})\\ &\quad\cdot\left[f(\mathrm{s},x_\mathrm{s}) + \int_0^{\mathfrak{J}} g(\mathfrak{J},\mathrm{s},x_\mathrm{s})\right]\mathrm{ds},\end{aligned} \tag{79}$$

for every $T \geq \mathfrak{J}_0$. Since x is continuous and $\mathscr{G}$ satisfies $(\mathbf{J}_2)$, by Lemma 9 and Remark 10,

$$\mathrm{s} \mapsto f(\mathrm{s},x_\mathrm{s}) + \int_0^{\mathfrak{J}} g(\mathfrak{J},\mathrm{s},x_\mathrm{s})\mathrm{ds}, \tag{80}$$

is an integrable function on $[\mathfrak{J}_0, T]$. By Remark 10, x is differentiable function and

$${}_0^c\mathscr{D}_q^H x(\mathfrak{J}) = f(\mathfrak{J},x_\mathfrak{J}) + \int_0^{\mathfrak{J}} g(\mathfrak{J},\mathrm{s},x_\mathrm{s})\mathrm{ds}, \tag{81}$$

for every $\mathfrak{J}_0 \geq T$. Theorem 25 is proved. □

5. Applications

5.1. Fuzzy Fractional Functional Evolution Equations with Distributed Delay. In below sections, we will look at class of delay fuzzy fractional functional evolution equations with distributed delay. Consider following delay fuzzy fractional functional differential equations with $m \in \mathbb{N}$ and $0 < \sigma_1 < \sigma_2 < \sigma_m < \sigma$ delay times:

$$\begin{gathered}{}_0^cD_q^H x(\mathfrak{T})=\int_{-\sigma}^{\mathfrak{T}}\left(\mathscr{G}_0(s,x_{\mathfrak{T}}(\mathfrak{T}+s))+\int_0^{\mathfrak{T}}g(\mathfrak{T},s,x_s)ds\right)ds+\sum_{i=1}^m\mathscr{G}_i(\mathfrak{T},x_{\mathfrak{T}}(\mathfrak{T}-\sigma_i)),\\ x(\mathfrak{T})=\psi(\mathfrak{T}-\mathfrak{T}_0)=\psi_0\in C_\sigma,\\ x'(\mathfrak{T})=\psi'(\mathfrak{T})=\psi_1,\end{gathered}\tag{82}$$

where $\mathscr{G}_i:[0,\infty)\times\mathbf{E^m}\longrightarrow\mathbf{E^m}, i=0,1,\cdots,m,$ are some functions. Let, function $\mathscr{G}_i:[0,\infty)\times C_\sigma\longrightarrow\mathbf{E^m}$ satisfies the following assumptions:

$(\mathbf{J_1}')$ There exist $L_i>0$,

$$\mathscr{D}[\mathscr{G}_i(\mathfrak{T},x),\mathscr{G}_i(\mathfrak{T},y)]\le L_i[x,y]\forall x,y\in\mathbf{E^m}\text{ and }\mathfrak{T}\ge 0.\tag{83}$$

$(\mathbf{J_2}')\mathscr{G}_i:[0,\infty)\times\mathbf{E^m}\longrightarrow\mathbf{E^m}$ is jointly continuous.

$(\mathbf{J_3}')$ There exist $M_i>0$ and $b_i>0$ that is

$$\mathscr{D}\left[\mathscr{G}_i(\mathfrak{T},0),\tilde{0}\right]\le M_i e^{b_i\mathfrak{T}}\forall\mathfrak{T}\ge 0.\tag{84}$$

