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Research Article

Novel Algorithms for Solving a System of Absolute Value Variational Inequalities

Safeera Batool,[1] **Muhammad Aslam Noor**,[2] **Riaz Ahmad**,[3] **Ilyas Khan**,[4] **and Mulugeta Andualem**[5]

[1]*Department of Basic Sciences, University of Engineering and Technology, Taxila, Pakistan*
[2]*Department of Mathematics, COMSATS University, Islamabad, Pakistan*
[3]*Faculty of Science, Yibin University, Yibin 644000, China*
[4]*Department of Mathematics, Faculty of Science in Zulfi, Majmaah University, 11952, Saudi Arabia*
[5]*Department of Mathematics, Bonga University, Bonga, Ethiopia*

Correspondence should be addressed to Safeera Batool; safeera.batool@uettaxila.edu.pk
and Mulugeta Andualem; mulugetaandualem4@gmail.com

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The goal of this paper is to study a new system of a class of variational inequalities termed as absolute value variational inequalities. Absolute value variational inequalities present a rational, pragmatic, and novel framework for investigating a wide range of equilibrium problems that arise in a variety of disciplines. We first develop a system of absolute value auxiliary variational inequalities to calculate the approximate solution of the system of absolute variational inequalities, and then by employing the projection technique, we prove the existence of solutions of the system of absolute value auxiliary variational inequalities. By utilizing an auxiliary principle and the existence result, we propose several new iterative algorithms for solving the system of absolute value auxiliary variational inequalities in the frame of four different operators. Furthermore, the convergence of the proposed algorithms is investigated in a thorough manner. The efficiency and supremacy of the proposed schemes is exhibited through some special cases of the system of absolute value variational inequalities and an illustrative example. The results presented in this paper are more general and rehash a number of some previously published findings in this field.

1. Introduction

The theory of variational inequalities, which was presented in the 1960s, exhibits an exceptional evolution as a fascinating and stimulating branch of applied mathematics that assumes a significant role in economics, finance, industry, transportation, optimization, and network analysis. Stampacchia [1] was the first to demonstrate the existence and uniqueness of variational inequality solutions. Variational inequalities have been utilized to examine problems that occurred in a variety of basic and applied sciences since their origin (see [2–6]). These significant applications prompted researchers to develop and broaden variational inequalities and associated optimization problems in various formations employing advanced and innovative methodologies, which include auxiliary principal technique, Wiener-Hopf equations, projection methods, and dynamical systems (see [7–10] and the references therein). It is noted that the operator must be Lipschitz continuous and strongly monotone for projection schemes to converge which is a very difficult set of requirements to verify. This fact led researchers to modify the projection method or to establish new ones. Extragradient-type methods address this difficulty as their convergence requires only the existence of solution and the Lipschitz continuity of the monotone operator. Various modified projection and extragradient-type algorithms have been proposed for finding the solution of variational inequalities. We would like to point out that the projection

technique is not appropriate for some variational inequality classes that include nonlinear functions which fail to be differentiable. These factors prompted us to employ the auxiliary principle technique, presented by Glowinski et al. [11]. They employed this method to investigate the existence of a mixed variational inequality solution. Adopting the fixed point approach, this strategy finds the auxiliary variational inequality and proves that the solution obtained from the auxiliary problem is the same as the solution of the underlying problem.

The system of variational inequalities is a natural and useful generalization of variational inequalities because it can be used to describe a variety of equilibrium problems, including traffic equilibrium, spatial equilibrium, the Nash equilibrium, and general equilibrium problems (see [12–16]). The emergence of this approach can be noticed as the simultaneous acquisition of two distinct figurations of research; that is, it validates the qualitative features of the solution of major types of problems, while also empowering us to build useful and effective new problem-solving strategies. Various iterative algorithms have been suggested to solve the different systems of variational inequalities. Agarwal et al. [17] considered a system of generalized nonlinear mixed quasi-variational inclusions and proved its associated sensitivity analysis. However, Pang [18] has shown that several equilibrium-type problems other than the Nash equilibrium problem may also be stated as a variational inequality problem that is equivalent to a system of variational inequalities. Thus, the variational inequality theory gives a natural, comprehensive, ordered, and effective framework for analyzing many linear and nonlinear problems.

In recent years, another remarkable extension of the variational inequalities known as absolute value variational inequalities is introduced and studied by Batool et al. [9]. They have shown that the absolute value variational inequalities can be transformed into a system of absolute value equations if the underlying domain is the entire space. The system of absolute value equations was proposed and analyzed by Mangasarian [19]. In fact, the system of absolute value equations has become an appealing direction for researchers as various mathematical and engineering problems including linear programs, quadratic programs, and bimatrix games can be reduced into an absolute system of equations (see [20–22] and the references therein). Inspired by the significant boost in this field, in this investigation, the approaches for constructing a novel system of absolute value variational inequalities in connection with the fixed point formulation were proposed with the help of the projection method. By equivalency, several new projection algorithms have been developed that are useful for solving the system of absolute value variational inequalities. Moreover, we examine the convergence of these algorithms under suitable constraints. Various special cases are also considered. A test example illustrates the graphical view of our proposed results. The suggested methods associate a variety of iterative algorithms in this direction. The findings in this study are more invigorating and can be viewed as an improvement and extension of the previously known results.

2. Results and Discussion

Let $\mathscr{H}$ be a real Hilbert space, whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let K_1 and K_2 be two closed and convex sets in $\mathscr{H}$. For given operators $T_1, T_2, B_1, B_2 : \mathscr{H} \longrightarrow \mathscr{H}$, consider the problem of finding $y \in K_1$ and $x \in K_2$ such that

$$\begin{cases} \langle T_1 x + B_1|x| - f_1, v - y\rangle \geq 0, & \forall v \in K_1, \\ \langle T_2 y + B_2|y| - f_2, v - x\rangle \geq 0, & \forall v \in K_2, \end{cases} \tag{1}$$

where f_1 and f_2 are the continuous functionals defined on $\mathscr{H}$ and $|.|$ contains the absolute values of components of $x, y \in \mathscr{H}$. The system (Equation (1)) is called a system of absolute value variational inequalities with four operators.

We will now discuss some special cases of the system of absolute value variational inequalities (Equation (1)).

