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Research Article

Fixed Points and Continuity Conditions of Generalized b-Quasicontractions

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In this research, we first check that the abstract cone b-metric is discontinuous in the case of normal cone by a counter example. We obtain several meaningful results about generalized b-quasicontraction and Ćirić-type b-quasicontraction in cone b-metric spaces over Banach algebras, weakening certain important conditions of the spaces and the mappings. Meanwhile, several valid examples are given to demonstrate the new notions and fixed point results, when the existing theorems in the literature are not applicable.

1. Introduction and Preliminaries

Since the concept of cone metric space was reintroduced by Huang and Zhang [1], a large number of fixed point results were gained in such spaces. In 2009, the authors in [2] defined quasicontraction in cone metric space with a normal cone. Subsequently, by removing the normality of the cone, Kadelburg et al. [3] established a fixed point theorem with a quasicontractive constant $k \in (0, 1/2)$ ([3], Theorem 2.2). Sequentially, Gajić and Rakočević [4] showed the result holds when $k \in [0, 1)$ in the same spaces. In 2011, the notion of cone b-metric space was given by Hussian and Shah [5], which generalized b-metric space and cone metric space. Afterwards, Huang and Xu [6] gave several fixed point results of different classes contraction in this spaces. In 2013, the scholars in [7] claimed the cone metric space over Banach algebra while the Banach space E is substituted with the Banach algebra $\mathscr{A}$. In their paper, the most important work was to verify that the fixed point conclusions in cone metric spaces over Banach algebras were not equivalent to those in metric spaces by a nontrivial example. In [8], the authors redefined cone b-metric space over Banach algebra. They obtained some fixed points of contractions in such spaces which were not equivalent to the corresponding work in b-metric spaces. Later on, numerous interesting fixed point theorems in these spaces were promoted to be studied by many scholars, see [9–19] and their references, but most of the results were established under the completeness of the spaces and some even required the continuity of b-metric (while cone metric and metric are continuous) (see [1–4, 6–8, 18–29]).

In 2018, Aleksić et al. [20] proved that b-metric is discontinuous in general by some examples, which is a generalization of metric. However, the fixed point conclusions of b-quasicontraction in b-metric spaces were also discussed under continuous b-metric and complete b-metric spaces. In order to improve these too strong conditions, we prove that the abstract cone b-metric is discontinuous even with a normal cone. Furthermore, we gain some fixed point results in cone b-metric spaces over Banach algebras when the cone b-metric is discontinuous. Some other conditions are weakened by giving several new concepts, such as T-orbital completeness, orbital continuity, and orbital compactness in these spaces. Our work develops and broadens some significant well-known theorems in the literature [8, 11, 20, 23, 28, 30]. Furthermore, some nontrivial examples are provided to demonstrate that the new concepts and main theorems in this paper are genuine developments and generalizations of some existing ones in the literature.

Now, we start our paper with some preliminary definitions in the literature.

Suppose $\mathscr{A}$ is a real Banach algebra and P is a cone over Banach algebra $\mathscr{A}$ with int $P \neq \varnothing$, the notation $\leq$ expresses the partial ordering in terms of P. For the definitions of Banach algebras and cones, the readers may refer to [24, 31].

Definition 1. (see [5, 8]). Suppose X is a nonempty set and $s \geq 1$ is a constant, the mapping $d : X \times X \longrightarrow \mathscr{A}$ is said to be a cone b-metric if

(d1) $\theta \leq d(\varpi, \eta)$ for all $\varpi, \eta \in X$ and $d(\varpi, \eta) = \theta$ if and only if $\varpi = \eta$

(d2) $d(\varpi, \eta) = d(\eta, \varpi)$ for all $\varpi, \eta \in X$

(d3) $d(\varpi, \eta) \leq s[d(\varpi, \zeta) + d(\zeta, \eta)]$ for all $\varpi, \eta, \zeta \in X$

The pair (X, d) is called a cone b-metric space over Banach algebra $\mathscr{A}$.

Definition 2. (see [8]). Suppose (X, d) is a cone b-metric space over Banach algebra $\mathscr{A}$, $\varpi \in X$, and $\{\varpi_n\}$ is a sequence in X, we say

(i) $\{\varpi_n\}$ converges to x if for each $c \in \mathscr{A}$ with $c \gg \theta$, there is an integer $N \geq 1$ such that $d(\varpi_n, \varpi) \ll c$ for all $n > N$

(ii) $\{\varpi_n\}$ is a Cauchy sequence if for each $c \in \mathscr{A}$ with $c \gg \theta$, there is an integer $N \geq 1$ such that $d(\varpi_n, \varpi_m) \ll c$ for all $n, m > N$

(iii) (X, d) is complete if each Cauchy sequence in X is convergent

It is significant to note that different from the usual metric and cone metric with a normal cone, cone b-metric is generally discontinuous even with a normal cone. Let us show an example.

Example 3. Take $\mathscr{A} = \mathbb{R}^2$ with a norm $\|\varpi_1, \varpi_2\| = |\varpi_1| + |\varpi_2|$. For any $\varpi = (\varpi_1, \varpi_2)$ and $\eta = (\eta_1, \eta_2)$ in $\mathscr{A}$, set the multiplication as

$$\varpi\eta = (\varpi_1, \varpi_2)(\eta_1, \eta_2) = (\varpi_1\eta_1, \varpi_1\eta_2 + \varpi_2\eta_1). \tag{1}$$

Let $P = \{(\varpi_1, \varpi_2) \in \mathbb{R}^2 : \varpi_1, \varpi_2 \geq 0\}$. Then, $\mathscr{A}$ is a real Banach algebra owing the unit element $e = (1, 0)$, and the cone $P \subseteq \mathscr{A}$ is normal. Set $X = \mathbb{N} \cup \{\infty\} \times \mathbb{N} \cup \{\infty\}$ and $d : X \times X \longrightarrow \mathscr{A}$ be defined as

$$d((\varpi_1, \varpi_2), (\eta_1, \eta_2)) = \begin{cases} (0, 0), & \varpi_1 = \eta_1, \varpi_2 = \eta_2\,; \\ \left(\left|\dfrac{1}{\varpi_1} - \dfrac{1}{\eta_1}\right|, \left|\dfrac{1}{\varpi_2} - \dfrac{1}{\eta_2}\right|\right), & \text{one of } (\varpi_1, \varpi_2) \text{ and } (\eta_1, \eta_2) \text{ is odd and the other is odd or } \infty; \\ (5, 5), & \text{one of } (\varpi_1, \varpi_2) \text{ and } (\eta_1, \eta_2) \text{ is even and the other is even or } \infty; \\ (2, 2), & \text{otherwise.} \end{cases} \tag{2}$$

In the above definition, (ϖ_1, ϖ_2) is odd if both ϖ_1 and ϖ_2 are odd; (ϖ_1, ϖ_2) is ∞ if both ϖ_1 and ϖ_2 are ∞. We can check that (X, d) is a cone b-metric space over Banach algebra $\mathscr{A}$ where $s = 5/2$.

Let $\zeta_m = (4m - 1, 4m + 1)$, $m \in \mathbb{N}^+$. We have

$$\begin{aligned} d(\zeta_m, \infty) &= d((4m - 1, 4m + 1), (\infty, \infty)) \\ &= \left(\left|\frac{1}{4m - 1} - \frac{1}{\infty}\right|, \left|\frac{1}{4m + 1} - \frac{1}{\infty}\right|\right) \\ &= \left(\frac{1}{4m - 1}, \frac{1}{4m + 1}\right) \longrightarrow (0, 0), \end{aligned} \tag{3}$$

which indicates $\zeta_m \longrightarrow \infty$ but

$$\begin{aligned} d(\zeta_m, 2) &= d((4m - 1, 4m + 1), (2, 2)) \\ &= (2, 2) \nrightarrow (5, 5) \\ &= d((\infty, \infty), (2, 2)) = d(\infty, 2). \end{aligned} \tag{4}$$

So, we have showed that the cone b-metric is discontinuous in the case of normal cone.

