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Research Article

Existence and Uniqueness of the Solution for an Inverse Problem of a Fractional Diffusion Equation with Integral Condition

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The solvability of the fractional partial differential equation with integral overdetermination condition for an inverse problem is investigated in this paper. We analyze the direct problem solution by using the "energy inequality" method. Using the fixed point technique, the existence and uniqueness of the solution of the inverse problem on the data are established.

1. Introduction

This work devoted to study the solvability of a pair of functions $\{u, f\}$ satisfying the following fractional parabolic problem:

$$ {}^{C}D_t^{\alpha} u - \Delta u + \beta u = f(t) g(x, t) \, ; x \in \Omega, t \in [0, T], \tag{1}$$

with the initial condition

$$u(x, 0) = \varphi(x), x \in \Omega, \tag{2}$$

the boundary condition

$$u(x, t) = 0, (x, t) \in \partial\Omega \times [0, T], \tag{3}$$

and the nonlocal condition

$$\int_{\Omega} v(x) u(x, t) dx = \theta(t), t \in [0, T]. \tag{4}$$

Here, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$.. The functions g, φ, and θ are known functions, and β is a positive constant. And $\Gamma(\cdot)$ denotes the gamma function. For any positive integer $0 < \alpha < 1$, the left Caputo derivative is defined as

$$ {}^{C}D_t^{\alpha} u(x, t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x, \tau)}{\partial \tau} \frac{1}{(t-\tau)^{\alpha}} d\tau. \tag{5}$$

Inverse parabolic equation problems occur naturally in many fields, and there is extensive literature on inverse heat equation problems (see [1–4], and references therein). The form (4) is an additional information of problem.

In engineering and physics, the parameter recognition in a partial differential equation from the data of the integral overdetermination condition plays an important role [5–10]. From a physical point of view, these conditions can be interpreted by a system averaging the domain of spatial variables as measurements of the temperature $u(x, t)$.

Note that nonlocal problems related with integral overdetermination [11, 12]. Studies have shown that when we deal with these kinds of nonclassical problems, classical approaches sometimes do not work [13, 14]. To date, different methods for addressing problems resulting from nonlocal problem have been suggested. The choice of approach depends on the form of nonlocal boundary value that are involved.

We note that several authors have studied the inverse parabolic problem with condition of type (4) and its special solubility (see, for example, [3, 4, 15–20]). There are also several articles dedicated to the study of the existence and uniqueness of inverse problem solutions for different parabolic equations with unknown source functions. Inverse problems related by determining unknown function in source term of a parabolic equation with overdetermination condition [21, 22].

In recent years, fractional differential equations have created growing interest from engineers and scientists and have great importance in modeling complex phenomena. Because FDEs have memory, nonlocal space, and time relationships, using these equations, complex phenomena can be modelled [23–28].

Namely, in the present paper, a new research on the inverse problem of a fractional parabolic equation is discussed, for which the solvability of the problem (1)–(4) is reduced to the concept of a fixed point technique. This work is divided into four sections; we start with an introduction then we give some definitions of function space and important lemmas. The third section is devoted to studying the solvability of the direct fractional parabolic problem. Finally, in the last section, we prove the existence and uniqueness of the solution to the main problem.

2. Functional Space

Definition 1. Let us introduce certain notations used below, we set

$$g^*(t) = \int_\Omega v(x)g(x,t)dx, Q_T = \Omega \times [0,T]. \tag{6}$$

We denote by $C((0,T), L_2(\Omega))$ the space is composed of all continuous functions on $(0,T)$ with values in $L_2(\Omega)$. For any $0<\alpha<1$, the Caputo and Riemann-Liouville derivatives are defined, respectively, as follows:

(i) The left Caputo derivatives:

$$^{C}D_t^\alpha u(x,t) \coloneqq \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u(x,\tau)}{\partial \tau}\frac{1}{(t-\tau)^\alpha}d\tau. \tag{7}$$

(ii) The left Riemann-Liouville derivatives:

$$^{R}D_t^\alpha u(x,t) \coloneqq \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_0^t \frac{u(x,\tau)}{(t-\tau)^\alpha}d\tau. \tag{8}$$

(iii) The right Riemann-Liouville derivatives:

$$_t^{R}D^\alpha u(x,t) \coloneqq \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_t^T \frac{u(x,\tau)}{(\tau-t)^\alpha}d\tau. \tag{9}$$

Many authors believe that the Caputo version is more natural because it makes it easier to manage inhomogeneous initial conditions. Then, the following relationship is related to the two concepts (7) and (8), which can be checked by a direct calculation:

$$^{R}D_t^\alpha u(x,t) = {}^{C}D_t^\alpha u(x,t) + \frac{u(x,0)}{\Gamma(1-\alpha)t^\alpha}. \tag{10}$$

