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Research Article

The Existence and Uniqueness of Solutions for Variable-Order Fractional Differential Equations with Antiperiodic Fractional Boundary Conditions

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In this paper, we discuss the existence and uniqueness of solutions for nonlinear fractional differential equations of variable order with fractional antiperiodic boundary conditions. The main results are obtained by using fixed point theorem.

1. Introduction

Fractional calculus has become one of the important tools for the development of modern society; the fractional differential equation with variable order has gained lots of interest [1–4]. Some researchers have investigated the physical background and numerical analysis of fractional differential equations of variable order [5–8]. In [9], Bushnaq et al. used Bernstein polynomials with nonorthogonal basis to establish operational matrices for variable-order integration and differentiation which convert the considered problem to some algebraic type matrix equations and obtained numerical solution to variable-order fractional differential equations by numerical simulation. In [10], Shah et al. proposed a new algorithm for numerical solutions to variable-order partial differential equations, used properties of shifted Legendre polynomials to establish some operational matrices of variable-order differentiation and integration, and got the numerical solution by numerical experiments.

In recent years, the antiperiodic boundary value problem of fractional differential equation has gradually become the focus of research, which have broad application in engineering and sciences such as physics, mechanics, chemistry, economics, and biology [11–17]. In [18], Ahmad and Nieto considered the following antiperiodic fractional boundary value problems:

$$\begin{cases} {}_0^C D_t^q x(t) = f(t, x(t)), & t \in [0, T], \\ x(0) = -x(T), \quad {}_0^C D_t^p x(0) = -{}_0^C D_t^p x(T), \end{cases} \tag{1}$$

where ${}^C D^p$ denotes the Caputo fractional derivative of order q and f is a given continuous function.

The problems related to the antiperiodic boundary value condition have been considered in [19–26], but the antiperiodic boundary value problem of fractional differential equation with variable order is almost not considered. In this paper, we investigate the existence of solutions for an antiperiodic fractional boundary value problem given by

$$\begin{cases} {}_0^C D_t^{q(t)} x(t) = f(t, x(t)), & t \in [0, T], \\ x(0) = -x(T), \quad {}_0^C D_t^p x(0) = -{}_0^C D_t^p x(T), \end{cases} \tag{2}$$

where ${}^C D^p$ denotes the Caputo fractional derivative of order p, $0 < p < 1$, ${}^C D^{q(t)}$ denotes the Caputo fractional derivative of variable order $q(t)$, $1 < q(t) \le 2$, T is a positive constant, and $f : [0, T] \times \mathfrak{R} \longrightarrow \mathfrak{R}$ is a given continuous function.

2. Preliminary Knowledge

In this section, we introduce some fundamental definitions and lemmas.

Definition 1 (see [27]). The Riemann-Liouville fractional integral of order q for a continuous function f: $[0,\infty) \longrightarrow \mathfrak{R}$ is defined as

$$ {}_0I_t^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds, \quad q > 0, \tag{3} $$

provided the integral exists.

Definition 2 (see [27]). For $(n-1)$ times absolutely continuous function f:$[0,\infty) \longrightarrow \mathfrak{R}$, the Caputo derivative of fractional order q is defined as

$$ {}_0^C D_t^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(n)}(s) ds, \quad n-1 < q < n, \; n = [q] + 1, \tag{4} $$

where $[q]$ denotes the integer part of the real number q.

Definition 3 (see [3]). The Riemann-Liouville fractional integral of variable order $q(t)$ for a continuous function f: $[0, \infty) \longrightarrow \mathfrak{R}$ is defined as

$$ {}_0I_t^{q(t)} f(t) = \int_0^t \frac{(t-s)^{q(t)-1}}{\Gamma(q(t))} f(s) ds, \quad q(t) > 0, \; t > 0, \tag{5} $$

provided that the right-hand side is pointwise defined.