Definition 4. (see [25]). Suppose P is a solid cone, $P \subseteq \mathscr{A}$. A sequence $\{\sigma_n\} \subset P$ is a c-sequence if for any $c \gg \theta$, there is $n_0 \in \mathbb{N}$ such that $\sigma_n \ll c$ for all $n \geq n_0$.

Lemma 5. (see [8]). *Suppose P is a solid cone, $P \subseteq \mathscr{A}$. If α, $\beta \in P$, $\{\varpi_n\}$, and $\{\eta_n\}$ are c-sequences in $\mathscr{A}$, then $\{\alpha\varpi_n + \beta\eta_n\}$ is a c-sequence in $\mathscr{A}$.*

Lemma 6. (see [31]). *Suppose $\mathscr{A}$ is a Banach algebra with a unit e and $\varpi \in \mathscr{A}$, if the spectral radius $\rho(\varpi)$ of ϖ satisfies*

$$\rho(\varpi) = \lim_{n \longrightarrow \infty} \|\varpi^n\|^{1/n} = \inf_{n \geq 1} \|\varpi^n\|^{1/n} < 1, \tag{5}$$

then $e - \varpi$ is invertible. Actually, $(e - \varpi)^{-1} = \sum_{i=0}^{\infty} \varpi^i$.

Throughout this paper, we always suppose (X, d) is a cone b-metric space over Banach algebra $\mathscr{A}$ with a unit e and $s \geq 1$.

2. Orbital Completeness

In this section, we give several fixed point theorems of generalized b-quasicontraction in orbitally complete cone b-metric spaces over Banach algebras. The cone is neither regular nor normal. The cone b-metric and the self mapping are not required continuous. At first, encouraged by the concepts of orbital continuity, Φ-orbital completeness [32], and k-continuity [21] in usual metric space, we provide the analogous concepts in cone b-metric space over Banach algebra $\mathscr{A}$, which are important in our proof.

Definition 7. Suppose (X, d) is a cone b-metric space over Banach algebra $\mathscr{A}$ and $\Phi : X \longrightarrow X$, take $\varpi \in X$ and $O_\Phi(\varpi) = \{\varpi, \Phi\varpi, \Phi^2\varpi, \Phi^3\varpi, \cdots\}$, namely, the orbit of ϖ under Φ.

The mapping Φ is orbitally continuous at an element $\zeta \in X$ if for any sequence $\{\varpi_n\} \subset O_\Phi(\varpi)$ (for all $\varpi \in X$), $\varpi_n \longrightarrow \zeta$ as $n \longrightarrow \infty$ implies $\Phi\varpi_n \longrightarrow \Phi\zeta$ as $n \longrightarrow \infty$. Note that each continuous mapping is orbitally continuous, but not the converse.

The mapping Φ is k-continuous for $k = 1, 2, \cdots$, if $\Phi^{k-1}\varpi_n \longrightarrow \zeta$ implies $\Phi^k\varpi_n \longrightarrow \Phi\zeta$ $(n \longrightarrow \infty)$. It is clear that Φ is 1-continuous if and only if it is continuous, and k-continuity implies $(k+1)$-continuity for any $k = 1, 2, \cdots$ but not the converse. Furthermore, continuity of Φ^k and k-continuity of Φ are independent when $k > 1$. See the following examples.

Example 8. Suppose $\mathscr{A} = C^1_{\mathbb{R}}[0, 1] \times C^1_{\mathbb{R}}[0, 1]$ with a norm,

$$\|\varpi_1, \varpi_2\| = \|\varpi_1\|_\infty + \|\varpi_2\|_\infty + \|\varpi'_1\|_\infty + \|\varpi'_2\|_\infty. \tag{6}$$

For any $\varpi = (\varpi_1, \varpi_2)$ and $\eta = (\eta_1, \eta_2)$ in $\mathscr{A}$, set the multiplication as

$$\varpi\eta = (\varpi_1, \varpi_2)(\eta_1, \eta_2) = (\varpi_1\eta_1, \varpi_1\eta_2 + \varpi_2\eta_1). \tag{7}$$

Let $P = \{(\varpi_1(z), \varpi_2(z)) \in \mathscr{A} : \varpi_1(z) \geq 0, \varpi_2(z) \geq 0, z \in [0, 1]\}$. It follows that there is a unit element $e = (1, 0)$ in the real Banach algebra $\mathscr{A}$. Let $X = [0, 4] \times [0, 4]$ and define $d : X \times X \longrightarrow \mathscr{A}$ by

$$d((\varpi_1, \varpi_2), (\eta_1, \eta_2))(z) = \left(|\varpi_1 - \eta_1|^2 \exp(z), |\varpi_2 - \eta_2|^2 \exp(z)\right), \tag{8}$$

for any $\varpi = (\varpi_1, \varpi_2), \eta = (\eta_1, \eta_2) \in X$. It is obvious that (X, d) is complete and $s = 2$. Suppose $\Phi : X \longrightarrow X$ is defined as

$$\Phi(\varpi_1, \varpi_2) = \begin{cases} \left(\sin\dfrac{\varpi_1}{2}, \sin\dfrac{\varpi_2}{3}\right), & (\varpi_1, \varpi_2) \in [0, 2] \times [0, 2]; \\ \left(\log\left(1 + \dfrac{\varpi_1}{2}\right), \log\left(1 + \dfrac{\varpi_2}{3}\right)\right), & \text{otherwise}. \end{cases} \tag{9}$$

For any $\zeta_0 = (\varpi_1, \varpi_2) \in X$, if $\zeta_n = \Phi\zeta_{n-1}, n \in \mathbb{N}$, then $\zeta_n \longrightarrow \theta$ implies $\Phi\zeta_n \longrightarrow \Phi\theta = \theta$ while $\theta = (0, 0)$. Clearly, Φ is an orbitally continuous mapping rather than continuous. Moreover, notice that Φ is 2-continuous but not continuous, that is, 2-continuity of Φ does not imply continuity of Φ^2. Furthermore, for each integer $k \geq 2$, Φ^k is discontinuous while Φ is k-continuous. This indicates that k-continuity of Φ does not imply continuity of Φ^k in usual situation.

In [12], we have given the following definitions of T-orbital completeness.

Definition 9. The space (X, d) is named Φ-orbitally complete, if each Cauchy sequence included in $O_\Phi(\varpi)$ for some $\varpi \in X$ converges in X. Each complete space (X, d) is Φ-orbitally complete for any Φ but not the converse.

For being convenient, we give the notion of generalized b-quasicontraction in (X, d). The mapping $\Phi : X \longrightarrow X$ is named a generalized b-quasicontraction if there exists $r \in P$ with $\rho(r) < 1/s$, and one has

$$d(\Phi\varpi, \Phi\eta) \preceq \mathrm{r}u(\varpi, \eta), (\varpi, \eta \in X), \tag{10}$$

where

$$u(\varpi, \eta) \in \{d(\varpi, \eta), d(\varpi, \Phi\varpi), d(\eta, \Phi\eta), d(\varpi, \Phi\eta), d(\eta, \Phi\varpi)\}. \tag{11}$$

When $s = 1$, we call it a generalized quasicontraction in cone metric space over Banach algebra. Before showing our main results, we give an important lemma without the assumptions of completeness and normality.