(1) If $B_1 = B_2 = B$, then system (Equation (1)) reduces to find $y \in K_1$ and $x \in K_2$ such that

$$\begin{cases} \langle T_1 x + B|x| - f_1, v - y\rangle \geq 0, & \forall v \in K_1, \\ \langle T_2 y + B|y| - f_2, v - x\rangle \geq 0, & \forall v \in K_2, \end{cases} \tag{2}$$

which is called a system of absolute value variational inequalities with three operators.

(2) If $K_1 = K_2 = K$, then system (Equation (2)) reduces to find $x, y \in K$ such that

$$\begin{cases} \langle T_1 x + B|x| - f_1, v - y\rangle \geq 0, & \forall v \in K, \\ \langle T_2 y + B|y| - f_2, v - x\rangle \geq 0, & \forall v \in K, \end{cases} \tag{3}$$

which is a system of absolute value variational inequalities.

(3) If $B_1|x| = B_2|y| = 0, \forall x, y \in \mathscr{H}$, then system (Equation (1)) is equivalent to find $y \in K_1$ and $x \in K_2$ such that

$$\begin{cases} \langle T_1 x - f_1, v - y\rangle \geq 0, & \forall v \in K_1, \\ \langle T_2 y - f_2, v - x\rangle \geq 0, & \forall v \in K_2, \end{cases} \tag{4}$$

which is called the system of variational inequalities.

(4) If $T_1 = T_2 = T, \forall x, y \in \mathscr{H}$, then system (Equation (1)) is equivalent to find $y \in K_1$ and $x \in K_2$ such that

$$\begin{cases} \langle Tx + B_1|x| - f_1, v - y\rangle \geq 0, & \forall v \in K_1, \\ \langle Ty + B_2|y| - f_2, v - x\rangle \geq 0, & \forall v \in K_2, \end{cases} \tag{5}$$

which is a system of absolute value variational inequalities with three operators.

(5) If $T_1 = T_2 = T$ and $K_1 = K_2 = K$, then system (Equation (2)) reduces to find $x \in K$ such that

$$\langle Tx + B|x| - f, y - x\rangle \geq 0, \quad \forall y \in K \tag{6}$$

is called an absolute value variational inequality.

(6) If $B|x| = 0, \forall x \in \mathscr{H}$, then Equation (4) collapses to find $x \in K$ such that

$$\langle Tx - f, y - x\rangle \geq 0, \quad \forall y \in K, \tag{7}$$

which are well-known classical variational inequalities, introduced by Lions and Stampacchia [23, 24] and have been studied extensively in many directions. Variational inequalities are useful to formulate various equilibrium problems.

(7) If $K^* = \{x \in K : \langle x, y\rangle \geq 0, y \in K\}$ is the polar cone of the closed and convex cone K in $\mathscr{H}$, then Equation (4) is equivalent to find $x \in K$ such that

$$x \in K, Tx + B|x| - f \in K^*, \langle Tx - B|x|, x\rangle = 0, \tag{8}$$

which is an absolute value complementarity problem. The absolute value complementarity problem was introduced and studied by Noor et al. [25].

(8) If $B|x| = 0, \forall x \in \mathscr{H}$, then Equation (6) reduces to find $x \in K$ such that

$$x \in K, Tx - f \in K^*, \langle Tx - f, x\rangle = 0, \tag{9}$$

which is called a complementarity problem. The complementarity problem was introduced and studied by Lemke [5] and has also been investigated by Cottle and Dantzig [26].

(9) If $T_1 = T_2 = T$, $B_1 = B_2 = B$, and $K_1 = K_2 = \mathscr{H}$, then Equation (1) is equivalent to find $x \in \mathscr{H}$ such that

$$Tx + B|x| = f, \tag{10}$$

which is known as the system of absolute value equations and is addressed in Reference [27]. The system of absolute value equations is widely applied in various branches of engineering and mathematics. Hence, the proper choice of operators and spaces may generate several known and new types of absolute value variational inequalities and its variant forms.

In order to obtain the main results of this paper, some basic definitions and results are needed which are essential for the further analysis.

Definition 1. $T : \mathscr{H} \longrightarrow \mathscr{H}$ is said to be strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in \mathscr{H}. \tag{11}$$

Definition 2. An operator $T : \mathscr{H} \longrightarrow \mathscr{H}$ is said to be Lipschitz continuous, if there exists a constant $\beta > 0$ such that

$$\|Tx - Ty\| \leq \beta\|x - y\|, \quad \forall x, y \in \mathscr{H}. \tag{12}$$

Definition 3. An operator $T : \mathscr{H} \longrightarrow \mathscr{H}$ is said to be monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0, \quad \forall x, y \in \mathscr{H}. \tag{13}$$

Definition 4. An operator $T : \mathscr{H} \longrightarrow \mathscr{H}$ is said to be pseudo-monotone if

$$\langle Tx, y - x\rangle \geq 0 \tag{14}$$

implies

$$\langle Ty, y - x\rangle \geq 0, \quad \forall x, y \in \mathscr{H}. \tag{15}$$

We now consider the well-known projection lemma which is due to Reference [4]. The projection lemma transforms the variational inequalities into a fixed point problem.

Lemma 5 (see [4]). *Let K be a closed and convex set in $\mathscr{H}$. Then, for a given $z \in \mathscr{H}, x \in K$ satisfies*

$$\langle x - z, y - x\rangle \geq 0, \quad \forall y \in K, \tag{16}$$

if and only if

$$x = P_K[z], \tag{17}$$

where P_K is the projection of $\mathscr{H}$ onto a closed and convex set K in $\mathscr{H}$.

It is notable that the projection operator P_K is a nonexpansive operator, that is

$$\|P_K[x] - P_K[y]\| \leq \|x - y\|, \quad \forall x, y \in \mathscr{H}. \tag{18}$$

The above lemma is important to obtain the main results of this paper.

Lemma 6 (see [28]). *If $\{\delta_n\}_{n=0}^{\infty}$ is a nonnegative sequence satisfying the following inequality*

$$\delta_{n+1} \leq (1 - \lambda_n)\delta_n + \sigma_n, \quad \forall n \geq 0, \tag{19}$$

with $0 \leq \lambda_n \leq 1$, $\sum_{n=0}^{\infty}\lambda_n = \infty$, and $\sigma_n = 0(\lambda_n)$, then $\lim_{n\longrightarrow\infty}\delta_n = 0$.

