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*Research Article*

# Regularization of Inverse Initial Problem for Conformable Pseudo-Parabolic Equation with Inhomogeneous Term

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The main goal of the paper is to approximate two types of inverse problems for conformable heat equation (or called parabolic equation with conformable operator); as follows, we considered two cases: the right hand side of equation such that $F(x, t)$ and $F(x, t) = \varphi(t) f(x)$. Up to now, there are very few surveys working on the results of regularization in $\mathscr{L}^p$ spaces. Our paper is the first work to investigate the inverse problem for conformable parabolic equations in such spaces. For the inverse source problem and the backward problem, use the Fourier truncation method to approximate the problem. The error between the regularized solution and the exact solution is obtained in $\mathscr{L}^p$ under some suitable assumptions on the Cauchy data.

## 1. Introduction

Partial differential equations (PDEs) have applications in many branches of science and engineering; see for example [1–8]. In this paper, for $s > 1$, we consider the initial value problem for the conformable heat equation (or called parabolic equation with conformable operator)

$$\begin{cases} \dfrac{{}^{\mathscr{C}}\partial^\beta}{\partial t^\beta}(y(x,t) - k\Delta y(x,t)) + (-\Delta)^s y(x,t) = F(x,t), & x \in \mathscr{D}, t \in (0,T) \\ y(x,t) = 0, & x \in \partial\mathscr{D}, t \in (0,T) \\ y(x,T) = y_T(x), & x \in \mathscr{D} \end{cases}. \tag{1}$$

Here, $\mathscr{D} \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with the smooth boundary $\partial\mathscr{D}$, and $T > 0$ is a given positive number. Here, ${}^{\mathscr{C}}\partial^\beta/\partial t^\beta$ is called the conformable time derivative with order $\beta \in (0,1)$ (Khalil et al. [9]) for a given function $f : [0, \infty) \longrightarrow \mathbb{R}$; the ${}^{\mathscr{C}}\partial^\beta/\partial t^\beta$ of order $\beta \in (0,1]$ is defined by

$$\frac{{}^{\mathscr{C}}\partial^\beta}{\partial t^\beta} f(t) = \lim_{\epsilon \longrightarrow 0} \frac{f\left(t + \epsilon t^{1-\beta}\right) - f(t)}{\epsilon}, \tag{2}$$

for all $t > 0$. For some $(0, t_0)$, $t_0 > 0$ and the $\lim_{t \longrightarrow t_0^+} ({}^{\mathscr{C}}\partial^\beta/\partial t^\beta) f(t)$ exist, then $({}^{\mathscr{C}}\partial^\beta/\partial t^\beta) f(t_0) = \lim_{t \longrightarrow t_0^+} ({}^{\mathscr{C}}\partial^\beta/\partial t^\beta) f(t)$. Some properties of ${}^{\mathscr{C}}\partial^\beta/\partial t^\beta$ can be found in more detail in [10, 11]. ${}^{\mathscr{C}}\partial^\beta/\partial t^\beta$ is a natural extension of usual derivative, it preserves basic properties of the classical derivative [10, 11], and it is a local and limit-based operator. In [10, 12, 13], we saw some applications. For the convenience of the reader, we will consider two models related to Problem (1) that most mathematicians often study.

(i) The first part of the paper deals with the final value problem for Problem (1) with a linear source function. The new feature of this part is the appearance of observed data, namely, $(y_{T,\delta}, F_\delta) \in \mathscr{L}^p(\mathscr{D}) \times L^\infty(0, T; \mathscr{L}^p(\mathscr{D}))$. This result is well described in Theorem 3. We investigated the problem of restoring the temperature function $y(x, t)$, in the fact that the couple $(y_T, F)$ are noised by the measurement data ( $y_{T,\delta}, F_\delta$) such that:

$$\begin{cases} \|y_{T,\delta} - y_T\|_{\mathscr{L}^p(\mathscr{D})} \le \delta \\ \|F_\delta - F\|_{\mathscr{L}^\infty(0,T;\mathscr{L}^p(\mathscr{D}))} \le \delta \end{cases}. \tag{3}$$

