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Research Article

Caristi's Fixed Point Theorem in Cone Metric Space

Fatemeh Lael,[1] **Naeem Saleem**,[2] **and Reny George**[3]

[1]*Imam Khomeini International University-Buin Zahra Higher Education Center of Engineering and Technology, Buin Zahra, Qazvin, Iran*
[2]*Department of Mathematics, University of Management and Technology, Lahore, Pakistan*
[3]*Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia*

Correspondence should be addressed to Naeem Saleem; naeem.saleem2@gmail.com and Reny George; renygeorge02@yahoo.com

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In this paper, we provide a short, comprehensive, and brief proof for Caristi-Kirk fixed point result for single and set-valued mappings in cone metric spaces. In addition, we partially addressed an open problem in which Caristi-Kirk fixed point result in cone metric spaces reduces to a classical result in metric spaces and provided a brief justification for a partial positive answer to this open problem using Caristi-Kirk fixed point theorem on uniform space. The proofs given to Caristi-Kirk's result vary and use different techniques.

1. Introduction and Preliminaries

Caristi-Kirk's fixed point theorem in [1] states that if X is a complete metric space and φ is a lower semicontinuous mapping from X into the nonnegative real numbers, then any mapping $T : X \longrightarrow X$ satisfying

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx), (x \in X) \tag{1}$$

has a fixed point.

Several researchers generalized the Caristi-Kirk's fixed point theorem in various directions, for details see [2–9].

Angelov [10] provided an extension of the Caristi-Kirk theorem to T_2-separated uniform spaces, the uniform space X is known as T_2-separated if each convergent sequence in X has a unique limit. As we know that every uniform space is generated by a family of pseudometrics $\{d_a(x, y): a \in A\}$, where A is an indexing set. Also, a sequence $(x_n) \in X$ is known as a Cauchy sequence, if for each $a \in A$, we have $\lim_{n,m\longrightarrow+\infty} d_a(x_n, x_m) = 0$, and a sequence $(x_n) \in X$ is convergent and converges to $x \in X$, if for each $a \in A$, we have $\lim_{n\longrightarrow+\infty} d_a(x_n, x) = 0$. Thus, a uniform space X is called complete if every Cauchy sequence is convergent in X.

In this regard, Angelov [10] generalizes the Caristi-Kirk fixed point theorem on uniform space, which stated as:

Theorem 1 [10]. *Let X be a T_2 -separated complete uniform space which is generated by a family of pseudometrics $\{d_a : a \in A\}$, where A is an indexing set. Let $T : X \longrightarrow X$ be a mapping and $\{\psi_a\}$ be a family of lower semicontinuous functionals. Suppose that the following inequality holds for each $a \in A$,*

$$d_{j(a)}(x, T(x)) \leq \psi_a(x) - \psi_a(T(x)), \tag{2}$$

where $x \in X$ and $j : A \longrightarrow A$ is a surjective mapping. Then, T has a fixed point in X.

The following theorem is a Banach fixed point theorem on uniform space, which stated as:

Theorem 2 [11]. *Let X be a T_2 -separated complete uniform space which is generated by a family of pseudometrics $\{d_a$*

: $a \in A\}$, where A is an indexing set. Suppose that $T : X \longrightarrow X$ is a mapping which is satisfying

$$d_a(T(x), T(y)) \leq k_a d_a(x, y), \tag{3}$$

for each $x, y \in X$ and $a \in A$. Then, T has a unique fixed point in X.

In 2007, Huang et al. [12] introduced the concept of cone metric space and proved some well-known fixed point results. The authors extended fixed point results proved for cone metric spaces which was just a simple reformulation of classical results presented in metric spaces. The obtained results are generalizations from classical results to cone metric spaces, for details see [13–16].