Then, function $\mathscr{G}:[0,\infty)\times C_\sigma\longrightarrow\mathbf{E^m}$ is defined as

$$\begin{aligned}\mathscr{G}(\mathfrak{T},\varphi)=&\int_{-\sigma}^{\mathfrak{T}}\left(\mathscr{G}_0(\mathfrak{T}_0,\varphi(\mathfrak{T}_0))+\int_0^{\mathfrak{T}}g(\mathfrak{T}_0,\mathrm{s},x_s)ds\right)ds\\&+\sum_{i=1}^m\mathscr{G}_i(\mathfrak{T},\varphi(\mathfrak{T}-\sigma_i)),\end{aligned}\tag{85}$$

and satisfies also assumptions $(\mathbf{J_1})-(\mathbf{J_2})$. $\mathscr{F}$ is jointly continuous. For all $i=0,1,\cdots,m$. For function $\mathscr{G}_i$, suppose L_i be Lipschitz constant. Now

$$\begin{aligned}&\mathscr{D}[\mathscr{G}(\mathfrak{T},\varphi),\mathscr{G}(\mathfrak{T},\varphi)]\\&\le\int_{-\sigma}^{\mathfrak{T}}\mathscr{D}[\mathscr{F}_0(\mathfrak{T}_0,\varphi(\mathfrak{T}_0)),\mathscr{F}_0(\mathfrak{T}_0,\psi(\mathfrak{T}_0))]d\mathfrak{T}_0\\&\quad+\sum_{i=1}^m\mathscr{D}[\mathscr{F}_i(\mathfrak{T},\varphi(-\sigma_i),\psi(-\sigma_i))]\\&\le\left(\sigma L_0+\sum_{i=1}^m L_i\right)\mathscr{D}_\sigma(\varphi,\psi),\end{aligned}\tag{86}$$

and $\mathscr{G}$ satisfies $(\mathbf{J_1})$. We obtain

$$\begin{aligned}\mathscr{D}\left[\mathscr{G}(\mathfrak{T},0),\tilde{0}\right]\le\mathscr{D}\left[a,\tilde{0}\right]&+\int_{-\sigma}^{\mathfrak{T}}\mathscr{D}\left[\mathscr{F}_0(\mathfrak{T}_0,0),\tilde{0}\right]d\mathfrak{T}_0\\&+\sum_{i=1}^m\mathscr{D}\left[\mathscr{G}_i(\mathfrak{T}_0,0),\tilde{0}\right]=\mathscr{D}\left[a,\tilde{0}\right]\\&+\frac{M_0}{b_0}\left(1-e^{b_0\sigma}\right)+\sum_{i=1}^m M_i e^{b_i\mathfrak{T}}.\end{aligned}\tag{87}$$

Now, we find $M_{m+1}>1$ and $b_{m+1}>1$,

$$\mathscr{D}\left[a,\tilde{0}\right]+\left(\frac{M_0}{b_0}\right)\left(1-e^{b_0\sigma}\right)\le M_{m+1}e^{b_{m+1}\mathfrak{T}}\forall\mathfrak{T}\ge 0,\tag{88}$$

and we get

$$\mathscr{D}\left[\mathscr{G}(\mathfrak{T},\varphi),\tilde{0}\right]\le Me^{b\mathfrak{T}_0}\forall\mathfrak{T}\ge 0,\tag{89}$$

where $M:=\max\{M_i\,;i=1,2,\cdots,m+1\}$ and $b=\max\{b_i\,;i=1,2,\cdots,m+1\}$. As a result, $\mathscr{G}$ satisfies $(\mathbf{J_3})$.

As a result, we get below result.

5.2. Fuzzy Population Models. First, we demonstrate how to use the following method to explain the initial problem for fuzzy fractional functional delay differential equation:

$$\begin{gathered}{}_0^cD_q^H x(\mathfrak{T})=f(\mathfrak{T},x_{\mathfrak{T}})+\int_0^{\mathfrak{T}}g(\mathfrak{T},s,x_s)ds,\mathfrak{T}\ge\mathfrak{T}_0,\mathfrak{T}\in[0,T],\\ x(\mathfrak{T})=\psi(\mathfrak{T}-\mathfrak{T}_0)=\psi_0\in C_\sigma,\mathfrak{T}_0\ge\mathfrak{T}\ge\mathfrak{T}_0-\sigma,\\ x'(\mathfrak{T})=\psi'(\mathfrak{T})=\psi_1,\end{gathered}\tag{90}$$