(ii) The second part of the paper deals with the final value problem for Problem (1) with $F$ is a linear source function as follows: $F(x, t) = \Phi(t)f(x)$, where both functions $(\Phi, f, g)$ are perturbed by $(\Phi_\delta, f_\delta, g_\delta)$ in $\mathscr{L}^p(0, T) \times \mathscr{L}^p(\mathscr{D}) \times \mathscr{L}^p(\mathscr{D})$, respectively

$$\begin{cases} \|\Phi_\delta - \Phi\|_{\mathscr{L}^p(0,T)} \le \delta \\ \|y_{T,\delta} - y_T\|_{\mathscr{L}^p(0,\mathscr{D}} \le \delta \\ \|f_\delta - f\|_{\mathscr{L}^p(\mathscr{D})} \le \delta \end{cases}. \tag{4}$$

The main contributions and novelties of this paper are stated as follows. As we know, two inverse problems are ill-posed in the sense of Hadamard. The well-posed problem satisfies three conditions above: the solution is existence, the solution is uniqueness, and the solution continues on data The problem that violates one of the above three conditions is an ill-posed problem. We need to regularize this problem, to give a good approximation. The number of works on the regularized problem with input data in $\mathscr{L}^2$ is quite abundant. The results of this study can be found in the following documents, attached to the regularization methods: the Tikhonov method, see [14, 15], the Fractional Tikhonov method, see [16], the fractional Landweber method, see [17, 18], the Quasi Boundary method, see [19], the truncation method, see [20], and their references.

However, for $p \neq 2$, results for regularized problem in $\mathscr{L}^p$ are quite rare. We confirm that our paper is the first result for the inverse problem for the conformable parabolic equation when the observed data is in the $\mathscr{L}^p$ space with $p \neq 2$. If the data is not in $\mathscr{L}^2$, the use of Parseval equality is not feasible. In this case, we used the embedding between $\mathscr{L}^p$ and Hilbert scales spaces $\mathbb{X}^s(\mathscr{D})$. These results are well described in Theorem 3 and Theorem 5. The main analytical technique in our paper is to use some embeddings and some analysis estimators related to Hölder inequality. To do this, we learn many interesting techniques from N.H. Tuan [21].

This paper is organized as follows. In Section 2, we state some function spaces and embeddings. In Section 3, we deal with the regularized solution for the inverse source problem for (1). Section 4 gives the mild solution of backward problem in case $F = 0$. After that, we solve two problems in the case of observed data in $\mathscr{L}^p$ space.

## 2. Preliminary Results

Let us recall that the spectral problem

$$\begin{cases} (-\Delta)^s e_m(x) = \lambda_m^s e_m(x), & x \in \mathscr{D} \\ e_m(x) = 0, & x \in \partial D \end{cases} \tag{5}$$

admits the eigenvalues $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_m \le \cdots$ with $\lambda_m \longrightarrow \infty$ as $m \longrightarrow \infty$. The corresponding eigenfunctions are $e_m \in H_0^1(\mathscr{D})$.

*Definition 1* (Hilbert scale space). We recall the Hilbert scale space, which is given as follows:

$$\mathscr{X}^n(\mathscr{D}) = \left\{ f \in \mathscr{L}^2(\mathscr{D}), \sum_{m=1}^{\infty} \lambda_m^{2n} \left( \int_{\mathscr{D}} f(x) e_m(x) dx \right)^2 < \infty \right\}, \tag{6}$$

for any $n \ge 0$. It is well-known that $\mathbb{X}^n(\mathscr{D})$ is a Hilbert space corresponding to the norm

$$\|f\|_{\mathscr{X}^n(\mathscr{D})} = \left( \sum_{m=1}^{\infty} \lambda_m^{2n} \left( \int_{\mathscr{D}} f(x) e_m(x) dx \right)^2 \right)^{1/2}, f \in \mathscr{X}^n(\mathscr{D}). \tag{7}$$

**Lemma 2** (See [22]). *The following statement is true:*

$$\left. \begin{array}{l} L^p(\mathscr{D}) \hookrightarrow \mathscr{X}^\mu(\mathscr{D}), \text{ if } -\dfrac{N}{4} < \mu \le 0, p \ge \dfrac{2N}{N - 4\mu} \\ \mathscr{X}^\mu(\mathscr{D}) \hookrightarrow L^p(\mathscr{D}), \text{ if } 0 \le \mu < \dfrac{N}{4}, p \le \dfrac{2N}{N - 4\mu} \end{array} \right\}. \tag{8}$$

