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Research Article

Ulam-Hyers-Rassias Stability of Nonlinear Differential Equations with Riemann-Liouville Fractional Derivative

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Fractional derivatives are used to model the transmission of many real world problems like COVID-19. It is always hard to find analytical solutions for such models. Thus, approximate solutions are of interest in many interesting applications. Stability theory introduces such approximate solutions using some conditions. This article is devoted to the investigation of the stability of nonlinear differential equations with Riemann-Liouville fractional derivative. We employed a version of Banach fixed point theory to study the stability in the sense of Ulam-Hyers-Rassias (UHR). In the end, we provide a couple of examples to illustrate our results. In this way, we extend several earlier outcomes.

1. Introduction

Fractional calculus (FC) has been appearing in a wide range of fields, such as chemistry, economics, polymer rheology, and aerodynamics. This is due to the existence of many nice tools (see e.g., [1, 2]) that are not available in the classical calculus. In particular, FC enables researches to model in an efficient way many complicated real world problems, e.g., COVID-19 (see [3]), Ebola virus (see [4]), and HIV (see [5]). Moreover, it has recent interesting applications in image processing (see [6]) and in diabetes (see [7]).

The stability problem named after Ulam is currently a research trend in many applications (see e.g., [8] for more references and details). It pupped up as a consequence of the famous question asked by Ulam at a conference held in Wisconsin University in the fall term of 1940 (see [9]). The mentioned Ulam's stability problem can be rewritten as follows:

Let (G^{**}, ρ) be a metric group and G^* be a group. Is it true that for some $\varepsilon_1 > 0$, there is a $\delta_1 > 0$ that satisfies if $T : G^* \longrightarrow G^{**}$ verifies

$$\rho(T(t_1 t_2), T(t_1)T(t_2)) < \delta_1, \quad (1)$$

for every $t_1, t_2 \in G^*$; thus, there exists a homomorphism $g : G^* \longrightarrow G^{**}$ fulfilling

$$\rho(T(t_1), g(t_1)) < \varepsilon_1, \quad (2)$$

for all $t_1 \in G^*$.

Answers have been introduced for the question of Ulam by many mathematicians. For instance, in 1941, Hyers gave an exact answer to Ulam's question. Afterwards, Rassias in 1978 (see [10]) introduced a general form of the result of Hyers. The famous result obtained by Rassias can be rewritten:

Theorem 1. [10]. *Assume that B^*, B^{**} are Banach spaces and assume some continuous mapping from $\mathbb{R}$ into B^{**}. Suppose that there exists $\omega \geq 0$ and $\vartheta \in [0, 1)$ such that*

$$\|h(b_1 + b_2) - h(b_1) - h(b_2)\| \leq \omega\left(\|b_1\|^\vartheta + \|b_2\|^\vartheta\right), \ b_1, b_2 \in B^* \setminus \{0\}. \tag{3}$$

Then, there is a unique solution $\Psi : B^* \longrightarrow B^{**}$ of the Cauchy equation $(h(b_1 + b_2) = h(b_1) + h(b_2))$ with

$$\|h(b_1) - \Psi(b_1)\| \leq \frac{2\omega\|b_1\|^\vartheta}{|2 - 2^\vartheta|}, \ b_1 \in B^* \setminus \{0\}. \tag{4}$$

Through the past six decades, the stability subject has been a common issue of investigations in many places (see, e.g., [12, 15, 22, 23, 9, 20, 21, 25, 27, 26, 28]). As a consequence of the interesting results presented in this direction, many articles devoted to this subject have been introduced ([24, 16, 29] and the references therein). In 2010, Jung employed a fixed point technique (FPT) to study the stability of the equation $\rho' = \rho(\theta, \lambda)$ (see [11]). It should be remarked that Jung in [11] generalized the work of Alsina and Ger to the nonlinear case. In 2012, Bojor (see [12]) used different assumptions to study the stability of

$$h'(x) + m(x)h(x) = r(x), \tag{5}$$

and improved the result of Jung in [11].