Definition 4 (see [3]). For $(n-1)$ times absolutely continuous function f: $[0,\infty) \longrightarrow \mathfrak{R}$, the Caputo fractional derivative of variable order $q(t)$ is defined as

$$ {}_0^C D_t^{q(t)} f(t) = \begin{cases} \displaystyle\int_0^t \frac{(t-s)^{n-q(t)-1}}{\Gamma(n-q(t))} f^{(n)}(s) ds, & n-1 < q(t) < n, \\ \displaystyle\frac{d^n f}{dt^n}, & n = q(t). \end{cases} \tag{6} $$

Definition 5 (see [25]). Let $I \subset \mathfrak{R}$, I is called a generalized interval if it is either an interval, or $\{a\}$ or $\varnothing$.

A finite set θ is called a partition of I if each x in I lies in exactly one of the generalized intervals ξ in θ.

A function f: $I \longrightarrow \mathfrak{R}$ is called piecewise constant with respect to partition θ of I if for any $\xi \in \theta$, f is constant on ξ.

Theorem 6 (see [27]). *Let E be a closed, convex, and nonempty subset of a Banach space X; let F: $E \longrightarrow E$ be a continuous mapping such that FE is a relatively compact subset of X. Then, F has at least one fixed point in E.*

Lemma 7 (see [27]). *Let $\alpha > 0$, and let $y(t) \in L_\infty(a, b)$ or $y(t) \in C[a, b]$, then ${}_a^C D_t^\alpha {}_a I_t^\alpha y(t) = y(t)$.*

Lemma 8 (see [27]). *Let $n = [\alpha] + 1$ for $\alpha \notin N_0$; $n = \alpha$ for $\alpha \in N_0$, if $y(t) \in AC^n[a, b]$ or $y(t) \in C^n[a, b]$, then*

$$ {}_a I_t^\alpha {}_a^C D_t^\alpha y(t) = y(t) - \sum_{k=0}^{n-1} \frac{y^k(a)}{k!} (t-a)^k. \tag{7} $$

3. Main Results

Let $J = [0, T]$. Denote $C(J, \mathfrak{R})$ be the Banach space of all continuous functions $x : J \longrightarrow \mathfrak{R}$ with the norm $\|x\| = \sup_{t \in J} |x(t)|$ and introduce the following assumption.

(H_1) Let $n \in N$ be an integer, $\theta = \{J_1 = [0, T_1], J_2 = (T_1, T_2], \cdots, J_n = (T_{n-1}, T_n]\}$ be a partition of the interval J, and $q(t)$: $J \longrightarrow (1, 2]$ be a piecewise constant function with respect to θ with the following forms:

$$ q(t) = \sum_{i=1}^n q_i I_i(t) = \begin{cases} q_1, & \text{if } t \in J_1, \\ q_2, & \text{if } t \in J_2, \\ \vdots \\ q_n, & \text{if } t \in J_n, \end{cases} \tag{8} $$

where $1 < q_i \le 2$ are constants, and I_i is the indicator of the interval $(T_{i-1}, T_i]$, $i = 1, 2, \cdots, n$ (with $T_0 = 0$, $T_n = T$), $I_i(t) = 1$ for $t \in (T_{i-1}, T_i]$, and $I_i(t) = 0$ for elsewhere.

Let $\Omega_i = C(J_i, \mathfrak{R})$ be the Banach space of all continuous functions $x : J_i \longrightarrow \mathfrak{R}$ with the norm $\|x\|_{\Omega_i} = \sup_{t \in J_i} |x(t)|$, $i = 1, 2, \cdots, n$ (with $T_0 = 0$, $T_n = T$).

The Caputo fractional derivative of variable order $q(t)$ for the function $x(t)$ could be presented as a sum of Caputo fractional derivatives of constant orders q_i by Definition 4, $i = 1, 2, \cdots, n$:

$$ {}_0^C D_t^{q(t)} x(t) = \int_0^{T_1} \frac{(t-s)^{1-q_1}}{\Gamma(2-q_1)} x^{(2)}(s) ds + \cdots + \int_{T_{i-1}}^t \frac{(t-s)^{1-q_i}}{\Gamma(2-q_i)} x^{(2)}(s) ds. \tag{9} $$

