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*Research Article*

# Some Fixed Point Theorems for Boyd and Wong Type Contraction Mapping in Ordered Partial Metric Spaces with an Application

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This paper establishes and proves a fixed point theorem for Boyd and Wong type contraction in ordered partial metric spaces. In doing so, we have extended several existing results into ordered complete partial metric spaces. An illustrative example is given to demonstrate the validity of our results. Finally, the existence of the solution of nonlinear integral equation is discussed as an application of the main result.

## 1. Introduction and Preliminaries

Banach's fixed point theorem [1] has been extensively studied to solve the problems in nonlinear analysis since many years. This theorem provides the existence and uniqueness of the solution. It states that, if $(M, \varrho)$ is a complete metric space and $T: M \longrightarrow M$ is a self-contractive mapping, then $T$ has a unique fixed point $u \in M$. Due to its usefulness and applications, this theorem has been massively investigated and generalized by different researchers. In 1969, Boyd and Wong [2] gave an important generalization of the Banach fixed point theorem by the application of control function in the Banach contraction condition. Boyd and Wong [2] took into account the condition as follows:

$$\varrho(Tu, Tv) \leq \chi(\varrho(u, v)), \quad \forall u, v \in M, \tag{1}$$

whereby $(M, \varrho)$ is a complete metric space, and a mapping $\chi: [0, \infty) \longrightarrow [0, \infty)$ is upper semicontinuous from the right on $[0, \infty)$ such that $\chi(t) < t$, $\forall t > 0$. Consequently, $T$ has a unique fixed point $z \in M$ and $\varrho(T^n u, z) \longrightarrow 0$ as $n \longrightarrow \infty$, $\forall u \in M$.

Imdad and Kumar [3] extended the existing results by relaxing "continuity" and lightening the "commutativity" requirement besides increasing the number of involved maps from "two" to "four." Several other researchers extended these results in different directions. Some of them are [4–6] and the references therein. A beautiful survey of the fixed point theory was given by Kumar [7]. Naziku and Kumar [8, 9] proved results using the Boyd and Wong type contractive condition. Recently, Kumar [10] established and proved a fixed point theorem for Boyd and Wong type contraction for a pair of maps in complete metric spaces. This theorem gives conditions for a pair of mappings that possess a fixed point but not continuous at the fixed point and can be applied for both continuous and discontinuous mappings.

In the last few decades, fixed point results in a partially ordered set have been revealed as a very important area of interest to many researchers. In particular, the existence of the fixed point in partially ordered sets has been massively considered in [11–16] and others as they appear in the literature.

In literature [17], Matthews introduced the study of partial metric spaces as the important subject in the approach of formalizing the meaning of programming languages by formulating mathematical objects called "denotations." Partial metric was introduced to ensure that partial order semantics should have a metric-based tools for program verification in which the notion of size of data object in a domain is used in quantifying how data object is well defined in the domain.

*Definition 1* (see [18]). Let $M$ be a nonempty set. A function $p: M \times M \longrightarrow [0, \infty)$ is called a partial metric on $M$ if it satisfies the followings:

(PM0): $0 \leq p(u, u) \leq p(u, v)$ (non-negativity and small self distance).

(PM1): $p(u, v) = p(u, u) = p(v, v) \Rightarrow u = v$ (indistancy implies equality).

(PM2): $p(u, v) = p(v, u)$ (symmetric).

(PM3): $p(u, v) + p(z, z) \leq p(u, z) + p(z, v)$ (triangularity), for all $u, v, z \in M$.

$(M, p)$ is called a partial metric space.

Note that $p(u, v) = 0$ implies $u = v$ (by PM0 through PM2); the converse is always not true. Therefore, a metric space is a partial metric space with all self distances zero.

Several researchers generalized the results of metric fixed point theory using partial metric space setting in different directions. Some of them are [19–22] and the references therein.

Now, we will recall some definitions and lemmas which will be utilized in the proof of main results of this paper.

*Definition 2* (see [18]). Let $\{u_n\}$ be a sequence in a partial metric space $(M, p)$; then,

(i) A sequence $u_n \longrightarrow u \in M$ if and only if $p(u, u) = \lim_{n \longrightarrow \infty} p(u, u_n) = \lim_{n \longrightarrow \infty} p(u_n, u_n)$.

(ii) A sequence $\{u_n\}$ is called a Cauchy sequence if there exists $\epsilon > 0$ such that for all $n, m > N$, we have $p(u_n, u_m) < \epsilon$ for some integers $N \geq 0$; that is $\lim_{n,m \longrightarrow +\infty} p(u_n, u_m)$ exists and it is finite.

(iii) A partial metric space $(M, p)$ is complete if every Cauchy sequence $\{u_n\}$ converges to a point $u \in M$ such that $p(u, u) = \lim_{n,m \longrightarrow +\infty} p(u_n, u_m)$.

*Definition 3* (see [18]). A contraction on a partial metric space $M$ is a function $f: M \longrightarrow M$ such that there exist a constant $0 \leq k < 1$ for all $u, u \in M$ satisfies that

$$p(f(u), f(v)) \leq k \times p(u, v). \tag{2}$$

*Definition 4* (see [17]). Let $M$ be a nonempty set. Partial ordering is a relation $\ll \subseteq M^2$ such that

(PO 1) for all $u \in M$, $u \ll u$ (reflexive).