Lemma 10. *Assume the mapping $\Phi : X \longrightarrow X$ is a generalized b-quasicontraction in the space (X, d), for each $\varpi_0 \in X$, let $\varpi_n = \Phi\varpi_{n-1}$. Then, for any integers $i, j \geq 1$, it holds that*

$$d\left(\varpi_i, \varpi_j\right) \preceq sr(e - sr)^{-1} d(\varpi_0, \varpi_1). \tag{12}$$

Proof. At first, we show that for any $n \geq 1$ and $1 \leq i \leq n$,

$$d(\varpi_i, \varpi_n) \preceq \mathrm{sr}(e - sr)^{-1} d(\varpi_0, \varpi_1). \tag{13}$$

If $n = 1$, the result is trivial. Assume $n = 2, i = 1$, then

$$d(\varpi_1, \varpi_2) = d(\Phi\varpi_0, \Phi\varpi_1) \preceq ru(\varpi_0, \varpi_1), \tag{14}$$

where

$$u(\varpi_0, \varpi_1) \in \{d(\varpi_0, \varpi_1), d(\varpi_0, \varpi_1), d(\varpi_1, \varpi_2), d(\varpi_0, \varpi_2), d(\varpi_1, \varpi_1)\}. \tag{15}$$

Obviously, $u(\varpi_0, \varpi_1) \neq d(\varpi_1, \varpi_2)$ and $u(\varpi_0, \varpi_1) \neq d(\varpi_1, \varpi_1)$; otherwise, there is a contradiction.

If $u(\varpi_0, \varpi_1) = d(\varpi_0, \varpi_1)$, then

$$d(\varpi_1, \varpi_2) \preceq \mathrm{rd}(\varpi_0, \varpi_1) \preceq \mathrm{sr}(e - sr)^{-1} d(\varpi_0, \varpi_1). \tag{16}$$

If $u(\varpi_0, \varpi_1) = d(\varpi_0, \varpi_2)$, then

$$d(\varpi_1, \varpi_2) \le rd(\varpi_0, \varpi_2) \le \mathrm{sr}[d(\varpi_0, \varpi_1) + d(\varpi_1, \varpi_2)], \quad (17)$$

which implies that $d(\varpi_1, \varpi_2) \le sr(e - sr)^{-1} d(\varpi_0, \varpi_1)$. Thus, (13) is true. Now, we suppose for all integers $n \ge 2$ and $1 \le i \le n$,

$$d(\varpi_i, \varpi_n) \le \mathrm{sr}(e - \mathrm{sr})^{-1} d(\varpi_0, \varpi_1). \quad (18)$$

We have to prove

$$d(\varpi_i, \varpi_{n+1}) \le \mathrm{sr}(e - \mathrm{sr})^{-1} d(\varpi_0, \varpi_1), \quad (19)$$

for $n + 1 \ge 2, 1 \le i \le n + 1$. Now, we prove

$$d(\varpi_1, \varpi_{n+1}) \le \mathrm{sr}(e - \mathrm{sr})^{-1} d(\varpi_0, \varpi_1). \quad (20)$$

By (10), we obtain

$$d(\varpi_1, \varpi_{n+1}) = d(\Phi\varpi_0, \Phi\varpi_n) \le \mathrm{ru}(\varpi_0, \varpi_n), \quad (21)$$

where

$$u(\varpi_0, \varpi_n) \in \{d(\varpi_0, \varpi_n), d(\varpi_0, \varpi_1), d(\varpi_n, \varpi_{n+1}), d(\varpi_0, \varpi_{n+1}), d(\varpi_n, \varpi_1)\}. \quad (22)$$

If $u(\varpi_0, \varpi_n) = d(\varpi_0, \varpi_n)$, then

$$\begin{aligned} d(\varpi_1, \varpi_{n+1}) &\le \mathrm{rd}(\varpi_0, \varpi_n) \\ &\le \mathrm{sr}[d(\varpi_0, \varpi_1) + d(\varpi_1, \varpi_n)] \\ &= \mathrm{sr}\left[d(\varpi_0, \varpi_1) + \mathrm{sr}(e - sr)^{-1} d(\varpi_0, \varpi_1)\right] \\ &= \mathrm{sr}\left(e + \sum_{i=1}^{+\infty} (\mathrm{sr})^i\right) d(\varpi_0, \varpi_1) \\ &= \mathrm{sr}(e - \mathrm{sr})^{-1} d(\varpi_0, \varpi_1). \end{aligned} \quad (23)$$

If $u(\varpi_0, \varpi_n) = d(\varpi_0, \varpi_1)$, then $d(\varpi_1, \varpi_{n+1}) \le \mathrm{rd}(\varpi_0, \varpi_1) \le \mathrm{sr}(e - \mathrm{sr})^{-1} d(\varpi_0, \varpi_1)$.

If $u(\varpi_0, \varpi_n) = d(\varpi_0, \varpi_{n+1})$, then

$$d(\varpi_1, \varpi_{n+1}) \le rd(\varpi_0, \varpi_{n+1}) \le \mathrm{sr}[d(\varpi_0, \varpi_1) + d(\varpi_1, \varpi_{n+1})], \quad (24)$$

which yields $(e - sr) d(\varpi_1, \varpi_{n+1}) \le \mathrm{srd}(\varpi_0, \varpi_1)$. That is, $d(\varpi_1, \varpi_{n+1}) \le sr(e - sr)^{-1} d(\varpi_0, \varpi_1)$.

If $u(\varpi_0, \varpi_n) = d(\varpi_1, \varpi_n)$, then

$$d(\varpi_1, \varpi_{n+1}) \le rd(\varpi_1, \varpi_n) \le r \cdot sr(e - sr)^{-1} d(\varpi_0, \varpi_1) \le sr(e - sr)^{-1} d(\varpi_0, \varpi_1). \quad (25)$$

At last, we only check that (20) is true when $u(\varpi_0, \varpi_n) = d(\varpi_n, \varpi_{n+1})$. That is,

$$d(\varpi_1, \varpi_{n+1}) \le rd(\varpi_n, \varpi_{n+1}) = rd(\Phi\varpi_{n-1}, \Phi\varpi_n) \le r^2 u(\varpi_{n-1}, \varpi_n), \quad (26)$$

where

$$u(\varpi_{n-1}, \varpi_n) \in \{d(\varpi_{n-1}, \varpi_n), d(\varpi_{n-1}, \varpi_n), d(\varpi_n, \varpi_{n+1}), d(\varpi_{n-1}, \varpi_{n+1}), d(\varpi_n, \varpi_n)\}. \quad (27)$$

Clearly, $u(\varpi_{n-1}, \varpi_n) \ne d(\varpi_n, \varpi_{n+1})$ and $u(\varpi_{n-1}, \varpi_n) \ne d(\varpi_n, \varpi_n)$. If $u(\varpi_{n-1}, \varpi_n) = d(\varpi_{n-1}, \varpi_n)$, then

$$\begin{aligned} d(\varpi_1, \varpi_{n+1}) &\le r^2 d(\varpi_{n-1}, \varpi_n) \\ &\le r^2 \cdot sr(e - sr)^{-1} d(\varpi_0, \varpi_1) \\ &\le sr(e - sr)^{-1} d(\varpi_0, \varpi_1). \end{aligned} \quad (28)$$