where $\mathscr{G}:[0,\infty)\times\mathbf{E^m}\longrightarrow\mathbf{E^m}$ is derived from the continuous function $F:[0,\infty)\times\mathbf{R^m}\longrightarrow\mathbf{R^m}$ using Zadeh's extension concept. Since $[\mathscr{G}(\mathfrak{T},x_{\mathfrak{T}})]^\beta=f(\mathfrak{T},[x_{\mathfrak{T}}]^\beta)\forall\beta\in[1,2]$ and $x\in\mathbf{E^m}$, then, Kaleva [10] denotes

$$\begin{gathered}[x(\mathfrak{T})]^\beta=\left[x_1^\beta(\mathfrak{T}),x_2^\beta(\mathfrak{T})\right],\left[x'(\mathfrak{T})\right]^\beta=\left[\left(x_1^\beta\right)'(\mathfrak{T}),\left(x_2^\beta\right)'(\mathfrak{T})\right],[\varphi(\mathfrak{T})]^\beta=\left[\varphi_1^\beta(\mathfrak{T}),\varphi_2^\beta(\mathfrak{T})\right],\\ [\mathscr{G}(\mathfrak{T},x(\mathfrak{T}-\sigma))]^\beta=\left[\mathscr{G}_1^\beta\left(\mathfrak{T},x_1^\beta(\mathfrak{T}-\sigma),x_2^\beta(\mathfrak{T}-\sigma)\right),\mathscr{G}_2^\beta\left(\mathfrak{T},x_1^\beta(\mathfrak{T}-\sigma),x_2^\beta(\mathfrak{T}-\sigma)\right)\right],\\ \mathscr{G}_1^\beta\left(\mathfrak{T},x_1^\beta(\mathfrak{T}-\sigma),x_2^\beta(\mathfrak{T}-\sigma)\right)=\min\left\{F(\mathfrak{T},u)\,;u\in\left[x_1^\beta(\mathfrak{T}-\sigma),x_2^\beta(\mathfrak{T}-\sigma)\right]\right\},\\ \mathscr{G}_2^\beta\left(\mathfrak{T},x_1^\beta(\mathfrak{T}-\sigma),x_2^\beta(\mathfrak{T}-\sigma)\right)=\max\left\{F(\mathfrak{T},u)\,;u\in\left[x_1^\beta(\mathfrak{T}-\sigma),x_2^\beta(\mathfrak{T}-\sigma)\right]\right\}.\end{gathered}\tag{91}$$

Problem (90) is now transformed into the following parameterized delay differential model using these notations:

$$\begin{gathered}\left(x_1^\beta\right)'(\mathfrak{T})=\mathscr{G}_1^\beta\left(\mathfrak{T},x_1^\beta(\mathfrak{T}-\sigma),x_2^\beta(\mathfrak{T}-\sigma)\right),\mathfrak{T}\ge 0,\\ \left(x_2^\beta\right)'(\mathfrak{T})=\mathscr{G}_2^\beta\left(\mathfrak{T},x_1^\beta(\mathfrak{T}-\sigma),x_2^\beta(\mathfrak{T}-\sigma)\right),\mathfrak{T}\ge 0,\end{gathered}\tag{92}$$

with initial conditions

$$\begin{gathered}\left(x_1^\beta\right)(\mathfrak{T})=\varphi_1^\beta,-\sigma\le\mathfrak{T}\le\mathfrak{T}_0,\\ \left(x_2^\beta\right)'(\mathfrak{T})=\varphi_2^\beta,-\sigma\le\mathfrak{T}\le\mathfrak{T}_0.\end{gathered}\tag{93}$$

We can solve the methods (92) and (93). If (x_1^β,x_2^β) is the solution (92) and (93), we can establish a fuzzy solution $x(\mathfrak{T})$ for Equation (90) using representation theorem of Negoita-Ralescu [35]:

$$[x(\mathfrak{T})]^\beta=\left[x_1^\beta,x_2^\beta\right]\forall\beta\in[1,2].\tag{94}$$

5.2.1. Fuzzy Fractional Functional Time-Delay Malthusian Model. Suppose the initial value problem for a Malthusian model with a fuzzy fractional functional time delay in the example:

$$\begin{aligned} \mathcal{N}'(\mathfrak{J}) &= r\mathcal{N}(\mathfrak{J}-1), \mathfrak{J} \geq 0, \\ \mathcal{N}(\mathfrak{J}) &= \mathcal{N}_0, -1 \leq \mathfrak{J} \leq 0, \end{aligned} \tag{95}$$

$$[\mathcal{N}_0]^\beta = (1-\beta)[-1,1], \beta \in [1,2] \text{ and } r > 1. \tag{96}$$