## 3. Regularization of Backward Problem

In order to find a precise formulation for solutions, we consider the mild solution in Fourier series $y(x, t) = \sum_{m=1}^{\infty} y_m(t) e_m(x)$, with $y_m(t) = \int_{\mathscr{D}} y(x, t) e_m(x) dx$. Taking the inner product of the equations of Problem (1) with $e_m$ gives

$$\begin{cases} \dfrac{{}^{\mathscr{C}}\partial^\beta}{\partial t^\beta} \langle y(.,t), e_m \rangle + k\lambda_m \dfrac{{}^{\mathscr{C}}\partial^\beta}{\partial t^\beta} \langle y(.,t), e_m \rangle - \lambda_m^s \langle y(.,t), e_m \rangle = \langle F(.,t), e_m \rangle, t \in (0,T) \\ \langle y(.,0), e_m \rangle = \langle y_0, e_m \rangle \end{cases}. \tag{9}$$

The first equation of (9) is a differential equation with a conformable derivative as follows:

$$\frac{{}^{\mathscr{C}}\partial^\beta}{\partial t^\beta} y_m(t) - \lambda_m^s (1+k\lambda_m)^{-1} y_m(t) = (1+k\lambda_m)^{-1} F_n(t). \quad (10)$$

Because of the result in [23], the solution of Problem (1) is

$$\begin{aligned}\langle y(.,t), e_m\rangle &= \exp\left(-\frac{\lambda_m^s}{1+k\lambda_m}\frac{t^\beta}{\beta}\right)\langle y(0), e_m\rangle \\ &+ \frac{1}{1+k\lambda_m}\int_0^t \theta^{\beta-1}\exp\left(\frac{\lambda_m^s}{1+k\lambda_m}\frac{\theta^\beta - t^\beta}{\beta}\right)\langle F(.,\theta), e_m\rangle d\theta.\end{aligned} \quad (11)$$

Letting $t=T$, we follow from (11) that

$$\begin{aligned}\left(\int_{\mathscr{D}} y_T(x)e_m(x)dx\right) &= \exp\left(-\frac{\lambda_m^s}{1+k\lambda_m}\frac{T^\beta}{\beta}\right)\left(\int_{\mathscr{D}} y_0(x)e_m(x)dx\right) \\ &+ \frac{1}{1+k\lambda_m}\int_0^T \theta^{\beta-1}\exp\left(\frac{\lambda_m^s}{1+k\lambda_m}\frac{\theta^\beta - T^\beta}{\beta}\right) \\ &\cdot \left(\int_{\mathscr{D}} F(.,\theta)e_m(x)dx\right)d\theta.\end{aligned} \quad (12)$$

From (12), we have

$$\begin{aligned}\left(\int_{\mathscr{D}} y_0(x)e_m(x)dx\right) &= \left(\exp\left(-\frac{\lambda_m^s}{1+k\lambda_m}\frac{T^\beta}{\beta}\right)\right)^{-1}\left[\left(\int_{\mathscr{D}} y_T(x)e_m(x)dx\right)\right. \\ &- \frac{1}{1+k\lambda_m}\int_0^T \theta^{\beta-1}\exp\left(\frac{\lambda_m^s}{1+k\lambda_m}\frac{\theta^\beta - T^\beta}{\beta}\right) \\ &\left.\cdot \left(\int_{\mathscr{D}} F(x,\theta)e_m(x)dx\right)d\theta\right].\end{aligned} \quad (13)$$

Substituting (13) into (12), we obtain

$$\begin{aligned}\langle y(.,t), e_m\rangle &= \exp\left(\frac{\lambda_m^s}{1+k\lambda_m}\frac{T^\beta - t^\beta}{\beta}\right)\int_{\mathscr{D}} y_T(x)e_m(x)dx \\ &- \frac{1}{1+k\lambda_m}\int_t^T \theta^{\beta-1}\exp\left(\frac{\lambda_m^s}{1+k\lambda_m}\frac{\theta^\beta - t^\beta}{\beta}\right) \\ &\cdot \left(\int_{\mathscr{D}} F(x,\theta)e_m(x)dx\right)d\theta\end{aligned} \quad (14)$$

