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Research Article

Numerical Solution of the Absolute Value Equations Using Two Matrix Splitting Fixed Point Iteration Methods

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The absolute value equations (AVEs) are significant *nonlinear* and *non-differentiable* problems that arise in the optimization community. In this article, we provide two new iteration methods for determining AVEs. These two approaches are based on the fixed point principle and splitting of the coefficient matrix with three extra parameters. The convergence of these procedures is also presented using some theorems. The validity of our methodologies is demonstrated via numerical examples.

1. Introduction

In the last few decades, the AVE has been identified as a type of NP-hard and non-differentiable problem, which can be equivalent to numerous mathematical problems, such as bimatrix games, linear and quadratic programming, contact problems, network prices, and network equilibrium problems; see [1–6] for more details.

We consider the AVE problem of finding an $x \in R^n$ such that

$$Ax - |x| = b. \tag{1}$$

Here $b \in R^n$, $A \in R^{n\times n}$ and $|x|$ signifies the absolute values of the components of $x \in R^n$. Note that Eq. (1) is a special case of the following generalized AVE:

$$Ax + B|x| = b, \tag{2}$$

where $B \in R^{n\times n}$ was familiarized by Rohn [1] and more studied in [7–11].

Furthermore, AVEs are equivalently reformulated into linear complementarity problems (LCPs). These formulations are discussed in [12–14] and the references therein. Taking the well-known LCP as an example: Let us assume that the LCP (f, M) consists of determining $y \in R^n$ such that

$$y \geq 0, \Gamma = \left(\bar{M}y + f\right) \geq 0, y^T\Gamma = 0, \tag{3}$$

where $f \in R^n$ and $\bar{M} \in R^{n\times n}$. The system (3) can be expressed as AVE

$$Ax - B|x| = f, \tag{4}$$

with

$$x = \frac{1}{2}(By + f), \tag{5}$$

where $B = (\bar{M} - I)$ and $A = (\bar{M} + I)$. Meszzadri [15] established the equivalence among horizontal LCPs and AVEs. Furthermore, the unique solvability of system (1) and its relation to mixed-integer programming and LCP have been discussed by Prokopyev [16].

Recently, the problem of determining the AVEs has enticed much consideration and has been studied in the literature. For instance, Ali et al. [17] introduced the generalized successive overrelaxation (GSOR) methods to determine

AVE (1) and provided the necessary conditions for the convergence of the methods. Zhang and Wei [18] introduced a generalized Newton approach for obtaining (1) and designated the global and finite convergence with the condition that $[A+I, A-I]$ is regular. Cruz et al. [19] established an inexact semi-smooth Newton approach for the AVE (1) and showed that the approach is globally convergent with the condition if $\|A^{-1}\| < 1/3$. Ke [20] presented the new iteration algorithm for determining (1) and proved the new convergence conditions under suitable assumptions. Moosaei et al. [21] presented two approaches for determining the NP-hard AVEs when the singular values of A exceed 1. Cacceta et al. [22] investigated the smoothing Newton method for obtaining (1) and discussed that this method is globally convergent with condition that $\|A^{-1}\| < 1$. Chen et al. [23] discussed the optimal parameter SOR-like iteration technique for Eq. (1). Wu as well as Li [24] used the shift splitting (SS) technique to develop an iterative shift splitting technique to find the AVE (1), and others; see [25–27] and the references therein.

Recent studies have revealed that Li and Dai [28] as well as Najafi and Edalatpanah [29] provide methods for determining LCPs utilizing the fixed point principle. The objective of this study is to apply this approach to AVEs based on the fixed point principle, and to suggest efficient approaches for calculating AVE (1). We have made the following contributions in our study:

(i) We divide the A matrix into various parts and then connect this splitting with the fixed point formula, which can accelerate the convergence of the proposed iterative procedures.

(ii) We consider the convergent conditions of newly designed approaches under various new situations.

The analysis is structured as follows. The offered strategies for defining AVE (1) are examined in Section 2. In addition, the numerical tests are discussed in Section 3, while the conclusion is presented in Section 4.

2. Suggested Methods

In this part, we propose strategies to determine AVE (1). We begin by discussing some symbols and auxiliary outcomes.

We illustrate the spectral radius, infinity norm, and tridiagonal matrix of A, respectively, as $\rho(A)$, $\|A\|_\infty$ and T diag (A).