In 2015, Tunç and Biçer in [13] improved the approach of Jung in [11] for the functional differential equation:

$$z'(x_1) = F(x_1, z(x_1), z(x_1 - \tau)). \tag{6}$$

In [14], Huang et al. investigated the stability of the following equation:

$$T^{(n)}(x_1) = F\left(x_1, T(x_1), T'(x_1), \cdots, T^{(n-1)}(x_1)\right). \tag{7}$$

In [15], Popa and Pugna studied the UH stability (UHS) of Euler's equation. In [16], Shen introduced Ulam stability for equations on time scales. In [17], the authors employed weakly Picard operator theory to investigate the UHS of some kind of equations in Banach Spaces. Furthermore, they obtained the UHR stability for such kind of equations via Pachpatte's integral inequalities. FPT has been employed in [18] to study the stability of a nonlinear Volterra integrodifferential equation with delay and in [19] to study the stability of impulsive Volterra integral equation.

The framework of the paper is as follows. In Section 2, we introduce some preliminaries; in Section 3, we present the stability results in UHR sense; in Section 4, we illustrate our results with two examples, and Section 5 is devoted to the conclusion.

2. Preliminaries

From now on, we use $\mathbb{R}$ to denote real numbers set and $\mathbb{C}$ to denote the complex numbers set. We define the generalized metric on a nonempty set S as follows.

Definition 2. [20]. The mapping $\sigma : S \times S \longrightarrow [0,\infty]$ is said to be a generalized metric on S if and only if σ fulfills the assertions:

$G_1 \sigma(r_1, r_2) = 0$ if and only if $r_1 = r_2$;
$G_2 \sigma(r_1, r_2) = \sigma(r_2, r_1)$ for all $r_1, r_2 \in S$;
$G_3 \sigma(r_1, r_3) \leq \sigma(r_1, r_2) + \sigma(r_2, r_3)$ for all $r_1, r_2, r_3 \in S$.

Now, we present the notion of UHR stability.

Definition 3. The following fractional differential equation

$$F\left(x, \nu, D_T^\lambda x(\nu)\right) = 0 \tag{8}$$

is UHR stable if for given $\epsilon > 0$ and a function $x(\nu)$ which satisfies

$$\left|F\left(x, \nu, D_T^\lambda x(\nu)\right)\right| \leq \epsilon \Pi(\nu). \tag{9}$$

There is a solution $x_0(\nu)$ of (8) with $|x(\nu) - x_0(\nu)| \leq \epsilon \Pi(\nu)\Phi(\nu)$, where $\Pi(\cdot)$ and $\Phi(\cdot)$ are some functions that do not depends on x and x_0.

The following theorem represents one of the central results of FPT (see [20]).

Theorem 4. *For a generalized complete metric space (Z, γ). Suppose an operator $\Gamma : Z \longrightarrow Z$ that is strictly contractive with some Lipschitz constant $L < 1$, if there exists an integer that is nonnegative k such that $\gamma(\Gamma^{k+1}y, \Gamma^k y) < \infty$ for some $y \in Z$, then the following are true:*

(a) The sequence $\Gamma^n y$ converges to a fixed point y^* of Γ

y^* is the unique fixed point of Γ in $Z^* := \{y_1 \in Z : \gamma(\Gamma^k y, y_1) < \infty\}$

(c) If $y_1 \in Z^*$, then $\gamma(y_1, y^*) \leq (1/1 - L)\gamma(\Gamma y_1, y_1)$

The current article is written to study the stability of the following differential equation with right-sided Riemann-Liouville fractional derivative

$$\left(D_T^\lambda x\right)(\nu) = f(\nu, x(\nu)), \text{ for all } \nu \in [0, T], \tag{10}$$

with initial conditions

$$\left(D_T^{\lambda-i} x\right)(T) = a_i \in \mathbb{R}, (i = 1, 2 \cdots, n-1), \quad a_n = I_T^{n-\lambda} x(T) = 0, \tag{11}$$

where $f : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is some continuous nonlinear function and $n = -[-\lambda]$, where $[\cdot]$ is the well-known greatest integer function.