Thus, according to (9), problem (2) can be written in the following form:

$$ {}_0^C D_t^{q(t)} x(t) = \int_0^{T_1} \frac{(t-s)^{1-q_1}}{\Gamma(2-q_1)} x^{(2)}(s) ds + \cdots + \int_{T_{i-1}}^t \frac{(t-s)^{1-q_i}}{\Gamma(2-q_i)} x^{(2)}(s) ds = f(t, x(t)). \tag{10} $$

Definition 9. The problem (2) has a solution, if there are functions x_i, so that $x_i \in C(J_i, \mathfrak{R})$, satisfy (10) and $x_i(0) = -x_i(T)$, ${}_0^C D_t^p x_i(0) = -{}_0^C D_t^p x_i(T)$.

Let the function $x \in C(J, \mathfrak{R})$ be such that $x(t) \equiv x(T_{i-1})$ on $[0, T_{i-1}]$, then consider (2) as the following form:

$$\begin{cases} {}_{T_{i-1}}^{C} D_t^{q_i} x(t) = f(t, x(t)), & t \in (T_{i-1}, T_i], \\ x(T_{i-1}) = -x(T_i), & {}_{T_{i-1}}^{C} D_t^{p} x(T_{i-1}) = -{}_{T_{i-1}}^{C} D_t^{p} x(T_i). \end{cases} \tag{11}$$

Proposition 10. *For any $x(t) \in \Omega_i$, $f(t, x(t)) \in C(J_i \times \mathfrak{R}, \mathfrak{R})$, $x(t)$ is a unique solution of problem (11) if and only if x satisfy the integral equation:*

$$x(t) = \int_{T_{i-1}}^{T_i} G_i(t,s) f(s, x(s)) ds, \tag{12}$$

where $G_i(t,s)$ is Green's function given by

$$G_i(t,s) = \begin{cases} \dfrac{(t-s)^{q_i-1}}{\Gamma(q_i)} + \dfrac{(T_i - s)^{q_i-p-1}\Gamma(2-p)(T_i - T_{i-1} - 2t)}{2(T_i - T_{i-1})^{1-p}\Gamma(q_i - p)} - \dfrac{(T_i - s)^{q_i-1}}{2\Gamma(q_i)}, & s \le t, \\ \dfrac{(T_i - s)^{q_i-p-1}\Gamma(2-p)(T_i - T_{i-1} - 2t)}{2(T_i - T_{i-1})^{1-p}\Gamma(q_i - p)} - \dfrac{(T_i - s)^{q_i-1}}{2\Gamma(q_i)}, & t \le s. \end{cases} \tag{13}$$

Proof. If $x(t) \in \Omega_i$ is a solution of problem (11), applying ${}_{T_{i-1}}I_t^{q_i}$ on both sides of (11), according to Lemma 8, we get

$$x(t) = {}_{T_{i-1}}I_t^{q_i} f(t, x(t)) + c_1 + c_2(t - T_{i-1}), \tag{14}$$

according to the facts that ${}_{T_{i-1}}^{C} D_t^p c_1 = 0$, ${}_{T_{i-1}}^{C} D_t^p t = (t - T_{i-1})^{1-p}/\Gamma(2-p)$, ${}_{T_{i-1}}^{C} D_t^p {}_{T_{i-1}}I_t^{q_i} x(t) = {}_{T_{i-1}}I_t^{q_i - p} x(t)$, and initial condition of problem (11), we get

$$\begin{aligned} c_2 &= -\frac{\Gamma(2-p)}{\Gamma(q_i - p)(T_i - T_{i-1})^{1-p}} \int_{T_{i-1}}^{T_i} (T_i - s)^{q_i-p-1} f(s, x(s)) ds, \\ c_1 &= \frac{(T_i - T_{i-1})\Gamma(2-p)}{2(T_i - T_{i-1})^{1-p}\Gamma(q_i - p)} \int_{T_{i-1}}^{T_i} (T_i - s)^{q_i-p-1} f(s, x(s)) ds \\ &\quad - \frac{1}{2\Gamma(q_i)} \int_{T_{i-1}}^{T_i} (T_i - s)^{q_i-1} f(s, x(s)) ds. \end{aligned} \tag{15}$$