In this paper, we aim to reformulate Caristi-Kirk's fixed point theorem for single and set-valued mappings in cone metric space and obtained a detailed answer to a question posed by Khamsi and Wojciechowski in ([17], Theorem 3-1) "whether the vectorial version and the classical version of Caristi-Kirk's fixed point theorem are equivalent." To address this particular answer, we defined a uniform space by considering cone metric space, and then, we addressed Theorem 1 in cone metric spaces and uniform spaces. Our proof is shorter, comprehensive, and easier than proof provided until now and our results generalize the existing results due to Khamsi and Wojciechowski in [17].

2. Cone Metric Version of Caristi-Kirk's Theorem

Suppose that P is a nonempty closed convex cone of a real Banach space E such that $P \neq \{\theta\}$, where θ is the null vector, $P \cap -P = \{\theta\}$ and $intP \neq \varnothing$.

In addition, P induces a partial order $\preceq$ on E which is defined as $x \preceq y$ if and only if $y - x \in P$ and we write $x \ll y$ if and only if $y - x \in intP$.

A convex subset $B \subset P$ is a base of P if $\theta \in \bar{B}$ and $P = \cup_{t \geq 0} tB$ and E^* is the topological dual space of E and $P^* = \{\psi \in E^* : \psi(x) \geq 0, \forall x \in P\}$ is known as dual cone of P. The dual cone P^* of a cone P in a Banach space E has a weak*-compact base B^*. A set $A \subset E$ is called bounded from above (below) if there exists $z \in E$ such that for all $a \in A$, $a \preceq z$ ($z \preceq a$). A cone is called regular if every nondecreasing (decreasing) sequence which is bounded from above (below) is convergent in norm. The cone P is called normal if there is a number $K > 0$, such that we have for all $x, y \in E$,

$$\theta \preceq x \preceq y \Longrightarrow \|x\| \leq K\|y\|. \tag{4}$$

The least positive number satisfying this inequality is called a normal constant of P.

The following lemma will be used in proving our main results.

Lemma 3 [18, 19]. *The weak*-compact base B^* satisfies:*

(1) any element $x \in P$ if and only if $\psi(x) \geq 0$, for all $\psi \in B^*$

(2) any element $x \in intP$ if and only if $\psi(x) > 0$, for all $\psi \in B^*$

Definition 4 [12]. Let X be a nonempty set. Consider a mapping $d : X \times X \longrightarrow E$ is satisfied as follows:

(1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$

(3) $d(x, y) \preceq d(x, z) + d(y, z)$ for all $x, y, z \in X$

Then, d is called a cone metric on X and (X, d) is named a cone metric space.

Definition 5 [12]. Let (x_n) be a sequence in a cone metric space (X, d) and some $x \in X$. A sequence (x_n) is as follows:

(1) a d-Cauchy sequence if for every $\theta \ll \varepsilon \in E$, there exists $N \in \mathbb{N}$, such that $d(x_m, x_n) \ll \varepsilon$, for all $m, n \geq N$

(2) a d-convergent and d-converges to $x \in X$ if for every $\theta \ll \varepsilon \in E$, there exists $N \in \mathbb{N}$, such that $d(x_n, x) \ll \varepsilon$, for all $n \geq N$, which is denoted as $x_n \longrightarrow x$

Definition 6 [12]. A cone metric space (X, d) is $d -$ complete if every $d -$ Cauchy sequence is d-convergent in (X, d).

Definition 7 [20]. Let (X, d) be a cone metric space. A mapping $\varphi : X \longrightarrow E$ is considered as a cone lower semicontinuous mapping at $x \in X$ if for any $\theta \ll \varepsilon \in E$, there exists a natural number $N_\varepsilon \in \mathbb{N}$ such that

$$\varphi(x) \preceq \varphi(x_n) + \varepsilon, \tag{5}$$

for all $n > N_\varepsilon$, where (x_n) is a sequence in X and $x_n \longrightarrow x$. If $E = \mathbb{R}$, then $P = \mathbb{R}^{\geq 0}$, (X, d) is a metric space and $\varphi : X \longrightarrow \mathbb{R}$, so φ is a lower semicontinuous mapping at $x \in X$, if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that for any $n > N_\varepsilon$, we have

$$\varphi(x) \leq \varphi(x_n) + \varepsilon, \tag{6}$$

where (x_n) is a convergent sequence and converges to x in a metric space (X, d).