3. Stability Results

This section is used to present the main findings of this article. In other words, we use it to prove the UHR stability of (10).

Let us first use $E = C([0, T], \mathbb{R})$ to denote the space of all continuous functions from the interval $[0, T]$ into the set of reals $\mathbb{R}$. In the next subsections, we investigate the stability of (10) when $0 < \lambda < 1$ and when $\lambda > 1$. We start with the case $0 < \lambda < 1$ as follows.

3.1. The Case $0 < \lambda < 1$. The following theorem represents the stability of (10) in the sense of UHR.

Theorem 5. *Assume that $f : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies*

$$|f(\nu, \rho_1) - f(\nu, \rho_2)| \leq K|\rho_1 - \rho_2|, \forall \nu \in [0, T], \rho_s \in \mathbb{R}, s = 1, 2. \tag{12}$$

If a continuous function $x \in E$ satisfies $I_T^{1-\lambda}x(T) = 0$, then

$$\left|\left(D_T^\lambda x\right)(\nu) - f(\nu, x(\nu))\right| \leq \epsilon\psi(\nu), \tag{13}$$

for all $\nu \in [0, T]$, where $\psi(\nu)$ is a nonincreasing function. Then, there is a unique function x_0 such that

$$\begin{aligned} x_0(\nu) &= \frac{1}{\Gamma(\lambda)}\int_\nu^T (s-\nu)^{\lambda-1} f(s, x_0(s))ds, \\ |x(\nu) - x_0(\nu)| &\leq \frac{\delta + K}{K}\frac{T^\lambda}{\Gamma(\lambda+1)}E_\lambda\left((K+\delta)T^\lambda\right)\epsilon\psi(\nu), \forall \nu \in [0, T], \end{aligned} \tag{14}$$

for any positive constants δ.

Proof. We start the proof by defining the metric on E in this manner

$$d(x_1, x_2) \coloneqq \inf\left\{M > 0 : \frac{|x_1(\nu) - x_2(\nu)|}{E_\lambda\left((K+\delta)(T-\nu)^\lambda\right)} \leq M\psi(\nu)\right\}. \tag{15}$$

We can prove that the space (E, d) is a complete generalized metric space (see Lemma 1 in [21]).

Define the operator $\mathscr{A} : E \longrightarrow E$ with

$$(\mathscr{A}u)(\nu) \coloneqq \frac{1}{\Gamma(\lambda)}\int_\nu^T (s-\nu)^{\lambda-1} f(s, u(s))ds. \tag{16}$$

Since we have $\mathscr{A}u \in E$ for all $u \in E$ and

$$\frac{|(\mathscr{A}u_0)(\nu) - u_0(\nu)|}{E_\lambda\left((K+\delta)(T-\nu)^\lambda\right)} < +\infty, \forall u_0 \in E, \nu \in [0, T], \tag{17}$$

Therefore, $d(\mathscr{A}u_0, u_0) < \infty$. Note also that we have $d(u_0, u) < \infty$, $\forall u \in E$, and then $\{u \in E : d(u_0, u) < \infty\} = E$.

In addition, for any $u_1, u_2 \in E$ we get

$$\begin{aligned} |(\mathscr{A}u_1)(\nu) - (\mathscr{A}u_2)(\nu)| &= \frac{1}{\Gamma(\lambda)}\left|\int_\nu^T (s-\nu)^{\lambda-1}[f(s, u_1(s)) - f(s, u_2(s)]ds\right| \\ &\leq \frac{K}{\Gamma(\lambda)}\int_\nu^T (s-\nu)^{\lambda-1}|u_1(s) - u_2(s)|ds \\ &\leq \frac{K}{\Gamma(\lambda)}\int_\nu^T (s-\nu)^{\lambda-1}\frac{|u_1(s) - u_2(s)|}{E_\lambda\left(K+\delta)(T-s)^\lambda\right)}E_\lambda\left((K+\delta)(T-s)^\lambda\right)ds \\ &\leq \frac{Kd(u_1, u_2)}{\Gamma(\lambda)}\int_\nu^T \psi(s)(s-\nu)^{\lambda-1}E_\lambda\left((K+\delta)(T-s)^\lambda\right)ds \\ &\leq \frac{Kd(u_1, u_2)}{\Gamma(\lambda)}\psi(\nu)\int_\nu^T (s-\nu)^{\lambda-1}E_\lambda\left((K+\delta)(T-s)^\lambda\right)ds, \end{aligned} \tag{18}$$

for all $\nu \in [0, T]$.