Definition 2 (see [29]). For any real $\sigma>0$, we define the space $^{l}H_0^\sigma(I)$ as the closure of $C_0^\infty(I)$ with respect to the following norm $\|\cdot\|_{^{l}H_0^\sigma(I)}$:

$$\|u\|_{^{l}H^\sigma(I)} \coloneqq \left(\|u\|_{L_2(I)}^2 + |u|_{lH_0^\sigma(I)}^2\right)^{1/2}, \tag{11}$$

where

$$|u|_{^{l}H^\sigma(I)}^2 \coloneqq \left\|{}_0^{R}D_t^\sigma u\right\|_{L_2(I)}^2. \tag{12}$$

Definition 3. For any real $\sigma>0$, we define the space $^{r}H_0^\sigma(I)$ as the closure of $C_0^\infty(I)$ with respect to the following norm $\|\cdot\|_{^{r}H_0^\sigma(I)}$:

$$\|u\|_{^{r}H_0^\sigma(I)} \coloneqq \left(\|u\|_{L_2(I)}^2 + |u|_{rH_0^\sigma(I)}^2\right)^{1/2}, \tag{13}$$

where

$$|u|_{^{r}H_0^\sigma(I)}^2 \coloneqq \left\|{}_t^{R}\partial_T^\sigma u\right\|_{L_2(I)}^2. \tag{14}$$

Lemma 4 (see [29, 30]). *For any real $\sigma\in\mathbb{R}_+$, if $u\in{}^{l}H^\alpha(I)$ and $v\in C_0^\infty(I)$, then*

$$\left({}^{R}D_t^\sigma u(t), v(t)\right)_{L^2(I)} = \left(u(t), {}_t^{R}D^\sigma v(t)\right)_{L^2(I)}. \tag{15}$$

Lemma 5 (see [29, 30]). *For $0<\sigma<2$, $\sigma\neq 1$, $u\in H_0^{\sigma/2}(I)$, on a*

$$^{R}D_t^\sigma u(t) = {}^{R}D_t^{\sigma/2}\,{}^{R}D_t^{\sigma/2}u(t). \tag{16}$$

Lemma 6 (see [29, 30]). *For $\sigma \in \mathbb{R}_+, \sigma \neq n + (1/2)$, the semi-norms $|.|_{{}^l H^\sigma(I)}, |.|_{{}^r H^\sigma(I)}$ and $|.|_{{}^c H^\sigma(I)}$ are equivalent. Then, we pose*

$$|.|_{{}^l H^\sigma(I)} = \tilde{} |.|_{{}^r H^\sigma(I)} = \tilde{} |.|_{{}^c H^\sigma(I)}. \tag{17}$$

Lemma 7 (see [29]). *For any real $\sigma > 0$, the space ${}^l H_0^\sigma(I)$ with respect to the norm (11) is complete.*

Definition 8. We denote by $L_2(0, T, L_2(0, d)) \coloneqq L_2(Q)$ the space of square functions, integrated with the scalar product in the Bochner sense,

$$(u, w)_{L_2(0,T,L_2(0,d))} = \int_0^T ((u, \cdot), (w, \cdot))_{L_2(0,d)} dt. \tag{18}$$

Since the space $L_2(0, d)$ is a Hilbert space, it can be shown that $L_2(0, T, L_2(0, d))$ is a Hilbert space as well. Let $C^\infty(0, T)$ denote the space of infinitely differentiable functions on $(0, T)$ and $C_0^\infty(0, T)$ denote the space of infinitely differentiable functions with compact support in $(0, T)$.

3. Solvability of the Direct Fractional Parabolic Problem

3.1. Position of Problem. In the rectangular domain $Q = (0, d) \times (0, T) = \Omega \times I$, with $d, T < \infty$ and $0 < \alpha < 1$, we shall study the existence and uniqueness of solutions $u = u(x, t)$ to the following fractional parabolic problem:

$$\begin{cases} {}^C D_t^\alpha u(x, t) - \left(\dfrac{\partial^2 u(x, t)}{\partial x^2} \right) + bu(x, t) = \tilde{f}(x, t) & \text{in } Q, \\ u(x, 0) = \varphi(x) & \forall x \in (0, d), \\ u(0, t) = u(d, t) = 0 & \forall t \in (0, T). \end{cases} \tag{19}$$

We consider the following fractional parabolic equation of the type

$$\mathscr{L} u = {}^C D_t^\alpha u - \frac{\partial^2 u}{\partial x^2} + bu = \tilde{f}, \tag{20}$$

with the initial condition

$$\ell u = u(x, 0) = \varphi(x), \forall x \in (0, d), \tag{21}$$

and Dirichlet condition

$$u(0, t) = u(d, t) = 0, \forall t \in (0, T), \tag{22}$$

where $b \in \mathbb{R}_*^+$; $\tilde{f}$ and φ are known functions.