Thus, the solution of problem (11) is

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(q_i)} \int_{T_{i-1}}^{t} (t-s)^{q_i-1} f(s, x(s)) ds + \frac{(T_i - T_{i-1} - 2t)\Gamma(2-p)}{2(T_i - T_{i-1})^{1-p}\Gamma(q_i - p)} \\ &\quad \cdot \int_{T_{i-1}}^{T_i} (T_i - s)^{q_i-p-1} f(s, x(s)) ds - \frac{1}{2\Gamma(q_i)} \int_{T_{i-1}}^{T_i} (T_i - s)^{q_i-1} f(s, x(s)) ds. \end{aligned} \tag{16}$$

Green's function can be written as

$$x(t) = \int_{T_{i-1}}^{T_i} G_i(t,s) f(s, x(s)) ds. \tag{17}$$

It implies that $x(t)$ is the solution to the integral equation (12). In turn, if $x(t) \in \Omega_i$ is the solution to the integral equation (12), according to Lemma 7, we deduce that $x(t)$ is the solution of the problem (11). Hence, we complete this proof. □

Theorem 11. *Assume that $(H_2) f(t, x(t)) \in C(J_i \times \mathfrak{R}, \mathfrak{R})$ and there exists a positive constant L such that $|f(t, x(t))| \le L$, for any $t \in (T_{i-1}, T_i]$, $x(t) \in \mathfrak{R}$. Then, problem (11) has at least a solution.*

Proof. According to Proposition 10, problem (11) is equivalent to the following integral equation:

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(q_i)} \int_{T_{i-1}}^{t} (t-s)^{q_i-1} f(s, x(s)) ds \\ &\quad + \frac{(T_i - T_{i-1} - 2t)\Gamma(2-p)}{2(T_i - T_{i-1})^{1-p}\Gamma(q_i - p)} \int_{T_{i-1}}^{T_i} (T_i - s)^{q_i-p-1} f(s, x(s)) ds \\ &\quad - \frac{1}{2\Gamma(q_i)} \int_{T_{i-1}}^{T_i} (T_i - s)^{q_i-1} f(s, x(s)) ds. \end{aligned} \tag{18}$$

Define operator $T:\Omega_i \longrightarrow \Omega_i$ by

$$Tx(t)=\frac{1}{\Gamma(q_i)}\int_{T_{i-1}}^{t}(t-s)^{q_i-1}f(s,x(s))ds+\frac{(T_i-T_{i-1}-2t)\Gamma(2-p)}{2(T_i-T_{i-1})^{1-p}\Gamma(q_i-p)}\cdot\int_{T_{i-1}}^{T_i}(T_i-s)^{q_i-p-1}f(s,x(s))ds-\frac{1}{2\Gamma(q_i)}\int_{T_{i-1}}^{T_i}(T_i-s)^{q_i-1}f(s,x(s))ds, \tag{19}$$

$B_{r_i}=\{x(t)\in\Omega_i, \|x\|_{\Omega_i}\le r_i, t\in J_i\}$,where $r_i\ge L(T_i-T_{i-1})^{q_i-1}[3\Gamma(q_i-p+1)(T_i-T_{i-1})+\Gamma(2-p)\Gamma(q_i+1)(T_i+T_{i-1})]/2\Gamma(q_i-p+1)\Gamma(q_i+1)$, observe that B_{r_i} is a closed, bounded, and convex subset of Banach space Ω_i. For any $x(t)\in\Omega_i$, we have