Lemma 1 (see [30]). *Suppose u, v be the two vectors $\in R^n$. Then $|u-v| \geq \|u| - |v\|$.*

In order to propose and examine the new iteration methods, the matrix A is divided as follows:

$$A = (N_A - M_A), \tag{6}$$

with

$$N_A = \frac{1}{\alpha}(\alpha D_A - \alpha U_A + \beta U_A^\star) \text{ and } M_A = \frac{1}{\alpha}(\alpha L_A + \beta U_A^\star), \tag{7}$$

where $0 < \alpha, \beta \leq 1$. Furthermore, D_A, U_A, $U_A^\star$ and L_A, are the diagonal, the strictly upper, the transpose of strictly upper and the strictly lower triangular parts of A, respectively. The AVE (1) is equivalent to the fixed point problem of solving

$$x = H(x), \tag{8}$$

such that

$$H(x) = x - \lambda E[Ax - |x| - b], \tag{9}$$

where $0 < \lambda \leq 1$ and $E \in R^{n\times n}$ is a positive diagonal matrix (then by choice of $E = D_A^{-1}$, see [31, 32] for more details). By utilizing splitting (6), we offer the following two new schemes to obtain AVEs (see Appendix A):

2.1. Method I.

$$x^{m+1} = x^m - \lambda E\left[-M_A x^{m+1} + N_A x^m - (|x^m| + b)\right].m = 0, 1, 2, \cdots, \tag{10}$$

2.2. Method II.

$$\begin{aligned} x^{m+1} = x^m + D_A^{-1} M_A x^{m+1} - \lambda E[Ax^m - |x^m| - b] \\ - D_A^{-1} M_A x^m .m = 0, 1, 2, \cdots, \end{aligned} \tag{11}$$

Now, we study the convergence investigation of the presented iteration methods.

Theorem 2. *Assume that the system (1) is solvable and $A = N_A - M_A$ be a splitting of A, then*

$$\left|x^{m+1} - x^\star\right| \leq R^{-1} Q \left|x^m - x^\star\right|, \tag{12}$$

where

$$R = I - \lambda E|M_A| \text{ and } Q = \lambda E + |I - \lambda E N_A|. \tag{13}$$

Moreover, if $\rho(R^{-1}Q) < 1$, the sequence $\{x^m\}$ designed by method I will lead to the unique solution $x^\star$ of the system (1).

Proof. Suppose $x^\star$ be a solution of system (1). Therefore,

$$x^\star = x^\star - \lambda E[-M_A x^\star + N_A x^\star - (|x^\star| + b)]. \tag{14}$$

After subtracting (14) from (10), we obtain

$$\begin{aligned} x^{m+1} - white\,|x^\star| &= x^m - x^\star - \lambda E\left[-M_A\left(x^{m+1} - x^\star\right) \right. \\ &\quad \left. + N_A(x^m - x^\star) - |x^m| + |x^\star|\right], \\ x^{m+1} - x^\star &= x^m - x^\star + \lambda E M_A\left(x^{m+1} - x^\star\right) - \lambda E N_A(x^m - x^\star) \\ &\quad + \lambda E(|x^m| - |x^\star|), \\ x^{m+1} - x^\star &= (I - \lambda E N_A)(x^m - x^\star) + \lambda E M_A\left(x^{m+1} - x^\star\right) \\ &\quad + \lambda E(|x^m| - |x^\star|). \end{aligned} \tag{15}$$

Considering the absolute values on each side, we have

$$\begin{aligned}\left|x^{m+1}-x^{\star}\right| &= \left|(I-\lambda EN_A)(x^m-x^{\star})+\lambda EM_A\left(x^{m+1}-x^{\star}\right)\right.\\ &\quad \left.+\lambda E(|x^m|-|x^{\star}|)\right|,\\ \left|x^{m+1}-x^{\star}\right| &\le |(I-\lambda EN_A)(x^m-x^{\star})|+\lambda E|M_A|\\ &\quad \cdot\left|x^{m+1}-x^{\star}\right|+\lambda E||x^m|-|x^{\star}||.\end{aligned} \tag{16}$$

Using Lemma 1, we get

$$\begin{aligned}\left|x^{m+1}-x^{\star}\right| &\le |I-\lambda EN_A||x^m-x^{\star}|+\lambda E|M_A|\left|x^{m+1}-x^{\star}\right|\\ &\quad +\lambda E|x^m-x^{\star}|,\\ \left|x^{m+1}-x^{\star}\right|&-\lambda E|M_A|\left|x^{m+1}-x^{\star}\right|\\ &\le |I-\lambda EN_A||x^m-x^{\star}|+\lambda E|x^m-x^{\star}|,\\ (I-\lambda E|M_A|)\left|x^{m+1}-x^{\star}\right| &\le (\lambda E+|I-\lambda EN_A|)|x^m-x^{\star}|.\end{aligned} \tag{17}$$

Since $(I-\lambda E|M_A|)$ is invertible. Therefore, $(I-\lambda E|M_A|)^{-1}$ exists as well as non-negative, we have

$$\left|x^{m+1}-x^{\star}\right| \le R^{-1}Q|x^m-x^{\star}|, \tag{18}$$

where

$$R = I-\lambda E|M_A| \text{ and } Q=\lambda E+|I-\lambda EN_A|. \tag{19}$$

Note that the matrix $R^{-1}Q$ is non-negative. Based on [31, 32], if $\rho(R^{-1}Q)<1$, then the sequence $\{x^m\}$ designed by Method I converges to the $x^{\star}$ solution of system (1).