We shall assume that the function φ satisfies a compatibility conditions, i.e.,

$$\varphi(0) = \varphi(d) = 0. \tag{23}$$

Now, introducing a new function

$$v(x, t) = u(x, t) - U(x) \Rightarrow u(x, t) = v(x, t) + U(x), \tag{24}$$

where

$$\varphi(x) = U(x). \tag{25}$$

So, we get

$$\begin{cases} {}^C D_t^\alpha v(x, t) - \left(\dfrac{\partial^2 v(x, t)}{\partial x^2} \right) + bv(x, t) = \tilde{f}(x, t) - \mathscr{L}\varphi(x) = f(x, t) & \text{in } Q, \\ v(x, 0) = 0 & \forall x \in (0, d), \\ v(0, t) = v(d, t) = 0 & \forall t \in (0, T). \end{cases} \tag{26}$$

Such that

$${}^C D_t^\alpha v(x, t) - \frac{\partial^2 v(x, t)}{\partial x^2} + bv(x, t) = f(x, t), \tag{27}$$

with the initial condition

$$\ell v = v(x, 0) = 0, \quad \forall \mathrm{x} \in (0, \mathrm{d}), \tag{28}$$

the boundary condition of Dirichlet type

$$v(0, t) = v(d, t) = 0, \quad \forall t \in (0, T), \tag{29}$$

where

$$f(x, t) = \tilde{f}(x, t) + \frac{\partial^2 \varphi(x)}{\partial x^2} - b\varphi(x). \tag{30}$$

3.2. A Priori Estimate. In this section, we illustrate the existence and uniqueness of the problem's solution (27)–(29) as a solution of the operator equation

$$Lv = \mathscr{F}, \tag{31}$$

where $L = (\mathscr{L}, \ell)$, with domain of definition B consisting of functions $v \in L^2(Q)$, such that v, ${}^{C}D_t^{\alpha} v, (\partial v/\partial x) \in L^2(Q)$, and v verify (29).

The operator L is considered from B to F, where B is the Banach space consisting of all functions $v(x, t)$ having a finite norm

$$\|v\|_B^2 = \left\| {}^{C}D_t^{\alpha/2} v \right\|_{L^2(Q)}^2 + \|v\|_{L^2(Q)}^2 + \left\| \frac{\partial v}{\partial x} \right\|_{L^2(Q)}^2, \tag{32}$$

and F is the Hilbert space consisting of all elements Fourier $= (f, 0)$ for which the norm $L^2(Q)$ is finite.

Theorem 9. *For any function $u \in B$, we have the inequality*

$$\|v\|_B \leq k \|Lv\|_{L^2(Q)}, \tag{33}$$

where k is a positive constant independent of v.

Proof. Multiplying equation (27) by the following function:

$$Mv = v(x, t), \tag{34}$$

and integrating over $Q = (0, d) \times (0, T)$, we get

$$\begin{aligned} &\int_Q \mathscr{L}v \cdot Mv dxdt \\ &= \int_Q {}^{C}D_t^{\alpha} v(x, t) \cdot v(x, t) dxdt - \int_Q \frac{\partial^2 v(x, t)}{\partial x^2} v(x, t) dxdt \\ &\quad + \int_Q b \cdot v^2(x, t) dxdt = \int_Q f(x, t) \cdot v(x, t) dxdt. \end{aligned} \tag{35}$$

As $v(x, 0) = 0$, so by applying Lemmas 4, 5, and 6 becomes

$$\begin{aligned} &\int_Q {}^{C}D_t^{\alpha} v(x, t).v(x, t) dxdt \\ &= \left({}^{C}D_t^{\alpha} v(x, t), v(x, t)\right)_{L^2(Q)} \\ &= \left({}^{R}D_t^{(\alpha/2)} {}^{R}D_t^{\alpha/2} v(x, t), \quad v(x, t)\right)_{L^2(Q)} \\ &\cdot \left({}^{R}D_t^{\alpha/2} v(x, t), \quad {}_t^{R}D^{\alpha/2} v(x, t)\right)_{L^2(Q)} \\ &= |u|_{{}^{c}H^{\alpha}(Q)}^2 2 \cong |u|_{{}^{l}H^{\alpha}(Q)}^2 \\ &\qquad \text{(According to Lemma 2)} \\ &= \left\| {}^{C}D_t^{\alpha/2} v \right\|_{L^2(Q)}^2, \quad \text{(According to Lemma 1)} \\ &\qquad \text{(According to Lemma 3)} \end{aligned} \tag{36}$$

and by integration by parts over $(0, d)$, we get

$$-\int_Q \frac{\partial^2 v(x, t)}{\partial x^2} v(x, t) dxdt = \int_Q \left(\frac{\partial v(x, t)}{\partial x}\right)^2 dxdt. \tag{37}$$