$$\begin{aligned}|Tx(t)|&=\left|\frac{1}{\Gamma(q_i)}\int_{T_{i-1}}^{t}(t-s)^{q_i-1}f(s,x(s))ds+\frac{(T_i-T_{i-1}-2t)\Gamma(2-p)}{2(T_i-T_{i-1})^{1-p}\Gamma(q_i-p)}\int_{T_{i-1}}^{T_i}(T_i-s)^{q_i-p-1}f(s,x(s))ds-\frac{1}{2\Gamma(q_i)}\int_{T_{i-1}}^{T_i}(T_i-s)^{q_i-1}f(s,x(s))ds\right|\\
&\le L\left[\frac{1}{\Gamma(q_i)}\int_{T_{i-1}}^{t}(t-s)^{q_i-1}ds+\frac{|T_i-T_{i-1}-2t|\Gamma(2-p)}{2(T_i-T_{i-1})^{1-p}\Gamma(q_i-p)}\int_{T_{i-1}}^{T_i}(T_i-s)^{q_i-p-1}ds+\frac{1}{2\Gamma(q_i)}\int_{T_{i-1}}^{T_i}(T_i-s)^{q_i-1}ds\right]\\
&\le L\left[\frac{(t-T_{i-1})^{q_i}}{\Gamma(q_i+1)}+\frac{(T_i-T_{i-1})^{q_i-p}}{\Gamma(q_i-p+1)}\frac{|T_i-T_{i-1}-2t|\Gamma(2-p)}{2(T_i-T_{i-1})^{1-p}}+\frac{(T_i-T_{i-1})^{q_i}}{2\Gamma(q_i+1)}\right]\\
&\le\frac{L(T_i-T_{i-1})^{q_i-1}[3\Gamma(q_i-p+1)(T_i-T_{i-1})+\Gamma(2-p)\Gamma(q_i+1)(T_i+T_{i-1})]}{2\Gamma(q_i-p+1)\Gamma(q_i+1)}\le r_i.\end{aligned} \tag{20}$$

It implies $T:\Omega_i\longrightarrow\Omega_i$ is well defined.

Now, we consider the continuity of operator T. Since $f(t,x(t))\in C(J_i\times\mathfrak{R},\mathfrak{R})$, given an arbitrary $\varepsilon>0$, for any $x(t), y(t)\in\Omega_i$,we can find$\delta>0$such that $|f(t,x(t))-f(t,y(t))|<2\Gamma(q_i-p+1)\Gamma(q_i+1)\varepsilon/(T_i-T_{i-1})^{q_i-1}[3\Gamma(q_i-p+1)(T_i-T_{i-1})+\Gamma(2-p)\Gamma(q_i+1)(T_i+T_{i-1})]$. When $|x(t)-y(t)|<\delta$ for $t\in J_i$, for any $t\in J_i$, we have

$$\begin{aligned}|Tx(t)-Ty(t)|&\le\frac{1}{\Gamma(q_i)}\int_{T_{i-1}}^{t}(t-s)^{q_i-1}|f(s,x(s))-f(s,y(s))|ds\\
&\quad+\frac{(T_i-T_{i-1}-2t)\Gamma(2-p)}{2(T_i-T_{i-1})^{1-p}\Gamma(q_i-p)}\int_{T_{i-1}}^{T_i}(T_i-s)^{q_i-p-1}|f(s,x(s))-f(s,y(s))|ds+\frac{1}{2\Gamma(q_i)}\int_{T_{i-1}}^{T_i}(T_i-s)^{q_i-1}|f(s,x(s))-f(s,y(s))|ds\\
&\le\left[\frac{1}{\Gamma(q_i)}\int_{T_{i-1}}^{t}(t-s)^{q_i-1}ds+\frac{|T_i-T_{i-1}-2t|\Gamma(2-p)}{2(T_i-T_{i-1})^{1-p}\Gamma(q_i-p)}\int_{T_{i-1}}^{T_i}(T_i-s)^{q_i-p-1}ds+\frac{1}{2\Gamma(q_i)}\int_{T_{i-1}}^{T_i}(T_i-s)^{q_i-1}ds\right]\\
&\quad\cdot\left[\frac{2\Gamma(q_i-p+1)\Gamma(q_i+1)\varepsilon}{(T_i-T_{i-1})^{q_i-1}[3\Gamma(q_i-p+1)(T_i-T_{i-1})+\Gamma(2-p)\Gamma(q_i+1)(T_i+T_{i-1})]}\right]\\
&\le\left[\frac{(t-T_{i-1})^{q_i}}{\Gamma(q_i+1)}+\frac{(T_i-T_{i-1})^{q_i-p}}{\Gamma(q_i-p+1)}\frac{|T_i-T_{i-1}-2t|\Gamma(2-p)}{2(T_i-T_{i-1})^{1-p}}+\frac{(T_i-T_{i-1})^{q_i}}{2\Gamma(q_i+1)}\right]\\
&\quad\times\left[\frac{2\Gamma(q_i-p+1)\Gamma(q_i+1)\varepsilon}{(T_i-T_{i-1})^{q_i-1}[3\Gamma(q_i-p+1)(T_i-T_{i-1})+\Gamma(2-p)\Gamma(q_i+1)(T_i+T_{i-1})]}\right]\\
&<\frac{(T_i-T_{i-1})^{q_i-1}[3\Gamma(q_i-p+1)(T_i-T_{i-1})+\Gamma(2-p)\Gamma(q_i+1)(T_i+T_{i-1})]}{2\Gamma(q_i-p+1)\Gamma(q_i+1)}\\
&\quad\times\left[\frac{2\Gamma(q_i-p+1)\Gamma(q_i+1)\varepsilon}{(T_i-T_{i-1})^{q_i-1}[3\Gamma(q_i-p+1)(T_i-T_{i-1})+\Gamma(2-p)\Gamma(q_i+1)(T_i+T_{i-1})]}\right]=\varepsilon.\end{aligned} \tag{21}$$