Uniqueness: suppose that $\bar{\bar{x}}$ represents another AVE solution. Based on the equations

$$\begin{aligned}Ax^{\star}-|x^{\star}|&=b,\\ A\bar{\bar{x}}-|\bar{\bar{x}}|&=b,\end{aligned} \tag{20}$$

presented as

$$\begin{aligned}x^{\star}&=x^{\star}-\lambda E[-M_Ax^{\star}+N_Ax^{\star}-(|x^{\star}|+b)],\\ \bar{\bar{x}}&=\bar{\bar{x}}-\lambda E[-M_A\bar{\bar{x}}+N_A\bar{\bar{x}}-(|\bar{\bar{x}}|+b)],\end{aligned} \tag{21}$$

we obtain

$$|x^{\star}-\bar{\bar{x}}| \le R^{-1}Q|x^{\star}-\bar{\bar{x}}|. \tag{22}$$

And since $\rho(R^{-1}Q)<1$, we get

$$x^{\star}=\bar{\bar{x}}. \tag{23}$$

The proof has been completed. □

Table 1: Convergence conditions of Theorem 2 and Theorem 3.

Examples	n	Method I $\rho(R^{-1}Q)$	Method II $\rho(G^{-1}J)$
3.1	100	0.8250	0.8929
	400	0.8250	0.8939
3.2	100	0.8167	0.8442
	400	0.8167	0.8448
3.3	1000	0.5024	0.5076
	3000	0.5074	0.5124
3.4	256	0.4027	0.4038
	4096	0.4033	0.3538
3.5	1000	0.3127	0.3173
	3000	0.3152	0.3195

Theorem 3. *Let $\{x^{m+1}\}$ and $\{x^m\}$ are the sequences generated by Method II, then*

$$\left|x^{m+1}-x^m\right| \le G^{-1}J\left|x^m-x^{m-1}\right|, \tag{24}$$

where

$$G=I-D_A^{-1}|M_A| \text{and } J=I+\lambda E-\left|\lambda EA+D_A^{-1}M_A\right|. \tag{25}$$

Moreover, if $\rho(G^{-1}J)<1$, the sequence $\{x^m\}$ designed by method II will lead to the unique solution $x^{\star}$ of the system (1).

Proof. Suppose $x^{\star}$ be a solution of system (1). Then

$$x^{\star}=x^{\star}+D_A^{-1}M_Ax^{\star}-\lambda E[Ax^{\star}-|x^{\star}|-b]-D_A^{-1}M_Ax^{\star}. \tag{26}$$

After subtracting (26) from (11), we obtain

$$\begin{aligned}x^{m+1}-x^{\star} &= x^m-x^{\star}+D_A^{-1}M_A\left(x^{m+1}-x^{\star}\right)-\lambda EA(x^m-x^{\star})\\ &\quad +\lambda E(|x^m|-|x^{\star}|)-D_A^{-1}M_A(x^m-x^{\star}),\end{aligned} \tag{27}$$

Taking absolute values on both sides and using Lemma 1, we have

$$\begin{aligned}\left|x^{m+1}-x^{\star}\right| &\le |x^m-x^{\star}|+D_A^{-1}|M_A|\left|x^{m+1}-x^{\star}\right|\\ &\quad -\left|\lambda EA+D_A^{-1}M_A\right||x^m-x^{\star}|\\ &\quad +\lambda E||x^m|-|x^{\star}||, \le |x^m-x^{\star}|\\ &\quad +D_A^{-1}|M_A|\left|x^{m+1}-x^{\star}\right|-\left|\lambda EA+D_A^{-1}M_A\right|\\ &\quad \cdot|x^m-x^{\star}|+\lambda E|x^m-x^{\star}|,\\ \left|x^{m+1}-x^{\star}\right| &\le \left(I+\lambda E-\left|\lambda EA+D_A^{-1}M_A\right|\right)|x^m-x^{\star}|\\ &\quad +D_A^{-1}|M_A|\left|x^{m+1}-x^{\star}\right|,\\ \left(I-D_A^{-1}|M_A|\right)&\left|x^{m+1}-x^{\star}\right|\\ &\le \left(I+\lambda E-\left|\lambda EA+D_A^{-1}M_A\right|\right)|x^m-x^{\star}|.\end{aligned} \tag{28}$$

Table 2: The outcomes of Example 4 with $\alpha = 0.5$, $\beta = 0.9$ and $\varphi = 4$.