If $u(\varpi_{n-1}, \varpi_n) = d(\varpi_{n-1}, \varpi_{n+1})$, then

$$d(\varpi_1, \varpi_{n+1}) \le rd(\varpi_n, \varpi_{n+1}) \le r^2 d(\varpi_{n-1}, \varpi_{n+1}) \le r^3 u(\varpi_{n-2}, \varpi_n), \quad (29)$$

where

$$u(\varpi_{n-2}, \varpi_n) \in \{d(\varpi_{n-2}, \varpi_n), d(\varpi_{n-2}, \varpi_{n-1}), d(\varpi_n, \varpi_{n+1}), d(\varpi_{n-2}, \varpi_{n+1}), d(\varpi_{n-1}, \varpi_n)\}. \quad (30)$$

Similarly, we also have $u(\varpi_{n-2}, \varpi_n) \ne d(\varpi_n, \varpi_{n+1})$. If $u(\varpi_{n-2}, \varpi_n)$ equals to one of $d(\varpi_{n-2}, \varpi_n), d(\varpi_{n-2}, \varpi_{n-1})$ and $d(\varpi_{n-1}, \varpi_n)$, then by the assumption (18), we have

$$d(\varpi_1, \varpi_{n+1}) \le r^3 \cdot sr(e - sr)^{-1} d(\varpi_0, \varpi_1) \le sr(e - sr)^{-1} d(\varpi_0, \varpi_1). \quad (31)$$

It remains to check (20) when $u(\varpi_{n-1}, \varpi_n) = d(\varpi_{n-2}, \varpi_{n+1})$; that is,

$$d(\varpi_1, \varpi_{n+1}) \le rd(\varpi_n, \varpi_{n+1}) \le r^2 d(\varpi_{n-1}, \varpi_{n+1}) \le r^3 d(\varpi_{n-2}, \varpi_{n+1}). \quad (32)$$

By a similar analysis, we can deduce that

$$\begin{aligned} d(\varpi_1, \varpi_{n+1}) &\le rd(\varpi_n, \varpi_{n+1}) \le r^2 d(\varpi_{n-1}, \varpi_{n+1}) \le \cdots \\ &\le r^{n-1} d(\varpi_2, \varpi_{n+1}) \le r^n u(\varpi_1, \varpi_n), \end{aligned} \quad (33)$$

where

$$u(\varpi_1, \varpi_n) \in \{d(\varpi_1, \varpi_n), d(\varpi_1, \varpi_2), d(\varpi_n, \varpi_{n+1}), d(\varpi_1, \varpi_{n+1}), d(\varpi_n, \varpi_2)\}. \quad (34)$$

Since $u(\varpi_1, \varpi_n) \ne d(\varpi_n, \varpi_{n+1})$ and $u(\varpi_1, \varpi_n) \ne d(\varpi_1, \varpi_{n+1})$, we know $u(\varpi_1, \varpi_n)$ equals to one of $d(\varpi_1, \varpi_n), d(\varpi_1, \varpi_2)$ and $d(\varpi_n, \varpi_2)$. Therefore, we finally obtain

$$\begin{aligned} d(\varpi_1, \varpi_{n+1}) &\le r^n d(\varpi_1, \varpi_n) \le r^n \cdot \mathrm{sr}(e - \mathrm{sr})^{-1} d(\varpi_0, \varpi_1) \\ &\le \mathrm{sr}(e - \mathrm{sr})^{-1} d(\varpi_0, \varpi_1) \end{aligned} \quad (35)$$

or

$$d(\varpi_1, \varpi_{n+1}) \leq r^n d(\varpi_1, \varpi_2) \leq r^n \cdot \mathrm{sr}(e - sr)^{-1} d(\varpi_0, \varpi_1) \leq \mathrm{sr}(e - sr)^{-1} d(\varpi_0, \varpi_1) \quad (36)$$

or

$$d(\varpi_1, \varpi_{n+1}) \leq r^n d(\varpi_n, \varpi_2) \leq r^n \cdot sr(e - sr)^{-1} d(\varpi_0, \varpi_1) \leq sr(e - sr)^{-1} d(\varpi_0, \varpi_1). \quad (37)$$

Hence, (20) is always true. Moreover, by (18), (20), and (33), we know

$$d(\varpi_n, \varpi_{n+1}) \leq r^{n-1} u(\varpi_1, \varpi_n) \leq r^{n-1} \cdot sr(e - sr)^{-1} d(\varpi_0, \varpi_1) \leq sr(e - sr)^{-1} d(\varpi_0, \varpi_1). \quad (38)$$

For $2 \leq i \leq n + 1$, we have

$$d(\varpi_i, \varpi_{n+1}) = d(\Phi\varpi_{i-1}, \Phi\varpi_n) \leq ru(\varpi_{i-1}, \varpi_n), \quad (39)$$

while

$$u(\varpi_{i-1}, \varpi_n) \in \{d(\varpi_{i-1}, \varpi_n), d(\varpi_{i-1}, \varpi_i), d(\varpi_n, \varpi_{n+1}), d(\varpi_{i-1}, \varpi_{n+1}), d(\varpi_n, \varpi_i)\}. \quad (40)$$

If $u(\varpi_{i-1}, \varpi_n)$ equals to one of $d(\varpi_{i-1}, \varpi_n), d(\varpi_{i-1}, \varpi_i), d(\varpi_n, \varpi_i)$ and $d(\varpi_n, \varpi_{n+1})$, then by (18) and (38),

$$d(\varpi_i, \varpi_{n+1}) \leq r \cdot \mathrm{sr}(e - sr)^{-1} d(\varpi_0, \varpi_1) \leq \mathrm{sr}(e - sr)^{-1} d(\varpi_0, \varpi_1). \quad (41)$$

If $u(\varpi_{i-1}, \varpi_n) = d(\varpi_{i-1}, \varpi_{n+1})$, by (18) and (20), (33), and (38), we conclude that

$$d(\varpi_i, \varpi_{n+1}) \leq rd(\varpi_{i-1}, \varpi_{n+1}) \leq r^{i-1} u(\varpi_1, \varpi_n) \leq r^{i-1} \cdot sr(e - sr)^{-1} d(\varpi_0, \varpi_1) \leq sr(e - sr)^{-1} d(\varpi_0, \varpi_1). \quad (42)$$

Therefore, (19) is true. The proof is finished. □

Now, we present and prove our main results without requiring the cone to be normal or d to be continuous.

Theorem 11. *Suppose $\Phi : X \longrightarrow X$ is a generalized b-quasi-contraction mapping in the Φ-orbitally complete space (X, d), if $\rho(r) < 1/s$, then the mapping Φ possesses one and only one fixed point $\zeta \in X$, and the sequence $\{\Phi^n \varpi_0\}$ converges to ζ for each $\varpi_0 \in X$.*