The following theorem is a cone metric version of the Caristi-Kirk's theorem with the some extra normal cone condition.

Theorem 8 [21]. *Let (X, d) be a cone metric space with normal and regular cone of a Banach space $(E, \|\cdot\|)$ such that $\lim_{m,n \longrightarrow \infty} \|d(x_m, x_n)\| = 0$ implies $\lim_{n \longrightarrow \infty} \|d(x_n, x)\| = 0$ for some $x \in X$. Also, $\varphi : X \longrightarrow P$ satisfies $\varphi(x) \preceq \liminf_n \varphi(x_n)$, for every*

$\lim_{n\longrightarrow\infty}\|d(x_n,x)\|=0$. *Suppose that the mapping* $T : X \longrightarrow X$ *satisfying the following condition:*

$$d(x,T(x)) \preceq \varphi(x)-\varphi(T(x)), \tag{7}$$

for all $x \in X$. *Then,* T *has a fixed point.*

Since then, some studies have focused on extending and improving the cone metric version of the Caristi-Kirk's fixed point theorem in many ways. In [16, 17], authors proved the Caristi-Kirk's fixed point theorem but the authors supposed that the cone is normal which is a strict condition and researchers did not accept it as a good condition. Further, the results are proved for regular and normal cone in [21].

In this paper, none of these conditions was considered for the cone. Now, we will omit the stronger conditions, the normality, and regularity of the cone in our main results and we will prove this result under the weaker condition as compared to the result proved in the literature under strict conditions.

3. Main Results

The following lemmas are handy tools that are used in the sequel.

Lemma 9. *Let* (X,d) *be a cone metric space and* $\psi\in B^*$. *Also suppose that* X *is a uniform space which is generated by a family of pseudometrics* $\{\psi\circ d:\psi\in B^*\}$. *Then,* X *is* T_2 *-separated.*

Proof. On contrary suppose that the sequence (x_n) has two different limits, i.e., $\lim x_n=x$ and $\lim x_n=y$ in the uniform space X. Then, according to definition, for each pseudometric $\psi\circ d$, we have $\lim_{n\longrightarrow\infty}\psi(d(x_n,x))=0$ and $\lim_{n\longrightarrow\infty}\psi(d(x_n,y))=0$. In addition, by the third property of the cone metric, we have $d(x,y)\preceq d(x_n,x)+d(x_n,y)$, (by lemma 3) we have $\psi(d(x,y))\le\psi(d(x_n,x))+\psi(d(x_n,y))$, for each $\psi\in B^*$. When $n\longrightarrow\infty$, we have that for each $\psi\in B^*$, $\psi(d(x,y))=0$. Thus, $\|d(x,y)\|=\sup_{\|\psi\|=1}|\psi(d(x,y))|=0$. Thus, $d(x,y)=\theta$, i.e., $x=y$. □

Lemma 10. *Let* (X,d) *be a cone metric space and* $\psi\in B^*$. *Also suppose that* X *is a uniform space which is generated by a family of pseudometrics* $\{\psi\circ d:\psi\in B^*\}$. *Then,* X *is a complete uniform space if and only if* (X,d) *is a* d *-complete cone metric space.*