n		λ	0.2	0.4	0.6	0.8	1
100	Method I	Iter	102	47	29	24	16
		Time	0.577	0.392	0.338	0.323	0.315
		RES	9.861e−07	9.234e−07	9.166e−07	5.999e−07	7.548e−07
	Method II	Iter	214	102	65	47	35
		Time	0.906	0.517	0.4247	0.398	0.354
		RES	9.264e−07	9.927e−07	9.523e−07	7.276e−07	9.448e−07
400	Method I	Iter	105	48	30	24	15
		Time	2.929	1.677	1.153	1.050	0.851
		RES	9.617e−07	9.974e−07	6.6810e−07	5.568e−07	9.702e−07
	Method II	Iter	230	111	71	51	38
		Time	13.115	6.426	4.291	3.153	2.476
		RES	9.700e−07	8.567e−07	7.950e−07	7.096e−07	9.720e−07

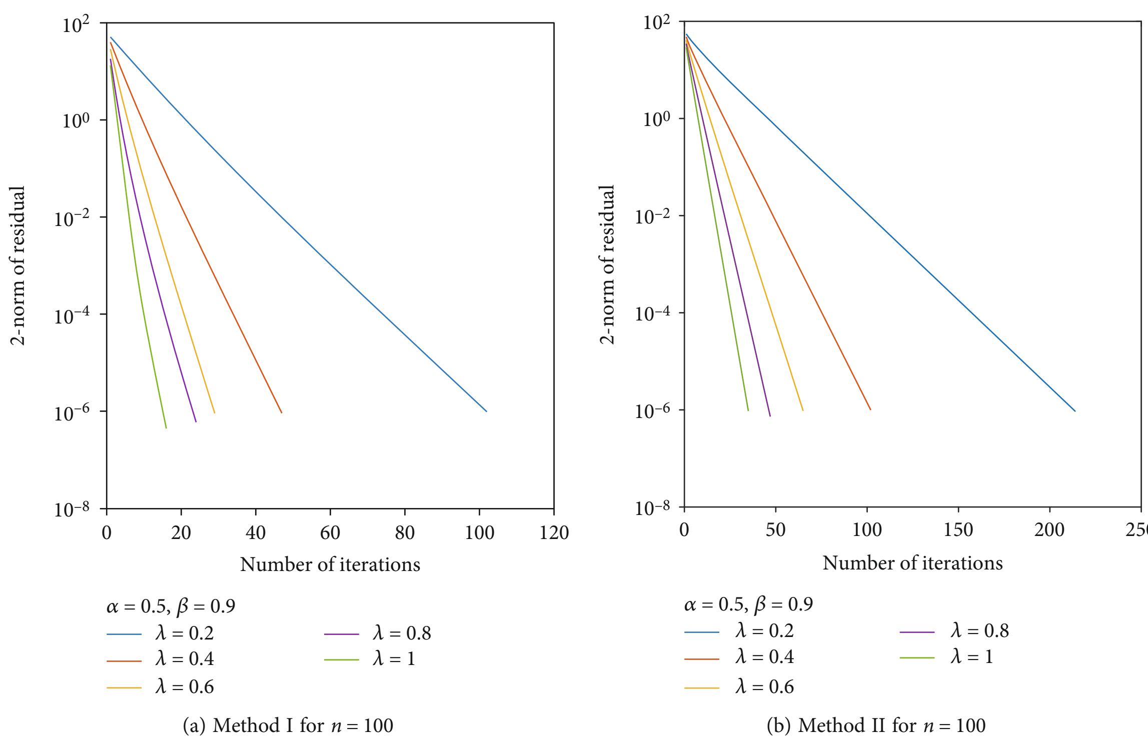


Figure 1: Graphs showing the convergence rates of the suggested approaches for various values of λ.

So,

$$\left|x^{m+1} - x^*\right| \leq G^{-1} J \left|x^m - x^*\right|. \quad (29)$$

Based on theorem 4.1 of [31] and theorem 3.1 of [28, 29], if $\rho(G^{-1}J) < 1$, the iteration sequence $\{x^m\}_{m=0}^{\infty}$ created by Method II is convergent.

The proof of the uniqueness is similar as the proof in Theorem 2 and is omitted here. □

3. Numerical Examples

In this unit, five examples are provided to analyze the performance of the proposed methods from three stances:

(i) 'Iter' indicates the iteration steps.