Zadeh's extension concept is used to obtain function $\mathcal{G} : \mathbf{E^m} \longrightarrow \mathbf{E^m}$ described by $\mathcal{G}(\mathcal{N}(\mathfrak{J}-1)) = r\mathcal{N}(\mathfrak{J}-1)$ from function $f(u) = ru, u \in \mathbf{R^m}$.

If $[\mathcal{N}(\mathfrak{J})]^\beta = [\mathcal{N}_1(\mathfrak{J}), \mathcal{N}_2(\mathfrak{J})]$, then

$$\begin{aligned} \left[\mathcal{N}'(\mathfrak{J})\right]^\beta &= \left[\mathcal{N}_1'(\mathfrak{J}), \mathcal{N}_2'(\mathfrak{J})\right], [r\mathcal{N}(\mathfrak{J}-1)]^\beta \\ &= [r\mathcal{N}_1(\mathfrak{J}-1), r\mathcal{N}_2(\mathfrak{J}-1)]. \end{aligned} \tag{97}$$

As a result, we solve fractional functional differential equations:

$$\begin{aligned} \mathcal{N}_1'(\mathfrak{J}) &= r\mathcal{N}_1(\mathfrak{J}-1), \mathfrak{J} \geq 0, \\ \mathcal{N}_2(\mathfrak{J}) &= -\alpha, -1 \leq \mathfrak{J} \leq 0, \end{aligned} \tag{98}$$

$$\begin{aligned} \mathcal{N}_2'(\mathfrak{J}) &= r\mathcal{N}_2(\mathfrak{J}-1), \mathfrak{J} \geq 0, \\ \mathcal{N}_1(\mathfrak{J}) &= \alpha, -1 \leq \mathfrak{J} \leq 0, \end{aligned} \tag{99}$$

where $\alpha = 1 - \alpha$. The system of steps is used to solve Equation (98). For $0 \leq \mathfrak{J} \leq 1$, we get

$$\begin{cases} -r\alpha, \\ \mathcal{N}_1(0) = -\alpha, \end{cases} \tag{100}$$

with solution $\mathcal{N}_1(\mathfrak{J}) = -\alpha - r\alpha\mathfrak{J}$ for $0 \leq \mathfrak{J} \leq 1$. For $1 \leq \mathfrak{J} \leq 2$, we get

$$\begin{cases} -r\alpha - r^2\alpha(\mathfrak{J}-1), \\ \mathcal{N}_1(1) = -\alpha - r\alpha, \end{cases} \tag{101}$$

with the solution $\mathcal{N}_1(\mathfrak{J}) = -\alpha - r\alpha - r\alpha\mathfrak{J} - (1/2)r^2\alpha(\mathfrak{J}-1)^2$ for $1 \leq \mathfrak{J} \leq 2$. For each $n \in \mathbb{N}$, the solution of (98) has polynomial form $\mathcal{N}_1(\mathfrak{J}) = \sum_{p=1}^{n+1} a_p \mathfrak{J}^p$ on $[n, n+1]$. Also, the solution of (99) has polynomial form when $\mathcal{N} - 2(\mathfrak{J}) = \sum_{p=1}^{n+1} b_p \mathfrak{J}^p$ on $[n, n+1]$. According to Negoita-Ralescu representation theorem [35], the solution of (95) has form on $[n, n+1]$:

$$[\mathcal{N}(\mathfrak{J})]^\beta = \left[\sum_{p=1}^{n+1} a_p \mathfrak{J}^p, \sum_{p=1}^{n+1} b_p \mathfrak{J}^p\right], \tag{102}$$

for every $\beta \in [1,2]$ and $n \in \mathbb{N}$.

