Hindawi
Journal of Function Spaces
Volume 2022, Article ID 7618683, 9 pages
https://doi.org/10.1155/2022/7618683

*Research Article*

# A New Iterative Algorithm for General Variational Inequality Problem with Applications

**Aysha Khan**[1] **M. Akram**[2] **M. Dilshad**[3] **and J. Shafi**[4]

[1]*Department of Mathematics, College of Arts and Science, Wadi-Ad-Dwasir, Prince Sattam Bin Abdulaziz University, Al-Kharj, Saudi Arabia*
[2]*Department of Mathematics, Faculty of Science, Islamic University of Madinah, Madinah 170, Saudi Arabia*
[3]*Computational & Analytical Mathematics and Their Applications Research Group, Department of Mathematics, Faculty of Science, University of Tabuk, Tabuk 71491, Saudi Arabia*
[4]*Department of Computer Science, College of Arts and Science, Wadi-Ad-Dwasir, Prince Sattam Bin Abdulaziz University, Al-Kharj, Saudi Arabia*

Correspondence should be addressed to Aysha Khan; aayshakhan055@gmail.com

Received 23 November 2021; Revised 23 January 2022; Accepted 27 January 2022; Published 29 March 2022

Academic Editor: Mohamed A. Taoudi



This study aims at investigation of a generalized variational inequality problem. We initiate a new iterative algorithm and examine its convergence analysis. Using this newly proposed iterative method, we estimate the common solution of generalized variational inequality problem and fixed points of a nonexpansive mapping. A numerical example is illustrated to verify our existence result. Further, we demonstrate that the considered iterative algorithm converges with faster rate than normal $S$-iterative scheme. Furthermore, we apply our proposed iterative algorithm to estimate the solution of a convex minimization problem and a split feasibility problem.

## 1. Introduction

All through this study, we presume that $\mathscr{H}$ is a real Hilbert space equipped with norm $\|\cdot\|$ induced by inner product $\langle\cdot,\cdot\rangle$. Let $\mathscr{C}$ be a nonempty closed convex subset of $\mathscr{H}$ and $f, g\colon \mathscr{C} \longrightarrow \mathscr{H}$ be nonlinear mappings. The generalized nonlinear variational inequality is to locate a point $x^* \in \mathscr{H}$ such that

$$\langle f(x^*), g(x^*) - g(y^*)\rangle \geq 0, \quad \forall y^* \in \mathscr{C}, g(x^*), g(y^*) \in \mathscr{H}, \tag{1}$$

which was introduced by Noor [18]. We denote the set of solutions of (1) by $\mathrm{Sol}(\mathscr{C}, f, g)$.

If $g = I$, then generalized nonlinear variational inequality (1) reduces to the classical variational inequality studied by Stampacchia [23], which is to allocate a point $x^* \in \mathscr{H}$, such that

$$\langle f(x^*), y^* - x^*\rangle \geq 0, \quad \forall y^* \in \mathscr{H}. \tag{2}$$

If $\mathscr{C}^* = \{x^* \in \mathscr{H}\colon \langle x^*, y^*\rangle \geq 0, \forall y^* \in \mathscr{C}\}$ is a dual cone of a convex cone $\mathscr{C}$, then generalized nonlinear variational inequality (1) coincides to generalized nonlinear complementarity problem which is to locate a point $x^* \in \mathscr{H}$ such that

$$\begin{aligned} &\langle f(x^*), g(x^*)\rangle = 0, \\ &g(x^*) \in \mathscr{C}, f(x^*) \in \mathscr{C}^*. \end{aligned} \tag{3}$$

It is worthy to adduce that variational inequalities which are unconventional and remarkable augmentation of variational principles provide well organized unified framework for figuring out a wide range of nonlinear problems arising in optimization, economics, physics, engineering science, operations research, and control theory, for example, [2, 8, 15, 20, 21, 24, 26, 33] and references cited therein.

Next, we recall the following definitions of a nonlinear mapping $f\colon \mathscr{C} \subset \mathscr{H} \longrightarrow \mathscr{H}$.

*Definition 1.* The mapping $f: \mathscr{C} \longrightarrow \mathscr{H}$ is said to be

(i) $a$-inverse strongly monotone or cocoercive if there exists a constant $a > 0$, such that

$$\langle f(x) - f(y), x - y \rangle \geq a \| f(x) - f(y) \|^2, \quad \forall x, y \in \mathscr{C}, \tag{4}$$

(ii) $L$-Lipschitz continuous if there exists a constant $L > 0$, such that

$$\| f(x) - f(y) \| \leq L \| x - y \|, \quad \forall x, y \in \mathscr{C}. \tag{5}$$

For $L = 1$, $f$ is nonexpansive, and if $0 < L < 1$, then $f$ is a contraction. Note that $a$-inverse strongly monotone mapping is $1/a$-Lipschitz continuous.

It is customary to mention that variational inequalities, variational inclusions, and related optimization problems can be posed as fixed-point problems. This unusual formulation plays a dominant role in studying variational inequalities and nonlinear problems by employing fixed-point iterative methods.

**Lemma 1.** *Let* $p_{\mathscr{C}}: \mathscr{H} \longrightarrow \mathscr{C}$ *be a projection mapping of* $\mathscr{H}$ *onto* $\mathscr{C}$. *For a given* $q \in \mathscr{H}$, $p \in \mathscr{C}$ *satisfies the inequality*

$$\langle p - q, r - p \rangle \geq 0, \quad \forall r \in \mathscr{C} \text{ if and only if } p = p_{\mathscr{C}}(q). \tag{6}$$

Note that the projection mapping $p_{\mathscr{C}}$ is nonexpansive [16]. For more details on projection mapping $p_{\mathscr{C}}$, we refer to [12]. By utilizing Lemma 1, the generalized nonlinear variational inequality (1) can be designed as a fixed-point problem as follows:

**Lemma 2** (see [17]). *Let* $p_{\mathscr{C}}: \mathscr{H} \longrightarrow \mathscr{C}$ *be a projection mapping. For any* $\rho > 0$, $x^* \in \mathscr{H}$, $g(x^*) \in \mathscr{C}$ *solves the generalized nonlinear variational inequality (1) if and only if*

$$g(x^*) = p_{\mathscr{C}}[g(x^*) - \rho f(x^*)]. \tag{7}$$

Relation (7) can be rescripted as

$$x^* = x^* - g(x^*) + p_{\mathscr{C}}[g - \rho f](x^*). \tag{8}$$

Let $T$ be a nonexpansive mapping and $F(T)$ denotes the set of fixed points of $T$. If $x^* \in F(T) \cap \mathrm{Sol}(\mathscr{H}, f, g)$, then

$$\begin{aligned} x^* &= T(x^*) = x^* - g(x^*) + p_{\mathscr{C}}[g - \rho f](x^*) \\ &= T\{x^* - g(x^*) + p_{\mathscr{C}}[g(x^*) - \rho f(x^*)]\}, \quad \rho > 0. \end{aligned} \tag{9}$$

It is significant to achieve better rate of convergence if two or more iterative algorithms converge to the same point for a given problem. We recall the following concepts which are versatile tools to find finer convergence rate for different iterative methods.