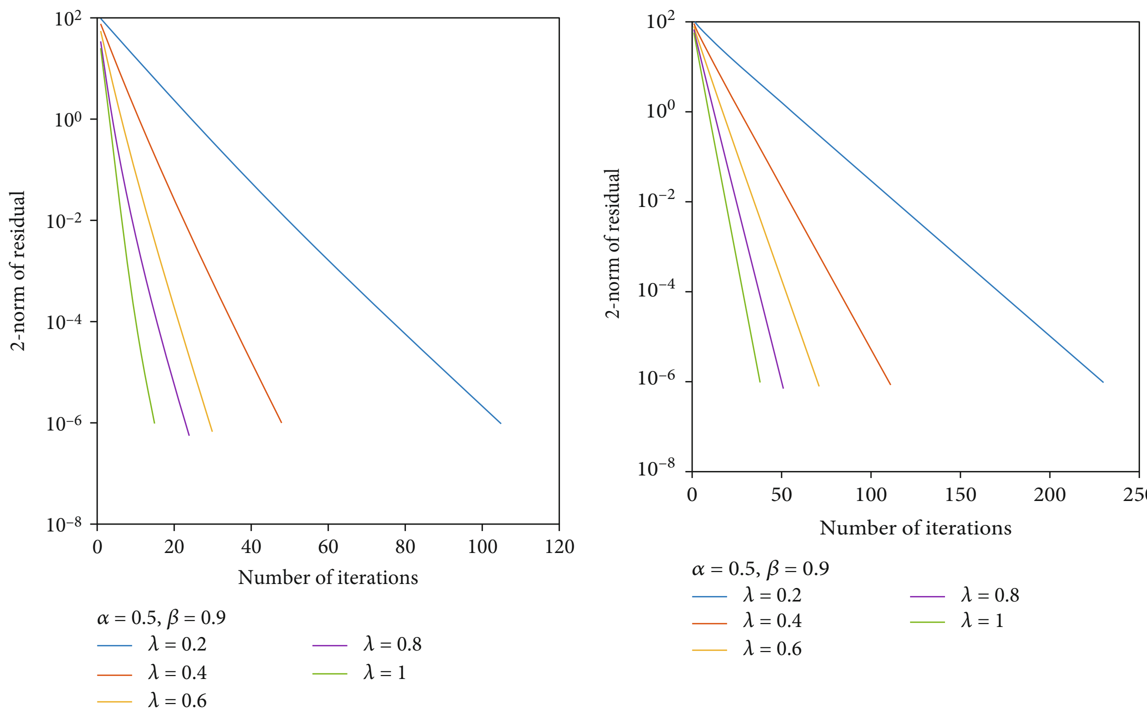
(ii) 'Time' implies the CPU time (s).

(iii) 'RES' signifies the 2-norm of residual vectors.

Here, 'RES' is determined by

$$RES := \| b + |x^m| - Ax^m \|_2 \leq 10^{-6}. \quad (30)$$

All calculations were done on a computer with an Intel(R) Core(TM) i5-3337U CPU @ 1.80 GHz processor and Memory 4GB using MATLAB R2016a. In Examples 4 and 5, the starting guess is supposed to be $x^{(0)} = (0, 0, 0, \cdots, 0)^T \in R^n$.



(a) Method I for $n = 400$

(b) Method II for $n = 400$

FIGURE 2: Graphs showing the convergence rates of the suggested approaches for various values of λ.

TABLE 3: The outcomes for Example 5 using $\alpha = 0.5$, $\beta = 0.9$ and $\varphi = 4$.

n		λ	0.2	0.4	0.6	0.8	1
100	Method I	Iter	90	40	23	14	10
		Time	0.645	0.370	0.3428	0.335	0.330
		RES	9.921e−07	9.6888e−07	9.284e−07	9.691e−07	1.650e−07
	Method II	Iter	117	54	33	22	16
		Time	0.717	0.447	0.391	0.364	0.351
		RES	9.392e−07	9.318e−07	8.321e−07	9.038e−07	4.232e−07
400	Method I	Iter	94	42	24	15	10
		Time	2.455	1.4193	1.0188	0.854	0.747
		RES	8.401e−07	7.318e−07	7.691e−07	4.637e−07	4.201e−07
	Method II	Iter	124	58	35	24	17
		Time	6.785	3.303	2.240	1.697	1.334
		RES	9.592e−07	7.878e−07	9.050e−07	6.450e−07	5.055e−07

First, we use numerical experiments to satisfy the convergence conditions $\rho(R^{-1}Q) < 1$ and $\rho(G^{-1}J) < 1$. Table 1 delivers the outcomes.