Proof. For each $\varpi_0 \in X$, take $\varpi_n = \Phi\varpi_{n-1}$. If there is some $n \in \mathbb{N}$ such that $\varpi_n = \varpi_{n+1} = \Phi\varpi_n$, then ϖ_n is the fixed point. Hence, we assume $\varpi_n \neq \varpi_{n+1}$ for all $n \in \mathbb{N}$. Let us show that $\{\varpi_n\}$ is a Cauchy sequence. For any $n > m > 1$, write $E(m, n) = \{d(\varpi_i, \varpi_j) : m \leq i < j \leq n\}$. From the concept of generalized b-quasicontraction, for any $u \in E(m, n)$, there exits $v \in E(m - 1, n)$ satisfying $u \leq rv$. Thus, it follows that

$$d(\varpi_m, \varpi_n) \leq ru_{m-1} \leq r^2 u_{m-2} \leq \cdots \leq r^{m-1} u_1 \leq r^{m-1} \cdot sr(e - sr)^{-1} d(\varpi_0, \varpi_1), \quad (43)$$

where $u_{m-1} \in E(m - 1, n), u_{m-2} \in E(m - 2, n), \cdots, u_1 \in E(1, n)$. The last inequality is obtained by Lemma 10. Since $\|s r^m (e - sr)^{-1} d(\varpi_0, \varpi_1)\| \longrightarrow 0$ as $m \longrightarrow \infty (\|r^m\| \longrightarrow 0$ as $m \longrightarrow \infty)$. Therefore, for any $c \in \mathscr{A}$ with $c \gg \theta$, there exists an integer $N \geq 1$ satisfying

$$d(\varpi_m, \varpi_n) \leq sr^m (e - sr)^{-1} d(\varpi_0, \varpi_1) \ll c, (n > m \geq N). \quad (44)$$

That is, $\{\varpi_n\}$ is a Cauchy sequence. By the Φ-orbital completeness of (X, d), we have $\zeta \in X$ such that $\varpi_n \longrightarrow \zeta$ as $n \longrightarrow \infty$. Next, we check $\Phi\zeta = \zeta$. By (10), we see

$$d(\varpi_{n+1}, \Phi\zeta) = d(\Phi\varpi_n, \Phi\zeta) \leq ru(\varpi_n, \zeta), \quad (45)$$

where

$$u(\varpi_n, \zeta) \in \{d(\varpi_n, \zeta), d(\varpi_n, \varpi_{n+1}), d(\zeta, \Phi\zeta), d(\varpi_n, \Phi\zeta), d(\zeta, \Phi\varpi_n)\}. \quad (46)$$

There are the following three cases:

Case 1. If $u(\varpi_n, \zeta)$ equals to one of $d(\varpi_n, \zeta), d(\varpi_n, \varpi_{n+1})$ and $d(\zeta, \Phi\varpi_n)$, then $d(\varpi_{n+1}, \Phi\zeta)$ is a c-sequence.

Case 2. If $u(\varpi_n, \zeta) = d(\zeta, \Phi\zeta)$, we get

$$d(\varpi_{n+1}, \Phi\zeta) \leq \mathrm{rd}(\zeta, \Phi\zeta) \leq sr[d(\zeta, \varpi_{n+1}) + d(\varpi_{n+1}, \Phi\zeta)], \quad (47)$$

that is, $d(\varpi_{n+1}, \Phi\zeta) \leq \mathrm{sr}(e - \mathrm{sr})^{-1} d(\varpi_{n+1}, \zeta)$. Thus, $d(\varpi_{n+1}, \Phi\zeta)$ is a c-sequence.

Case 3. If $u(\varpi_n, \zeta) = d(\varpi_n, \Phi\zeta)$, we gain

$$d(\varpi_{n+1}, \Phi\zeta) \leq \mathrm{rd}(\varpi_n, \Phi\zeta) \leq \mathrm{sr}[d(\varpi_n, \varpi_{n+1}) + d(\varpi_{n+1}, \Phi\zeta)], \quad (48)$$

that is, $d(\varpi_{n+1}, \Phi\zeta) \leq \mathrm{sr}(e - \mathrm{sr})^{-1} d(\varpi_n, \varpi_{n+1})$. Then, $d(\varpi_{n+1}, \Phi\zeta)$ is a c-sequence.

In summary, $d(\varpi_{n+1}, \Phi\zeta)$ is always a c-sequence. This gives $\varpi_n \longrightarrow \Phi\zeta$ as $n \longrightarrow \infty$. According to the uniqueness of the limit, we know $\zeta = \Phi\zeta$.

It remains to prove the uniqueness of ζ. We assume there exists another fixed point $\widehat{\zeta}$ such that $\Phi\widehat{\zeta} = \widehat{\zeta}$, and then

$$d\left(\zeta, \widehat{\zeta}\right) = d\left(\Phi\zeta, \Phi\widehat{\zeta}\right) \leq ru\left(\zeta, \widehat{\zeta}\right), \quad (49)$$

where

$$u\left(\zeta,\widehat{\zeta}\right)\in\left\{d\left(\zeta,\widehat{\zeta}\right),d(\zeta,\Phi\zeta),d\left(\widehat{\zeta},\Phi\widehat{\zeta}\right),d\left(\zeta,\Phi\widehat{\zeta}\right),d\left(\widehat{\zeta},\Phi\zeta\right)\right\}$$
$$=\left\{d\left(\zeta,\widehat{\zeta}\right),\theta\right\}. \tag{50}$$

It is a contradiction. In conclusion, the fixed point ζ is unique, and the sequence $\{\Phi^n\varpi_0\}$ converges to ζ. The proof is finished. □

Taking $s=1$, we obtain the fixed point results in cone metric spaces over Banach algebras.

Corollary 12. *Suppose $T:X\longrightarrow X$ is a generalized quasicontraction mapping in Φ-orbitally complete cone metric space over Banach algebra $\mathscr{A}$ with a unit e, if $\rho(r)<1$, then the mapping Φ possesses one and only one fixed point $\zeta\in X$, and the sequence $\{\Phi^n\varpi_0\}$ converges to ζ for every $\varpi_0\in X$.*

Remark 13. Theorem 11 greatly improves Theorem 3.1 in [20], while Theorem 3.1 in [20] depends strongly on the continuity of b-metric. It also generalizes the condition sh <1 (i.e. $h<1/s$) of Theorem 2.13 in [23] to $\rho(r)<1/s$. The assumption of completeness in Theorem 3.1 of [11] and Theorem 2.13 in [23] is relaxed by Φ-orbital completeness. Corollary 12 mainly improves and generalizes Theorem 9 in [28], while the results rely on the conditions that the cone is normal and the d is continuous.

Turning to the next theorem, we show that another type of b-quasicontraction in the space (X,d) has a unique fixed point when $1/s<\rho(r)<1$. Before giving the related result, we require an important lemma in [26].

Lemma 14. *Suppose $\{\varpi_n\}$ is a sequence in (X,d) satisfying*

$$d(\varpi_n,\varpi_{n+1})\le rd(\varpi_{n-1},\varpi_n), \tag{51}$$

for some $r\in P$ with $\rho(r)<1$ and $n\in\mathbb{N}$. Then, $\{\varpi_n\}$ is a Cauchy sequence in (X,d).

Theorem 15. *Suppose the space (X,d) is Φ-orbitally complete, Assume the mapping $\Phi:X\longrightarrow X$ satisfies*

$$d(\Phi\varpi,\Phi\eta)\le ru(\varpi,\eta), \tag{52}$$

for all $\varpi,\eta\in X$ and $\rho(r)\in[(1/s),1)$, where

$$u(\varpi,\eta)\in\left\{d(\varpi,\eta),d(\varpi,\Phi\varpi),d(\eta,\Phi\eta),\frac{d(\varpi,\Phi\eta)}{2s},\frac{d(\eta,\Phi\varpi)}{2s}\right\}. \tag{53}$$

If Φ is k-continuous for some $k\ge1$ or orbitally continuous, then the mapping Φ possesses one and only one fixed point $\zeta\in X$, and the sequence $\{\Phi^n\varpi_0\}$ converges to ζ for every $\varpi_0\in X$.