We get the operator T is continuous.

For each $x(t) \in \Omega_i$, we prove that if $t_1, t_2 \in J_i$, and $0 < t_2 - t_1 < \delta$, then $\|Tx(t_2) - Tx(t_1)\| < \varepsilon$:

$$\begin{aligned}
&\|Tx(t_2) - Tx(t_1)\| \\
&= \left| \frac{1}{\Gamma(q_i)} \int_{T_{i-1}}^{t_2} (t_2 - s)^{q_i - 1} f(s, x(s)) ds - \frac{1}{\Gamma(q_i)} \int_{T_{i-1}}^{t_1} (t_1 - s)^{q_i - 1} f(s, x(s)) ds \right. \\
&\quad \left. + \frac{(t_1 - t_2)\Gamma(2-p)}{(T_i - T_{i-1})^{1-p}\Gamma(q_i - p)} \int_{T_{i-1}}^{T_i} (T_i - s)^{q_i - p - 1} f(s, x(s)) ds \right| \\
&\le \frac{1}{\Gamma(q_i)} \int_{T_{i-1}}^{t_1} (t_2 - s)^{q_i - 1} - (t_1 - s)^{q_i - 1} |f(s, x(s))| ds \\
&\quad + \frac{1}{\Gamma(q_i)} \int_{t_1}^{t_2} (t_2 - s)^{q_i - 1} |f(s, x(s))| ds + \frac{(t_2 - t_1)\Gamma(2-p)}{(T_i - T_{i-1})^{1-p}\Gamma(q_i - p)} \\
&\quad \times \int_{T_{i-1}}^{T_i} (T_i - s)^{q_i - p - 1} |f(s, x(s))| ds \\
&\le L \left[\frac{(t_2 - T_{i-1})^{q_i} - (t_1 - T_{i-1})^{q_i}}{\Gamma(q_i + 1)} + \frac{(t_2 - t_1)(T_i - T_{i-1})^{q_i - 1}\Gamma(2-p)}{\Gamma(q_i - p + 1)} \right].
\end{aligned} \tag{22}$$

By the mean value theorem, we have

$$\begin{aligned}
\|Tx(t_2) - Tx(t_1)\| &\le L \left[\frac{(t_2 - T_{i-1})^{q_i} - (t_1 - T_{i-1})^{q_i}}{\Gamma(q_i + 1)} \right. \\
&\quad \left. + \frac{(t_2 - t_1)(T_i - T_{i-1})^{q_i - 1}\Gamma(2-p)}{\Gamma(q_i - p + 1)} \right] \\
&\le \left[L \frac{(T_i - T_{i-1})^{q_i - 1}}{\Gamma(q_i)} + L \frac{(T_i - T_{i-1})^{q_i - 1}\Gamma(2-p)}{\Gamma(q_i - p + 1)} \right] \delta < \varepsilon.
\end{aligned} \tag{23}$$