(PO 2) for all $u, v \in M$, $u \ll v$, and $v \ll u \Rightarrow u = v$ (antisymmetric).

(PO 3) for all $u, v, z \in M$, $u \ll v$, and $v \ll z \Rightarrow u \ll z$ (transitivity).

*Definition 5* (see [17]). For each partial metric space $p: M \times M \longrightarrow [0, \infty)$, $\ll_p \subseteq M^2$ is a binary relation such that for all $u, v \in M$, $u \ll_p v \Leftrightarrow p(u, u) = p(u, v)$.

Note that for each partial metric space $p$, $\ll_p$ is a partial ordering.

*Definition 6* (see [23]). Let $(M, \varrho)$ be a metric space. Let a mapping $T: M \longrightarrow M$ to be injective (one to one) and continuous (ICS) mapping with the property that if $\{Tu_n\}$ is convergent, then the sequence $\{u_n\}$ is also convergent for all sequences $\{u_n\} \in M$.

*Definition 7* (see [23]). Let $\Phi$ be the set of functions $\phi: [0, \infty) \longrightarrow [0, \infty)$ satisfying,

(i) $\phi(t) < t$ for all $t > 0$.

(ii) $\phi$ is an upper semicontinuous from right; that is, for any sequence $\{t_n\} \in [0, \infty)$ such that $t_n \longrightarrow t$ as $n \longrightarrow \infty$ as $t_n > t$, we have $\limsup_{n \longrightarrow \infty} \phi(t_n) \leq \phi(t)$.

Aydi and Karapinar [24] generalized results of Harjani et al. [11] and Luong and Thun [13] by using an ICS mapping and involved Boyd and Wong type contractive condition and provided the following theorem:

**Theorem 1** (see [23]). *Let $(M, \ll)$ be a partially ordered set. Suppose there exists a metric $\varrho$ such that $(M, \varrho)$ is a complete metric space. Let $f, T: M \longrightarrow M$ be a mapping such that $T$ is an ICS mapping and $f$ is a nondecreasing mapping satisfying, $\varrho(Tfu, Tfv) \leq \phi(N(u, v))$ for all $u, v \in M$ with $u \leq v$ where $\phi \in \Phi$ and*

$$N(u, v) = \max\left\{\frac{\varrho(Tu, Tfu)\varrho(Tv, Tfv)}{\varrho(Tu, Tv)}, \varrho(Tu, Tv)\right\}. \tag{3}$$

Also assume that either

(i) $f$ is continuous, or

(ii) If the sequence $\{u_n\}$ is nondecreasing in $M$ such that $u_n \longrightarrow u$, then $u = \sup\{u_n\}$.

If there exists a point $u_0 \in M$ such that $u_0 \leq fu_0$, then $f$ has a unique fixed point.

In the next section, the letter $\mathbb{N}$ will be used to refer to the set of all positive integer numbers.

## 2. Main Results

We now present an extension of Definition 6 in partial metric spaces.

*Definition 8.* Let $(M, p)$ be a partial metric space. Let a mapping $T: M \longrightarrow M$ to be injective (one to one) and continuous (ICS) mapping with the property that if $\{Tu_n\}$ is convergent, then $\{u_n\}$ is also convergent for all sequences $\{u_n\} \in M$.

Corresponding to Theorem 1, we state and prove our main results and then provide an illustrative example to demonstrate our results.

**Theorem 2.** *Let $(M, \leq)$ be a partially ordered set (Poset). Let $p$ be a partial metric such that $(M, p)$ is a complete partial metric space. Also, let $T: M \longrightarrow M$ be an ICS mapping, and $f: M \longrightarrow M$ be a nondecreasing mapping satisfying*

$$p(Tfu, Tfv) \leq \phi(\lambda(u, v)), \tag{4}$$

*with $u \leq v$ for all distinct $u, v \in m$, where $\phi \in \Phi$ and*

$$\lambda(u,v) = \max\left\{\frac{p(Tu,Tfu)p(Tv,Tfv)}{p(Tu,Tv)}, p(Tu,Tv)\right\}. \quad (5)$$

Furthermore, we assume that either

(i) a mapping $f$ is continuous, or

(ii) If a sequence $\{u_n\} \in M$ is a nondecreasing sequence such that $u_n \longrightarrow u$ as $n \longrightarrow \infty$, then $u = \sup_n\{u_n\}$.

Therefore, if there exists $u_0 \in M$ such that $u_0 \leq fu_0$, then $f$ has a unique fixed point $u_0$ with $p(u_0,u_0) = 0$.

Proof. Let a point $u_0 \in M$ such that $u_1 = fu_0$. We define a sequence $\{u_n\} \in M$ as

$$u_n = fu_{n-1}, \quad (6)$$

for all integers $n \geq 1$. Since $f$ is a nondecreasing mapping and $(M, \leq)$ is a Poset, then we can have

$$u_0 \leq u_1 = fu_0 \leq u_2 = fu_1 \leq \cdots \leq u_{n-1} \leq u_n = fu_{n-1} \leq \cdots, \quad (7)$$

for all integers $n \geq 1$. By induction, we can have that

$$u_0 \leq u_1 \leq u_2 \leq \cdots \leq u_{n-1} \leq u_n \leq u_{n+1} \leq \cdots. \quad (8)$$

If we suppose that there exists $n \in \mathbb{N}$ such that $u_n = u_{n+1} = fu_n$, then $f$ has a fixed point $u_n$, which ends the proof.