Since the projection-type techniques could not be used to suggest iterative algorithms for mixed variational

inequalities, Glowinski et al. [11] suggested a new technique for solving the variational inequalities. It is called auxiliary principle technique which proved to be useful as it does not depend on the projection. Also, it is worth mentioning that unified descent algorithms for variational inequalities can be suggested by using an auxiliary principle technique.

Hence, Equation (1) can easily be written in an equivalent form by using the auxiliary principle technique, that is to find $y \in K_1$ and $x \in K_2$ such that

$$\begin{cases} \langle \gamma_1 T_1 x + \gamma_1 B_1 |x| - \gamma_1 f_1 + y - x, v - y \rangle \geq 0, & \forall v \in K_1, \\ \langle \gamma_2 T_2 y + \gamma_2 B_2 |y| - \gamma_2 f_2 + x - y, v - x \rangle \geq 0, & \forall v \in K_2, \end{cases} \tag{20}$$

where $\gamma_1 > 0$ and $\gamma_2 > 0$ are the constants.

We use this equivalent system to suggest some new iterative algorithms for solving the system of absolute value variational inequalities and its alternative systems.

2.1. Main Results. In this section, we establish the equivalence between system of absolute value Equation (20) and the fixed point problems. We use this equivalent formulation to suggest some iterative algorithms for solving the system of absolute value equations. The convergence analysis of the proposed methods is also demonstrated.

Lemma 7. *The system of absolute value variational inequalities (Equation (20)) has a solution $y \in K_1$ and $x \in K_2$ if and only if $y \in K_1$ and $x \in K_2$ satisfy the relations:*

$$y = P_{K_1}[x - \gamma_1 T_1 x - \gamma_1 B_1 |x| + \gamma_1 f_1], \tag{21}$$

$$x = P_{K_2}[y - \gamma_2 T_2 y - \gamma_2 B_2 |y| + \gamma_2 f_2], \tag{22}$$

where $\gamma_1 > 0$ and $\gamma_2 > 0$ are constants.

It is clear from Lemma 7 that the system (Equation (20)) is equivalent to the fixed point problems (Equations (21) and (22)). This equivalent formulation is very important from theoretical as well as from the numerical point of view (see [29]). We propose and analyze some iterative schemes by using the composition (Equations (21) and (22)).

Equations (21) and (22) can be rewritten in the following equivalent forms:

$$y = (1 - \eta_n) y + \eta_n P_{K_1}[x - \gamma_1 T_1 x - \gamma_1 B_1 |x| + \gamma_1 f_1], \tag{23}$$

$$x = (1 - \zeta_n) x + \zeta_n P_{K_2}[y - \gamma_2 T_2 y - \gamma_2 B_2 |y| + \gamma_2 f_2], \tag{24}$$

where $0 \leq \zeta_n, \eta_n \leq 1$ for all $n \geq 0$.

We use this equivalent formulation to suggest the following iterative algorithms for solving the system of absolute value variational inequalities (Equation (20)) and its related formations.

Algorithm 1. For given $y_0 \in K_2$ and $x_0 \in K_1$, compute x_{n+1} and y_{n+1} by the iterative schemes:

$$y_{n+1} = (1 - \eta_n) y_n + \eta_n P_{K_1}[x_n - \gamma_1 T_1 x_n - \gamma_1 B_1 |x_n| + \gamma_1 f]_1, \tag{25}$$

$$x_{n+1} = (1 - \zeta_n) x_n + \zeta_n P_{K_2}[y_n - \gamma_2 T_2 y_n - \gamma_2 B_2 |y_n| + \gamma_2 f_2], \tag{26}$$

where $0 \leq \zeta_n, \eta_n \leq 1$ for all $n \geq 0$.

Algorithm 1 is known as a parallel algorithm which can be considered as the Jacobi method for solving the system of absolute value equations. It is proved that parallel algorithms outperform the sequential schemes.

We now discuss some of the special cases of Algorithm 1.

(1) If $B_1 = B_2 = B$, then Algorithm 1 reduces to the following parallel algorithm to find the solution of the system (Equation (2))

Algorithm 2. For given $y_0 \in K_2$ and $x_0 \in K_1$, compute x_{n+1} and y_{n+1} by the iterative schemes:

$$\begin{aligned} y_{n+1} &= (1 - \eta_n) y_n + \eta_n P_{K_1}[x_n - \gamma_1 T_1 x_n - \gamma_1 B |x_n| + \gamma_1 f_1], \\ x_{n+1} &= (1 - \zeta_n) x_n + \zeta_n P_{K_2}[y_n - \gamma_2 T_2 y_n - \gamma_2 B |y_n| + \gamma_2 f_2], \end{aligned} \tag{27}$$

where $0 \leq \zeta_n, \eta_n \leq 1$ for all $n \geq 0$.

(2) If $K_1 = K_2 = K$, then Algorithm 2 reduces to the following projection algorithm to solve the system of absolute value variational inequalities (Equation (3))

Algorithm 3. For given $x_0, y_0 \in K$, compute x_{n+1} and y_{n+1} by the iterative schemes:

$$\begin{aligned} y_{n+1} &= (1 - \eta_n) y_n + \eta_n P_{K_1}[x_n - \gamma_1 T_1 x_n - \gamma_1 B |x_n| + \gamma_1 f_1], \\ x_{n+1} &= (1 - \zeta_n) x_n + \zeta_n P_{K_2}[y_n - \gamma_2 T_2 y_n - \gamma_2 B |y_n| + \gamma_2 f_2], \end{aligned} \tag{28}$$

where $0 \leq \zeta_n, \eta_n \leq 1$ for all $n \geq 0$.

(3) If $B_1 |x| = B_2 |y| = 0, \forall x, y \in \mathscr{H}$, then Algorithm 2 reduces to the following projection algorithm to solve the system of variational inequalities (Equation (4))

Algorithm 4. For given $y_0 \in K_2$ and $x_0 \in K_1$, compute x_{n+1} and y_{n+1} by the iterative schemes:

$$\begin{aligned} y_{n+1} &= (1-\eta_n)y_n + \eta_n P_{K_1}[x_n - \gamma_1 T_1 x_n + \gamma_1 f_1], \\ x_{n+1} &= (1-\zeta_n)x_n + \zeta_n P_{K_2}[y_n - \gamma_2 T_2 y_n + \gamma_2 f_2], \end{aligned} \tag{29}$$

where $0 \le \zeta_n, \eta_n \le 1$ for all $n \ge 0$.