Proof. From Theorem 11, we obtain a sequence $\{\varpi_n\}$ by $\varpi_{n+1}=\Phi\varpi_n$ and suppose $\varpi_n\ne\varpi_{n+1}$ for all $n\in\mathbb{N}$ and $n\ge0$. In view of (52), we see that

$$d(\varpi_n,\varpi_{n+1})=d(\Phi\varpi_{n-1},\Phi\varpi_n)\le ru(\varpi_{n-1},\varpi_n), \tag{54}$$

where

$$u(\varpi_{n-1},\varpi_n)\in\left\{d(\varpi_{n-1},\varpi_n),d(\varpi_{n-1},\varpi_n),d(\varpi_n,\varpi_{n+1}),\frac{d(\varpi_{n-1},\varpi_{n+1})}{2s},\frac{d(\varpi_n,\varpi_n)}{2s}\right\}$$
$$=\left\{d(\varpi_{n-1},\varpi_n),d(\varpi_n,\varpi_{n+1}),\frac{d(\varpi_{n-1},\varpi_{n+1})}{2s},\theta\right\}. \tag{55}$$

We immediately get $u(\varpi_{n-1},\varpi_n)\ne d(\varpi_n,\varpi_{n+1})$ and $u(\varpi_{n-1},\varpi_n)\ne\theta$. If $u(\varpi_{n-1},\varpi_n)=d(\varpi_{n-1},\varpi_n)$, then

$$d(\varpi_n,\varpi_{n+1})\le rd(\varpi_{n-1},\varpi_n). \tag{56}$$

If $u(\varpi_{n-1},\varpi_n)=d(\varpi_{n-1},\varpi_{n+1})/2s$, then

$$d(\varpi_n,\varpi_{n+1})\le r\cdot\frac{s[d(\varpi_{n-1},\varpi_n)+d(\varpi_n,\varpi_{n+1})]}{2s}=\frac{r[d(\varpi_{n-1},\varpi_n)+d(\varpi_n,\varpi_{n+1})]}{2}, \tag{57}$$

which gives

$$d(\varpi_n,\varpi_{n+1})\le r(2e-r)^{-1}d(\varpi_{n-1},\varpi_n). \tag{58}$$

In fact,

$$\rho\left(r(2e-r)^{-1}\right)\le\rho(r)\rho\left((2e-r)^{-1}\right)\le\frac{\rho(r)}{2-\rho(r)}<1. \tag{59}$$

So, the conditions of Lemma 14 are satisfied for each case. By Lemma 14, $\{\varpi_n\}$ is a Cauchy sequence. In view of Φ-orbital completeness of (X,d), we have $\zeta\in X$ such that $\varpi_n\longrightarrow\zeta$.

We are now in a position to show that $\Phi\zeta=\zeta$. If Φ is k-continuous, then $\Phi^k\varpi_n\longrightarrow\Phi\zeta$ by k-continuity of Φ and $\Phi^{k-1}\varpi_n\longrightarrow\zeta$ as $n\longrightarrow\infty$. According to the uniqueness of the limit, we have $\Phi\zeta=\zeta$.

If Φ is orbitally continuous, then $\Phi\varpi_n\longrightarrow\Phi\zeta$ due to the fact that $\varpi_n\longrightarrow\zeta$. This yields $\zeta=\Phi\zeta$.

Uniqueness of the fixed point follows immediately from (52). □

Once we replace the set (53) by the following set (61), then the result is true without any continuity of the mapping Φ.

Theorem 16. *Suppose (X,d) is the same as in Theorem 15, assume the mapping $\Phi:X\longrightarrow X$ that satisfies*

$$d(\Phi\varpi,\Phi\eta)\le ru(\varpi,\eta), \tag{60}$$

for all $\varpi,\eta\in X$ and $\rho(r)\in[(1/s),1)$, where

$$u(\varpi,\eta)\in\left\{d(\varpi,\eta),d(\varpi,\Phi\varpi),\frac{d(\eta,\Phi\eta)}{s},\frac{d(\varpi,\Phi\eta)}{2s},d(\eta,\Phi\varpi)\right\}. \tag{61}$$

Then, the mapping Φ possesses one and only one fixed point $\zeta\in X$, and the sequence $\{\Phi^n\varpi_0\}$ converges to ζ for every $\varpi_0\in X$.

Proof. The proof is analogous to Theorem 15. We at first gain a sequence $\varpi_n=\Phi\varpi_{n-1}=\Phi^n\varpi_0$, $n\ge 1$ and suppose that $\varpi_n\ne\varpi_{n+1}$, $\forall n\in\mathbb{N}$. By an analogous analysis with Theorem 15, $\varpi_n\longrightarrow\zeta$ for some $\zeta\in X$. We proceed to show that $\zeta=\Phi\zeta$. By (60), we see that

$$d(\varpi_n,\Phi\zeta)=d(\Phi\varpi_{n-1},\Phi\zeta)\le ru(\varpi_{n-1},\zeta), \tag{62}$$

where

$$u(\varpi_{n-1},\zeta)\in\left\{d(\varpi_{n-1},\zeta),d(\varpi_{n-1},\varpi_n),\frac{d(\zeta,\Phi\zeta)}{s},\frac{d(\varpi_{n-1},\Phi\zeta)}{2s},d(\zeta,\varpi_n)\right\}. \tag{63}$$

There are the following three cases.

Case 1. If $u(\varpi_{n-1},\zeta)$ equals to one of $d(\varpi_{n-1},\zeta)$, $d(\varpi_{n-1},\varpi_n)$ and $d(\zeta,\varpi_n)$, then $\{d(\varpi_n,\Phi\zeta)\}$ is a c-sequence.

Case 2. If $u(\varpi_{n-1},\zeta)=d(\zeta,\Phi\zeta)/s$, we have

$$d(\varpi_n,\Phi\zeta)\le\frac{rd(\zeta,\Phi\zeta)}{s}\le\frac{sr[d(\zeta,\varpi_n)+d(\varpi_n,\Phi\zeta)]}{s}, \tag{64}$$

that is, $d(\varpi_n,\Phi\zeta)\le r(e-r)^{-1}d(\varpi_n,\zeta)$. Thus, $\{d(\varpi_n,\Phi\zeta)\}$ is a c-sequence.

Case 3. If $u(\varpi_{n-1},\zeta)=d(\varpi_{n-1},\Phi\zeta)/2s$, we have

$$d(\varpi_n,\Phi\zeta)\le\frac{rd(\varpi_{n-1},\Phi\zeta)}{2s}\le\frac{sr[d(\varpi_{n-1},\varpi_n)+d(\varpi_n,\Phi\zeta)]}{2s}, \tag{65}$$

that is, $d(\varpi_n,\Phi\zeta)\le r(2e-r)^{-1}d(\varpi_{n-1},\varpi_n)$. Thus, $\{d(\varpi_n,\Phi\zeta)\}$ is a c-sequence.

In summary, we always deduce that $\{d(\varpi_n,\Phi\zeta)\}$ is a c-sequence. This gives $\varpi_n\longrightarrow\Phi\zeta$ as $n\longrightarrow\infty$. Since the limit is unique, we know $\zeta=\Phi\zeta$. The remaining proof is analogous to Theorem 11. □

Remark 17. In Theorem 15, we complement and perfect Theorem 11 in [8], which obtained the conclusions under the condition $\rho(r)<1/s$ in complete spaces (X,d). Moreover, the conditions in Theorem 15 are much weaker than Theorem 2.1 in [20], since we obtain the results by k-continuity for some $k\ge 1$ or orbital continuity of the mapping, without appealing the continuity of cone b-metric or the mapping Φ.