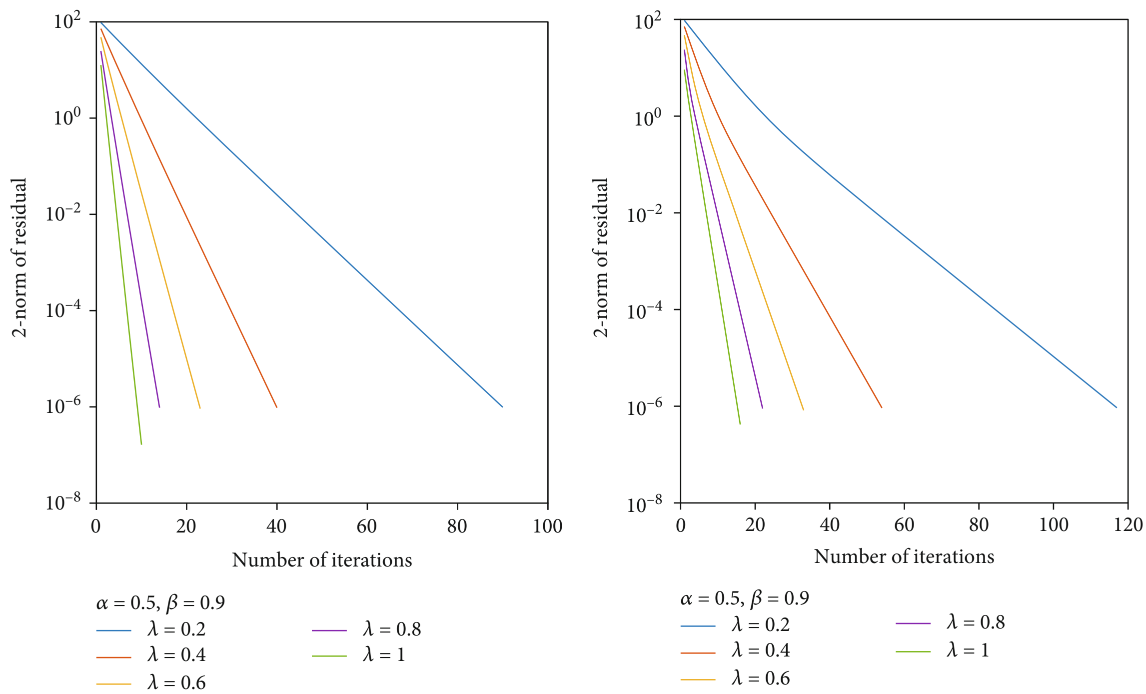
In Table 1, we performed the convergence conditions of both theorems using numerical experiments. Obviously, these two methods meet these conditions. To examine the implementation of our novel methods, we consider the following tests.

Example 4. Assume that $A = M + \varphi I \in R^{n\times n}$ and $b = Ax^\star - |x^\star| \in R^n$, such that.

$$M = T\,\mathrm{diag}\,(-I, S, -I) \in R^{n\times n}, \text{and}\, x^\star = (1, 2, 1, 2, \cdots,)^T \in R^n. \tag{31}$$

Here, $S = T\,\mathrm{diag}\,(-1, 4, -1) \in R^{\Delta\times\Delta}$, $I \in R^{\Delta\times\Delta}$ and $n = \Delta^2$. The outcomes are shown in Table 2, and the graphs are displayed in Figures 1 and 2, respectively.

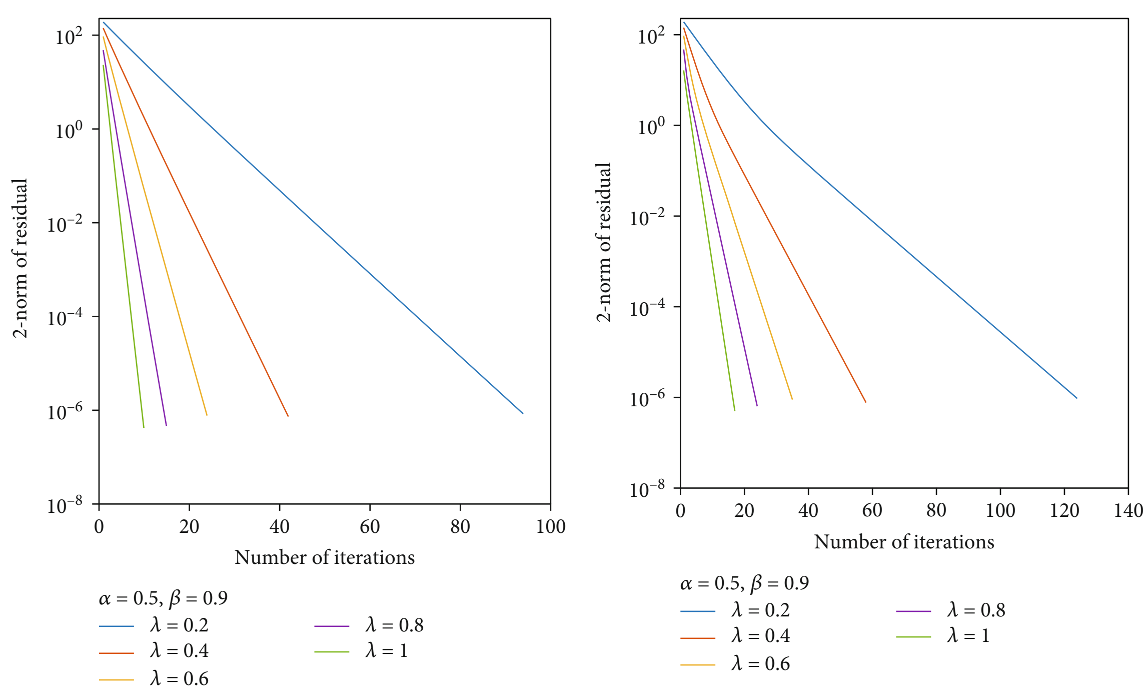
In Table 2, the given methods calculate the AVE solution for different α, β and λ values. We notice that if we increase λ, the convergence of the given strategies becomes quicker. The curves in Figures 1 and 2 display the effectiveness of the given procedures. Graphically sketch demonstrates that



(a) Method I for $n = 100$

(b) Method II for $n = 100$

FIGURE 3: Graphs showing the convergence rates of the suggested approaches for various values of λ.