*Definition 2* (see [3]). Let $\{p_n\}$ and $\{q_n\}$ be two real sequences converging to $p$ and $q$, respectively. Suppose that $\lim_{n \longrightarrow \infty} \| p_n - p \| / \| q_n - q \| = l$ exists. Then,

(i) $\{p_n\}$ converges faster than $\{q_n\}$ if $l = 0$

(ii) $\{p_n\}$ and $\{q_n\}$ converges with identical rates if $0 < l < \infty$

*Definition 3* (See [3]). Let $\{p_n\}$ and $\{q_n\}$ be two real sequences converging to the same fixed point $t$. If $\{u_n\}$ and $\{v_n\}$ are two sequences of positive real numbers converging to 0 such that $\| p_n - t \| \leq u_n$ and $\| q_n - t \| \leq v_n$ for all $n \in \mathbb{N}$. Then, $\{p_n\}$ converges to $t$ faster than $\{q_n\}$ if $\{u_n\}$ converges faster than $\{v_n\}$.

**Lemma 3** (see [4]). *Let* $\{\varphi_n\}$ *and* $\{\psi_n\}$ *be nonnegative sequences of real numbers satisfying*

$$\varphi_{n+1} \leq \sigma \varphi_n + \psi_n, \quad \forall n \in \mathbb{N}, \tag{10}$$

*where* $\sigma \in (0, 1)$ *and* $\lim_{n \longrightarrow \infty} \psi_n = 0$. *Then* $\lim_{n \longrightarrow \infty} \varphi_n = 0$.

**Lemma 4** (see [31]). *Let* $\{\phi_n\}$, $\{\varphi_n\}$, *and* $\{\psi_n\}$ *be nonnegative sequences of real numbers satisfying*

$$\phi_{n+1} \leq (1 - \varphi_n) \phi_n + \psi_n, \quad \forall n \in \mathbb{N}, \tag{11}$$

*where* $\varphi_n \in (0, 1)$, $\sum_{n=1}^{\infty} \varphi_n = \infty$, *and* $\psi_n = o(\varphi_n)$. *Then,* $\lim_{n \longrightarrow \infty} \phi_n = 0$.

Mann, Ishikawa, and Halpern iterative methods are fundamental tools for solving fixed-point problems of nonexpansive mappings. In recent past, a number of fixed point iterative methods have been constructed and implemented to solve various classes of nonlinear mappings [2, 9, 10, 19, 22, 25, 28–30, 34]. Agarwal and others [1] introduced the $S$-iteration method which converges faster than some well-known iterative algorithms such as Mann, Ishikawa, and Picard for contraction as well as nonexpansive mappings. Due to the super convergence rate, it attracted number of researchers to study fixed-point problems, minimization problems, variational inclusions, variational inequalities, and alternate points problems in different settings. In [18], Noor utilized formulation (9) to propose following iterative algorithm:

$$\begin{cases} u_0 \in \mathscr{C}, \\ u_{n+1} = (1 - a_n) u_n + a_n T\{u_n - g(u_n) + p_{\mathscr{C}}[g(u_n) - \rho f(u_n)]\}, \end{cases} \tag{12}$$

where $\{a_n\}$ is a sequence in $[0, 1]$. The author proved strong convergence of the proposed iterative algorithm. Furthermore, it is customary that the normal $S$-iterative algorithm converges faster than the Mann and Picard iterative algorithm. Owing to its uncomplicated nature and faster convergence rate, Gursoy and others [14] investigated the following normal $S$-iterative algorithm to examine (1) as follows:

$$\begin{cases} p_0 \in \mathscr{C}, \\ p_{n+1} = T\{q_n - g(q_n) + p_{\mathscr{C}}[g(q_n) - \rho f(q_n)]\}, \\ q_n = (1 - \xi_n) p_n + \xi_n T\{p_n - g(p_n) + p_{\mathscr{C}}[g(p_n) - \rho f(p_n)]\}, \xi_n \in [0, 1]. \end{cases} \tag{13}$$

Recently, Ullah and Arshad [27] introduced a more efficient iterative algorithm called the $M$-iterative method for Suzuki's generalized nonexpansive mappings as follows:

$$\begin{cases} u_1 \in \mathscr{C}, \\ w_n = (1-a_n)u_n + a_n T u_n, \\ v_n = T w_n, \\ u_{n+1} = T v_n, \end{cases} \tag{14}$$

where $\{a_n\}$ is a sequence in $(0,1)$. They analyzed convergence and showed that their iterative procedure converges faster than the Picard $S$ [13] and $S$-iteration process [1]. In recent work, Garodia and Uddin [11] developed a new iterative algorithm for Suzuki's generalized nonexpansive mappings as follows:

$$\begin{cases} u_1 \in \mathscr{C}, \\ w_n = T u_n, \\ v_n = T((1-a_n)w_n + a_n T w_n), \\ u_{n+1} = T v_n, \end{cases} \tag{15}$$

where $\{a_n\}$ is a sequence in $(0,1)$. The authors approximated fixed-points and inspected the convergence. Also, they proved that the posed iterative method converges with faster rate than that of the $M$-iterative method.

Stimulated by the work discussed in above-mentioned references, in this study, we investigate algorithm (15) to estimate the common solution of fixed points of a nonexpansive mapping $T$ and the generalized nonlinear variational inequality (1) as follows:

$$\begin{cases} u_1 \in \mathscr{C}, \\ w_n = T\{u_n - g(u_n) + \mathrm{p}_{\mathscr{C}}[g(u_n) - \rho f(u_n)]\}, \\ r_n = (1-a_n)w_n + a_n T\{w_n - g(w_n) + \mathrm{p}_{\mathscr{C}}[g(w_n) - \rho f(w_n)]\}, \\ v_n = T\{r_n - g(r_n) + \mathrm{p}_{\mathscr{C}}[g(r_n) - \rho f(r_n)]\}, \\ u_{n+1} = T\{v_n - g(v_n) + \mathrm{p}_{\mathscr{C}}[g(v_n) - \rho f(v_n)]\}, \end{cases} \tag{16}$$

where $\{a_n\}$ is a sequence in $(0,1)$ satisfying certain assumptions. We analyze strong convergence of our proposed iterative algorithm (16) under some mild assumptions. We also pose a modified form of our iterative algorithm (16) to investigate convex optimization and split feasibility problems. Theoretical findings are validated by an illustrative numerical example. Our existence and convergence results can be seen as generalizations and prevalent of some known results.