(4) If $T_1 = T_2 = T$, then Algorithm 1 reduces to the following algorithm

Algorithm 5. For given $y_0 \in K_2$ and $x_0 \in K_1$, compute x_{n+1} and y_{n+1} by the iterative schemes:

$$\begin{aligned} y_{n+1} &= (1-\eta_n)y_n + \eta_n P_{K_1}[x_n - \gamma_1 T x_n - \gamma_1 B_1|x_n| + \gamma_1 f_1], \\ x_{n+1} &= (1-\zeta_n)x_n + \zeta_n P_{K_2}[y_n - \gamma_2 T y_n - \gamma_2 B_2|y_n| + \gamma_2 f_2], \end{aligned} \tag{30}$$

where $0 \le \zeta_n, \eta_n \le 1$ for all $n \ge 0$.

(5) If $T_1 = T_2 = T$ and $B_1 = B_2 = B$, then Algorithm 2 reduces to the following parallel algorithm

Algorithm 6. For given $y_0 \in K_2$ and $x_0 \in K_1$, compute x_{n+1} and y_{n+1} by the iterative schemes:

$$\begin{aligned} y_{n+1} &= (1-\eta_n)y_n + \eta_n P_{K_1}[x_n - \gamma_1 T x_n - \gamma_1 B|x_n| + \gamma_1 f_1], \\ x_{n+1} &= (1-\zeta_n)x_n + \zeta_n P_{K_2}[y_n - \gamma_2 T y_n - \gamma_2 B|y_n| + \gamma_2 f_2], \end{aligned} \tag{31}$$

where $0 \le \zeta_n, \eta_n \le 1$ for all $n \ge 0$.

Several new and known iterative schemes can be suggested for solving absolute value variational inequalities and the associated problems by making proper and appropriate choice for operators and spaces.

We now examine the convergence analysis of Algorithm 1 which is the key motivation of the next result.

Theorem 8. *Let the operators $T_1, T_2 : \mathscr{H} \longrightarrow \mathscr{H}$ be strongly monotone with constants $\alpha_{T_1} > 0, \alpha_{T_2} > 0$ and Lipschitz continuous with constants $\beta_{T_1} > 0, \beta_{T_2} > 0$ and the operators $B_1, B_2 : \mathscr{H} \longrightarrow \mathscr{H}$ be Lipschitz continuous with constants $\beta_{B_1} > 0, \beta_{B_2} > 0$, respectively. If the following conditions*

(1) $\phi_{T_1} = \sqrt{1 - 2\gamma_1\alpha_{T_1} + \gamma_1^2\beta_{T_1}^2} < 1$

(2) $\phi_{T_2} = \sqrt{1 - 2\gamma_2\alpha_{T_2} + \gamma_2^2\beta_{T_2}^2} < 1$

(3) $0 \le \zeta_n, \eta_n \le 1 \forall n \ge 0$,

$$\begin{aligned} \zeta_n - \eta_n\left(\phi_{T_1} + \phi_{B_1}\right) &\ge 0, \\ \eta_n - \zeta_n\left(\phi_{T_2} + \phi_{B_2}\right) &\ge 0, \end{aligned} \tag{32}$$

such that

$$\begin{aligned} \sum_{n=0}^{\infty}\left(\zeta_n - \eta_n\left(\phi_{T_1} + \phi_{B_1}\right)\right) &= \infty, \\ \sum_{n=0}^{\infty}\left(\eta_n - \zeta_n\left(\phi_{T_2} + \phi_{B_2}\right)\right) &= \infty, \end{aligned} \tag{33}$$

where

$$\begin{aligned} \phi_{T_1} &= \sqrt{1 - 2\gamma_1\alpha_{T_1} + \gamma_1^2\beta_{T_1}^2}, \\ \phi_{T_2} &= \sqrt{1 - 2\gamma_2\alpha_{T_2} + \gamma_2^2\beta_{T_2}^2}, \\ \phi_{B_1} &= \gamma_2\beta_{B_2}, \\ \phi_{B_2} &= \gamma_1\beta_{B_1}, \end{aligned} \tag{34}$$

hold, then sequences $\{x_n\}$ and $\{y_n\}$ obtained from Algorithm 1 converge to x and y, respectively.

Proof. Let $x, y \in \mathscr{H}$ such that $y \in K_1$ and $x \in K_2$ be a solution of the system (Equation (20)). Then, from Equation (23) and Equation (26), we have the following:

$$\begin{aligned} \|x_{n+1} - x\| &= \|(1-\zeta_n)x_n + \zeta_n P_{K_2}[y_n - \gamma_2 T_2 y_n - \gamma_2 B_2|y_n| + \gamma_2 f_2]\} - (1-\zeta_n) \\ &\quad \cdot x - \zeta_n P_{K_2}[y - \gamma_2 T_2 y - \gamma_2 B_2|y| + \gamma_2 f_2]\| \le (1-\zeta_n)\|x_n - x\| \\ &\quad + \zeta_n\|P_{K_2}[y_n - \gamma_2 T_2 y_n - \gamma_2 B_2|y_n| + \gamma_2 f_2] - P_{K_2}[y - \gamma_2 T_2 y - \gamma_2 B_2|y| \\ &\quad + \gamma_2 f_2]\| \le (1-\zeta_n)\|x_n - x\| + \zeta_n\|(y_n - y) - \gamma_2(T_2 y_n - T_2 y)\| \\ &\quad + \zeta_n\gamma_2\||B_2|y_n| - B_2|y|\|. \end{aligned} \tag{35}$$

□

Since the operator T_2 is strongly monotone and Lipschitz continuous with constants $\alpha_{T_2} > 0$ and $\beta_{T_2} > 0$, respectively, then we have the following:

$$\begin{aligned} &\|(y_n - y) - \gamma_2(T_2 y_n - T_2 y)\| \le \|y_n - y\|^2 \\ &\quad - 2\gamma_2\langle T_2 y_n - T_2 y, y_n - y\rangle + \|T_2 y_n - T_2 y\|^2 \\ &\le \sqrt{1 - 2\gamma_2\alpha_{T_2} + \gamma_2^2\beta_{T_2}^2}\|y_n - y\|^2. \end{aligned} \tag{36}$$