Proof. First, we suppose that X is a complete uniform space. Let (x_n) be a d-Cauchy sequence in the cone metric space (X,d). Then, for each $\theta\ll\varepsilon/k$, where $k\in\mathbb{N}$, there exists $N\in\mathbb{N}$ such that for each $m,n>N$, we have $d(x_m,x_n)\ll\varepsilon/k$. Using Lemma 3 part (2), for each $\psi\in B^*$, we have $\psi(\varepsilon)/k>0$ and

$$\psi(d(x_m,x_n)<\frac{\psi(\varepsilon)}{k}. \tag{8}$$

For any $\varepsilon>0$ and $\psi\in B^*$, there is a $k\in\mathbb{N}$ such that $\psi(\varepsilon)/k<\varepsilon$. For $m,n>N$, inequality (8) implies that $\psi(d(x_m,x_n))<\varepsilon$. Using the definition, (x_n) is a Cauchy sequence in a complete uniform space. Therefore, (x_n) is convergent and converges to x which belongs to the uniform space X. Then, for each $\varepsilon>0$ and $\psi\in B^*$, there exists $N\in\mathbb{N}$, such that for each $n>N$,

$$\psi(d(x_n,x))<\varepsilon. \tag{9}$$

Now, we demonstrate that the sequence (x_n) is d-convergent. On the contrary, suppose that there is some $\theta\ll\varepsilon$, such that for each $N\in\mathbb{N}$ there is $n>N$ such that $d(x_n,x)\succeq\varepsilon$. From lemma 3 part (8), for each $\psi\in B^*$, $\psi(\varepsilon)>0$, and $\psi(d(x_n,x))>\psi(\varepsilon)$, which is a contradiction, if $\varepsilon=\psi(\varepsilon)$.

On contrary suppose that (X,d) is a d-complete space. Let (x_n) be a Cauchy sequence in a uniform space X, for each $\varepsilon>0$ and $\psi\in B^*$, there is $N\in\mathbb{N}$, such that for each m, $n>N$,

$$\psi(d(x_m,x_n))<\varepsilon. \tag{10}$$

Now, we show that (x_n) is a $d-$Cauchy sequence. On the contrary, suppose that (x_n) is not a $d-$Cauchy sequence. Then, there is $\theta\ll\varepsilon$ such that for each $N\in\mathbb{N}$, there are $m,n>N$ such that $d(x_m,x_n)\succeq\varepsilon$. Thus, by using lemma 3 for each $\psi\in B^*$, we have $\psi(\varepsilon)>0$ and $\psi(d(x_m,x_n))\ge\psi(\varepsilon)>0$, which is a contradiction, as $\varepsilon=\psi(\varepsilon)$. Therefore, (x_n) is a $d-$Cauchy sequence, and accordingly, it is a d-convergent and converges to some x (from definition 5) for each $\theta\ll\varepsilon/k$ where $k\in\mathbb{N}$, there is $N\in\mathbb{N}$ such that for each $n>N$, we have $d(x_n,x)\ll\varepsilon/k$, (using lemma 3) for each $\psi\in B^*$, we have $\psi(\varepsilon)/k>0$ and

$$\psi(d(x_n,x))<\frac{\psi(\varepsilon)}{k}. \tag{11}$$

For each $\varepsilon>0$ and $\psi\in B^*$, there is $k\in\mathbb{N}$ such that $\psi(\varepsilon)/k<\varepsilon$ thus (11) implies that for $n>N$, we have

$$\psi(d(x_n,x))<\frac{\psi(\varepsilon)}{k}<\varepsilon. \tag{12}$$

Thus, (x_n) is convergent to x in the uniform space X. □

Lemma 11. *Let* (X,d) *be a cone metric space and* $x\in X$. *Then,* $d(x,\cdot): X\longrightarrow E$ *is a cone lower semicontinuous mapping.*

Proof. Let $\theta\ll\varepsilon\in E$ and (y_n) be a sequence in X such that $y_n\longrightarrow y\in X$. There exists $N\in\mathbb{N}$ such that $d(y_n,y)\ll\varepsilon$ for all $n\ge N$. Then, $d(x,y)\ll d(x,y_n)+d(y_n,y)\ll d(x,y_n)+\varepsilon$, for all $n\ge N$. Thus, $d(x,\cdot)$ is a cone lower semicontinuous mapping. □