## 2. Convergence Results

**Theorem 1.** *Let $f, g: \mathscr{C} \longrightarrow \mathscr{H}$ be $a_1$, $a_2$-inverse strongly monotone mappings, respectively, and $T: \mathscr{H} \longrightarrow \mathscr{C}$ be a nonexpansive mapping such that $F(T) \cap Sol(\mathscr{C}, f, g) \neq \phi$. Suppose that the assumption*

$$|\rho - a_1| < a_1(1 - \Upsilon), \tag{17}$$

*holds, where $\Upsilon = 2|a_2 - 1/a_2|$. Then, the iterative sequence $\{u_n\}$ defined by (16) converges strongly to $u^* \in F(T) \cap Sol(\mathscr{C}, f, g)$ with the following error estimates:*

$$\|u_{n+1} - u^*\| \le \zeta^{3(n+1)} \|u_0 - u^*\| \prod_{k=0}^{n} [1 - a_k(1-\zeta)], \quad \forall n \in \mathbb{N}, \tag{18}$$

*where*

$$\zeta = 2\left|\frac{a_2 - 1}{a_2}\right| + \left|\frac{a_1 - \rho}{a_1}\right|. \tag{19}$$

*Proof.* Note that

$$u^* = T\{u^* - g(u^*) + \mathrm{p}_{\mathscr{C}}[g(u^*) - \rho f(u^*)]\}. \tag{20}$$

Since $f$ being $a_1$-inverse strongly monotone is $1/a_1$-Lipschitz continuous mapping, $T$ and $\mathrm{p}_{\mathscr{C}}$ are the nonexpansive mappings. Then, from (16 and 20), we obtain

$$\begin{aligned} \|w_n - u^*\| &= \|T\{u_n - g(u_n) + \mathrm{p}_{\mathscr{C}}[g(u_n) - \rho f(u_n)]\} \\ &\quad - T\{u^* - g(u^*) + \mathrm{p}_{\mathscr{C}}[g(u^*) - \rho f(u^*)]\}\| \\ &\le 2\|u_n - u^* - (g(u_n) - g(u^*))\| \\ &\quad + \|u_n - u^* - \rho(f(u_n) - f(u^*))\|. \end{aligned} \tag{21}$$

Since $f$ is $a_1$-inverse strongly monotone mapping, then we have

$$\begin{aligned} &\|u_n - u^* - \rho(f(u_n) - f(u^*))\|^2 \\ &= \|u_n - u^*\|^2 + \rho^2\|f(u_n) - f(u^*)\|^2 \\ &\quad - 2\rho\langle u_n - u^*, f(u_n) - f(u^*)\rangle \\ &\le \|u_n - u^*\|^2 + \frac{\rho^2}{a_1^2}\|u_n - u^*\|^2 - 2\rho a_1\|f(u_n) - f(u^*)\|^2 \\ &\le \left(\frac{a_1 - \rho}{a_1}\right)^2 \|u_n - u^*\|^2. \end{aligned} \tag{22}$$

Also, $g$ is $a_2$-inverse strongly monotone mapping; then we have

$$\begin{aligned} &\|u_n - u^* - (g(u_n) - g(u^*))\|^2 \\ &= \|u_n - u^*\|^2 + \|g(u_n) - g(u^*)\|^2 \\ &\quad - 2\langle u_n - u^*, g(u_n) - g(u^*)\rangle \\ &\le \|u_n - u^*\|^2 + \frac{1}{a_2^2}\|u_n - u^*\|^2 - 2a_2\|g(u_n) - g(u^*)\|^2 \\ &\le \left(\frac{a_2 - 1}{a_2}\right)^2 \|u_n - u^*\|^2. \end{aligned} \tag{23}$$

Thus, from (21) to (23), we have

$$\|w_n - u^*\| \le \left(2\left|\frac{a_2 - 1}{a_2}\right| + \left|\frac{a_1 - \rho}{a_1}\right|\right)\|u_n - u^*\| = \zeta\|u_n - u^*\|, \tag{24}$$

where $\zeta$ is defined by (19), and from (17), we have $\zeta < 1$. Again, following the same steps (21)–(24) and from (16), we obtain

$$\|v_n - u^*\| \le \zeta \|r_n - u^*\|. \tag{25}$$

Next, we estimate

$$\begin{aligned}\|r_n - u^*\| &= \|(1-a_n)w_n + a_n T\{w_n - g(w_n) \\ &\quad + p_{\mathscr{C}}[g(w_n) - \rho f(w_n)]\} - u^*\| \\ &\le (1-a_n)\|w_n - u^*\| + a_n\zeta\|w_n - u^*\| \\ &= 1 - a_n(1-\zeta)\|w_n - u^*\|,\end{aligned} \tag{26}$$

which amounts to say

$$\begin{aligned}&\|v_n - u^*\| \le \zeta[1 - a_n(1-\zeta)]\|w_n - u^*\|, \\ &\|u_{n+1} - u^*\| = \|T\{v_n - g(v_n) + p_{\mathscr{C}}[g(v_n) - \rho f(v_n)]\} \\ &\quad - T\{u^* - g(u^*) + p_{\mathscr{C}}[g(u^*) - \rho f(u^*)]\}\| \\ &\le \zeta\|v_n - u^*\| \le \zeta^3[1 - a_n(1-\zeta)]\|u_n - u^*\|.\end{aligned} \tag{27}$$

Since, $1 - a_n(1-\zeta) < 1$. Therefore, we get $\|u_{n+1} - u^*\| \le \zeta^3\|u_n - u^*\|, \forall n \in \mathbb{N}$. By repeating the process in this fashion, we obtain

$$\|u_{n+1} - u^*\| \le \zeta^{3(n+1)}\|u_0 - u^*\|, \quad \forall n \in \mathbb{N}, \tag{28}$$

which gives that $\lim_{n\longrightarrow\infty}\|u_n - u^*\| = 0$. □

Now, we exemplify the existence of solution.