So, we obtain

$$\begin{aligned} &\int_Q \left({}^{R}D_t^{\alpha} v(x, t) - \frac{\partial^2 v(x, t)}{\partial x^2} + bv(x, t)\right) \cdot Mv dxdt \\ &\cong \left\| RD_t^{\alpha/2} v \right\|_{L^2(Q)}^2 + \int_Q \left(\frac{\partial v(x, t)}{\partial x}\right)^2 dxdt + \int_Q bv^2(x, t) dxdt \\ &\leq \frac{1}{2\varepsilon} \int_Q |f(x, t)|^2 dxdt + \frac{\varepsilon}{2} \int_Q |v(x, t)|^2 dxdt. \end{aligned} \tag{38}$$

So, we get

$$\begin{aligned} &\left\| {}^{C}D_t^{\alpha/2} v \right\|_{L^2(Q)}^2 + \int_Q \left(\frac{\partial v(x, t)}{\partial x}\right)^2 dxdt + \int_Q \left(b - \frac{\varepsilon}{2}\right) v^2(x, t) dxdt \\ &\leq \frac{1}{2\varepsilon} \int_Q |f(x, t)|^2 dxdt, \end{aligned} \tag{39}$$

which give

$$\begin{aligned} &\left\| {}^{C}D_t^{\alpha/2} v \right\|_{L^2(Q)}^2 + \int_Q \left(\frac{\partial v(x, t)}{\partial x}\right)^2 dxdt + \int_Q \left(b - \frac{\varepsilon}{2}\right) v^2(x, t) dxdt \\ &\leq \frac{1}{2\varepsilon} \int_Q |f(x, t)|^2 dxdt. \end{aligned} \tag{40}$$

So, we have

$$\left\| {}^{C}D_t^{\alpha/2} v \right\|_{L^2(Q)}^2 \leq \frac{1}{2\varepsilon} \|f\|_{L^2(Q)}^2. \tag{41}$$

On the other hand, we have

$$\left\| \frac{\partial v}{\partial x} \right\|_{L^2(Q)}^2 \leq \frac{1}{2\varepsilon} \|f\|_{L^2(Q)}^2. \tag{42}$$

Also, we have

$$\|v\|_{L^2(Q)}^2 \leq \frac{1}{2\varepsilon(b - (\varepsilon/2))} \|f\|_{L^2(Q)}^2. \tag{43}$$

By combining (41), (42), and (43), for $\varepsilon < b/2$, we get

$$\begin{aligned} &\left\| {}^{C}D_t^{\alpha/2} v \right\|_{L^2(Q)}^2 + \left\| \frac{\partial v}{\partial x} \right\|_{L^2(Q)}^2 + \|v\|_{L^2(Q)}^2 \\ &\leq \frac{1}{2\varepsilon} \left(1 + \frac{1}{(b - (\varepsilon/2))}\right) \|f\|_{L^2(Q)}^2. \end{aligned} \tag{44}$$

Finally, it follows that

$$\left\| {}^{C}D_{t}^{\alpha/2} v \right\|_{L^2(Q)}^{2} + \left\| \frac{\partial v}{\partial x} \right\|_{L^2(Q)}^{2} + \|v\|_{L^2(Q)}^{2} \le C \|f\|_{L^2(Q)}^{2}, \quad (45)$$

with

$$C = \frac{1}{2\varepsilon}\left(1 + \frac{1}{(b-(\varepsilon/2))}\right). \quad (46)$$

Therefore, we obtain that

$$\|v\|_{B} \le k\|Lv\|_{F}, \text{where } k = \sqrt{C}. \quad (47)$$

Hence, the uniqueness of the solution. □

Remark 10. This inequality $\|v\|_B \le k\|Lv\|_F$ gives the uniqueness of the solution, indeed:

Let v_1 and v_2 two solutions, so

$$\begin{cases} Lv_1 = \mathscr{F} \\ Lv_2 = \mathscr{F} \end{cases} \Rightarrow L(v_1 - v_2) = 0, \quad (48)$$

then

$$\|v_1 - v_2\|_B \le k\|0\|_F \Rightarrow \|v_1 - v_2\|_B \le 0 \Rightarrow v_1 - v_2 = 0, \quad (49)$$

which gives the uniqueness of the solution.