(a) Method I for $n = 400$

(b) Method II for $n = 400$

FIGURE 4: Graphs showing the convergence rates of the suggested approaches for various values of λ.

the convergence of the presented approaches is faster when the value of λ is bigger.

Example 5. Assume that $A = M + \varphi I \in R^{n\times n}$ and $b = Ax^\star - |x^\star| \in R^n$, such that

$$M = Tdiag(-1.5I, S, -0.5I) \in R^{n\times n} \in R^{n\times n}, x^\star = \left((-1)^l, l = 1, 2, \cdots, n\right)^T \in R^n, \tag{32}$$

where $S = Tdiag(-1.5, 8, -0.5) \in R^{\Delta\times\Delta}$, $I \in R^{\Delta\times\Delta}$ as well as

Table 4: The outcomes for Example 6 using $\alpha = \beta = 1$ and $\lambda = 0.97$.

Techniques	n	1000	2000	3000	4000
	Iter	19	19	19	19
AM	Time	3.1590	14.2079	36.5739	74.4664
	RES	7.880e-07	7.884e-07	7.896e-07	7.896e-07
	Iter	14	14	14	14
SS	Time	2.7501	9.5131	19.2935	33.4672
	RES	8.913e-07	8.924e-07	8.928e-07	8.930e-07
	Iter	11	11	11	11
Method I	Time	1.739	5.5728	13.3846	25.4884
	RES	9.531e-07	9.537e-07	9.539e-07	9.540e-07
	Iter	12	12	12	12
Method II	Time	2.1002	7.4991	16.8944	31.8679
	RES	3.688e-07	3.667e-07	3.661e-07	3.657e-07

Table 5: The results for Example 7 using $\alpha = \beta = 1$ and $\lambda = 0.97$.

Methods	V	256	1296	2401	4096
	Iter	17	18	18	18
SRM	Time	0.3607	4.5834	22.8075	117.7810
	RES	6.804e–09	3.735e–09	5.072e–09	6.619e–09
	Iter	14	15	15	15
GSOR	Time	0.4744	16.7242	88.9636	58.7786
	RES	3.269e–09	1.805e–09	2.712e–09	3.808e–09
	Iter	11	11	12	12
Method I	Time	0.3423	4.6869	24.6612	125.0572
	RES	3.548e–09	7.540e–09	1.070e–09	1.383e–09
	Iter	11	11	12	12
Method II	Time	0.5307	11.0436	11.9317	57.3148
	RES	4.297e–09	9.345e–09	1.357e–09	1.760e–09

$n = \Delta^2$. The outcomes are presented in Table 3, and the graphical representations are shown in Figures 3 and 4, respectively.

In Table 3, we presented the convergence behavior of the given methods using the values of α, β and λ. Obviously, if the value of λ is larger, the convergence of the given approaches grows faster. The graphical representation is illustrated in Figures 3 and 4. These curves explain the efficiency of the suggested approaches at various λ values.

Example 6. Assume that

$$A = \begin{cases} 4, & \text{for} j = i \\ -1, & \text{for} \begin{cases} j = i + 1, i = 1, 2, \cdots, n - 1, \\ j = i - 1, i = 2, \cdots, n, \end{cases} \\ 0, & \text{otherwise.} \end{cases} \quad (33)$$

and $b = Ax^\star - |x^\star| \in R^n$, such that $x^\star = ((-1)^w, w = 1, 2, \cdots, n)^T \in R^n$. This example uses the same stopping criteria and starting guess as shown in [24]. Moreover, we compare the new techniques with the procedure described in [20] (symbolized by AM) as well as the shift splitting iteration method reported in [24] (represented by SS). These outcomes are presented in Table 4.

The results of Table 4 show that all methods are capable of determining the problem efficiently and precisely. Our techniques are more valuable than existing strategies, such as the AM and SS methods, in terms of iterations (Iter) and solving time (Time).