*Example 1.* Let $\mathscr{H} = \mathbb{R}, \mathscr{C} = [1,2]$ be equipped with norm $\|u\| = |u|$ and inner product $\langle u, v\rangle = u.v$. Let $f, g, T\colon [1,2] \longrightarrow \mathbb{R}$ be defined by

$$f(u) = u^2, g(u) = \frac{u^3}{4} + \frac{3}{4}, T(u) = \frac{u^2 + u^3}{16} + \frac{7}{8}. \tag{29}$$

Then, for all $u, v \in \mathscr{C}$, observe that

$$\begin{aligned}&\langle f(u) - f(v), u - v\rangle = (u-v)^2(u+v) \ge 2|u-v|^2, \\ &\langle g(u) - g(v), u - v\rangle = \frac{1}{4}(u-v)^2(u^2 + uv + v^2) \ge \frac{3}{4}|u-v|^2, \\ &|T(u) - T(v)| = \frac{1}{16}|u-v||u^2 + uv + v^2 + u + v| \le |u-v|.\end{aligned} \tag{30}$$

Then, $f$ and $g$ are 2 and 3/4-inverse strongly monotone mapping, respectively, and $T$ is nonexpansive mapping. One can easily verify that $u^* = 1 \in \mathscr{C}$ is the unique fixed point of $T$. Also,

$$\langle f(u^*), g(v) - g(u^*)\rangle = \frac{v^3 - 1}{4} \ge 0, \quad \text{for all } v \in \mathscr{C}. \tag{31}$$

Thus, we have $u^* = 1 \in F(T) \cap \mathrm{Sol}(\mathscr{C}, f, g)$.

**Theorem 2.** *Let $\mathscr{H}$ be a real Hilbert space and $\mathscr{C}$ be a nonempty closed convex subset of $\mathscr{H}$. Let $f, g, T$, and $\zeta$ be same as defined in Theorem 1. Let $\{p_n\}$ and $\{u_n\}$ be the sequences defined by (13) and (16), respectively. Suppose that (17) holds and $F(T) \cap (\mathscr{C}, f, g) \ne \phi$. Then, the following statements hold:*

*(i) If $\{(1+\zeta^3)/\xi_n\}$ is bounded and $\sum_{n=0}^{\infty} a_n = \infty$, then the sequence $\{u_n - p_n\}$ converges strongly to 0 with following error estimates:*

$$\begin{aligned}\|u_{n+1} - p_{n+1}\| &\le [1 - \xi_n(1-\zeta)]\|u_n - p_n\| \\ &\quad + (1+\zeta^3)\|u_n - u^*\|, \quad \forall n \in \mathbb{N}.\end{aligned} \tag{32}$$

*$\{p_n\}$ converges strongly to $u^* \in F(T) \cap Sol(\mathscr{C}, f, g)$.*

*(ii) If $\{p_n\}$ converges strongly to $u^* \in F(T) \cap Sol(\mathscr{C}, f, g)$, then $\{p_n - u_n\}$ converges strongly to 0 with following error estimates:*

$$\begin{aligned}\|p_{n+1} - u_{n+1}\| &\le \zeta^3\|p_n - u_n\| \\ &\quad + (1+\zeta^3)\|p_n - u^*\|, \quad \forall n \in \mathbb{N}.\end{aligned} \tag{33}$$

*Proof*

(i) It follows from Theorem 1 that $\lim_{n\longrightarrow\infty}\|u_n - u^*\| = 0$. Next, we prove that $\lim_{n\longrightarrow\infty}\|p_n - u^*\| = 0$. Following (13) and (16) and the steps as in (21)–(24), we obtain

$$\begin{aligned}\|u_{n+1} - p_{n+1}\| &= \|T\{v_n - g(v_n) + p_{\mathscr{C}}[g(v_n) - \rho f(v_n)]\} \\ &\quad - T\{q_n - g(q_n) + p_{\mathscr{C}}[g(q_n) - \rho f(q_n)]\}\| \\ &\le \zeta\|v_n - q_n\|,\end{aligned} \tag{34}$$

where $\zeta$ is same as in (19). Again, utilizing (13), (16), and (34), we have

$$\begin{aligned}\|u_{n+1}-p_{n+1}\| &\le \zeta\|v_n-(1-\xi_n)p_n-\xi_n T\{p_n-g(p_n)+p_{\mathscr{C}}[g(p_n)-\rho f(p_n)]\}\| \\ &\le \zeta\|v_n-u^*\|+(1-\xi_n)\zeta\|p_n-u^*\|+\xi_n\zeta\|T\{p_n-g(p_n) \\ &\quad +p_{\mathscr{C}}[g(p_n)-\rho f(p_n)]\} \\ &\quad -T\{u^*-g(u^*)+p_{\mathscr{C}}[g(u^*)-\rho f(u^*)]\}\| \\ &\le \zeta\|v_n-u^*\|+(1-\xi_n)\zeta\|p_n-u^*\|+\xi_n\zeta^2\|p_n-u^*\| \\ &= \zeta\left[\|v_n-u^*\|+1-\xi_n(1-\zeta)\|p_n-u^*\|\right] \\ &\le \zeta\left[\zeta^2(1-a_n(1-\zeta))\|u_n-u^*\|+1-\xi_n(1-\zeta)\|p_n-u^*\|\right] \\ &= \zeta\left[\zeta^2(1-a_n(1-\zeta))\|u_n-u^*\|+1-\xi_n(1-\zeta)\|u_n-u^*\|\right. \\ &\quad \left.+1-\xi_n(1-\zeta)\|u_n-p_n\|\right] \\ &\le \zeta[1-\xi_n(1-\zeta)]\|u_n-p_n\|+\left(1+\zeta^3\right)\max\{1-a_n(1-\zeta), \\ &\quad 1-\xi_n(1-\zeta)\}\|u_n-u^*\| \\ &\le [1-\xi_n(1-\zeta)]\|u_n-p_n\|+\left(1+\zeta^3\right)\|u_n-u^*\|.\end{aligned} \tag{35}$$

Let $\phi_n=\|u_n-p_n\|$, $\varphi_n=\xi_n(1-\zeta)$, $\psi_n=(1+\zeta^3)\|u_n-u^*\|$, and $\delta_n=\|u_n-u^*\|, \forall n\in\mathbb{N}$. It follows from assumption of the theorem that $\left\{(1+\zeta^3)/\xi_n\right\}$ is bounded; therefore, $\left\{(1+\zeta^3)/\xi_n(1-\zeta)\right\}$ is also bounded. Then, there exists a constant $M>0$, such that $|(1+\zeta^3)/\xi_n(1-\zeta)|<M, \forall n\in\mathbb{N}$. Since $\lim_{n\longrightarrow\infty}\delta_n=0$ and $\left\{(1+\zeta^3)/\xi_n(1-\zeta)\right\}$ is bounded, therefore, $\left\{(1+\zeta^3)/\xi_n(1-\zeta)\delta_n\right\}\longrightarrow 0$ as $n\longrightarrow\infty$, i.e., $\lim_{n\longrightarrow\infty}(\psi_n/\varphi_n)=0$, which amounts to say that $\psi_n=o(\varphi_n)$. Thus, all the assumptions of Lemma 4 are fulfilled. Hence, $\lim_{n\longrightarrow\infty}\|u_n-p_n\|=0$ and $\|p_n-u^*\|\le\|u_n-p_n\|+\|u_n-u^*\|$. Thus, we have $\lim_{n\longrightarrow\infty}\|p_n-u^*\|=0$.