Proposition 11. *The operator L from B to F admits a closure.*

Proof. Let $(v_n)_{n\in\mathbb{N}} \subset D(L)$ be a sequence such that:

$$\begin{aligned} v_n &\longrightarrow 0 \text{ in } B, \\ Lv_n &\longrightarrow \mathscr{F} \text{ in } F, \end{aligned} \quad (50)$$

it must be shown that

$$f \equiv 0. \quad (51)$$

The convergence of v_n toward 0 in B entails that

$$v_n \longrightarrow 0 \,\text{in} (C_0^\infty(Q_T))'. \quad (52)$$

As the continuity of the fractional derivation(2) and the derivation of the first order (as a particular case of the fractional derivative) of $(C_0^\infty(Q_T))'$ in $(C_0^\infty(Q_T))'$, then (52) implies

$$\mathscr{L}u_n \longrightarrow 0 \,\text{in} (C_0^\infty(Q_T))'. \quad (53)$$

On the other hand, the convergence of $\mathscr{L}v_n$ to f in $F = L^2(Q_T)$ implies that

$$\mathscr{L}u_n \longrightarrow f \text{in} (C_0^\infty(Q_T))'. \quad (54)$$

By virtue of the uniqueness of the limit in $(C_0^\infty(Q_T))'$, we conclude between (53) and (54) that

$$f \equiv 0. \quad (55)$$

Hence, the operator L is closable. □

Definition 12. Let $\bar{L}$ the closure of L and $D(\bar{L})$ the definition domain of $\bar{L}$. The solution of the equation

$$\bar{L}v = \mathscr{F}, \quad (56)$$

is called generalized strong solution of the problem (27)–(29).

Theorem 9 is valid for a generalized strong solution, i.e., we have the following inequality:

$$\|v\|_B \le k\left\|\bar{L}v\right\|_F, \forall v \in D(\bar{L}). \quad (57)$$

Consequently, this last inequality entails the following corollaries:

Corollary 13. *The strong solution of the problem (27)–(29) is unique and depends continuously on $f \in F$.*

Corollary 14. *The range $R(\bar{L})$ of the operator $\bar{L}$ is equal to the closure $\overline{R(L)}$ of $R(L)$.*

Proof. Let $z \in \overline{R(L)}$, then there exists a Cauchy sequence $(z_n)_n$ in F consists of the elements of the set $R(L)$ such that

$$\lim_{n\longrightarrow+\infty} z_n = z. \quad (58)$$

So there is a corresponding sequence $(v_n)_n \subset D(L)$ such that

$$Lv_n = z_n. \quad (59)$$

From the estimate (41), we obtain

$$\left\|v_p - v_q\right\|_B \le k\left\|Lv_p - Lv_q\right\|_F \longrightarrow 0, \text{ when } p, q \longrightarrow +\infty. \quad (60)$$

We can deduce that $(v_n)_n$ is a Cauchy sequence in B, so there is $v \in B$

$$\lim_{n\longrightarrow+\infty} v_n = v \text{ in } B. \quad (61)$$

By virtue of the definition of $\bar{L}$ ($\lim_{n\longrightarrow+\infty} v_n = v$ in B; if $\lim_{n\longrightarrow+\infty} Lv_n = \lim_{n\longrightarrow+\infty} z_n = z$, so $\lim_{n\longrightarrow+\infty} \bar{L}v_n = z$ and as $\bar{L}$ is closed so $\bar{L}v = z$), the function v verifies that

$$v \in D(\bar{L}), \quad \bar{L}v = z. \quad (62)$$

Thus, $z \in R(\bar{L})$, then

$$\overline{R(L)} \subset R(\bar{L}). \tag{63}$$

So we conclude here that $R(\bar{L})$ is closed because it is complete (any complete subspace of a metric space (not necessarily complete) is closed).

It remains to show the opposite inclusion.

Let $z \in R(\bar{L})$, then there is a sequence of $(z_n)_n$ in F consists of the elements of the set $R(\bar{L})$ such that

$$\lim_{n \longrightarrow +\infty} z_n = z, \tag{64}$$

where $z \in R(\bar{L})$, because $R(\bar{L})$ is closed subset of a complete space F; then, $R(\bar{L})$ is complete.