Also, using the Lipschitz continuity of the operator B_2 with constant $\beta_{B_2} > 0$, we have the following:

$$\||B_2|y_n| - B_2|y|\| \le \beta_{B_2}\|y_n - y\|. \tag{37}$$

Combining Equations (35), (36), and (37), we obtain the

following:

$$\begin{aligned}\|x_{n+1}-x\| &\le (1-\zeta_n)\|x_n-x\| \\ &\quad + \zeta_n\left(\sqrt{1-2\gamma_2\alpha_{T_2}+\gamma_2^2\beta_{T_2}^2}+\gamma_2\beta_{B_2}\right)\|y_n-y\| \\ &= (1-\zeta_n)\|x_n-x\| + \zeta_n\left(\phi_{T_2}+\phi_{B_2}\right)\|y_n-y\|.\end{aligned} \tag{38}$$

In a similar way, from Equation (23) and Equation (25), we have the following:

$$\begin{aligned}\|y_{n+1}-y\| &= \big\|(1-\eta_n)y_n+\eta_n P_{K_1}[x_n-\gamma_1T_1x_n-\gamma_1B_1|x_n| \\ &\quad +\gamma_1f_1]-(1-\eta_n)y-\eta_nP_{K_1}[x-\gamma_1T_1x-\gamma_1B_1|x| \\ &\quad +\gamma_1f_1]\big\| \le (1-\eta_n)\|y_n-y\|+\eta_n\big\|P_{K_1}[x_n-\gamma_1T_1x_n \\ &\quad -\gamma_1B_1|x_n|+\gamma_1f_1]-P_{K_1}[x-\gamma_1T_1x-\gamma_1B_1|x| \\ &\quad +\gamma_1f_1]\big\| \le (1-\eta_n)\|y_n-y\|+\eta_n\|(x_n-x) \\ &\quad -\gamma_1(T_1x_n-T_1x)\|+\eta_n\gamma_1\||B_1|x_n|-B_1|x|\| \\ &\le (1-\eta_n)\|y_n-y\|+\eta_n\sqrt{1-2\gamma_1\alpha_{T_1}+\gamma_1^2\beta_{T_1}^2}\|x_n \\ &\quad -x\|+\eta_n\gamma_1\beta_{B_1}\|x_n-x\| = (1-\eta_n)\|y_n-y\|+\eta_n \\ &\quad \cdot\left(\sqrt{1-2\gamma_1\alpha_{T_1}+\gamma_1^2\beta_{T_1}^2}+\gamma_1\beta_{B_1}\right)\|x_n-x\| \\ &= (1-\eta_n)\|y_n-y\|+\eta_n\left(\phi_{T_1}+\phi_{B_1}\right)\|x_n-x\|,\end{aligned} \tag{39}$$

where we have used the strong monotonicity of T_1 with constant $\alpha_{T_1}>0$ and Lipschitz continuity of the operators T_1 and B_1 with constants $\beta_{T_1}>0$ and $\beta_{B_1}>0$, respectively.

Adding Equations (38) and (39), we have the following:

$$\begin{aligned}&\|x_{n+1}-x\|+\|y_{n+1}-y\| \\ &\le \left(1-\zeta_n+\eta_n\left(\phi_{T_1}+\phi_{B_1}\right)\|x_n-x\|+\left(1-\eta_n+\zeta_n\left(\phi_{T_2}+\phi_{B_2}\right)\right)\right. \\ &\quad \cdot\|y_n-y\| \le \max\left\{\left(1-\zeta_n+\eta_n\left(\phi_{T_1}+\phi_{B_1}\right)\right),\left(1-\eta_n+\zeta_n\left(\phi_{T_2}+\phi_{B_2}\right)\right)\right\} \\ &\quad \cdot(\|x_n-x\|+\|y_n-y\|) = \max(\xi_1,\xi_2)(\|x_n-x\|+\|y_n-y\|) \\ &= \phi(\|x_n-x\|+\|y_n-y\|),\end{aligned} \tag{40}$$

where

$$\begin{aligned}\phi &= \max(\xi_1,\xi_2), \\ \xi_1 &= 1-\zeta_n+\eta_n\left(\phi_{T_1}+\phi_{B_1}\right), \\ \xi_2 &= 1-\eta_n+\zeta_n\left(\phi_{T_2}+\phi_{B_2}\right).\end{aligned} \tag{41}$$

From assumption (iii), it follows that $\phi<1$. Hence, using Lemma 6, we obtain from Equation (44) the following:

$$\lim_{n\longrightarrow\infty}[\|x_n-x\|+\|y_n-y\|]=0. \tag{42}$$

This further implies that

$$\begin{aligned}\lim_{n\longrightarrow\infty}\|x_n-x\| &= 0, \\ \lim_{n\longrightarrow\infty}\|y_n-y\| &= 0,\end{aligned} \tag{43}$$

which is the required result.

We now propose and examine some new iterative schemes for solving system of absolute value variational inequalities, by employing a useful substitution.

It can easily be shown, by using Lemma 5, that $x,y\in\mathscr{H}$ such that $y\in K_1$ and $x\in K_2$ is a solution of the system of absolute value variational inequalities (Equation (1)), if and only if $x,y\in\mathscr{H}: y\in K_1, x\in K_2$ satisfies the following:

$$y=P_{K_1}[z], \tag{44}$$

$$x=P_{K_2}[w], \tag{45}$$

$$z=x-\gamma_1T_1x-\gamma_1B_1|x|+\gamma_1f_1, \tag{46}$$

$$w=y-\gamma_2T_2y-\gamma_2B_2|y|+\gamma_2f_2. \tag{47}$$

By using this alternative formation, we can propose and examine the following iterative schemes to solve the system (Equation (1)).