Now suppose that $u_n \neq u_{n+1}$ for all integers $n \in \mathbb{N}$, then (8) becomes

$$u_0 < u_1 < u_2 < \cdots < u_{(n-1)} < u_n < u_{(n+1)} < \cdots. \quad (9)$$

From (4), we can have that

$$\begin{aligned} p(Tu_n,Tu_{n+1}) &= p(Tfu_{n-1},Tfu_n) \\ &\leq \phi(\lambda(u_{n-1},u_n)), \end{aligned} \quad (10)$$

where

$$\begin{aligned} \lambda(u_{n-1},u_n) &= \max\left\{\frac{p(Tu_{n-1},Tfu_{n-1})p(Tu_n,Tfu_n)}{p(Tu_{n-1},Tu_n)}, p(Tu_{n-1},Tu_n)\right\} \\ &= \max\{p(Tu_n,Tu_{n+1}), p(Tu_{n-1},Tu_n)\}. \end{aligned} \quad (11)$$

Suppose that $\lambda(u_{n-1},u_n) = p(Tu_n,Tu_{n+1})$ for some integers $n \geq 1$, then (10) becomes,

$$p(Tu_n,Tu_{n+1}) \leq \phi(p(Tu_n,Tu_{n+1})) \quad (12)$$

From Definition 7 (i), we see that (12) becomes

$$p(Tu_n,Tu_{n+1}) < p(Tu_n,Tu_{n+1}), \quad (13)$$

which is a contradiction. Hence, $\lambda(u_{n-1},u_n) = p(Tu_{n-1},Tu_n)$ for all integers $n \geq 1$; hence (10) becomes

$$\begin{aligned} p(Tu_n,Tu_{n+1}) &= p(Tfu_{n-1},Tfu_n) \\ &\leq \phi(\lambda(u_{n-1},u_n)) \\ &= \phi(p(Tu_{n-1},Tu_n)) \\ &< p(Tu_{n-1},Tu_n). \end{aligned} \quad (14)$$

Therefore, from the above equation, we can observe that the sequence $\{p(Tu_{n-1},Tu_n)\}$ is a decreasing sequence, and it is bounded below.

Let $c_n = p(Tu_{n-1},Tu_n)$, for all integers $n \geq 1$. Therefore, there exists a real number $c \geq 0$ such that $\lim_{n\longrightarrow\infty} c_n = c$.

We claim that $c = 0$. In contrary, we suppose that $c > 0$, then by the semicontinuity property of $\phi$ and considering (10) above, we can have that

$$\begin{aligned} 0 < c &= \limsup_{n\longrightarrow\infty} p(Tu_n,Tu_{n+1}) = \limsup_{n\longrightarrow\infty} p(Tfu_{n-1},Tfu_n) \\ &\leq \limsup_{n\longrightarrow\infty} \phi(\lambda(u_{n-1},u_n)) = \limsup_{n\longrightarrow\infty} \phi(p(Tu_{n-1},Tu_n)) \\ &= \phi(c) < c, \end{aligned} \quad (15)$$

which is a contradiction. Hence,

$$\lim_{n\longrightarrow\infty} p(Tu_{n-1},Tu_n) = c = 0. \quad (16)$$

Now, we need to prove that the sequence $\{Tu_n\} \in X$ is a Cauchy sequence. For the sake of contradiction, we suppose that $\exists\epsilon > 0$ and the sequence of integers $n(r), m(r) \geq r$ for some $r \geq 0$ such that

$$p\left(Tu_{m(r)},Tu_{n(r)}\right) \geq \epsilon. \quad (17)$$

Furthermore, suppose that $n(r)$ is chosen as the smallest integer such that (17) above holds so that we can have

$$p\left(Tu_{m(r)},Tu_{n(r)-1}\right) < \epsilon. \quad (18)$$

Thus, by triangle inequality, we obtain

$$\begin{aligned} \epsilon &\leq p\left(Tu_{m(r)},Tu_{n(r)}\right) \leq p\left(Tu_{m(r)},Tu_{n(r)-1}\right) \\ &\quad + p\left(Tu_{n(r)-1},Tu_{n(r)}\right) - p\left(Tu_{n(r)-1},Tu_{n(r)-1}\right) \\ &\leq p\left(Tu_{m(r)},Tu_{n(r)-1}\right) + p\left(Tu_{n(r)-1},Tu_{n(r)}\right). \end{aligned} \quad (19)$$

As $r \longrightarrow \infty$ in (19) above and considering (16), we obtain

$$\epsilon \leq \lim_{r \longrightarrow \infty} p\left(Tu_{m(r)}, Tu_{n(r)}\right) < \epsilon. \quad (20)$$

Thus,

$$\lim_{r \longrightarrow \infty} p\left(Tu_{m(r)}, Tu_{n(r)}\right) = \epsilon. \quad (21)$$