Now, using the fact that

$$\begin{aligned} &\int_\nu^T (s-\nu)^{\lambda-1}E_\lambda\left((K+\delta)(T-s)^\lambda\right)ds \\ &\quad \leq \frac{\Gamma(\lambda)}{K+\delta}E_\lambda\left((K+\delta)(T-\nu)^\lambda\right), \end{aligned} \tag{19}$$

for all $\nu \in [0, T]$.

Then,

$$|(\mathscr{A}u_1)(\nu) - (\mathscr{A}u_2)(\nu)| \leq \frac{K}{K+\delta}d(u_1, u_2)E_\lambda\left((K+\delta)(T-\nu)^\lambda\right)\psi(\nu), \tag{20}$$

which implies that

$$d(\mathscr{A}u_1, \mathscr{A}u_2) \leq \frac{K}{K+\delta}d(u_1, u_2), \tag{21}$$

which proves that $\mathscr{A}$ is a strictly contractive. Following the same way as in the proof of Theorem 7.1 in [22], we get

$$\begin{aligned} &\left|x(\nu) - \frac{1}{\Gamma(\lambda)}\int_\nu^T (s-\nu)^{\lambda-1}f(s, x(s))ds\right| \\ &\quad \leq \frac{1}{\Gamma(\lambda)}\int_\nu^T (s-\nu)^{\lambda-1}\epsilon\psi(s)ds \leq \frac{\epsilon}{\Gamma(\lambda)}\psi(\nu)\int_\nu^T (s-\nu)^{\lambda-1}ds \\ &\quad \leq \frac{\epsilon}{\Gamma(\lambda+1)}\psi(\nu)(T-\nu)^\lambda, \end{aligned} \tag{22}$$

which means that

$$d(x, \mathscr{A}x) \leq \frac{\epsilon}{\Gamma(1+\lambda)} T^{\lambda}. \quad (23)$$

Now, in view of Theorem 4 there is a solution x^* with

$$d(x^*, x) \leq \frac{\delta + K}{\delta} \frac{\epsilon T^{\lambda}}{\Gamma(1+\lambda)}, \quad (24)$$

and then

$$|x^*(\nu) - x(\nu)| \leq \frac{\delta + K}{\delta} \frac{T^{\lambda}}{\Gamma(\lambda+1)} E_{\lambda}\left((K+\delta)T^{\lambda}\right) \epsilon\psi(\nu), \quad (25)$$

for all $\nu \in [0, T]$. This results prove that in view of Definition 3, (10) is UHR stable. □

Remark 6. In the current work, we do not assume any constrains on K unlike the case of results in [22] where the assumption $0 < \text{KLM} < 1$ is a basic condition.

Now, we investigate the stability of (10) in the case where $\lambda > 1$ as follows.

3.2. The Case $\lambda > 1$

Theorem 7. *Assume that $f : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies*

$$|f(\nu, \rho_1) - f(\nu, \rho_2)| \leq K|\rho_1 - \rho_2|, \forall \nu \in [0, T], \rho_i \in \mathbb{R}, i = 1, 2. \quad (26)$$