(ii) Next, we estimate that $\{p_n-u_n\}\longrightarrow 0$. Since $\{p_n\}$ converges to $u^*\in F(T)\cap\mathrm{Sol}(\mathscr{C},f,g)$, then following the same arguments as in (34) and (35), we obtain

$$\begin{aligned}\|p_{n+1}-u_{n+1}\| &\le \zeta[1-\xi_n(1-\zeta)]\|p_n-u^*\|+\zeta\left[\zeta^2(1-a_n(1-\zeta))\right]\|u_n-u^*\| \\ &\le \zeta^3[1-a_n(1-\zeta)]\|p_n-u_n\|+\zeta^3[1-a_n(1-\zeta)]\|p_n-u^*\| \\ &\quad +\zeta[1-\xi_n(1-\zeta)]\|p_n-u^*\| \\ &\le \zeta^3[1-a_n(1-\zeta)]\|p_n-u_n\|+\zeta^3\|p_n-u^*\|+\|p_n-u^*\| \\ &\le \zeta^3\|p_n-u_n\|+\left(1+\zeta^3\right)\|p_n-u^*\|.\end{aligned} \tag{36}$$

Let $\phi_n'=\|p_n-u_n\|, \psi_n'=(1+\zeta^3)\|p_n-u^*\|, \forall n\in\mathbb{N}$. By the assumption $\{p_n\}$ converges to $u^*$ and utilizing the fact that $(1+\zeta^3)$ is bounded, we obtain that $\psi_n'\longrightarrow 0$ as $n\longrightarrow\infty$. Thus, all the assumptions of Lemma 3 are fulfilled. Hence, $\lim_{n\longrightarrow\infty}\|p_n-u_n\|=0$. Also, we know that $\|u_n-u^*\|\le\|p_n-u_n\|+\|p_n-u^*\|, \forall n\in\mathbb{N}$. Thus, $\lim_{n\longrightarrow\infty}\|u_n-u^*\|=0$. Hence, $\{p_n-u_n\}\longrightarrow 0$ as $n\longrightarrow\infty$. □

**Theorem 3.** *Let $\mathscr{H}$ be a real Hilbert space and $\mathscr{C}$ be a closed convex subset of $\mathscr{H}$. Suppose $f, g, T$, and $\zeta$ are identical as in Theorem 1. Let $\{p_n\}$ and $\{u_n\}$ be sequences defined by (13) and (16), respectively. Suppose that assumption (17) holds and $F(T)\cap Sol(\mathscr{C},f,g)\neq\phi$. If $p_0=u_0$, then $\{u_n\}$ converges faster than $\{p_n\}$ to $u^*$, such that $u^*\in F(T)\cap Sol(\mathscr{C},f,g)$.*

*Proof.* It follows from (27) that

$$\|u_{n+1}-u^*\|\le\zeta^3[1-a_n(1-\zeta)]\|u_n-u^*\|. \tag{37}$$

Since $\{a_n\}$ is a sequence in $(0,1)$, we can choose a constant $a\in\mathbb{R}$, such that $0<a\le a_n<1, \forall n\in\mathbb{N}$. Then,

$$\|u_{n+1}-u^*\|\le\zeta^3[1-a(1-\zeta)]\|u_n-u^*\|. \tag{38}$$

By repeating the process, we obtain

$$\|u_{n+1}-u^*\|\le\zeta^{3(n+1)}[1-a(1-\zeta)]^{n+1}\|u_0-u^*\|, \quad \forall n\in\mathbb{N}. \tag{39}$$

Also, it follows from (13) that

$$\begin{aligned}\|p_{n+1}-u^*\| &= \|T\{q_n - g(q_n) + p_{\mathscr{C}}[g(q_n) - \rho f(q_n)]\} - T\{u^* - g(u^*) + p_{\mathscr{C}}[g(u^*) - \rho f(u^*)]\}\| \\ &\le \|q_n - u^* - (g(q_n) - g(u^*))\| + \|g(q_n) - g(u^*) - \rho(f(q_n) - f(u^*))\| \\ &\le 2\|q_n - u^* - (g(q_n) - g(u^*))\| + \|q_n - u^* - \rho(f(q_n) - f(u^*))\|.\end{aligned} \tag{40}$$

By following the arguments as discussed from (21)to(24), we have

$$\|p_{n+1}-u^*\| \le \zeta\|q_n - u^*\|. \tag{41}$$

Also,

$$\begin{aligned}\|q_n-u^*\| &= \|(1-\xi_n)p_n + \xi_n T\{p_n - g(p_n) + p_{\mathscr{C}}[g(p_n) - \rho f(p_n)]\} - u^*\| \\ &\le (1-\xi_n)\|p_n-u^*\| + 2\xi_n\|p_n - u^* - (g(p_n) - g(u^*))\| \\ &\quad + \xi_n\|p_n - u^* - \rho(f(p_n) - f(u^*))\| \\ &\le (1-\xi_n)\|p_n-u^*\| + \xi_n\zeta\|p_n-u^*\| \\ &= [1-\xi_n(1-\zeta)]\|p_n-u^*\|.\end{aligned} \tag{42}$$

By combining (41) and (42), we get

$$\|p_{n+1}-u^*\| \le \zeta[1-\xi_n(1-\zeta)]\|p_n-u^*\|. \tag{43}$$

Since $\{\xi_n\}$ is a sequence in $[0,1]$, we can choose a constant $\xi \in \mathbb{R}$, such that $0<\xi\le\xi_n<1, \forall n\in\mathbb{N}$. Then,

$$\|p_{n+1}-u^*\| \le \zeta[1-\xi(1-\zeta)]\|p_n-u^*\|. \tag{44}$$

Thus, by repeating the process, we obtain

$$\|p_{n+1}-u^*\| \le \zeta^{(n+1)}[1-\xi(1-\zeta)]^{n+1}\|p_0-u^*\|, \quad \forall n\in\mathbb{N}, \tag{45}$$