Similarly, we have

$$\begin{aligned} \epsilon \leq p\left(Tu_{m(r)-1}, Tu_{n(r)-1}\right) &\leq p\left(Tu_{m(r)-1}, Tu_{m(r)}\right) \\ &+ p\left(Tu_{m(r)}, Tu_{n(r)-1}\right) - p\left(Tu_{m(r)}, Tu_{m(r)}\right) \\ &\leq p\left(Tu_{m(r)-1}, Tu_{m(r)}\right) + p\left(Tu_{m(r)}, Tu_{n(r)-1}\right). \end{aligned} \quad (22)$$

Also, as $r \longrightarrow \infty$ in (22) and again considering (26), we obtain

$$\epsilon \leq \lim_{r \longrightarrow \infty} p\left(Tu_{m(r)-1}, Tu_{n(r)-1}\right) < \epsilon. \quad (23)$$

Thus,

$$\lim_{r \longrightarrow \infty} p\left(Tu_{m(r)-1}, Tu_{n(r)-1}\right) = \epsilon. \quad (24)$$

Now from (1) and for all positive integers $n(r) > m(r)$, we obtain

$$\begin{aligned} p\left(Tu_{m(r)}, Tu_{n(r)}\right) &= p\left(Tfu_{m(r)-1}, Tfu_{n(r)-1}\right) \\ &\leq \phi\left(\lambda\left(u_{m(r)-1}, u_{n(r)-1}\right)\right), \end{aligned} \quad (25)$$

where

$$\begin{aligned} &\lambda\left(u_{m(r)-1}, u_{n(r)-1}\right) \\ &= \max\left\{\frac{p\left(Tu_{m(r)-1}, Tfu_{m(r)-1}\right)p\left(Tu_{n(r)-1}, Tfu_{n(r)-1}\right)}{p\left(Tu_{m(r)-1}, Tu_{n(r)-1}\right)}, p\left(Tu_{m(r)-1}, Tu_{n(r)-1}\right)\right\} \\ &= \max\left\{\frac{p\left(Tu_{m(r)-1}, Tu_{m(r)}\right)p\left(Tu_{n(r)-1}, Tu_{n(r)}\right)}{p\left(Tu_{m(r)-1}, Tu_{n(r)-1}\right)}, p\left(Tu_{m(r)-1}, Tu_{n(r)-1}\right)\right\}. \end{aligned} \quad (26)$$

As $r \longrightarrow \infty$ in (25) and in (26) and by considering (16), (21), and (24), then (25) becomes

$$\epsilon \leq \phi(\max\{0, \epsilon\}) = \phi(\epsilon) < \epsilon, \quad (27)$$

which is a contradiction. Hence, the sequence $\{Tu_n\} \in M$ is a Cauchy sequence.

Since $(M, p)$ is a complete partial metric space; therefore, there exists a point $u_0 \in M$ such that a sequence $Tu_n$ converges to a point $u_0$.

Given that $T$ is an ICS mapping, and a sequence $Tu_n$ converges, then there exists a point $u \in M$ such that

$$u = \lim_{n \longrightarrow \infty} u_n. \quad (28)$$

Furthermore, since $T$ is also continuous, then

$$Tu = \lim_{n \longrightarrow \infty} Tu_n = u_0. \quad (29)$$

Now, we need to prove that $u$ is a fixed point for a mapping $f$.

(i) We suppose that the first assumption of Theorem 2 holds; that is, $f$ is a continuous mapping. Therefore,

$$u = \lim_{n \longrightarrow \infty} u_n = \lim_{n \longrightarrow \infty} fu_{n-1} = f\left(\lim_{n \longrightarrow \infty} u_{n-1}\right) = fu. \quad (30)$$

Hence, $u$ is a fixed point for a mapping $f$.

(ii) Now, we suppose that the second assumption of Theorem 2 holds. Given that a sequence $u_n$ is a nondecreasing sequence such that $u_n \longrightarrow u$ as $n \longrightarrow \infty$ and $\sup_n u_n = u$, then for all integers $n \geq 0$, we have that $u_n \leq u$.

Since $f$ is a nondecreasing mapping, consequently we obtain that $fu_n \leq fu$, And respectively we have

$$u_n \leq u_{n+1} \leq \cdots \leq fu, \quad (31)$$

for all integers $n \geq 0$.

Since $u_n \longrightarrow u$ and $\sup_n u_n = u$ as $n \longrightarrow \infty$; hence,

$$u \leq fu. \quad (32)$$

Now, we construct a new sequence $v_n \in M$ such that $v_n \longrightarrow v$ as $n \longrightarrow \infty$ and $\sup_n v_n = v$ which is defined as follows:

$$\begin{aligned} v_0 &= u, \\ v_n &= fv_{n-1}, \end{aligned} \quad (33)$$

$\forall n \in \mathbb{N}$.

Since the mapping $T$ is continuous, then

$$\lim_{n \longrightarrow \infty} Tv_n = Tv. \quad (34)$$

Consequently, $v_0 \leq fv_0$ since $u \leq fu$ and $u = v_0$. Similar to the above discussion, we can conclude that a sequence $\{Tv_n\}$ is a Cauchy sequence.