So there is a corresponding sequence $(v_n)_n \subset D(\bar{L})$ such that

$$\bar{L}v_n = z_n. \tag{65}$$

From the estimate (57), we obtain

$$\left\| v_p - v_q \right\|_B \leq k \left\| \bar{L}v_p - \bar{L}v_q \right\|_F \longrightarrow 0, \text{if } p, q \longrightarrow +\infty. \tag{66}$$

We can deduce that $(v_n)_n$ is a Cauchy sequence in B, so there is $v \in B$

$$\lim_{n \longrightarrow +\infty} v_n = v \text{in} B. \tag{67}$$

Once more, there is a corresponding sequence $(L(v_n))_n \in R(L)$ such that

$$Lv_n = \bar{L}v_n \text{ over } R(L), \forall n. \tag{68}$$

Then

$$\lim_{n \longrightarrow +\infty} Lv_n = z. \tag{69}$$

Consequently, $z \in \overline{R(L)}$, and then, we conclude that

$$R(\bar{L}) \subset \overline{R(L)}. \tag{70}$$

□

3.3. Existence of Solution. To show the existence of solutions, we prove that $R(L)$ is dense in F for all $u \in B$ and for arbitrary $\mathscr{F} = (f, 0) \in F$.

Theorem 15. *The problem (27)–(29) admits a solution.*

Proof. The scalar product of F is defined by

$$(Lv, W)_F = \int_{Q_T} \mathscr{L}v \cdot w dxdt, \quad \text{where} \quad W = (w, 0) \in D(L). \tag{71}$$

If we put $w \in R(L)^{\perp}$, we have

$$\int_{Q_T} \left({}^C D_t^{\alpha} v(x,t) - \frac{\partial^2 v(x,t)}{\partial x^2} + bv(x,t) \right) \cdot w(x,t) dxdt = 0, \tag{72}$$

where ${}^C D_t^{\alpha} v$, $\partial v / \partial x$, $v \in L^2(Q_T)$, with v satisfies the boundary conditions of (27)–(29). From (72), we get the equality

$$\begin{aligned} &\int_{Q_T} {}^C D_t^{\alpha} v(x,t) \cdot w(x,t) dxdt - \int_{Q_T} \frac{\partial^2 v(x,t)}{\partial x^2} \cdot w(x,t) dxdt \\ &\quad + b \int_{Q_T} v(x,t) \cdot w(x,t) dxdt = 0. \end{aligned} \tag{73}$$

And from the equality (73), we give the function w in terms of v as follows:

$$w = v, \tag{74}$$

then $w \in L^2(Q_T)$.

Replacing w in (73) by its representation (74) and integrating by parts each term of (73) and by taking the condition of v, we obtain

$$\begin{aligned} &\int_{Q_T} \left(CD_t^{\alpha/2} v(x,t) \right)^2 dxdt + \int_{Q_T} bv^2(x,t) dxdt \\ &\quad \leq -\int_{Q_T} \left(\frac{\partial v(x,t)}{\partial x} \right)^2 dxdt \leq 0, \end{aligned} \tag{75}$$

then

$$\left\| {}^C D_t^{\alpha/2} v \right\|^2_{L^2(Q_T)} + b \|v\|^2_{L^2(Q_T)} \leq 0. \tag{76}$$

Hence

$$\|v\|_{L^2(Q_T)} = 0. \tag{77}$$

And thus, $v = 0$ in Q_T which gives $w = 0$ in Q_T. This proves Theorem 15. So $\overline{R(L)} = F$. □

4. Existence and Uniqueness of the Solution of Main Problem

We are finding a solution in the form of the original inverse problem. $\{u, f\} = \{z, f\} + \{y, 0\}$ where y is the solution of the direct problem

$${}^C D_t^{\alpha} y - \Delta y + \beta y = 0. \quad (x,t) \in Q_T, \tag{78}$$

$$y(x, 0) = \varphi(x), \quad x \in \Omega, \tag{79}$$

$$y(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0, T], \tag{80}$$

while the pair $\{z, f\}$ is the solution of the inverse problem

$$^{C}D_t^{\alpha} z - \Delta z + \beta z = f(t) g(x, t). \quad (x, t) \in Q, \tag{81}$$

$$z(x, 0) = 0, \quad x \in \Omega, \tag{82}$$

$$z(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, T], \tag{83}$$

$$\int_{\Omega} v(x) z(x, t) dx = E(t), t \in [0, T], \tag{84}$$

where

$$E(t) = \theta(t) - \int_{\Omega} v(x) y(x, t) dx. \tag{85}$$

We will assume that the functions that appear in the problem data are measurable and fulfill the following conditions:

$$\begin{cases} g \in C((0, T), L_2(\Omega)), v \in W_2^1(\Omega), E \in W_2^2(0, T), \\ \|g(x, t)\| \leqslant m\,; |g^*(\mathrm{t})| \geqslant p > 0, \quad \text{for} \quad p \in \mathbb{R}, (\mathrm{x}, \mathrm{t}) \in \mathrm{Q_T}, \\ \varphi(x) \in W_2^1(\Omega) \quad \text{where} \quad g^* \quad \text{is} \quad \text{defind} \quad \text{in} \quad (5). \end{cases} \tag{86}$$