Set $a_n = \zeta^{3(n+1)}[1-a(1-\zeta)]^{n+1}\|u_0-u^*\|$ and $b_n = \zeta^{(n+1)}[1-\xi(1-\zeta)]^{n+1}\|p_0-u^*\|$; then,

$$A_n = \frac{a_n}{b_n} = \frac{\zeta^{3(n+1)}[1-a(1-\zeta)]^{n+1}\|u_0-u^*\|}{\zeta^{(n+1)}[1-\xi(1-\zeta)]^{n+1}\|p_0-u^*\|} \longrightarrow 0 \text{ as } n\longrightarrow\infty. \tag{46}$$

Hence, $\{u_n\}$ converges faster that $\{p_n\}$. □

## 3. Applications

*3.1. Convex Minimization Problem.* Now, we solve convex minimization problem as an application of Theorem 1.

Let $\mathscr{C}$ be a closed convex subset of a real Hilbert space $\mathscr{H}$, $\rho_{\mathscr{C}}: \mathscr{H}\longrightarrow\mathscr{C}$ be a projection, and $F: \mathscr{C}\longrightarrow\mathbb{R}$ be a convex, Frechet differentiable mapping. We consider the following convex minimization problem:

$$\min_{u^*\in\mathscr{C}} F(u^*). \tag{47}$$

Clearly, $u^*\in\mathscr{C}$ is a solution of $p_{\mathscr{C}}(I-\rho\nabla F)$ if and only if

$$\langle\nabla F(u^*), u-u^*\rangle \ge 0, \quad \forall u\in\mathscr{C}. \tag{48}$$

More precisely, $u^*\in\mathscr{C}$ solves problem (47) if and only if $u^*$ is a fixed point of the projection mapping $p_{\mathscr{C}}(I-\rho\nabla F)$, i e.,

$$u^* = \rho_{\mathscr{C}}[u^*-\rho\nabla F(u^*)], \tag{49}$$

where $\nabla F$ is the gradient of mapping $F$. This formulation is known as gradient projection, which plays a key role in solving problem (47). So far, several iterative methods have been employed to solve minimization problems [7, 26, 32]. By considering $f: = \nabla F$ and assuming $T=g=I$, the identity mapping, we propose the following modified gradient projection algorithm for solving $p_{\mathscr{C}}(I-\rho\nabla F)$ as follows:

$$\begin{cases} u_1\in\mathscr{C}, \\ w_n = p_{\mathscr{C}}[u_n-\rho\nabla F(u_n)], \\ r_n = (1-a_n)w_n + a_n p_{\mathscr{C}}[w_n-\rho\nabla F(w_n)], \\ v_n = p_{\mathscr{C}}[r_n-\rho\nabla F(r_n)], \\ u_{n+1} = p_{\mathscr{C}}[v_n-\rho\nabla F(v_n)], \end{cases} \tag{50}$$

where $\{a_n\}$ is a sequence in $(0,1)$. Now, we approximate the proposed algorithm (50) to estimate the solution of (47).

**Theorem 4.** *Let $\mathscr{C}$ be a nonempty closed convex subset of real Hilbert space $\mathscr{H}$. Let $F: \mathscr{C}\longrightarrow\mathbb{R}$ be a convex, Freschet differentiable mapping, and $\nabla F$ is a-inverse strongly monotone mapping. Suppose that the convex minimization problem (47) has a solution and condition (17) holds. Then, the sequence $\{u_n\}$ generated by (50) converges strongly to $u^*$ which solves convex minimization problem (47) with the following error estimates:*

$$\|u_{n+1}-u^*\| \le \zeta^{3(n+1)} \prod_{k=0}^{n} \left[1-a_k(1-\zeta)\right]\|u_0-u^*\|, \quad \forall n \in \mathbb{N}, \tag{51}$$

*where*

$$\zeta = \left|\frac{a_1-\rho}{a_1}\right|. \tag{52}$$

*Proof.* The desired conclusion is accomplished by taking $f = \nabla F$ and $T, g = I$ in Theorem 1. □

*Example 2.* Let $\mathscr{H} = L^2[0,1] = \{\mathscr{G}: [0,1] \longrightarrow \mathbb{R}: \int_0^1 \mathscr{G}^2(u)\mathrm{d}u < \infty\}$. Then, $(\mathscr{H}, \|\cdot\|_2)$ is a Hilbert space given by

$$\|\mathscr{G}(u)\|_2^2 = \langle \mathscr{G}(u), \mathscr{G}(u)\rangle = \int_0^1 \mathscr{G}^2(u)\mathrm{d}u. \tag{53}$$

Consider a closed convex subset $\mathscr{C} = \{\mathscr{G} \in L^2[0,1]: \|\mathscr{G}(u)\|_2^2 \le 1\}$ of $\mathscr{H}$. Define $F: \mathscr{C} \longrightarrow \mathbb{R}$ by $F(\mathscr{G}) = \|\mathscr{G}(u)\|_2^2$. Then, $\mathscr{G}(u) = 0$ is a unique minimum of a convex function $f$, and $f$ is the Frechet differentiable at $\mathscr{G}$. The gradient $\nabla F: \mathscr{C} \longrightarrow \mathscr{H}$ is evaluated as $\nabla F(\mathscr{G}) = 2\mathscr{G}$. Then, for all $\mathscr{G}_1, \mathscr{G}_2 \in \mathscr{C}$, we get

$$\begin{aligned}\langle \nabla F(\mathscr{G}_1) - \nabla F(\mathscr{G}_2), \mathscr{G}_1 - \mathscr{G}_2\rangle &= \int_0^1 (2\mathscr{G}_1(u) - 2\mathscr{G}_2(u))(\mathscr{G}_1 - \mathscr{G}_2)\mathrm{d}u \\ &= 2\int_0^1 (\mathscr{G}_1(u) - \mathscr{G}_2(u))^2\mathrm{d}u \\ &\ge -\frac{1}{4}\int_0^1 (2\mathscr{G}_1(u) - 2\mathscr{G}_2(u))^2\mathrm{d}u \\ &= -\frac{1}{4}\|\nabla F(\mathscr{G}_1) - \nabla F(\mathscr{G}_2)\|_2^2,\end{aligned} \tag{54}$$

i.e., $\nabla F$ is 1/4 inverse strongly monotone. Also, $\zeta < 1$ for $\rho = 1/4$. Thus, all the assumptions of Theorem 4 are satisfied, and for $a_n = 1/n+1$, the sequence $\{u_n\}$ generated by (50) is given as