Since $\sup_n v_n = v$, then $v_n \leq v$. Therefore, from (32), we obtain

$$u = v_0 \leq fu = fv_0 \leq v_2 \leq v_3 \leq \cdots \leq v_n \leq v_{n+1} \leq \cdots \leq v, \quad (35)$$

for all integers $n \geq 0$. If we suppose $u = v$, then

$$u = u_0 \leq fu = fu_0 \leq u_2 \leq \cdots \leq u_n \leq u_{n+1} \leq \cdots \leq u, \quad (36)$$

and hence, $u \leq fu \leq u$, which ends the proof that is $fu = u$.

Otherwise, suppose that $u \neq v$, then $Tu \neq Tv$ since $T$ is an injective map. Therefore, $p(Tu, Tv) > 0$.

From (1), we have

$$p(Tu_{n+1}, Tv_{n+1}) = p(Tfu_n, Tfv_n) \leq \phi(\lambda(u_n, v_n)), \quad (37)$$

where

$$\begin{aligned}\lambda(u_n, v_n) &= \max\left\{\frac{p(Tu_n, Tfu_n)p(Tv_n, Tfv_n)}{p(Tu_n, Tv_n)}, p(Tu_n, Tv_n)\right\} \\ &= \max\left\{\frac{p(Tu_n, Tu_{n+1})p(Tv_n, Tv_{n+1})}{p(Tu_n, Tv_n)}, p(Tu_n, Tv_n)\right\}.\end{aligned} \quad (38)$$

Considering (16), (29), and (34) and by letting $n \longrightarrow \infty$ in (37) and (38), then (37) becomes

$$p(Tu, Tv) \leq \phi(p(Tu, Tv)) < p(Tu, Tv), \quad (39)$$

which is a contradiction; thus $u = v$, and therefore we have $u \leq fu \leq u$; hence $u = fu$.

Therefore, $u$ is a unique fixed point of the mapping $f$.

*Remark 1.* If we let $\phi(t) = kt$ for all $t \in [0, \infty)$ and $k \in [0, 1)$ in Theorem 2, we get the following corollary:

**Corollary 1.** *Let $(M, \leq)$, $(M, p)$, $T$, and $f$ be the same as in Theorem 2 such that*

$$p(Tfu, Tfv) \leq k\lambda(u, v), \quad (40)$$

*with $u \leq v$ for all distinct $u, v \in M$, and $\lambda(u, v)$ is defined as in Theorem 2.*

Also assume either

(i) $f$ is continuous, or

(ii) If a sequence $\{u_n\} \in M$ is a nondecreasing sequence such that $u_n \longrightarrow u$ as $n \longrightarrow \infty$, then $u = \sup_n\{u_n\}$.

Therefore, if there exists a point $u_0 \in M$ such that $u_0 \leq fu_0$, then $f$ has a unique fixed point.

*Remark 2.* If we take $k = a + b$ in Corollary 1 with $a, b \in (0, 1)$ such that $a + b < 1$, we get the following corollary:

**Corollary 2.** *Let $(M, \leq)$, $(M, p)$, $T$, and $f$ be the same as in Theorem 2 such that*

$$p(Tfu, Tfv) \leq a\frac{p(Tu, Tfu)p(Tv, Tfv)}{p(Tu, Tv)} + bp(Tu, Tv), \quad (41)$$

*for all $u, v \in M$ with $u \leq v$.*

Assume either

(i) $f$ is continuous, or

(ii) If a sequence $\{u_n\} \in M$ is a nondecreasing sequence such that $u_n \longrightarrow u$ as $n \longrightarrow \infty$, then $u = \sup_n\{u_n\}$.

Therefore, if there exists a point $u_0 \in M$ such that $u_0 \leq fu_0$, then $f$ has a unique fixed point.

*Remark 3.* If we take the mapping $Tu = u$ in Corollary 2, we get an extension to the work of Harjani et al. [11] in partial metric spaces which runs as follows:

**Corollary 3.** *Let $(M, \leq)$, $(M, p)$, and $f$ be the same as in Theorem 2 such that*

$$p(fu, fv) \leq a\frac{p(u, fu)p(v, fv)}{p(u, v)} + bp(u, v), \quad (42)$$

*for all $u, v \in M$ with $u \leq v$.*

Assume either

(i) $f$ is continuous, or

(ii) If a sequence $\{u_n\} \in M$ is a nondecreasing sequence such that $u_n \longrightarrow u$ as $n \longrightarrow \infty$, then $u = \sup_n\{u_n\}$.

Therefore, if there exists a point $u_0 \in M$ such that $u_0 \leq fu_0$, then $f$ has a unique fixed point.

*Remark 4.* If we set $\phi(t) = kt$ and set $T$ to be the identity mapping in Theorem 2, then (4), which is

$$p(Tfu, Tfv) \leq \phi(\lambda(u, v)), \quad (43)$$

becomes

$$p(fu, fv) \leq k\lambda(u, v), \quad (44)$$

which leads to the extension of the work of Ran and Reurings [15] and Nieto and Rodriguez-Lopez [14] in partial metric spaces which runs as follows:

**Corollary 4.** *Let $(M, \leq)$, $(M, p)$, and $f$ be the same as in Theorem 2 such that*

$$p(fu, fv) \leq k\lambda(u, v), \quad (45)$$

*for all $u, v \in M$ with $u \leq v$.*

Assume either

(i) $f$ is continuous, or

(ii) If a sequence $\{u_n\} \in M$ is a nondecreasing sequence such that $u_n \longrightarrow u$ as $n \longrightarrow \infty$, then $u = \sup_n\{u_n\}$.