Algorithm 7. For given $y_0\in K_1$ and $x_0\in K_2$, find x_{n+1} and y_{n+1} by the iterative schemes:

$$y_{n+1}=(1-\eta_n)y_n+\eta_nP_{K_1}[z_n], \tag{48}$$

$$x_{n+1}=(1-\zeta_n)x_n+\zeta_nP_{K_2}[w_n], \tag{49}$$

$$z_n=x_n-\gamma_1T_1x_n-\gamma_1B_1|x_n|+\gamma_1f_1, \tag{50}$$

$$w_n=y_n-\gamma_2T_2y_n-\gamma_2B_2|y_n|+\gamma_2f_2, \tag{51}$$

where $0\le\zeta_n,\eta_n\le1$ for all $n\ge0$.

By choosing the useful operators and proper spaces, one can have various new as well as known iterative schemes for solving the system of absolute value variational inequalities and its variant forms. Now, we examine the convergence analysis of Algorithm 7 by employing the approach of Theorem 8.

Theorem 9. *Let the operators $T_1,T_2:\mathscr{H}\longrightarrow\mathscr{H}$ be strongly monotone with constants $\alpha_{T_1}>0,\alpha_{T_2}>0$ and Lipschitz continuous with constants $\beta_{T_1}>0,\beta_{T_2}>0$ and the operators $B_1,B_2:\mathscr{H}\longrightarrow\mathscr{H}$ be Lipschitz continuous with constants $\beta_{B_1}>0,\beta_{B_2}>0$, respectively. If the following conditions*

(i) $\phi_{T_1}=\sqrt{1-2\gamma_1\alpha_{T_1}+\gamma_1^2\beta_{T_1}^2}<1$

(ii) $\phi_{T_2}=\sqrt{1-2\gamma_2\alpha_{T_2}+\gamma_2^2\beta_{T_2}^2}<1$

(iii) $0\le\zeta_n,\eta_n\le1\forall n\ge0$

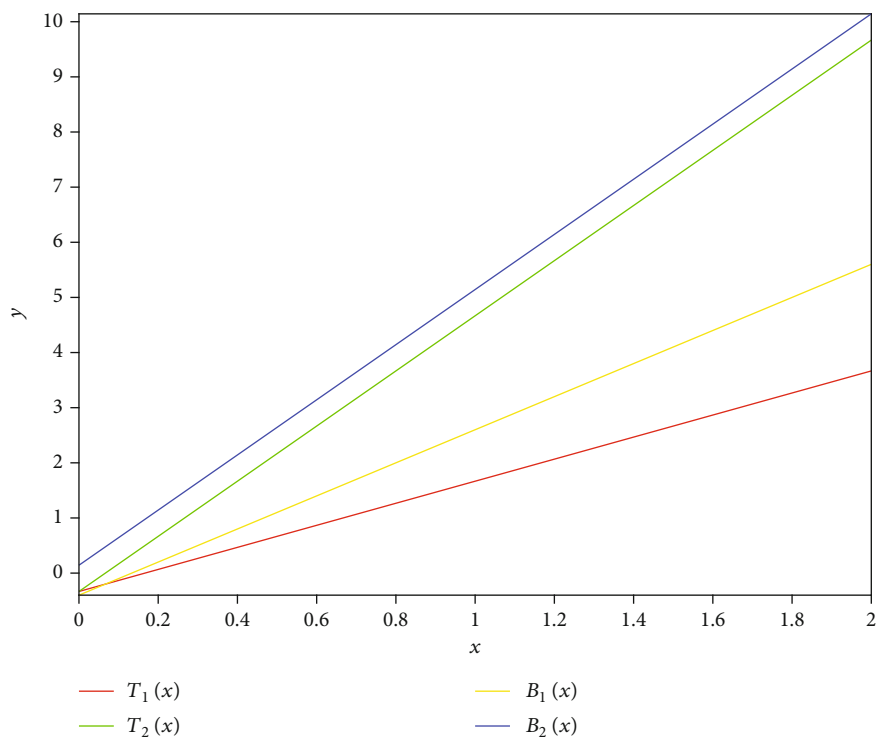

FIGURE 1: Graphical behaviour of $T_1(x)$, $T_2(x)$, $B_1(x)$, and $B_2(x)$.

$$\begin{aligned} \zeta_n - \eta_n\left(\phi_{T_1} + \phi_{B_1}\right) &\geq 0, \\ \eta_n - \zeta_n\left(\phi_{T_2} + \phi_{B_2}\right) &\geq 0, \end{aligned} \tag{52}$$

such that

$$\begin{aligned} \sum_{n=0}^{\infty}\left(\zeta_n - \eta_n\left(\phi_{T_1} + \phi_{B_1}\right)\right) &= \infty, \\ \sum_{n=0}^{\infty}\left(\eta_n - \zeta_n\left(\phi_{T_2} + \phi_{B_2}\right)\right) &= \infty, \end{aligned} \tag{53}$$

where

$$\begin{aligned} \phi_{T_1} &= \sqrt{1 - 2\gamma_1\alpha_{T_1} + \gamma_1^2\beta_{T_1}^2}, \\ \phi_{T_2} &= \sqrt{1 - 2\gamma_2\alpha_{T_2} + \gamma_2^2\beta_{T_2}^2}, \\ \phi_{B_1} &= \gamma_2\beta_{B_2}, \\ \phi_{B_2} &= \gamma_1\beta_{B_1}, \end{aligned} \tag{54}$$

hold, then sequences $\{x_n\}$ and $\{y_n\}$ obtained from Algorithm 7 converge to x and y, respectively.