Example 1. One of the deficiency of population models in time-delay Malthusian model is that in every case, when population change instantly, birth rate is supposed to change. Moreover, when members of the population hit a certain age before giving birth, we should assume time delay in the model [34].

$$\mathcal{N}'(\mathfrak{J}) = r\mathcal{N}(\mathfrak{J}-\sigma), \tag{103}$$

where population growth rate at time $\mathfrak{J}$ is determined by population at time $\mathfrak{J} - \sigma$.

Also, assume, for time-delay Malthusian model, a more realistic approach should take into account both effect of a time delay and changing of environment. Therefore, it is interesting and necessary to study the general delay-distributed equation:

$$\mathcal{N}'(\mathfrak{J}) = \sum_{p=1}^{n} r_p \mathcal{N}\left(\mathfrak{J} - \sigma_p\right) + \int_{-\sigma}^{\mathfrak{J}} r\mathcal{N}(\mathfrak{J} + s) ds. \tag{104}$$

5.2.2. Fuzzy Fractional Functional Ehrlich Ascites Tumor Model. To explain tumor model of fuzzy fractional function Ehrlich Ascites, consider fuzzy delay equation:

$$\begin{aligned} \mathcal{N}'(\mathfrak{J}) &= r\mathcal{N}(\mathfrak{J}-1)(1-\mathcal{N}(\mathfrak{J}-1)), \mathfrak{J} \geq 0, \\ \mathcal{N}(\mathfrak{J}) &= \mathcal{N}_0, -1 \leq \mathfrak{J} \leq 0, \end{aligned} \tag{105}$$

where $[\mathcal{N}_0]^\beta = \alpha[-1,1], \alpha = ((1-\beta)/2), \beta \in [0,1]$. Assume that $r \in (0,2]$. The function $\mathcal{G} : \mathbf{E^m} \longrightarrow \mathbf{E^m}$, defined by $\mathcal{G}(\mathcal{N}(\mathfrak{J}-1)) = r\mathcal{N}(\mathfrak{J}-1)(1-\mathcal{N}(\mathfrak{J}-1))$, is obtained from function $f(u) = ru(1-u), u \in \mathbf{R^m}$, using Zadeh's extension principle. We get

$$\left[\mathcal{N}'(\mathfrak{J})\right]^\beta = [r\mathcal{N}(\mathfrak{J}-1)(1-\mathcal{N}(\mathfrak{J}-1))]^\beta, \beta \in [0,1]. \tag{106}$$

We remark, function $f(u) = ru(1-u)$ is increasing on $(-\infty, 1/2)$ and decreasing on $(1/2, \infty)$, and $\max_{u \in \mathfrak{R}} f(u) = r/4$. Using the procedure of steps [36] and Negoita-Ralescu representation theorem [35], we can obtain the solution to (105). If $0 \leq \mathfrak{J} \leq 1$, now, we have

$$\begin{aligned} \mathcal{N}'(\mathfrak{J}) &= r\mathcal{N}_0(2-\mathcal{N}_0), \mathfrak{J} \geq 0, \\ \mathcal{N}(0) &= \alpha, -1 \leq \mathfrak{J} \leq 1. \end{aligned} \tag{107}$$

Since $\alpha \leq 1/2$, then, for $0 \leq \mathfrak{J} \leq 1$,

$$\begin{aligned} \left[\mathcal{N}'(\mathfrak{J})\right]^\beta &= [r\mathcal{N}_0(1-\mathcal{N}_0)]^\beta = \left[\min_{-\alpha \leq u \leq \alpha} f(u), \max_{-\alpha \leq u \leq \alpha} f(u)\right] \\ &= [-r\alpha(1+\alpha), r\alpha(1-\alpha)]. \end{aligned} \tag{108}$$

As a result, we solve differential equations on $[0,1]$

$$\left[\mathcal{N}'(\mathfrak{J})\right]^\beta = [-r\alpha(1+\alpha), r\alpha(1-\alpha)], \tag{109}$$

with initial condition

$$[\mathcal{N}(0)]^{\beta} = [-\alpha, \alpha]. \tag{110}$$

Further, for (52) on [0,1], the solution

$$[\mathcal{N}(\mathfrak{J})]^{\beta} = [\mathcal{N}_{11}(\mathfrak{J}), \mathcal{N}_{21}(\mathfrak{J})], \mathfrak{J} \in [0, 1], \tag{111}$$

where

$$\begin{aligned}\mathcal{N}_{11}(\mathfrak{J}) &= -\alpha - r\alpha(1+\alpha)\mathfrak{J}, \mathcal{N}_{21}(\mathfrak{J}) \\ &= \alpha + r\alpha(1-\alpha)\mathfrak{J}, \beta \in [0, 1].\end{aligned} \tag{112}$$