Therefore, $\|Tx(t_2) - Tx(t_1)\| < \varepsilon$. According to the previous analysis, we know that is equicontinuous and uniformly bounded. We know by the Arzela-Ascoli theorem that T is compact on B_{r_i}, so the operator T is completely continuous. So, Theorem 11 implies that the antiperiodic boundary value problem of variable order (11) has at least a solution on J_i. This completes the proof. □

Theorem 12. *Assume that* $(H_3) f(t, x(t)) \in C(J_i \times \mathfrak{R}, \mathfrak{R})$ *such that* $|f(t, x(t)) - f(t, y(t))| \le K|x(t) - y(t)|$, *for any* $t \in J_i, x(t), y(t) \in \mathfrak{R}$, *and if* $K(T_i - T_{i-1})^{q_i - 1}[3\Gamma(q_i - p + 1)(T_i - T_{i-1}) + \Gamma(2-p)\Gamma(q_i + 1)(T_i + T_{i-1})]/\Gamma(q_i - p + 1)\Gamma(q_i + 1) < 1$, *then problem (11) has a unique solution.*

Proof. Let $\sup_{t \in J_i} |f(t, 0)| = M < \infty$, $B_{r_i} = \{x(t) \in \Omega_i, \|x\|_{\Omega_i} \le r_i, t \in J_i\}$, where

$$r_i \ge \frac{M(T_i - T_{i-1})^{q_i - 1}[3\Gamma(q_i - p + 1)(T_i - T_{i-1}) + \Gamma(2-p)\Gamma(q_i + 1)(T_i + T_{i-1})]}{\Gamma(q_i - p + 1)\Gamma(q_i + 1)}. \tag{24}$$

For any $x(t) \in B_{r_i}$, we have

$$\begin{aligned}
|Tx(t)| &\le \frac{1}{\Gamma(q_i)} \int_{T_{i-1}}^{t} (t - s)^{q_i - 1} |f(s, x(s))| ds + \frac{|T_i - T_{i-1} - 2t|\Gamma(2-p)}{2(T_i - T_{i-1})^{1-p}\Gamma(q_i - p)} \int_{T_{i-1}}^{T_i} (T_i - s)^{q_i - p - 1} |f(s, x(s))| ds \\
&\quad + \frac{1}{2\Gamma(q_i)} \int_{T_{i-1}}^{T_i} (T_i - s)^{q_i - 1} |f(s, x(s))| ds \le \frac{1}{\Gamma(q_i)} \int_{T_{i-1}}^{t} (t - s)^{q_i - 1} (|f(s, x(s)) - f(s, 0)| + |f(s, 0)|) ds \\
&\quad + \frac{|T_i - T_{i-1} - 2t|\Gamma(2-p)}{2(T_i - T_{i-1})^{1-p}\Gamma(q_i - p)} \int_{T_{i-1}}^{T_i} (T_i - s)^{q_i - p - 1} (|f(s, x(s)) - f(s, 0)| + |f(s, 0)|) ds \\
&\quad + \frac{1}{2\Gamma(q_i)} \int_{T_{i-1}}^{T_i} (T_i - s)^{q_i - 1} (|f(s, x(s)) - f(s, 0)| + |f(s, 0)|) ds \\
&\le (Kr_i + M) \left[\frac{1}{\Gamma(q_i)} \int_{T_{i-1}}^{t} (t - s)^{q_i - 1} ds + \frac{|T_i - T_{i-1} - 2t|\Gamma(2-p)}{2(T_i - T_{i-1})^{1-p}\Gamma(q_i - p)} \int_{T_{i-1}}^{T_i} (T_i - s)^{q_i - p - 1} ds + \frac{1}{2\Gamma(q_i)} \int_{T_{i-1}}^{T_i} (T_i - s)^{q_i - 1} ds \right] \\
&\le (Kr_i + M) \left[\frac{(T_i - T_{i-1})^{q_i - 1}[3\Gamma(q_i - p + 1)(T_i - T_{i-1}) + \Gamma(2-p)\Gamma(q_i + 1)(T_i + T_{i-1})]}{2\Gamma(q_i - p + 1)\Gamma(q_i + 1)} \right] \le r_i.
\end{aligned} \tag{25}$$

It implies that $TB_{r_i} \subseteq B_{r_i}$.