Lemma 12. *Let* (X,d) *be a cone metric space,* $\varphi : X\longrightarrow E$ *be a cone lower semicontinuous mapping and* $\psi\in B^*$. *Then,* $\psi\circ\varphi : X\longrightarrow\mathbb{R}$ *is a lower semicontinuous function.*

Proof. Let $\theta \ll \varepsilon \in E$ be fixed. For any $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that $\psi(\varepsilon/m) < \varepsilon$, ($\psi$ is linear) and φ is cone lower semicontinuous and $x_n \longrightarrow x$. Thus, there exists $N \in \mathbb{N}$ such that

$$\varphi(x) \preceq \varphi(x_n) + \frac{1}{m}\varepsilon, \tag{13}$$

for all $n \geq N$, and so

$$\psi(\varphi(x)) \leq \psi(\varphi(x_n)) + \psi\left(\frac{1}{m}\varepsilon\right) \leq \psi(\varphi(x_n)) + \varepsilon, \tag{14}$$

for all $n \geq N$. This relation indicates the lower semicontinuity of $\psi \circ \varphi$. □

As is shown in [14], all fixed point results in cone metric spaces obtained recently, in which the assumption that the underlying cone is normal and with the nonempty interior is present, can be reduced to the corresponding results in metric spaces. On the other hand, when we deal with non-normal cones, this is not possible.

Theorem 13 is a cone metric version of Caristi-Kirk's theorem without extra conditions normality and regularity which are always put in cone metric theorems, so our results are original. To prove this theorem, we show that the cone metric space is uniform too; then, it will be proved by applying Theorem 1. We know that a T_2-separated uniform space is metrizable if its uniformity can be defined by a countable family of pseudometrics. Indeed, such uniformity can be defined by a single pseudometric, which is necessarily a metric. This implies that a cone metric version of Caristi-Kirk's theorem may be derived from the classical one if B^* which is defined in Section 2 is countable. This is a partial answer to the open question mentioned before.

Theorem 13. *Let (X, d) be a d -complete cone metric space and $\varphi : X \longrightarrow P$ be a cone lower semicontinuous mapping. Suppose that the self-mapping $T : X \longrightarrow X$ satisfying the following condition:*

$$d(x, T(x)) \preceq \varphi(x) - \varphi(T(x)), \tag{15}$$

for all $x \in X$. Then, T has a fixed point.

Proof. We provide the conditions of Theorem 1 to conclude that T has a fixed point. It is easily shown that $\{\psi \circ d : \psi \in B^*\}$ is a family of pseudometrics on X, and X will be a uniform space with the topology generated by these pseudometrics. By Lemma 9, the uniform space X is T_2-separated. Using lemma 10, X is a $d-$complete cone metric space, since X is a complete uniform space. By lemma 12, $\psi \circ \varphi$ is a lower semicontinuous mapping. Further, lemma 3 and assumption

$$d(x, T(x)) \preceq \varphi(x) - \varphi(T(x)) \tag{16}$$

imply that for each $\psi \in B^*$, $\psi(d(x, T(x))) \leq \psi(\varphi(x)) - \psi(\varphi(T(x)))$. By considering j as an identity mapping, all assumptions are satisfied. □

In Theorem 13, the regularity of the cone, which is an essential condition in [21] is omitted. So, our theorem is a real generalization of Theorem 8.

For example, we cannot even conclude from Theorem 8 that the identity mapping has a fixed point but it is possible by Theorem 8. The following example is presented in the support of the theorem 13.