Moreover, $\mathcal{N}_{11}(\mathfrak{J}) \leq 0$ and $1/2 \leq \mathcal{N}_{21}(\mathfrak{J}) \leq 1$ on $[0, 1]$, for $1 \leq \mathfrak{J} \leq 2$,

$$\begin{aligned}\left[\mathcal{N}'(\mathfrak{J})\right]^{\beta} &= [r\mathcal{N}(\mathfrak{J}-1)(1-\mathcal{N}(\mathfrak{J}-1))]^{\beta} \\ &= \left[\min_{\mathcal{N}_{11}(\mathfrak{J}-1)\leq u \leq \mathcal{N}_{21}(\mathfrak{J}-1)} f(u), \max_{\mathcal{N}_{11}(\mathfrak{J}-1)\leq u \leq \mathcal{N}_{21}(\mathfrak{J}-1)} f(u)\right] \\ &= \left[-r\alpha + r\alpha(1+\alpha)(\mathfrak{J}-1)(1+\alpha+r\alpha(1+\alpha)(\mathfrak{J}-1)), \frac{r^2}{4}\left(1-\frac{r}{4}\right)\right].\end{aligned} \tag{113}$$

As it follows, we solve the differential equation on [1 , 2]:

$$\left[\mathcal{N}'(\mathfrak{J})\right]^{\beta} = \left[-r\alpha + r\alpha(1+\alpha)(\mathfrak{J}-1)(1+\alpha+r\alpha(1+\alpha)(\mathfrak{J}-1)), \frac{r^2}{4}\left(1-\frac{r}{4}\right)\right]. \tag{114}$$

As a result, we get (105) on $[1, 2]$, as follows:

$$[\mathcal{N}(\mathfrak{J})]^{\beta} = [\mathcal{N}_{12}(\mathfrak{J}), \mathcal{N}_{22}(\mathfrak{J})], \beta \in [1, 2], \tag{115}$$

where

$$\begin{aligned}\mathcal{N}_{12}(\mathfrak{J}) &= -\alpha - 2r\alpha(1+\alpha) - r\alpha(1+\alpha)(\mathfrak{J}-1) \\ &\quad - r^2\alpha(1+\alpha)(2+\alpha)\frac{(\mathfrak{J}-1)^2}{2} - r^3\alpha^2\frac{(\mathfrak{J}-1)^3}{3},\end{aligned}$$

$$\mathcal{N}_{22}(\mathfrak{J}) = \alpha + r\alpha(1-\alpha) + \frac{r^2}{4}\left(1-\frac{r}{4}\right)\mathfrak{J}, \mathfrak{J} \in [0, 1]. \tag{116}$$

This procedure can be continued on [21, 37].

Example 2. To explain the Ehrlich ascities tumor, the following logistic equation was suggested in [9]:

$$\mathcal{N}'(\mathfrak{J}) = r\mathcal{N}(\mathfrak{J}-\sigma)\left(1 - \frac{\mathcal{N}(\mathfrak{J}-\sigma)}{\mathcal{K}}\right). \tag{117}$$

The delay associated cell cycle [37] is represented by σ, where r is net tumor replication and $\mathcal{K}$ is caring capacity. This equation differs from the traditional Verhulst-Hutchinson equation [38], which has only one delay expression.

Many independent characteristics of state variables can affect population dynamics: natural and social resources, medical care, job environment, and crime, habitations. Classically, the exact value of these attributes cannot always be calculated and evaluated since they are unknown and can only be conjectured. As a result, the Ehrlich ascities tumor model should be a more realistic solution.