Proof. Let $x, y \in \mathscr{H}$ such that $y \in K_1$ and $x \in K_2$ be a solution of the system (Equation (20)). Then, from Equation (45) and Equation (49), we have the following:

$$\begin{aligned} \|x_{n+1} - x\| &= \left\|(1-\zeta_n)x_n + \zeta_n P_{K_2}[w_n] - (1-\zeta_n)x - \zeta_n P_{K_2}[w]\right\| \\ &\leq (1-\zeta_n)\|x_n - x\| + \zeta_n\left\|P_{K_2}[w_n] - P_{K_2}[w]\right. \\ &\leq (1-\zeta_n)\|x_n - x\| + \zeta_n\|w_n - w\|. \end{aligned} \tag{55}$$

□

In a similar way, from Equation (44) and Equation (48), we have the following:

$$\begin{aligned} \|y_{n+1} - y\| &= \left\|(1-\eta_n)y_n + \eta_n P_{K_1}[z_n] - (1-\eta_n)y - \eta_n P_{K_1}[z]\right\| \\ &\leq (1-\eta_n)\|y_n - y\| + \eta_n\left\|P_{K_1}[z_n] - P_{K_1}[z]\right. \\ &\leq (1-\eta_n)\|y_n - y\| + \eta_n\|z_n - z\|. \end{aligned} \tag{56}$$

From Equations (36), (37), (47), and (51), we have the following:

$$\begin{aligned} \|w_n - w\| &= \|(y_n - y) - \gamma_2(T_2 y_n - T_2 y)\| + \|B_2|y_n| - B_2|y|\| \\ &\leq \left(\phi_{T_2} + \phi_{B_2}\right)\|y_n - y\|. \end{aligned} \tag{57}$$

Also, from Equations (39), (46), and (50), we have the following:

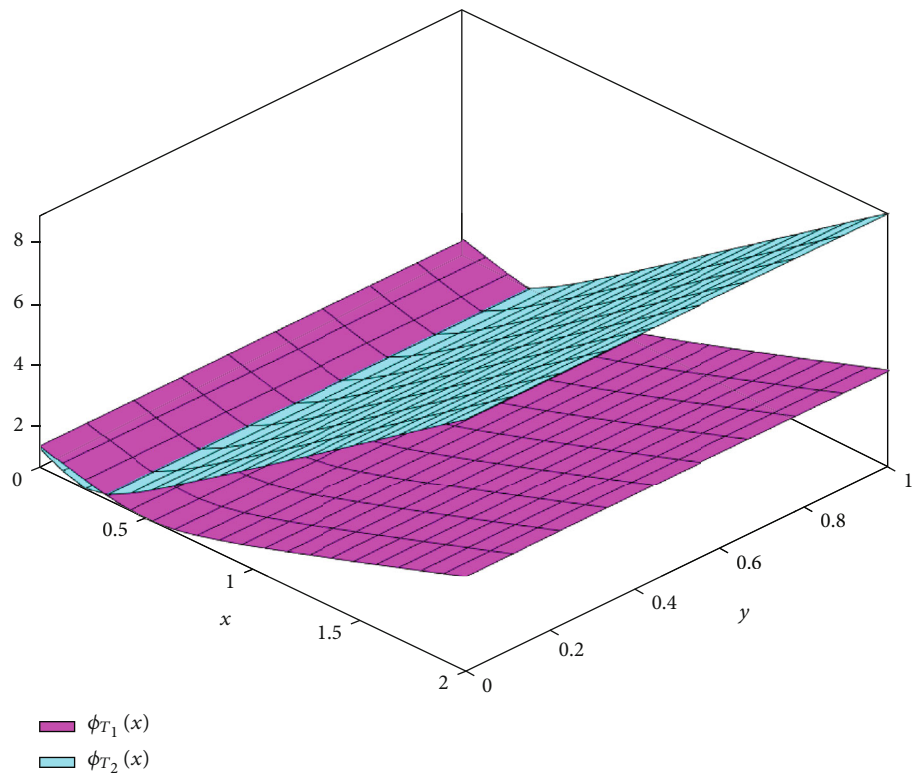

Figure 2: Behaviour of Theorem 8 via a three-dimensional plot satisfies Example 10.

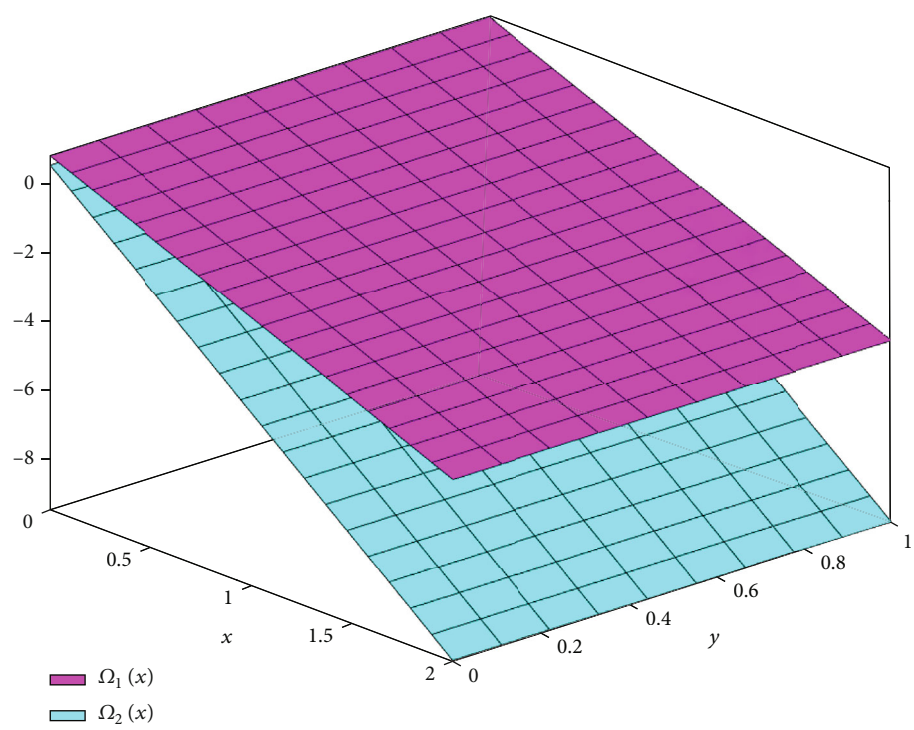

Figure 3: Behaviour of Theorem 9 via a three-dimensional plot satisfies Example 10 approach. The equivalence between the system of absolute value variational inequalities and the system of fixed point problems has been established. Further, we suggested several innovative iterative algorithms for solving our considered system of variational inequalities and the convergence of the proposed schemes has been analyzed under some suitable conditions. Finally, to demonstrate the existence and convergence results, a numerical example was given. We also discussed some special cases of the system of absolute value variational inequalities. The concept and technique of this paper may encourage researchers to analyze the innovative and unique applications of the system of absolute value variational inequalities and its associated optimization problems. In futuristic research, we extend this study to exponential absolute value variational inequalities and their variant forms.