Example 14. Consider the Banach space $\ell_\infty(R)$ with its cone $P = \{(x_n) \in \ell_\infty(R)\colon x_n \geq 0 \text{ for all } n \in N\}$. It is not difficult to see that $\ell_\infty(R)$ is complete and P is normal with nonempty interior. Let B be a subset of $\ell_\infty(R)$ consisting of all (x_n) which are nondecreasing and converging to 1 with $1/2 \leq x_n \leq 1$, for all $n \in N$. Define $d : B \times B \longrightarrow P$ as, $d((x_n), (y_n)) = (|x_1 - y_1|, \cdots, |x_n - y_n|, \cdots)$, for every $(x_n), (y_n) \in B$. It is not hard to check that (B, d) is a d-complete space. Now, define the mapping $T : B \longrightarrow B$ by $T((x_n)) = (x_n)$ and $\varphi : B \longrightarrow P$ is the inclusion mapping. It is clear that φ is cone lower semicontinuous and T satisfies $d(x, T(x)) \preceq \varphi(x) - \varphi(T(x))$, since $d(x, Tx) = d(x, x) = \theta$. But P is not regular because the sequence (a_n) that $a_n = (\underbrace{1, \cdots, 1}_{n}, 0, 0, \cdots)$, for each $n \in N$, is nondecreasing and bounded from above but it is not convergent.

Thus, one of the conditions of Theorem 8 is not satisfied, although $T(x) = x$, for all $x \in B$. But Theorem 13 implies that T has a fixed point.

Remark 15. In example 14, one of conditions of Theorem 8 is not satisfied, as P *is* not regular although $T(x) = x$, for all $x \in B$. In the example 14, all the conditions of the Theorem 13 are satisfied, and hence, the underlying mapping T has a fixed point. This shows that Theorem 13 is a real and proper generalization of Theorem 8.

The following example shows that the cone lower semicontinuity of φ is essential in Theorem 13 and may not be dropped.

Example 16. Let $X \subseteq \ell_\infty(R)$ be a family of the sequences $\bar{0} = \{0, 0, \cdots, 0, \cdots\}$, $\tilde{1} = \{1, 1, \cdots, 1, \cdots\}$, $\bar{1} = \{1/2, 1/3, \cdots, (1/m), \cdots\}$, $\bar{2} = \{(1/2)^2, (1/3)^2, \cdots, (1/m)^2, \cdots\}$, $\bar{n} = \{(1/2)^n, (1/3)^n, \cdots, (1/m)^n, \cdots\}$ and P is defined as same as the cone defined in Example 14 and the cone metric is

$$d(\bar{x}, \bar{y}) = \left\{\left|\frac{1}{2^{x+1}} - \frac{1}{2^{y+1}}\right|, \left|\frac{1}{3^{x+1}} - \frac{1}{3^{y+1}}\right|, \cdots, \left|\frac{1}{m^{x+1}} - \frac{1}{m^{y+1}}\right|, \cdots\right\}, \tag{17}$$

for every $\bar{x}, \bar{y} \in X$. Define the mapping $T : X \longrightarrow X$ and $\varphi : X \longrightarrow P$ in the following way:

$$T(\bar{n}) = \overline{n+1}, T(\bar{0}) = \tilde{1}, T(\tilde{1}) = \bar{0}, \tag{18}$$

and $\varphi = T$. Obviously $d(\bar{x}, T(\bar{x})) = \varphi(\bar{x}) - \varphi(T(\bar{x}))$ but φ is not cone lower semicontinuous map because $\lim \bar{n} = \bar{0}$ does imply $\tilde{1} = \varphi(\bar{0}) \npreceq \liminf \varphi(\bar{n}) = \bar{0}$, and therefore, one of conditions of Theorem 13 is not satisfied. It is clear that T has no fixed point because $T(\bar{n}) = \overline{n+1} \neq \bar{n}$.

In the next theorem, we give a short proof for a set-valued version of Caristi-Kirk's fixed point theorem in cone metric space. An element $x \in X$ is considered as a fixed point of set-valued mapping $f : X \twoheadrightarrow X$ if $x \in f(x)$.