6. Conclusion

The solution to fuzzy fractional functional differential equations possesses global uniqueness and existence, as shown in this paper. We have used the successive approximation method to prove a local uniqueness and existence result. Future research on fuzzy neutral fractional functional differential equations could benefit from the findings of this study. Other alternative research approaches include a fuzzy fractional functional differential equation approach based on other fuzzy differentiability concepts (see [8, 11]).

Data Availability

No new data were created this study.

Conflicts of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References

[1] B. Ahmad and J. J. Nieto, "Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions," *Computers & Mathematics with Applications*, vol. 58, no. 9, pp. 1838–1843, 2009.

[2] B. Ahmad, S. K. Ntouyas, R. P. Agarwal, and A. Alsaedi, "On fractional differential equations and inclusions with nonlocal and average-valued (integral) boundary conditions," *Advances in Difference Equations*, vol. 2016, 18 pages, 2016.

[3] S. S. Mansouri, M. Gachpazan, and O. S. Fard, "Existence, uniqueness and stability of fuzzy fractional differential equations with local Lipschitz and linear growth conditions," *Advances in Difference Equations*, vol. 2017, 13 pages, 2017.

[4] S. S. Chang and L. A. Zadeh, "On fuzzy mapping and control," *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi a Zadeh*, pp. 180–184, 1996.

[5] M. Chen and C. Han, "Some topological properties of solutions to fuzzy differential systems," *Information Sciences*, vol. 197, pp. 207–214, 2012.

[6] O. Kaleva, "The Cauchy problem for fuzzy differential equations," *Fuzzy Sets and Systems*, vol. 35, no. 3, pp. 389–396, 1990.

[7] J. J. Nieto, "The Cauchy problem for continuous fuzzy differential equations," *Fuzzy Sets and Systems*, vol. 102, no. 2, pp. 259–262, 1999.

[8] J. J. Buckley and T. Feuring, "Fuzzy differential equations," *Fuzzy Sets and Systems*, vol. 110, no. 1, pp. 43–54, 2000.

[9] S. Song and C. Wu, "Existence and uniqueness of solutions to Cauchy problem of fuzzy differential equations," *Fuzzy Sets and Systems*, vol. 110, no. 1, pp. 55–67, 2000.

[10] X. Xiaoping and F. Yongqiang, "On the structure of solutions for fuzzy initial value problem," *Fuzzy Sets and Systems*, vol. 157, no. 2, pp. 212–229, 2006.

[11] P. Diamond, "Stability and periodicity in fuzzy differential equations," *IEEE Transactions on Fuzzy Systems*, vol. 8, no. 5, pp. 583–590, 2000.

[12] E. Hüllermeier, "An approach to modelling and simulation of uncertain dynamical systems," *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol. 5, no. 2, pp. 117–137, 1997.

[13] B. Bede and S. G. Gal, "Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations," *Fuzzy Sets and Systems*, vol. 151, no. 3, pp. 581–599, 2005.

[14] J. Y. Park, S. Y. Lee, and J. U. Jeong, "The approximate solutions of fuzzy functional integral equations," *Fuzzy Sets and Systems*, vol. 110, no. 1, pp. 79–90, 2000.

[15] J. Y. Park, I. H. Jung, and M. J. Lee, "Almost periodic solutions of fuzzy systems," *Fuzzy Sets and Systems*, vol. 119, no. 3, pp. 367–373, 2001.

[16] P. Balasubramaniam and S. Muralisankar, "Existence and uniqueness of fuzzy solution for the nonlinear fuzzy integro-differential equations," *Applied Mathematics Letters*, vol. 14, no. 4, pp. 455–462, 2001.

[17] M. Guo, X. Xue, and R. Li, "Impulsive functional differential inclusions and fuzzy population models," *Fuzzy Sets and Systems*, vol. 138, no. 3, pp. 601–615, 2003.

[18] A. Abbas, R. Shafqat, M. B. Jeelani, and N. H. Alharthi, "Significance of chemical reaction and Lorentz force on third-grade fluid flow and heat transfer with Darcy–Forchheimer law over an inclined exponentially stretching sheet embedded in a porous medium," *Symmetry*, vol. 14, no. 4, p. 779, 2022.