Example 7. Consider

$$A = I \otimes T + \chi \otimes I \in R^{\Theta \times \Theta}. \quad (34)$$

Here, $I \in R^{\Theta \times \Theta}$ is a unit matrix, and $\otimes$ indicates the Kronecker product. In addition, T and $\chi \in R^{n \times n}$, as shown below.

$$\begin{cases} T = Tdiag\left[\left(\frac{2+\phi}{8}\right), 8, \left(\frac{2-\phi}{8}\right)\right], \\ \chi = Tdiag\left[\left(\frac{1+\phi}{4}\right), 4, \left(\frac{1-\phi}{4}\right)\right], \\ \phi = 1/n\,; \Theta = n^2. \end{cases} \quad (35)$$

Here, $b = Ax^\star - |x^\star| \in R^\Theta$, where $x^\star = ones(\Theta, 1) \in R^\Theta$. In Examples 7 and 8, using the same starting guess as well as the stopping criterion as given in [23]. Moreover, we compare the recommended techniques with the process shown in [23] (exposed by SRM) and the iteration scheme introduced in [17] (represented by GSOR). These data are explained in Table 5.

All techniques in Table 5 consider the solution $x^\star$ for various numbers of V. Based on the data in Table 5, we can identify that our recommended procedures provide better results than both SRM and GSOR procedures.

Table 6: The outcomes for Example 8 using $\alpha = \beta = 1$ and $\lambda = 0.97$.

Methods	n	1000	2000	3000	4000
	Iter	20	20	20	20
SRM	Time	2.8619	12.8195	37.2905	75.2874
	RES	4.125e-09	5.834e-09	7.146e-09	8.252e-09
	Iter	17	17	18	18
GSOR	Time	2.6841	7.5746	11.3116	17.7786
	RES	6.846e–09	9.711e–09	2.808e–09	3.244e–09
	Iter	13	13	13	13
Method I	Time	1.5501	3.1790	7.9509	12.6772
	RES	5.343e–09	7.518e–09	9.191e–09	1.652e–09
	Iter	13	13	13	13
Method II	Time	2.4768	5.3038	11.7203	16.3148
	RES	5.766e–09	8.047e–09	9.812e–09	1.760e–09

Example 8. We consider the AVE (1) with

$$A = Tdiag(-1, 8, -1) \in R^{n\times n}, x^\star = \left((-1)^l, l = 1, 2, \cdots, n\right)^T \in R^n, \quad (36)$$

and $b = Ax^\star - |x^\star|$. The data are reported in Table 6.

Based on Table 6, we perceive that all procedures can determine the problem efficiently and precisely. We can clearly distinguish that our techniques are more beneficial than existing processes, such as SRM and GSOR, with respect to the iteration steps (Iter) and the solving time (Time).

4. Conclusion

We have introduced two novel iterative techniques for obtaining the AVE (1) and demonstrated that the offered approaches converge to the system (1) under proper selections of the involved parameters. The effectiveness of the recommended methods has also been evaluated numerically. The numerical outcomes indicate that the suggested strategies are effective for large and sparse AVEs. For future research, the theoretical comparison and analysis of these iteration methods are of great interest.

Appendix

Here, we describe how to perform the suggested methods.

A. Method I

$$x^{m+1} = x^m - \lambda E\left[-M_A x^{m+1} + N_A x^m - (|x^m| + b)\right]. \quad (A.1)$$

B. Method II

$$x^{m+1} = x^m + D_A^{-1} M_A x^{m+1} - \lambda E[Ax^m - |x^m| - b] - D_A^{-1} M_A x^m. \quad (A.2)$$

In both methods, the right-hand side also contains x^{m+1} which is the unknown. From $Ax - |x| = b$, we have

$$x = A^{-1}(|x| + b). \quad (A.3)$$

Therefore, x^{m+1} can be approximated as follows:

$$x^{m+1} \approx A^{-1}(|x^m| + b). \quad (A.4)$$

This technique is named the Picard technique [27]. Here, we present the algorithms for the proposed methods.

B.1. Algorithms for Method I and Method II. Step 1: Select the parameters $0 < \alpha, \beta, \lambda \le 1$, an starting vector $x^0 \in R^n$ and fix $m = 0$.

Step 2: Compute $y^m = x^{m+1} \approx A^{-1}(|x^m| + b)$,

Step 3: Calculate (Method I)

$$x^{m+1} = x^m - \lambda E[-M_A y^m + N_A x^m - (|x^m| + b)]. \quad (A.5)$$

Step 4: Calculate (Method II)

$$x^{m+1} = x^m + D_A^{-1} M_A y^m - \lambda E[Ax^m - |x^m| - b] - D_A^{-1} M_A x^m. \quad (A.6)$$

Step 5: Stop if $x^{m+1} = x^m$. Otherwise, set $m = m + 1$ and return to step 2.

Note that the idea behind considering certain types of structures in Method I and Method II comes from [28, 29]. Several authors have discussed the use of these types of methods for the solution of LCPs; see [33, 34] and the reference therein. In our study, we applied this concept to AVEs. In addition, the concept of $E = D_A^{-1}$, whose diagonal contains positive entries, is inspired by the work of [31, 32].

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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