According to the proof of Theorem 16 and the symmetry of the cone b-metric d, we see at once that (61) can be replaced by

$$u(\varpi,\eta)\in\left\{d(\varpi,\eta),\frac{d(\varpi,\Phi\varpi)}{s},d(\eta,\Phi\eta),d(\varpi,\Phi\eta),\frac{d(\eta,\Phi\varpi)}{2s}\right\}, \tag{66}$$

for all $\varpi,\eta\in X$.

Example 18. Set $\mathscr{A}=C^1_{\mathbb{R}}[0,2]$ with a norm $\|\varpi\|=\|\varpi\|_\infty+\|\varpi'\|_\infty$, for any $\varpi\in\mathscr{A}$. The multiplication in $\mathscr{A}$ is taken as the pointwise multiplication. We conclude that $\mathscr{A}$ is a Banach algebra owing the unit element $e=1$. Let $X=[0,2)$. For all $\varpi,\eta\in X$, define

$$d(\varpi,\eta)(z)=\begin{cases}|\varpi-\eta|^2\psi, & \varpi,\eta\in[0,1];\\ 2|\varpi-\eta|^2\psi, & \text{otherwise},\end{cases} \tag{67}$$

where $\psi\in P=\{\varphi(z)\in\mathscr{A}:\varphi(z)\ge 0, z\in[0,2]\}$. Note that the cone P is nonnormal, and the cone b-metric d is discontinuous. Indeed, let $\varpi_n=1+(1/n)$, $n\in\mathbb{N}$, and then

$$d(\varpi_n,1)(z)=d\left(1+\frac{1}{n},1\right)(z)=2\left|1+\frac{1}{n}-1\right|^2\psi\longrightarrow 0. \tag{68}$$

So, $\varpi_n\longrightarrow 1$. However,

$$d\left(\varpi_n,\frac{1}{2}\right)(z)=d\left(1+\frac{1}{n},\frac{1}{2}\right)(z)=2\left|1+\frac{1}{n}-\frac{1}{2}\right|^2\psi\longrightarrow\frac{1}{2}\psi,$$
$$d\left(1,\frac{1}{2}\right)(z)=\left|1-\frac{1}{2}\right|^2\psi=\frac{1}{4}\psi, \tag{69}$$

that is,

$$d\left(\varpi_n,\frac{1}{2}\right)(z)\nrightarrow d\left(1,\frac{1}{2}\right)(z). \tag{70}$$

Thus, the cone b-metric d is discontinuous.

The mapping $\Phi: X\longrightarrow X$ is defined as

$$\Phi\varpi=\begin{cases}\dfrac{\varpi}{3}, & \varpi\in[0,1],\\ 0, & \varpi\in(1,2).\end{cases} \tag{71}$$

It suffices to show that Φ is orbitally continuous rather than continuous (one also can check that Φ is k-continuous for each integer $k\ge 2$ but Φ^k is discontinuous for each $k\ge 1$). In addition, (X,d) is Φ-orbitally complete but not complete cone b-metric space over Banach algebra $\mathscr{A}$ with the coefficient $s=4$. In fact, for $\varpi_n=2-1/n$, $n\in\mathbb{N}$, we get

$$d(\varpi_n,\varpi_m)=2\left|\frac{1}{n}-\frac{1}{m}\right|^2\psi\longrightarrow 0, n,m\longrightarrow\infty, \tag{72}$$

but there exists no $\varpi \in X$ satisfying $d(\varpi_n, \varpi) \longrightarrow 0$. Hence, (X, d) is not complete. Take $r(z) = (z/4) + 1/2$. We can calculate that $\rho(r) = 3/4 \in [(1/s), 1)$. Now, there are the following three cases to verify the inequality (52).

For all $\varpi, \eta \in [0, 1]$, we get $d(\Phi\varpi, \Phi\eta)(z) = |\varpi/3 - \eta/3|^2\psi$ and

$$u(\varpi, \eta)(z) \in \left\{ |\varpi - \eta|^2\psi, \left|\varpi - \frac{\varpi}{3}\right|^2 \psi, \left|\eta - \frac{\eta}{3}\right|^2 \psi, \frac{1}{8}\left|\varpi - \frac{\eta}{3}\right|^2 \psi, \frac{1}{8}\left|\eta - \frac{\varpi}{3}\right|^2 \psi \right\}. \tag{73}$$

Then, (52) holds by taking $u(\varpi, \eta)(z) = |\varpi - \eta|^2\psi$.

For all $\varpi \in [0, 1], \eta \in (1, 2)$, we gain $d(\Phi\varpi, \Phi\eta)(z) = |\varpi/3 - 0|^2\psi = (\varpi^2/9)\psi$ and

$$\begin{aligned} u(\varpi, \eta)(z) &\in \left\{ 2|\varpi - \eta|^2\psi, \left|\varpi - \frac{\varpi}{3}\right|^2 \psi, 2|\eta - 0|^2\psi, \frac{1}{8}|\varpi - 0|^2\psi, \frac{1}{8}\cdot 2\left|\eta - \frac{\varpi}{3}\right|^2 \psi \right\} \\ &= \left\{ 2|\varpi - \eta|^2\psi, \frac{4\varpi^2}{9}\psi, 2\eta^2\psi, \frac{\varpi^2}{8}\psi, \frac{1}{4}\left|\eta - \frac{\varpi}{3}\right|^2 \psi \right\}. \end{aligned} \tag{74}$$

Then, (52) holds by taking $u(\varpi, \eta)(z) = (4\varpi^2/9)\psi$.

For all $\varpi, \eta \in (1, 2)$, then $\Phi\varpi = \Phi\eta = 0$. We observe that $d(\Phi\varpi, \Phi\eta)(z) = 0$ and

$$\begin{aligned} u(\varpi, \eta)(z) &\in \left\{ 2|\varpi - \eta|^2\psi, 2|\varpi - 0|^2\psi, 2|\eta - 0|^2\psi, \frac{1}{8}\cdot 2|\varpi - 0|^2\psi, \frac{1}{8}\cdot 2|\eta - 0|^2\psi \right\} \\ &= \left\{ 2|\varpi - \eta|^2\psi, 2\varpi^2\psi, 2\eta^2\psi, \frac{\varpi^2}{4}\psi, \frac{\eta^2}{4}\psi \right\}. \end{aligned} \tag{75}$$

So, (52) holds trivially. In the same manner, we can prove that (52) is true for all $\varpi \in (1, 2), \eta \in [0, 1]$. Therefore, Φ possesses a unique fixed point $0 \in X$, and the sequence $\{\Phi^n\varpi\}$ converges to 0 for each $\varpi \in X$ by Theorem 15.

Furthermore, there is no $r \in P$ with $\rho(r) \in [0, 1)$ satisfying $(\varpi^2/9)\psi \leq 2r|\varpi - \eta|^2\psi$ for

$$\varpi = 1 \in [0, 1], \eta = 1 + \frac{1}{9} \in (1, 2) \text{ and } \psi(z) > 0, \tag{76}$$

in Case 2, which means that Φ is not a Banach-type b-contraction. The work from b-metric space, cone b-metric space, or cone b-metric space over Banach algebra which requires completeness, continuity, or Banach-type b-contraction is not applicable here (see [6, 8, 11, 18–20, 23, 26]). Due to the continuity of metric and cone metric, the corresponding theorems from such spaces (see [1–4, 7, 22, 24, 28]) cannot be used in this example, either.

3. Orbital Compactness

Garai et al. [29] and Haokip and Goswami [33] defined Φ-orbitally compact metric spaces and Φ-orbitally compact b-metric spaces, respectively, which extend sequentially compact metric (b-metric) spaces. Now, the similar definition of Φ-orbital compactness and a fixed point theorem of Ćirić-type b-quasicontraction in cone b-metric spaces over Banach algebras is showed.