For any $x(t), y(t) \in \Omega_i, t \in J_i$,

$$\begin{aligned}
&|Tx(t) - Ty(t)| \\
&\le \frac{1}{\Gamma(q_i)} \int_{T_{i-1}}^{t} (t-s)^{q_i-1} |f(s, x(s)) - f(s, y(s))| ds + \frac{|T_i - T_{i-1} - 2t| \Gamma(2-p)}{2(T_i - T_{i-1})^{1-p} \Gamma(q_i - p)} \int_{T_{i-1}}^{T_i} (T_i - s)^{q_i - p - 1} |f(s, x(s)) - f(s, y(s))| ds \\
&\quad + \frac{1}{2\Gamma(q_i)} \int_{T_{i-1}}^{T_i} (T_i - s)^{q_i-1} |f(s, x(s)) - f(s, y(s))| ds \\
&\le \left[\frac{1}{\Gamma(q_i)} \int_{T_{i-1}}^{t} (t-s)^{q_i-1} ds + \frac{|T_i - T_{i-1} - 2t| \Gamma(2-p)}{2(T_i - T_{i-1})^{1-p} \Gamma(q_i - p)} \int_{T_{i-1}}^{T_i} (T_i - s)^{q_i-p-1} ds + \frac{1}{2\Gamma(q_i)} \int_{T_{i-1}}^{T_i} (T_i - s)^{q_i-1} ds \right] K \|x - y\| \\
&\le \left[\frac{K(T_i - T_{i-1})^{q_i-1} [3\Gamma(q_i - p + 1)(T_i - T_{i-1}) + \Gamma(2-p)\Gamma(q_i + 1)(T_i + T_{i-1})]}{2\Gamma(q_i - p + 1)\Gamma(q_i + 1)} \right] \|x - y\| < \frac{1}{2} \|x - y\|.
\end{aligned} \tag{26}$$

It follows that T is a contraction mapping. Thus, the Banach fixed point theorem yields that T has a unique fixed point which is the unique solution of the antiperiodic boundary value problem (11). □

Theorem 13. *Assume that (H_1), (H_2) hold for all $i = 1, 2, \cdots, n$, then the problem (2) has at least a solution in $C(J, \mathfrak{R})$.*

Proof. According to Theorem 11, problem (11) has at least a solution $x^* \in \Omega_i$. Define the function

$$x_i = \begin{cases} x_i^*(T_{i-1}), & t \in [0, T_{i-1}], \\ x_i^*, & t \in J_i. \end{cases} \tag{27}$$

Thus, the function $x_i \in C([0, T_i], \mathfrak{R})$ satisfy (10) and $x_i(0) = -x_i(T)$, ${}_0^C D_t^p x_i(0) = -{}_0^C D_t^p x_i(T)$. Then, the function

$$x(t) = \begin{cases} x_1(t), & t \in J_1, \\ x_2(t) = \begin{cases} x_2^*(T_1), & t \in J_1, \\ x_2^*, & t \in J_2, \end{cases} \\ \vdots \\ x_n(t) = \begin{cases} x_n^*(T_{n-1}), & t \in [0, T_{n-1}], \\ x_n^*, & t \in J_n, \end{cases} \end{cases} \tag{28}$$

is a solution of problem (2) in $C(J, \mathfrak{R})$. □

Theorem 14. *Assume that (H_1), (H_3) hold for all $i = 1, 2, \cdots, n$, then problem (2) has a unique solution in $C(J, \mathfrak{R})$.*

The proof of Theorem 14 is similar to Theorem 13.

4. Conclusion

This paper is devoted to considering the existence of solutions to the antiperiodic fractional boundary value problems for nonlinear differential equations of variable order, which is a piecewise constant function based on the essential difference about the variable order. Based on the fixed point theory, the results are obtained. It is also worth considering fractional differential equations of variable-order problems related to thermodynamics, fluid mechanics, resonance, etc.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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