$$\begin{cases} u_0 \in \mathscr{C}, \\ w_n = \mathrm{p}_{\mathscr{C}}\left[\frac{1}{2}u_n\right], \\ r_n = \left(1-\frac{1}{n+1}\right)w_n + \frac{1}{n+1}\mathrm{p}_{\mathscr{C}}\left[\frac{1}{2}w_n\right], \\ v_n = \mathrm{p}_{\mathscr{C}}\left[\frac{1}{2}r_n\right], \\ u_{n+1} = \mathrm{p}_{\mathscr{C}}\left[\frac{1}{2}v_n\right], \end{cases} \tag{55}$$

where $\mathrm{p}_{\mathscr{C}} = \begin{cases} \mathscr{G}, & \mathscr{G} \in \mathscr{C}, \\ \mathscr{G}/\|\mathscr{G}\|, & \mathscr{G} \notin \mathscr{C}. \end{cases}$ Then, the sequence $\{u_n\}$ generated by (50) converges to 0 function.

*3.2. Split Feasibility Problem.* This subsection is devoted to utilization of Theorem 1 to examine a split feasibility problem (SFP). Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be nonempty closed convex subsets of real Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. Let $\mathscr{A}: \mathscr{H}_1 \longrightarrow \mathscr{H}_2$ be a bounded linear operator. The SFP is to locate a point $u^*$, such that

$$u^* \in \mathscr{C}_1: \mathscr{A}u^* \in \mathscr{C}_2. \tag{56}$$

Let $\Gamma$ denotes the solution set of SFP (56); then,

$$\Gamma =: \{u^* \in \mathscr{C}_1: \mathscr{A}u^* \in \mathscr{C}_2\} = \mathscr{C}_1 \cap \mathscr{A}^{-1}\mathscr{C}_2. \tag{57}$$

A class of inverse problems has been solved by using SFP, for example, [6]. In [32], Xu established the relationship between SFP (56) and the fixed point of problem $\mathrm{p}_{\mathscr{C}_1}[I-\rho\mathscr{A}^*(I-\mathrm{p}_{\mathscr{C}_2})\mathscr{A}]$. More precisely, for $\rho > 0$, $u^* \in \mathscr{C}_1$ solves SFP (56) if and only if $\mathrm{p}_{\mathscr{C}_1}[I-\rho\mathscr{A}^*(I-\mathrm{p}_{\mathscr{C}_2})\mathscr{A}](u^*) = u^*$. Byrne [5] posed the following iterative algorithm for solving SFP (56) as follows:

$$u_{n+1} = \mathrm{p}_{\mathscr{C}_1}\left[I-\rho\mathscr{A}^*\left(I-\mathrm{p}_{\mathscr{C}_2}\right)\mathscr{A}\right](u_n), \quad \forall n \ge 0, \tag{58}$$

where $0 < \rho < 2/\|\mathscr{A}\|^2$, $\mathscr{A}^*$ is the adjoint of operator $\mathscr{A}$, and $\mathrm{p}_{\mathscr{C}_1}$ and $\mathrm{p}_{\mathscr{C}_2}$ are the projections onto $\mathscr{C}_1$ and $\mathscr{C}_2$, respectively.

Note that the operator $\mathrm{p}_{\mathscr{C}_1}[I-\rho\mathscr{A}^*(I-\mathrm{p}_{\mathscr{C}_2})\mathscr{A}]$ with $0 < \rho < 2/\|\mathscr{A}\|^2$ is nonexpansive. Now, we propose following iterative algorithm to solve SFP (56):

$$\begin{cases} u_1 \in \mathscr{C}_1, \\ w_n = \mathrm{p}_{\mathscr{C}_1}\left[I-\rho\mathscr{A}^*\left(I-\mathrm{p}_{\mathscr{C}_2}\right)\mathscr{A}\right](u_n), \\ r_n = (1-a_n)w_n + a_n\mathrm{p}_{\mathscr{C}_1}\left[I-\rho\mathscr{A}^*\left(I-\mathrm{p}_{\mathscr{C}_2}\right)\mathscr{A}\right](w_n), \\ v_n = \mathrm{p}_{\mathscr{C}_1}\left[I-\rho\mathscr{A}^*\left(I-\mathrm{p}_{\mathscr{C}_2}\right)\mathscr{A}\right](r_n), \\ u_{n+1} = \mathrm{p}_{\mathscr{C}_1}\left[I-\rho\mathscr{A}^*\left(I-\mathrm{p}_{\mathscr{C}_2}\right)\mathscr{A}\right](v_n), \end{cases} \tag{59}$$

where $\{a_n\}$ is a sequence in $(0,1)$ and $0 < \rho < 2/\|\mathscr{A}\|^2$.

**Theorem 5.** *Suppose that $\Gamma \neq \phi$ and condition (17) holds. Then, the sequence $\{u_n\}$ initiated in (59) converges weakly to $u^*$, which solves SFP (56) with following error estimates:*

$$\|u_{n+1} - u^*\| \leq \zeta^{3(n+1)} \prod_{k=0}^{n} [1 - a_k(1-\zeta)] \|u_0 - u^*\|, \quad \forall n \in \mathbb{N}, \quad (60)$$

*where*

$$\zeta = \left| \frac{a_1 - \rho}{a_1} \right|. \quad (61)$$

*Proof.* The desired conclusion follows by taking $\nabla F = \mathscr{A}^*(I - \mathrm{p}_{\mathscr{C}_2})\mathscr{A}]$ and $T, g = I$ in Theorem 1. □

## 4. Conclusion

In this study, a new iterative algorithm (16) has been proposed and employed to explore convergence analysis. Using this newly constructed iterative procedure, a common solution of the generalized variational inequality problem and fixed points of nonexpansive mapping is investigated, and theoretical findings are verified by a numerical example. Furthermore, we have shown that our iteration algorithm converges faster than the normal $S$-iteration process for contraction mapping. Finally, we applied our newly constructed iterative algorithm to investigate the convex optimization problem and split feasibility problem.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Acknowledgments

The first and fourth authors would like to thank the Deanship of Scientific Research, Prince Sattam bin Abdulaziz University, for supporting this work.

## References

[1] R. P. Agarwal, D. O. Regan, and D. R. Sahu, "Iterative construction of fixed points of nearly asymptotically nonexpansive mappings," *J. Nonlinear Convex Anal.*vol. 8, no. 1, pp. 61–79, 2007.