The correspondence between f and z can be seen as one way of defining the linear operator.

$$A : L_2(0, T) \longrightarrow L_2(0, T), \tag{87}$$

with the values

$$(Af)(t) = \frac{1}{g^*} \left\{ \int_{\Omega} \nabla z \nabla v dx \right\}. \tag{88}$$

In this view, the linear equation of the second form for the function is rational to refer to f over the space $L_2(0, T)$:

$$f = Af + W, \tag{89}$$

where

$$W = \frac{CD_t^{\alpha} E + \beta E}{g^*}. \tag{90}$$

Remark 16. As $\{u, f\} = \{z, f\} + \{y, 0\}$ where y is the solution of the direct problem (78)–(80). Obviously, the previous section implies that y exists and is unique, but instead of demonstrating the solvability of the initial problem (1)–(4), we demonstrate the existence and uniqueness of the inverse problem (81)–(84) solution.

Theorem 17. *Suppose the input of the inverse problem data (81)–(84) satisfies* (H)*. Then, the following assertions are valid: (i) if the inverse problem (81)–(84) is solvable, then so is equation (89). And (ii) if equation (89) has a solution and the condition of compatibility has*

$$E(0) = 0, \tag{91}$$

holds, then a solution to the inverse problem exists.

Proof.

(i) Suppose that the inverse problem (81)–(84) is solvable. We denote its solution by $\{z, f\}$. Multiplying the function v scalarly in $L_2(\Omega)$ both sides of (81), we get

$$^{C}D_t^{\alpha} \int_{\Omega} zv dx + \int_{\Omega} \nabla z \nabla v dx + \beta \int_{\Omega} zv dx = \int_{\Omega} f(t) g^*(x, t). \tag{92}$$

With (84) and (88), from (92), it follows that $f = Af + ($ $(CD_t^{\alpha} E + \beta E)/g^*)$. This gives that f solves equation (89).

(ii) Equation (89) has a solution in space, according to the assumption, $L_2(0, T)$, say f. The resulting relationship (81)–(83) can be viewed as a direct problem with a unique solution $z \in W_2^1(Q_T)$ when inserting this function in (81). Let us show that the z function also satisfies the condition of integral overdetermination (84). By equation (92), the function z is subject to the following relation

$$^{C}D_t^{\alpha} E + \beta E + \int_{\Omega} \nabla z \nabla v dx = f(t) g^*(t). \tag{93}$$

Subtracting equation (92) from equation (93), we get

$$^{C}D_t^{\alpha} \int_{\Omega} zv dx + \beta \int_{\Omega} zv dx = {}^{C}D_t^{\alpha} E + \beta E. \tag{94}$$

Integrating the preceding differential equation and taking into account the compatibility condition (89), we find that the overdetermination condition (84) is satisfied by z and the function pair $\{z, f\}$ is a solution to the inverse problem (81)–(84).

This completes the theorem's proof. □

Now, we are touching on some properties of operator A.

Lemma 18. *Let the condition* (H) *hold. Then, there exists a positive* ε *for which* A *is a contracting operator in* $L_2(0, T)$.

Proof. Obviously, (88) yields the estimate

$$\|Af\|_{L_2(0,T)} \leq \frac{k}{p} \left(\int_0^T \|\nabla z(.,\tau)\|_{L_2(\Omega)}^2 d\tau \right)^{1/2}, \tag{95}$$

where

$$k = \|\nabla v\|_{L_2(\Omega)}. \tag{96}$$

Multiplying both sides of (81) by z scalarly in $L_2(Q_T)$ and integrating the resulting by parts with use of (82), we get

$$\begin{aligned}&\left\|{}^C D_t^{\alpha/2} z\right\|_{L_2(Q_T)}^2 + \|\nabla z\|_{L_2(Q_T)}^2 + \beta\|z\|_{L_2(Q_T)}^2\\&\quad = \int_0^T \left(f(t)\int_\Omega g(x,t) z dx\right) dt.\end{aligned} \tag{97}$$