[19] A. Abbas, R. Shafqat, M. B. Jeelani, and N. H. Alharthi, "Convective heat and mass transfer in third-grade fluid with Darcy–Forchheimer Relation in the presence of thermal-diffusion and diffusion-thermo effects over an exponentially inclined stretching sheet surrounded by a porous medium: a CFD study," *Processes*, vol. 10, no. 4, p. 776, 2022.

[20] A. U. K. Niazi, J. He, R. Shafqat, and B. Ahmed, "Existence, uniqueness, and Eq–Ulam-type stability of fuzzy fractional differential equation," *Fractal and Fractional*, vol. 5, no. 3, p. 66, 2021.

[21] N. Iqbal, A. U. K. Niazi, R. Shafqat, and S. Zaland, "Existence and uniqueness of mild solution for fractional-order controlled fuzzy evolution equation," *Journal of Function Spaces*, vol. 2021, Article ID 5795065, 8 pages, 2021.

[22] R. Shafqat, A. U. K. Niazi, M. B. Jeelani, and N. H. Alharthi, "Existence and uniqueness of mild solution where $\alpha \in (1,2)$ for fuzzy fractional evolution equations with uncertainty," *Fractal and Fractional*, vol. 6, no. 2, p. 65, 2022.

[23] K. Abuasbeh, R. Shafqat, A. U. K. Niazi, and M. Awadalla, "Local and global existence and uniqueness of solution for time-fractional fuzzy Navier–Stokes equations," *Fractal and Fractional*, vol. 6, no. 6, p. 330, 2022.

[24] A. S. Alnahdi, R. Shafqat, A. U. K. Niazi, and M. B. Jeelani, "Pattern formation induced by fuzzy fractional-order model of COVID-19," *Axioms*, vol. 11, no. 7, p. 313, 2022.

[25] Y. Guo, M. Chen, X. B. Shu, and F. Xu, "The existence and Hyers-Ulam stability of solution for almost periodical fractional stochastic differential equation with fBm," *Stochastic Analysis and Applications*, vol. 39, no. 4, pp. 643–666, 2021.

[26] Y. Shi, "A study on the mild solution of impulsive fractional evolution equations," *Applied Mathematics and Computation*, vol. 273, pp. 465–476, 2016.

[27] V. Lupulescu, "On a class of fuzzy functional differential equations," *Fuzzy Sets and Systems*, vol. 160, no. 11, pp. 1547–1562, 2009.

[28] O. Kaleva, "Fuzzy differential equations," *Fuzzy Sets and Systems*, vol. 24, no. 3, pp. 301–317, 1987.

[29] C. G. Gal and S. G. Gal, "Semigroups of operators on spaces of fuzzy-number-valued functions with applications to fuzzy differential equations," 2013, https://arxiv.org/abs/1306.3928.

[30] D. Otrocol and V. Ilea, "Ulam stability for a delay differential equation," *Open Mathematics*, vol. 11, no. 7, pp. 1296–1303, 2013.

[31] R. J. Aumann, "Integrals of set-valued functions," *Journal of Mathematical Analysis and Applications*, vol. 12, no. 1, pp. 1–12, 1965.

[32] I. Podlubny, *Fractional differential equations*, Academic Press, 1999.

[33] J. J. Nieto and R. R. López, "Applications of contractive-like mapping principles to fuzzy equations," *Revista Matemática Complutense*, vol. 19, no. 2, pp. 361–383, 2006.

[34] C. C. Travis and G. F. Webb, "Cosine families and abstract nonlinear second order differential equations," *Acta Mathematica Hungarica*, vol. 32, no. 1-2, pp. 75–96, 1978.

[35] C. V. Negoiţă and D. A. Ralescu, *Applications of Fuzzy Sets to Systems Analysis*, Birkhäuser, Basel, Switzerland, 1975.

[36] Y. Kuang, *Delay Differential Equations: With Applications in Population Dynamics*, Academic press, 1993.

[37] K. H. Jack, "Theory of functional differential equations," *Applied Mathematical Sciences*, vol. 3, 1981.

[38] R. Schuster and H. Schuster, "Reconstruction models for the Ehrlich ascites tumor of the mouse," *Mathematical population dynamics*, vol. 2, pp. 335–348, 1995.