Therefore, if there exists a point $u_0 \in M$ such that $u_0 \leq fu_0$, then $f$ has a unique fixed point.

The following theorem results are from the additional assumption made on the assumptions to the hypotheses of Theorem 2.

**Theorem 3.** *Let $M$ be a partially ordered set, and $p$ be a partial metric on $M$ such that $(M, p)$ is a complete partial metric space. Also, let $T\colon M \longrightarrow M$ be an ICS mapping, and $f\colon M \longrightarrow M$ be a nondecreasing mapping satisfying*

$$p(Tfu, Tfv) \leq \phi(\lambda(u, v)), \tag{46}$$

*with $u \leq v$ for all distinct $u, v \in M$, where $\phi \in \Phi$, and*

$$\lambda(u, v) = \max\left\{\frac{p(Tu, Tfu)p(Tv, Tfv)}{p(Tu, Tv)}, p(Tu, Tv)\right\}. \tag{47}$$

Furthermore, assume that either

(i) A mapping $f$ is continuous, or

(ii) If a sequence $\{u_n\} \in M$ is a nondecreasing sequence such that $u_n \longrightarrow u$ as $n \longrightarrow \infty$, then $u = \sup_n\{u_n\}$

(iii) For every $u, v \in M$, there exists $z \in M$, that is comparable to $u$ and $v$

Therefore, if there exists $u_0 \in M$ such that $u_0 \leq fu_0$, then $f$ has a unique fixed point $u_0$.

Proof. In contrary, suppose that a mapping $f$ has two distinct fixed points, say $u$ and $v$.

From assumption (iii) of Theorem 3, there exists $z \in M$ which is comparable to $u$ and $v$. Without the lose of generality, we choose $z \leq u$.

Now, we construct a sequence $\{z_n\}$ as follows:

$z_n = fz_{n-1}$ for all $n \in \mathbb{N}$ such that $z = z_0$.

Since $f$ is a nondecreasing mapping, then $z \leq u$ implies that

$z_1 = fz_0 = fz \leq fz_1 \leq fz_2 \leq \cdots \leq fu = u$. Inductively, we obtain $z_n \leq u$.

If we suppose that there exists $n_0 \in \mathbb{N}$ such that $u = z_{n_0}$, then $z_n = fz_{n-1} = fu = u$, for all $n \geq n_0 - 1$.

This implies that $\lim_{n\longrightarrow\infty} z_n = u$. Similarly, $\lim_{n\longrightarrow\infty} z_n = v$, which completes the proof.

Now, if we suppose that $u \neq z_n$, for all integers $n \geq 0$, then $p(Tu, Tz_n) > 0$ since the mapping $T$ is one to one.

From (46), we have

$$p(Tu, Tz_n) = p(Tfu, Tfz_{n-1}) \leq \phi(\lambda(u, z_{n-1})), \tag{48}$$

where

$$\begin{aligned}\lambda(u, z_{n-1}) &= \max\left\{\frac{p(Tu, Tfu)p(Tz_{n-1}, Tfz_{n-1})}{p(Tu, Tz_{n-1})}, p(Tu, Tz_{n-1})\right\} \\ &= \max\left\{\frac{p(Tu, Tu)p(Tz_{n-1}, Tz_n)}{p(Tu, Tz_{n-1})}, p(Tu, Tz_{n-1})\right\}.\end{aligned} \tag{49}$$

Analogously to Theorem 2, we have

$$\lambda(u, z_{n-1}) = p(Tu, Tz_{n-1}). \tag{50}$$

Thus, (48) becomes

$$\begin{aligned}p(Tu, Tz_n) = p(Tfu, Tfz_{n-1}) &\leq \phi(p(Tu, Tz_{n-1})) \\ &< p(Tu, Tz_{n-1}).\end{aligned} \tag{51}$$

We observe that the sequence $p(Tu, Tz_{n-1})$ is the decreasing sequence which is bounded below. Therefore, there exists a constant $c \geq 0$ such that $\lim_{n\longrightarrow\infty} p(Tu, Tz_{n-1}) = c$. We claim that $c = 0$. In contrary, suppose that $c > 0$.