Thus, by using the Cauchy's ε-inequality, we obtain

$$\begin{aligned}&\left\|{}^C D_t^{\alpha/2} z\right\|_{L_2(Q_T)}^2 + \|\nabla z\|_{L_2(Q_T)}^2 + \beta\|z\|_{L_2(Q_T)}^2\\&\quad \le \frac{m|\Omega|}{2\varepsilon}\int_0^T |f(t)|^2 dt + \frac{\varepsilon}{2}\|z\|_{L_2(Q_T)}^2.\end{aligned} \tag{98}$$

Choosing $0 < \varepsilon < 2\beta$, we get

$$\begin{aligned}&\left\|{}^C D_t^{\alpha/2} z\right\|_{L_2(Q_T)}^2 + \|\nabla z\|_{L_2(Q_T)}^2 + \left(\beta - \frac{\varepsilon}{2}\right)\|z\|_{L_2(Q_T)}^2\\&\quad \le \frac{m|\Omega|}{2\varepsilon}\int_0^T |f(\tau)|^2 dt.\end{aligned} \tag{99}$$

Omitting some terms on the left-hand side (99) leads to

$$\|\nabla z\|_{L_2(Q_T)}^2 = \int_0^T \|\nabla z(.,\tau)\|_{L_2(\Omega)}^2 d\tau \le \frac{m|\Omega|}{2\varepsilon}\int_0^T |f(\tau)|^2 dt. \tag{100}$$

According to (95) and (100), we can obtain the following estimate:

$$\|Af\|_{L_2(0,T)} \le \delta\int_0^T |f(\tau)|^2 dt, \quad 0 \leqslant t \leqslant T, \tag{101}$$

where

$$\delta = \frac{k\sqrt{m}|\Omega|}{p\sqrt{2\varepsilon}}. \tag{102}$$

So, we obtain

$$\|Af\|_{L_2(0,T)} \le \delta\|f\|_{L_2(0,T)}. \tag{103}$$

It is obvious from the above that there is positive ε such that

$$\delta < 1. \tag{104}$$

Inequality (103) shows that the operator A is a contracting mapping on $L_2(0,T)$. □

The following result may be useful with respect to the particular solvability of the inverse problem concerned.

Theorem 19. *Let the compatibility condition (91) and the condition (H) hold. Then, the inverse problem (81)–(84) has a unique solution $\{z, f\}$.*

Proof. This means that the equation (89) has a unique solution f in $L_2(0,T)$.

The existence of a solution to the inverse problem (81)–(84) is verified, according to Lemma 6.

The uniqueness of this solution has yet to be proven.

Suppose the contrary that there are two distinct solutions $\{z_1, f_1\}$ and $\{z_2, f_2\}$ of the under consideration inverse problem.

Also, the linear operator A is contracting on $L_2(0,T)$ from Lemma 18, which gives that $f_1 = f_2$; then, by the theorem of the uniqueness of the solution of main direct problem (78)–(80), we will just have $z_1 = z_2$. □

Corollary 20. *The solution f to equation (91) depends continuously, under the conditions of Theorem 19, on the data W.*

Proof. Let V_1 and V_2 twosets of data that satisfy Theorem 19's assumptions.

Let f and g be solutions of the equation (89) corresponding to the data V_1 and V_2, respectively. According to (103), we have

$$\begin{aligned}f &= Af + V_1,\\ g &= Ag + V_2.\end{aligned} \tag{105}$$

Let us estimate the difference first, $f - g$. It is easy to see with the use of (103) that

$$\begin{aligned}\|f-g\|_{L_2(0,T)} &= \|(Af + V_1) - (Ag + V_2)\|_{L_2(0,T)}\\ &= \|A(f-g) + (V_1 - V_2)\|_{L_2(0,T)}\\ &\leqslant \delta\|f-g\|_{L_2(0,T)} + \|(V_1 - V_2)\|_{L_2(0,T)},\end{aligned} \tag{106}$$

so, we get

$$\|f-g\|_{L_2(0,T)} \leqslant \frac{1}{(1-\delta)}\|(V_1 - V_2)\|_{L_2(0,T)}. \tag{107}$$

□

5. Conclusion and Perspectives

This work contains a new inverse problem by investigating the fractional derivatives where we develop the method of fixed point and energy inequality method for proving the solvability of an inverse fractional problem. We note that our work extends to the existence of open problems as a study of the nonlinear case of this problem and the numerical part.

Data Availability

No data were used to support this study.

Disclosure

An earlier version of this manuscript has been presented as online conference in Modern Fractional Calculus and Its Applications (OCMFCA-2020) Biruni University Istanbul Turkey.

Conflicts of Interest

The authors declare no conflict of interest.

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