[2] M. Akram, A. Khan, and M. Dilshad, "Convergence of some iterative algorithms for system of generalized set-valued variational inequalities," *J. Funct. Spaces*, vol. 2021, 2021.

[3] V. Berinde, "Picard iteration converges faster than Mann iteration for a class of quasi-contractive operators," *Fixed Point Theory and Applications*, vol. 2004, no. 2, pp. 97–105, 2004.

[4] V. Berinde, "On a family of first order difference inequalities used in the iterative approximation of fixed points," *Creat. Math. Inform.*vol. 18, pp. 110–122, 2009.

[5] C. Byrne, "A unified treatment of some iterative algorithms in signal processing and image reconstructions," *Inverse Problems*, vol. 20, no. 1, 2003.

[6] Y. Censor and T. Elfving, "A multiprojection algorithm using Bregman projections in a product space," *Numerical Algorithms*, vol. 8, no. 24, pp. 221–239, 1994.

[7] S. S. Chang, M. Liu, L. Zhao, and J. Tang, "A modified hybrid algorithm for solving a composite minimization problem in Banach spaces," *J. Nonlinear Var. Anal.*vol. 3, pp. 357–364, 2019.

[8] P. L. Combettes and V. R. Wajs, "Signal recovery by proximal forward-backward splitting," *Multiscale Modeling and Simulation*, vol. 4, no. 4, pp. 1168–1200, 2005.

[9] M. Dilshad, A. Khan, A. Khan, and M. Akram, "Splitting type viscosity methods for inclusion and fixed point problems on Hadamard manifolds," *AIMS Mathematics*, vol. 6, no. 5, pp. 5205–5221, 2021.

[10] M. Dilshad, A. F. Aljohani, and M. Akram, "Iterative scheme for split variational inclusion and a fixed-point problem of a finite collection of nonexpansive mappings," *J. Funct. Spaces*, vol. 2020, 2020.

[11] C. Garodia and I. Uddin, "A new fixed point algorithm for finding the solution of a delay differential equation," *AIMS Mathematics*, vol. 5, no. 4, pp. 3182–3200, 2020.

[12] K. Goebel and S. Reich, *Uniform Covexity, Hyperbolic Geometry and Nonexpansive Mappings*, Marcel Deccer, New York, 1984.

[13] F. Gürsoy and V. Karakaya, "A Picard-*S* hybrid type iteration method for solving a differential equation with retarded argument," 2014, https://arxiv.org/abs/1403.2546v2.

[14] F. Gürsoy, M. Ertürk, and M. Abbas, "A Picard-type iterative algorithm for general variational inequalities and nonexpansive mappings," *Numerical Algorithms*, vol. 83, no. 3, pp. 867–883, 2020.

[15] M. C. Ferris and J. S. Pang, "Engineering and economic applications of complementarity problems," *SIAM Review*, vol. 39, no. 4, pp. 669–713, 2006.

[16] D. Kinderlehrar and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980.

[17] M. A. Noor, "General variational inequalities," *Applied Mathematics Letters*, vol. 1, no. 2, pp. 119–122, 1988.

[18] M. A. Noor, "General variational inequalities and nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 2, pp. 810–822, 2007.

[19] G. A. Okeke, M. Abbas, M. De la Sen, and H. Iqbal, "Accelerated modified tseng's extragradient method for solving variational inequality problems in Hilbert spaces," *Axioms*, vol. 10, no. 4, p. 248, 2021.

[20] J. Puangpee and S. Suantai, "A new accelerated viscosity iterative method for an infinite family of nonexpansive mappings with applications to image restoration problems," *Mathematics*, vol. 8, no. 4, p. 615, 2020.

[21] D. R. Sahu, A. Pitea, and M. Verma, "A new iteration technique for nonlinear operators as concerns convex programming and feasibility problems," *Numerical Algorithms*, vol. 83, no. 2, pp. 421–449, 2019.

[22] Y. Shehu, "An iterative method for fixed point problems, variational inclusions and generalized equilibrium problems," *Mathematical and Computer Modelling*, vol. 54, no. 5-6, pp. 1394–1404, 2011.

[23] G. Stampacchia, "Formes bilineaires coercivites sur les ensembles convexes," *Comptes Rendus de l'Académie des Sciences*, vol. 258, pp. 4413–4416, 1964.

[24] Y.-F. Sun, Z. Zeng, Z. Zeng, and J. Song, "Quasilinear iterative method for the boundary value problem of nonlinear

fractional differential equation," *Numerical Algebra, Control and Optimization*, vol. 10, no. 2, pp. 157–164, 2020.

[25] B. Tan, S. Li, and X. Qin, "Self-adaptive inertial single projection methods for variational inequalities involving non-Lipschitz and Lipschitz operators with their applications to optimal control problems," *Applied Numerical Mathematics*, vol. 170, pp. 219–241, 2021.

[26] S. Treanta, "Well posedness of new optimization problems with variational inequality constraints," *Fract*, vol. 5, p. 123, 2021.

[27] K. Ullah and M. Arshad, "Numerical reckoning fixed points for Suzuki's generalized nonexpansive mappings via new iteration process," *Filomat*, vol. 32, no. 1, pp. 187–196, 2018.

[28] Q.-W. Wang, J.-L. Guan, L.-C. Ceng, and B. Hu, "General iterative methods for systems of variational inequalities with the constraints of generalized mixed equilibria and fixed point problem of pseudocontractions," *Journal of Inequalities and Applications*, vol. 2018, no. 1, p. 315, 2018.

[29] F. Wang and H. Pham, "On a new algorithm for solving variational inequality and fixed point problems," *J. Nonlinear Var. Anal.*vol. 3, pp. 225–233, 2019.

[30] S. Wang, "On fixed point and variational inclusion problems," *Filomat*, vol. 29, no. 6, pp. 1409–1417, 2015.

[31] X. Weng, "Fixed point iteration for local strictly pseudo-contractive mapping," *Proceedings of the American Mathematical Society*, vol. 113, no. 3, p. 727, 1991.

[32] H. K. Xu, "Fixed point Algorithm in convex optimization," in *Topics in Nonlinear Analysis and Optimization*, Q. H. Ansari, Ed., pp. 87–136, World Education, Delhi, India, 2012.

[33] Y. Yao, Y.-C. Liou, and S. M. Kang, "Two-step projection methods for a system of variational inequality problems in Banach spaces," *Journal of Global Optimization*, vol. 55, no. 4, pp. 801–811, 2013.

[34] Y. Yao, M. Postolache, and J. C. Yao, "An iterative algorithm for solving generalized variational inequalities and fixed points problems," *Mathematics*, vol. 7, no. 1, 2019.