From (51), we can have that

$$\begin{aligned}\limsup_{n\longrightarrow\infty} p(Tu, Tz_n) &\leq \limsup_{n\longrightarrow\infty} \phi(p(Tu, Tz_{n-1})) \\ &< \limsup_{n\longrightarrow\infty} p(Tu, Tz_{n-1}),\end{aligned} \tag{52}$$

which leads to

$$0 < c \leq \phi(c) < c, \tag{53}$$

which is a contradiction since $\phi$ is an upper semicontinuous mapping. Hence, $c = 0$. Therefore, we conclude that

$$\lim_{n\longrightarrow\infty} p(Tu, Tz_n) = 0. \tag{54}$$

Similarly,

$$\lim_{n\longrightarrow\infty} p(Tv, Tz_n) = 0. \tag{55}$$

Since $u$ and $v$ are distinct fixed points of $f$ and $T$ is one to one mapping, then

$$0 < p(Tu, Tv) = p(Tfu, Tfv) \leq \phi(\lambda(u, v)) < \lambda(u, v), \tag{56}$$

where

$$\begin{aligned}\lambda(u, v) &= \max\left\{\frac{p(Tu, Tfu)p(Tv, Tfv)}{p(Tu, Tv)}, p(Tu, Tv)\right\} \\ &= \max\left\{\frac{p(Tu, Tu)p(Tv, Tv)}{p(Tu, Tv)}, p(Tu, Tv)\right\} \\ &= p(Tu, Tv).\end{aligned} \tag{57}$$

Therefore, from (56), we have

$$p(Tu, Tv) < \lambda(u, v) = p(Tu, Tv), \tag{58}$$

which is a contradiction. Hence, $u = v$, which ends the proof.

*Example 1.* Let $M = [0, \infty)$ be a set equipped with a partial metric $p(u, v) = \max\{u, v\}$ for all $u, v \in M$. Let the order $\ll_p$ be defined by $u \ll_p v \Leftrightarrow u = v$, for all $u, v$.

It is easy to check that $(M, \ll_p)$ is a partially ordered set (Poset) by proving PO1 through PO3 from Definition 4.

$Tu = (1/2)e^u$, for all $u \in M$, is defined. Also,

$$fu = \begin{cases} \frac{1}{2}u^2, & u \in [0,1], \\ u - \frac{1}{2}, & u \in (1, \infty). \end{cases} \tag{59}$$

is defined. It is easy to see that a mapping $T$ is an ICS mapping as it is injective and continuous. Also, we observe that a function $fu$ is continuous and nondecreasing $f$.

Now, we show that inequality (4) holds.

From left hand side of (4), for all $u, v \in [0, 1]$, we have

$$\begin{aligned} p(Tfu, Tfv) &= \max\{T(fu), T(fv)\} \\ &= \max\left\{T\left(\frac{1}{2}u^2\right), T\left(\frac{1}{2}v^2\right)\right\} \\ &= \max\left\{\frac{1}{2}e^{u^2/2}, \frac{1}{2}e^{v^2/2}\right\} \\ &\le \max\left\{\frac{1}{2}e^{u^2}, \frac{1}{2}e^{v^2}\right\} \\ &\le \max\left\{\frac{e^u}{2}, \frac{e^v}{2}\right\}. \end{aligned} \tag{60}$$

From the right hand side of (4), we have

$$\lambda(u, v) = \max\left\{\frac{p(Tu, Tfu)p(Tv, Tfv)}{p(Tu, Tv)}, p(Tu, Tv)\right\}, \tag{61}$$

where

$$\begin{aligned} p(Tu, Tfu) &= \max\{Tu, T(fu)\} \\ &= \max\left\{\frac{e^u}{2}, T\left(\frac{u^2}{2}\right)\right\} \\ &= \max\left\{\frac{e^u}{2}, \frac{e^{(u^2/2)}}{2}\right\} = \frac{e^u}{2}. \end{aligned} \tag{62}$$

Similarly,

$$p(Tv, Tfv) = \frac{e^v}{2}, \tag{63}$$

and $p(Tu, Tv) = \max\{Tu, Tv\} = \max\{(e^u/2), (e^v/2)\}$.

*Case 1.* Suppose, $\max\{(e^u/2), (e^v/2)\} = (e^u/2)$.

Then, $\lambda(u, v) = (e^u/2)$. Hence, $p(Tfu, Tfv) \le (e^u/2) = \lambda(u, v)$.

*Case 2.* Suppose, $\max\{(e^u/2), (e^v/2)\} = (e^v/2)$.

Then, $\lambda(u, v) = (e^v/2)$. Hence, $p(Tfu, Tfv) \le (e^v/2) = \lambda(u, v)$.

From left hand side of (4), for all $u, v \in (1, \infty)$, we have

$$\begin{aligned} p(Tfu, Tfv) &= \max\{T(fu), T(fv)\} \\ &= \max\left\{T\left(u - \frac{1}{2}\right), T\left(v - \frac{1}{2}\right)\right\} \\ &= \max\left\{\frac{1}{2}e^{u-(1/2)}, \frac{1}{2}e^{v-(1/2)}\right\} \\ &\le \max\left\{\frac{1}{2}e^u, \frac{1}{2}e^v\right\}. \end{aligned} \tag{64}$$

From the right hand side of (4), we have

$$\lambda(u, v) = \max\left\{\frac{p(Tu, Tfu)p(Tv, Tfv)}{p(Tu, Tv)}, p(Tu, Tv)\right\}, \tag{65}$$

where

$$\begin{aligned} p(Tu, Tfu) &= \max\{Tu, T(fu)\} \\ &= \max\left\{\frac{e^u}{2}, T\left(u - \frac{1}{2}\right)\right\} \\ &= \max\left\{\frac{e^u}{2}, \frac{e^{u-(1/2)}}{2}\right\} = \frac{e^u}{2}. \end{aligned} \tag{66}$$

Similarly,

$$p(Tv, Tfv) = \frac{e^v}{2}, \tag{67}$$

and $p(Tu, Tv) = \max\{Tu, Tv\} = \max\{(e^u/2), (e^v/2)\}$.