Theorem 17. *Let (X, d) be a d -complete cone metric space, $\varphi : X \longrightarrow P$ be a cone lower semicontinuous mapping and there exists $y \in f(x)$ for a set-valued mapping $f : X \twoheadrightarrow X$ such that*

$$d(x, y) \preceq \varphi(x) - \varphi(y), \tag{19}$$

for each $x \in X$. Then, f has a fixed point.

Proof. By assumption, for each $x \in X$, the set $\{y \in f(x): d(x, y) \preceq \varphi(x) - \varphi(y)\}$ is nonempty. Using the axiom of choice, there is a single-valued mapping $T : X \longrightarrow X$ such that $d(x, T(x)) \preceq \varphi(x) - \varphi(T(x))$, for each $x \in X$. Theorem 13 is applied for T to find a fixed point x (say) of T. Since $T(x) \in f(x)$, we have $x \in f(x)$. □

Additionally, Khamsi (2010) proved the Theorem 18, which is the cone metric version of the Banach fixed point theorem. In Theorem 19, we improve it by removing the stronger condition of "normal cone."

Theorem 18 [16]. *Let (X, d) be a d -complete cone metric space over the Banach space $(E, \|\cdot\|)$ with the cone P which is normal. Suppose that for some $0 < \alpha < 1$, the mapping $T : X \longrightarrow X$ satisfies*

$$\|d(T(x), T(y))\| \leq \alpha \|d(x, y)\|, \tag{20}$$

for all $x, y \in X$. Then, T has a unique fixed point.

Theorem 19. *Let (X, d) be a d -complete cone metric space and for some $0 < \alpha < 1$, the mapping $T : X \longrightarrow X$ satisfying*

$$d(T(x), T(y)) \preceq \alpha d(x, y), \tag{21}$$

for all $x, y \in X$. Then, T has a unique fixed point.

Proof. We provide the conditions of Theorem 2 to conclude that T has a fixed point. We know that $\{\psi \circ d : \psi \in B^*\}$ is a family of pseudometrics on X, and X will be a uniform space with the topology generated by these pseudometrics. By lemma 9, this uniform space is T_2-separated. Using lemma 10, X is a complete uniform space since (X, d) is a $d-$ complete cone metric space. In addition, lemma 3 and assumption

$$d(T(x), T(y)) \preceq \alpha d(x, y), \tag{22}$$

imply that for each $\psi \in B^*$, $\psi(d(T(x), T(y))) \leq \alpha \psi(d(x, y))$. Thus, Theorem 2 implies that T has a fixed point. □

Remark 20. It is worth noting that Theorem 19 is a generalization of the Theorem 18. We used cone metric space with a nonnormal cone in our main results. Therefore, our theorems are the strict generalizations of the results which are proved in [16, 17, 21].

Theorem 21. *Let (X, d) be a d -complete cone metric space, and $f : B \twoheadrightarrow B$ be a set-valued mapping that for each $x, y \in X$ and $z \in f(x)$, there exists $w \in f(y)$ such that*

$$d(z, w) \preceq \alpha d(x, y). \tag{23}$$

Then, f has a fixed point.

Proof. It is direct consequence of the Theorem 19. □

In this article, we provided a brief proof for the Caristi-Kirk's fixed point result for single and set-valued mappings in cone metric spaces. Also, we partially addressed an open problem in which Caristi-Kirk's fixed point resulted in cone metric spaces. We improved the already existing results on Caristi-Kirk's fixed point in cone metric spaces by improving and removing the extra and strict conditions on the underlying spaces and mappings as well. Further, we provided a brief justification as a partial positive answer to this open problem using Caristi-Kirk's fixed point theorem on uniform space. We further provided a short proof in the cone metric version of the Banach fixed point theorem by using a short and comprehensive approach.

Data Availability

No data were used.

Conflicts of Interest

The authors declare no conflict of interest.

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