If a continuous function $x \in E$ satisfies $(D_T^{\lambda-i}x)(T) = a_i$, $(i = 1, 2 \cdots, n-1)$, $a_n = I_T^{n-\lambda}x(T) = 0, n = -[-\lambda]$, then

$$\left|\left(D_T^{\lambda}x\right)(\nu) - f(\nu, x(\nu))\right| \leq \epsilon\psi(\nu), \quad (27)$$

for all $\nu \in [0, T]$, where $\psi(\nu)$ is a nonincreasing function. Then, there is a unique function x_0 with

$$x_0(\nu) = \sum_{j=1}^{n-1} \frac{(-1)^{n-j}a_j}{\Gamma(\lambda - j + 1)} (T-\nu)^{\lambda-j} + \frac{1}{\Gamma(\lambda)} \int_{\nu}^{T} (s-\nu)^{\lambda-1} f(s, x_0(s)) ds,$$
$$|x(\nu) - x_0(\nu)| \leq \frac{T^{\lambda} e^{(K+\delta)T}}{(1-c)\Gamma(\lambda+1)} \epsilon\psi(\nu), \quad \forall \nu \in [0, T], \quad (28)$$

where $c = KT^{\lambda-1}/\Gamma(\lambda)(K+\delta)$ and some positive constant δ such that $0 < c < 1$.

Proof. We start by defining the metric on E by the form

$$d(x_1, x_2) := \inf\left\{M > 0 : \frac{|x_1(\nu) - x_2(\nu)|}{e^{(K+\delta)(T-\nu)}} \leq M\psi(\nu)\right\}, \quad (29)$$

and we define the operator $\mathscr{A} : E \longrightarrow E$ such that

$$(\mathscr{A}u)(\nu) := \sum_{j=1}^{n-1} \frac{(-1)^{n-j}a_j}{\Gamma(\lambda - j + 1)} (T-\nu)^{\lambda-j} + \frac{1}{\Gamma(\lambda)} \int_{\nu}^{T} (s-\nu)^{\lambda-1} f(s, u(s)) ds. \quad (30)$$

Since we have $\mathscr{A}u \in E$ for all $u \in E$ and

$$\frac{|(\mathscr{A}u_0)(\nu) - u_0(\nu)|}{e^{(K+\delta)(T-\nu)}} < +\infty, \forall u_0 \in E, \nu \in [0, T], \quad (31)$$

so that it is clear that $d(\mathscr{A}u_0, u_0) < \infty$. Note also that we have $d(u_0, u) < \infty$, $\forall u \in E$, and then $\{u \in E : d(u_0, u) < \infty\} = E$.

In addition, for any $u_1, u_2 \in E$, we get

$$\begin{aligned} |(\mathscr{A}u_1)(\nu) - (\mathscr{A}u_2)(\nu)| &= \frac{1}{\Gamma(\lambda)} \left|\int_{\nu}^{T} (s-\nu)^{\lambda-1} [f(s, u_1(s)) - f(s, u_2(s)] ds\right| \\ &\leq \frac{K}{\Gamma(\lambda)} \int_{\nu}^{T} (s-\nu)^{\lambda-1} |u_1(s) - u_2(s)| ds \\ &\leq \frac{K(T)^{\lambda-1} d(u_1, u_2)}{\Gamma(\lambda)} \psi(\nu) \frac{e^{(K+\delta)(T-\nu)}}{K+\delta}. \end{aligned} \quad (32)$$

Then, (using $0 < c < 1$)

$$d(\mathscr{A}u_1, \mathscr{A}u_2) \leq \frac{KT^{\lambda-1}}{\Gamma(\lambda)(K+\delta)} d(u_1, u_2) \leq cd(u_1, u_2), \quad (33)$$

which proves that the operator $\mathscr{A}$ is strictly contractive. Following the same way as in the proof of Theorem 5 in [22], we have

$$\begin{aligned} |x(\nu) - (\mathscr{A}x)(\nu)| &\leq \frac{1}{\Gamma(\lambda)} \int_{\nu}^{T} (s-\nu)^{\lambda-1} \epsilon\psi(s) ds \\ &\leq \frac{\epsilon}{\Gamma(\lambda+1)} \psi(\nu)(T-\nu)^{\lambda}. \end{aligned} \quad (34)$$

Then,

$$d(x, \mathscr{A}x) \leq \frac{\epsilon}{\Gamma(\lambda+1)} T^{\lambda}. \quad (35)$$

Now, there is a solution x^* (due to Theorem 4) with

$$d(x^*, x) \leq \frac{1}{(1-c)} \frac{\epsilon T^{\lambda}}{\Gamma(\lambda+1)}, \quad (36)$$

and then

$$|x^*(\nu) - x(\nu)| \leq \frac{T^{\lambda} e^{(K+\delta)T}}{(1-c)\Gamma(\lambda+1)} \epsilon\psi(\nu), \quad (37)$$

for all $\nu \in [0, T]$. □

Remark 8. Notice that in the current work, we do not assume any condition on K unlike the case of Theorem 7in [22] where the condition $0 < KLM < 1$ is a basic condition.