In the similar way, we find that $p(Tfu, Tfv) \le (e^u/2) = \lambda(u, v)$.

Now, if we define $\phi(t) = (1/2)t$, for all $t = \lambda(u, v) \in [0, \infty)$, we obtain that $p(Tfu, Tfv) \le \phi(\lambda(u, v))$. Therefore, (4) holds, and $u_0 = 0$ is the fixed point of a mapping $f$.

## 3. The Existence Solution of Nonlinear Integral Equations

In this section, we studied the existence of solutions for nonlinear integral equations, as an application to the fixed point theorems proved in the previous section.

The following integral equations were inspired by [25, 26]. An unknown function $x$ is considered. Now, an application of Theorem 3 is presented as a study of the existence and uniqueness of solution to nonlinear integral equations.

$$u(t) = h(t) + \int_a^b G(t, s, )K(s, u(s))\mathrm{d}s, \tag{68}$$

where $K$: $[a,b] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $G$: $[a,b]^2 \longrightarrow [0,\infty)$ are given continuous functions.

Let $X$ be the set $C[a,b]$ of real continuous function on $[a,b]$ and let $p$: $X \times X \longrightarrow \mathbb{R}^+$ be given by

$$p(u,v) = \max_{a\le t\le b}\{u,v\}. \tag{69}$$

It is easy to see that $p$ is a partial metric and that $(X, p)$ is a complete partial metric space.

Next, we prove a theorem to establish the existence of a common fixed point for a pair of self mappings:

**Theorem 4.** *Let us consider the integral equation (68) as above. Also, suppose that it satisfies the following conditions:*

*(i) for $t, s \in [a,b]$ and $u, v \in X$, there exists a nondecreasing function $\phi \in \Phi$ such that the following inequality holds:*

$$|K(t,s,u(s)) - K(t,s,v(s))| \le \phi|\lambda(u,v)|, \tag{70}$$

*where*

$$\lambda(u,v) = \max\left\{\frac{p(Tu,Tfu)p(Tv,Tfv)}{p(Tu,Tv)}, p(Tu,Tv)\right\}. \tag{71}$$

*(ii)*

$$\max_{a\le t\le b}\int_a^b G(t,s)\mathrm{d}s \le \frac{1}{b-a}. \tag{72}$$

Then, the integral equation (68) has a unique common solution $u_0 \in X$.

Proof. Let $T, f$: $X \longrightarrow X$ be a mapping defined by

$$Tfu(t) = h(t) + \int_a^b G(t,s)K(t,s,u(s))\mathrm{d}s. \tag{73}$$

This implies that $f \in T$ and $f \in X$ possess a fixed point $u_0 \in Tf$. To prove the existence of the fixed point of $Tf$, we prove that $Tf$ is a contraction. On contrary, we assume that $Tfu \neq Tfv$, for all $u, v \in [a,b]$. Using conditions (i) and (ii) of Theorem 4, we have

$$\begin{aligned}
p(Tfu,Tfv) &= \max_{a\le t\le b}\{Tfu(t), Tfv(t)\},\\
&\le \max_{a\le t\le b}\left\{h(t) + \int_a^b G(t,s)K(t,s,u(s)), h(t) + \int_a^b G(t,s)K(t,s,v(s))\mathrm{d}s\right\},\\
&\le \max_{a\le t\le b}\left\{\int_a^b G(t,s)\mathrm{d}s\,(h(t) + K(t,s,u(s)), h(t) + K(t,s,v(s)))\right\},\\
&\le \int_a^b G(t,s)\mathrm{d}s \max_{a\le t\le b}\{h(t) + K(t,s,u(s)), h(t) + K(t,s,v(s))\},\\
&\le \frac{1}{b-a}\max_{a\le t\le b}\{h(t) + K(t,s,u(s)), h(t) + K(t,s,v(s))\},\\
&\le \frac{1}{b-a}\max\left\{\frac{p(Tu,Tfu)p(Tv,Tfv)}{p(Tu,Tv)}, p(Tu,Tv)\right\},\\
&\le \frac{1}{b-a}\phi|\lambda(u,v)|.
\end{aligned} \tag{74}$$

For $b = 1$, $a = 0$, we have

$$p(Tfu,Tfv) \le \phi\,|\,\lambda(u,v)|, \tag{75}$$

which is a contradiction. Hence, $u$ is a common fixed of $T$ and $f$, also a solution to integral equation (74). Thus, Theorem 3 is satisfied.

## 4. Conclusion

The main contribution of this paper to fixed point theory is Definition 8, Theorems 2, and 3. Here, the results have proved for Boyd and Wong type contraction in ordered partial metric spaces. Several existing results in the literature are generalized and extended into ordered complete partial metric spaces. Suitable examples are given to demonstrate the validity of the results. Finally, the existence of the solution of nonlinear integral equation is discussed as an application of the main result.

## Data Availability

No data were used to support this study.

## Ethical Approval

Not applicable.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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