Definition 19. The mapping $\Phi : X \longrightarrow X$ is named a *Ćirić*-type b-quasicontraction in (X, d), if for all $\varpi, \eta \in X$ with $\varpi \neq \eta$, we have

$$d(\Phi\varpi, \Phi\eta) < u(\varpi, \eta), \tag{77}$$

where

$$u(\varpi, \eta) \in \left\{ d(\varpi, \eta), d(\varpi, \Phi\varpi), d(\eta, \Phi\eta), \frac{d(\varpi, \Phi\eta) + d(\eta, \Phi\varpi)}{2s} \right\}. \tag{78}$$

Definition 20. Suppose the mapping $\Phi : X \longrightarrow X$, the set X is Φ-orbitally compact, if each sequence in $O_\Phi(\varpi)$ has a convergent subsequence for all $\varpi \in X$. Clearly, every Φ-orbitally compact cone b-metric space over Banach algebra does not need to be complete.

Example 21. Suppose the Banach algebra $\mathscr{A}$ and cone P are the same as in Example 8, take $X = [0, 2) \times [0, 2)$ and $\Phi : X \longrightarrow X$ be given by $\Phi(\varpi_1, \varpi_2) = ((\varpi_1/3), (\varpi_2/5))$. Then, X is Φ-orbitally compact rather than complete.

In the last theorem, suppose that (X, d) owes a regular cone P with $d(\varpi, \eta) \in \text{int}\, P$, where $\varpi, \eta \in X$ and $\varpi \neq \eta$, the cone b-metric d is continuous.

Theorem 22. *Suppose (X, d) is Φ-orbitally compact, if $\Phi : X \longrightarrow X$ is a Ćirić-type b-quasicontraction and orbitally continuous, then the mapping Φ possesses one and only one fixed point $\zeta \in X$.*

Proof. For any $\varpi_0 \in X$, set $\varpi_n = \Phi\varpi_{n-1} = T^n\varpi_0, n \geq 1$. Note that $\varpi_n \neq \varpi_{n+1}$ for all $n \in \mathbb{N}$. In fact, if $\varpi_n = \varpi_{n+1} = \Phi\varpi_n$ for some $n \in \mathbb{N}$, then ϖ_n is a fixed point of Φ. Let $l_n = d(\varpi_n, \varpi_{n+1})$ for every $n \in \mathbb{N}$. From (77), we know

$$l_n = d(\varpi_n, \varpi_{n+1}) = d(\Phi\varpi_{n-1}, \Phi\varpi_n) < u(\varpi_{n-1}, \varpi_n), \tag{79}$$

where

$$u(\varpi_{n-1}, \varpi_n) \in \left\{ d(\varpi_{n-1}, \varpi_n), d(\varpi_{n-1}, \varpi_n), d(\varpi_n, \varpi_{n+1}), \frac{d(\varpi_{n-1}, \varpi_{n+1}) + d(\varpi_n, \varpi_n)}{2s} \right\}. \tag{80}$$

Indeed, $u(\varpi_{n-1}, \varpi_n) \neq d(\varpi_n, \varpi_{n+1})$; so, it remains the following two cases.

Case 1. When $u(\varpi_{n-1}, \varpi_n) = d(\varpi_{n-1}, \varpi_n)$, then $d(\varpi_n, \varpi_{n+1}) < d(\varpi_{n-1}, \varpi_n)$.

Case 2. When $u(\varpi_{n-1}, \varpi_n) = d(\varpi_{n-1}, \varpi_{n+1}) + d(\varpi_n, \varpi_n)/2s$, then

$$d(\varpi_n, \varpi_{n+1}) < \frac{d(\varpi_{n-1}, \varpi_{n+1}) + d(\varpi_n, \varpi_n)}{2s} \leq \frac{s[d(\varpi_{n-1}, \varpi_n) + d(\varpi_n, \varpi_{n+1})]}{2s}, \quad (81)$$

which means $d(\varpi_n, \varpi_{n+1}) < d(\varpi_{n-1}, \varpi_n)$. Thus, by repeating the above process, we deduce that

$$\theta < l_n = d(\varpi_n, \varpi_{n+1}) < l_{n-1} < \cdots < l_0 = d(\varpi_1, \varpi_0). \quad (82)$$

By the regularity of the cone, there exists $c_0 \in \mathscr{A}, c_0 \geq \theta$ such that $l_n \longrightarrow c_0$ as $n \longrightarrow \infty$. Because X is Φ-orbitally compact, we have a convergent subsequence $\{\varpi_{n_i}\}$ of $\{\varpi_n\}$ and a point $\zeta \in X$ satisfying $\varpi_{n_i} \longrightarrow \zeta \; as \, n \longrightarrow \infty$, according to orbital continuity of Φ, $\Phi\varpi_{n_i} \longrightarrow \Phi\zeta$.

When $c_0 > \theta$, we gain

$$\theta < c_0 = \lim_{i \longrightarrow \infty} d\left(\varpi_{n_i}, \Phi\varpi_{n_i}\right) = d(\zeta, \Phi\zeta). \quad (83)$$

Moreover, since the cone is regular, we see that

$$\theta < c_0 = \lim_{i \longrightarrow \infty} l_{n_i} = \lim_{i \longrightarrow \infty} d\left(\Phi\varpi_{n_i}, \Phi^2\varpi_{n_i}\right) = d\left(\Phi\zeta, \Phi^2\zeta\right) < u(\zeta, \Phi\zeta), \quad (84)$$

where

$$u(\zeta, \Phi\zeta) \in \left\{ d(\zeta, \Phi\zeta), d(\zeta, \Phi\zeta), d\left(\Phi\zeta, \Phi^2\zeta\right), \frac{d\left(\zeta, \Phi^2\zeta\right) + d(\Phi\zeta, \Phi\zeta)}{2s} \right\}. \quad (85)$$

It is evidence to get

$$c_0 = d\left(\Phi\zeta, \Phi^2\zeta\right) < d(\zeta, \Phi\zeta) = c_0, \quad (86)$$

a contradiction. So, $c_0 = \theta$ and $\zeta = \Phi\zeta$, namely, ζ is a fixed point.

Finally, let us prove that the fixed point is unique by (77). Otherwise, if there is another fixed point η, then

$$d(\eta, \zeta) = d(\Phi\eta, \Phi\zeta) < u(\eta, \zeta), \quad (87)$$

where

$$u(\eta, \zeta) \in \left\{ d(\eta, \zeta), d(\eta, \Phi\eta), d(\zeta, \Phi\zeta), \frac{d(\eta, \Phi\zeta) + d(\zeta, \Phi\eta)}{2s} \right\} = \left\{ d(\eta, \zeta), \theta, \frac{d(\eta, \Phi\zeta) + d(\zeta, \Phi\eta)}{2s} \right\}. \quad (88)$$

If $u(\eta, \zeta) = d(\eta, \zeta)$ or $u(\eta, \zeta) = \theta$, then this is a contradiction. If $u(\eta, \zeta) = [d(\eta, \Phi\zeta) + d(\zeta, \Phi\eta)]/2s$, then

$$d(\eta, \zeta) < \frac{d(\eta, \Phi\zeta) + d(\zeta, \Phi\eta)}{2s} \leq \frac{s[d(\eta, \zeta) + d(\zeta, \Phi\zeta)] + s[d(\zeta, \eta) + d(\eta, \Phi\eta)]}{2s} = d(\eta, \zeta), \quad (89)$$

a contradiction too. The result follows. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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