4. Examples

The following examples are used to illustrate our findings.

Example 9. Consider equation (10) for $\lambda = 0.5$, $T = 10$, and $f(\nu, x) = \nu^2 \cos(x)$.

We have

$$\left|\nu^2 \cos(x_1) - \nu^2 \cos(x_2)\right| \le 100|x_1 - x_2|, \quad \forall \nu \in [0, 10], \quad x_1, x_2 \in \mathbb{R}. \tag{38}$$

Then, $K = 100$.

Suppose that $x \in C([0, 10], \mathbb{R})$ satisfies $I_{10}^{0.5}x(10) = 0$ and

$$\left|\left(D_{10}^{0.5}x\right)(\nu) - \nu^2 \cos(x(\nu))\right| \le 1, \tag{39}$$

for all $\nu \in [0, 10]$.

Here, $\epsilon = 1$ and $\psi(\nu) = 1$. Using Theorem 5, there is a continuous function x_0 such that

$$\begin{aligned} x_0(\nu) &= \frac{1}{\Gamma(0.5)} \int_\nu^{10} (s - \nu)^{-0.5} s^2 \cos(x_0(s)) ds, \\ |x(\nu) - x_0(\nu)| &\le \frac{11 E_{0.5}(11)}{10\Gamma(1.5)}, \quad \forall \nu \in [0, 10]. \end{aligned} \tag{40}$$

Example 10. Consider equation (10) for $\lambda = 1.5$, $T = 5$, and $f(\nu, x) = \nu^3 \sin(x)$.

We have

$$\left|\nu^3 x_1 - \nu^3 x_2\right| \le 125|x_1 - x_2|, \quad \forall \nu \in [0, 5], \quad x_1, x_2 \in \mathbb{R}. \tag{41}$$

Then, $K = 125$.

Suppose that $x \in C([0, 5], \mathbb{R})$ satisfies $(D_5^{0.5}x)(5) = a \in \mathbb{R}$, $I_5^{0.5}x(5) = 0$ and

$$\left|\left(D_5^{1.5}x\right)(\nu) - \nu^3 \sin(x(\nu))\right| \le 1, \tag{42}$$

for all $\nu \in [0, 5]$.

Here, $\epsilon = 1$ and $\psi(\nu) = 1$. Using Theorem 7, there is a continuous function x_0 such that

$$\begin{aligned} x_0(\nu) &= -\frac{a}{\Gamma(1.5)}(5 - \nu)^{0.5} + \frac{1}{\Gamma(1.5)} \int_\nu^5 (s - \nu)^{0.5} s^3 \sin(x_0(s)) ds, \\ |x(\nu) - x_0(\nu)| &\le \frac{5^{1.5} e^{1600}}{\Gamma(2.5)\left(1 - 125\sqrt{5}/320\Gamma(1.5)\right)}, \quad \forall \nu \in [0, 5]. \end{aligned} \tag{43}$$

5. Conclusion

A version of Banach's contraction principle has been successfully utilized in this work to study the UHR stability of nonlinear differential equations with Riemann-Liouville fractional derivatives. In this way, under specific assumptions and conditions, the stability results have been obtained. In our analysis, we get rid of some constrains that have been posed on the lipschitz constants in some interesting recent related works. Two illustrative examples are given at the end to apply our theoretical results and show its validity. Potential future directions of our work can be dedicated to applying our obtained results to some practical applications. Some possible extensions and generalizations of our obtained results can also be our future investigations.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Ethical Approval

The authors declare the ethical standards are taken into consideration.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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