$$\begin{aligned}\|z_n - z\| &= \|(x_n - x) - \gamma_1(T_1x_n - T_1x)\| + \||B_1|x_n| - B_1|x|\| \\ &\le \left(\phi_{T_1} + \phi_{B_1}\right)\|x_n - x\|.\end{aligned} \tag{58}$$

Combining Equations (55), (56), (57), and (58), we have the following:

$$\|x_{n+1} - x\| \le (1 - \zeta_n)\|x_n - x\| + \zeta_n\left(\phi_{T_2} + \phi_{B_2}\right)\|y_n - y\|, \tag{59}$$

$$\|y_{n+1} - y\| \le (1 - \eta_n)\|y_n - y\| + \eta_n\left(\phi_{T_1} + \phi_{B_1}\right)\|x_n - x\|. \tag{60}$$

Addition of Equations (59) and (60) implies

$$\begin{aligned}\|x_{n+1} - x\| + \|y_{n+1} - y\| &\le \left(1 - \zeta_n + \eta_n\left(\phi_{T_1} + \phi_{B_1}\right)\right. \\ &\cdot \|x_n - x\| + \left(1 - \eta_n + \zeta_n\left(\phi_{T_2} + \phi_{B_2}\right)\right)\|y_n - y\| \\ &\le \max\left\{\left(1 - \zeta_n + \eta_n\left(\phi_{T_1} + \phi_{B_1}\right)\right), \left(1 - \eta_n + \zeta_n\left(\phi_{T_2} + \phi_{B_2}\right)\right)\right\} \\ &\cdot (\|x_n - x\| + \|y_n - y\|) = \max(\xi_1, \xi_2)(\|x_n - x\| + \|y_n - y\|) \\ &= \phi(\|x_n - x\| + \|y_n - y\|),\end{aligned} \tag{61}$$

where

$$\begin{aligned}\phi &= \max(\xi_1, \xi_2), \\ \xi_1 &= 1 - \zeta_n + \eta_n\left(\phi_{T_1} + \phi_{B_1}\right), \\ \xi_2 &= 1 - \eta_n + \zeta_n\left(\phi_{T_2} + \phi_{B_2}\right).\end{aligned} \tag{62}$$

From assumption (iii), it follows that $\phi < 1$. Hence, using Lemma 6, we obtain from Equation (23) the following:

$$\lim_{n\longrightarrow\infty}[\|x_n - x\| + \|y_n - y\|] = 0. \tag{63}$$

This further implies that

$$\begin{aligned}\lim_{n\longrightarrow\infty}\|x_n - x\| &= 0, \\ \lim_{n\longrightarrow\infty}\|y_n - y\| &= 0,\end{aligned} \tag{64}$$

which is the required result.

Example 10. $\mathscr{H} = \mathbb{R}, K_1 = (-\infty, 0]$ and $K_2 = [0, \infty)$.. Let T_1, $T_2 : \mathscr{H} \longrightarrow \mathscr{H}$ be the single-valued mappings defined by the following:

$$T_1(x) = \frac{2x - 1}{3}, T_2(x) = \frac{5x - 2}{6}, \quad \forall x \in \mathscr{H}. \tag{65}$$

Also, the mappings $B_1, B_2 : \mathscr{H} \longrightarrow \mathscr{H}$ are defined by the following:

$$B_1(x) = \frac{3x - 2}{5}, B_2(x) = \frac{5x + 1}{7}, \quad \forall x \in \mathscr{H}. \tag{66}$$

Then, it can easily be verified that for each $i = 1, 2$, T_i is strongly monotone and Lipschitz continuous with $\alpha_{T_1} = 2/3 = \beta_{T_1}$ and $\alpha_{T_2} = 5/6 = \beta_{T_2}$, and B_i is strongly monotone and Lipschitz continuous with $\alpha_{B_1} = 3/5 = \beta_{B_1}$ and $\alpha_{B_2} = 5/7 = \beta_{B_2}$.

Then, for $\gamma_1, \gamma_2 = 1$,

$$\begin{aligned}\phi_{T_1} &= \sqrt{1 - 2\gamma_1\alpha_{T_1} + \gamma_1^2\beta_{T_1}^2} = \sqrt{1 - 2\left(\frac{2}{3}\right) + \frac{2^2}{3}} = \frac{2}{3} < 1, \\ \phi_{T_2} &= \sqrt{1 - 2\gamma_2\alpha_{T_2} + \gamma_2^2\beta_{T_2}^2} = \sqrt{1 - 2\left(\frac{5}{6}\right) + \frac{5^2}{6}} = \frac{1}{6} < 1.\end{aligned} \tag{67}$$

Also, for each $n = 1, 2$,

$$\begin{aligned}0 &< \frac{1}{2} = \zeta_n, \\ \eta_n &= \frac{1}{2} < 1,\end{aligned} \tag{68}$$

we have the following:

$$\begin{aligned}\Omega_1 &= \zeta_n - \eta_n\left(\phi_{T_1} + \phi_{B_1}\right) = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{3} + \frac{3}{5}\right) = \frac{1}{30} > 0, \\ \Omega_2 &= \eta_n - \zeta_n\left(\phi_{T_2} + \phi_{B_2}\right) = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{6} + \frac{5}{7}\right) = \frac{5}{84} > 0,\end{aligned} \tag{69}$$

where $\phi_{B_1} = \gamma_1\beta_{B_1} = 3/5$ and $\phi_{B_2} = \gamma_2\beta_{B_2} = 5/7$. Clearly, we see that all the assumptions of Theorem 8 and Theorem 9 are satisfied. Hence, by using Algorithm 1 and Algorithm 7, the conclusions of Theorem 8 and Theorem 9 follow.

Figure 1 is the graphical representation of the operators defined in Example 10. Figure 2 depicts the behaviour of Theorem 8 satisfying Example 10. Similarly, Figure 3 interprets the behaviour of Theorem 9 via a three-dimensional plot satisfying Example 10.

3. Conclusion

In this paper, we have introduced a new system of variational inequalities, called the system of absolute value variational inequalities. To determine the approximate solution of the system of absolute value variational inequalities, we first built a system of absolute value auxiliary variational inequalities. We demonstrate the existence of a solution to the system of absolute value auxiliary variational inequalities using the projection.

Data Availability

The (graphs and an example) data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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