This leads to

$$\begin{aligned}y(x,t) &= \sum_{m=1}^{+\infty}\exp\left(\frac{\lambda_m^s}{1+k\lambda_m}\frac{T^\beta - t^\beta}{\beta}\right)\left(\int_{\mathscr{D}} y_T(x)e_m(x)dx\right)e_m(x) \\ &- \sum_{m=1}^{+\infty}\frac{1}{1+k\lambda_m}\left(\int_t^T \theta^{\beta-1}\exp\left(\frac{\lambda_m^s}{1+k\lambda_m}\frac{\theta^\beta - t^\beta}{\beta}\right)\right. \\ &\left.\cdot\left(\int_{\mathscr{D}} F(x,\theta)e_m(x)dx\right)d\theta\right)e_m(x).\end{aligned} \quad (15)$$

## 4. The Mild Solution of Backward Problem in Case $F=0$

In this section, we investigate the existence and regularity of mild solutions of Problem (1). Firstly, we consider the following initial value problem

$$\begin{cases}\dfrac{{}^{\mathscr{C}}\partial^\beta}{\partial t^\beta}(y(x,t) - k\Delta y(x,t)) + (-\Delta)^s y(x,t) = 0, & x\in\mathscr{D}, t\in(0,T) \\ y(x,t) = 0, & x\in\partial\mathscr{D}, t\in(0,T) \\ y(x,T) = y_T(x), & x\in\mathscr{D}, t\in(0,T)\end{cases}. \quad (16)$$

According to (15), in this case, we have

$$y(x,t) = \sum_{m=1}^{+\infty}\exp\left(\frac{\lambda_m^s}{1+k\lambda_m}\frac{T^\beta - t^\beta}{\beta}\right)\left(\int_{\mathscr{D}} y_T(x)e_m(x)dx\right)e_m(x). \quad (17)$$

*4.1. The Ill-Posedness of Problem (1).* In order to prove that the solution to the backward problem is unstable at$F(x,t)=0$, let us take the perturbed final data$y_{T,j}(x)\in\mathscr{L}^2(\mathscr{D})$, by choosing$y_{T,j}(x) = e_j(x)\lambda_j^{-1/2}$. For $s > 3/2$, let us choose input final data $y_T(x)=0$; we know that an error in $L^2(\mathscr{D})$ norm between two input final data as follows:

$$\begin{aligned}\left\|y_{T,j} - y_T\right\|_{\mathscr{L}^2(\mathscr{D})} &= \left\|e_j\lambda_j^{-1/2}\right\|_{\mathscr{L}^2(\mathscr{D})} \\ &= \lambda_j^{-1/2} \text{ this leads to } \lim_{j\longrightarrow\infty}\left\|y_{T,j} - y_T\right\|_{\mathscr{L}^2(\mathscr{D})} \\ &= \lim_{j\longrightarrow\infty}\lambda_j^{-1/2} = 0.\end{aligned} \quad (18)$$

Therefore, we obtain

$$y_j(x,t) = \exp\left(\frac{\lambda_j^s}{1+k\lambda_j}\frac{T^\beta - t^\beta}{\beta}\right)y_{T,j}(x). \quad (19)$$

First of all, we have $\lambda_j^s/(1+k\lambda_j) = (\lambda_j^{s-1})/((1/\lambda_j)+k) \geq (\lambda_j^{s-1})/((1/\lambda_j)+k)$, and $((T^\beta - t^\beta)/\beta)\geq 0$; this implies that

$$\exp\left(\frac{\lambda^s}{1+k\lambda_j}\frac{T^\beta - t^\beta}{\beta}\right) \geq \exp\left(\frac{\lambda_j^{s-1}}{\lambda_1^{-1}+k}\frac{T^\beta - t^\beta}{\beta}\right). \quad (20)$$

Next, using the inequality $\exp(x) \geq x$, for $x > 0$, this leads to:

$$\begin{aligned}\left\|y_j(\cdot,t)\right\|_{\mathscr{L}^2(\mathscr{D})} &\geq \left\|\exp\left(\frac{\lambda_j^{s-1}}{\lambda_1^{-1}+k}\frac{T^\beta - t^\beta}{\beta}\right)\frac{1}{\lambda_j^{1/2}}\right\|_{\mathscr{L}^2(\mathscr{D})} \\ &\geq \frac{\lambda_j^{s-3/2}}{\lambda_1^{-1}+k}\frac{T^\beta - t^\beta}{\beta}.\end{aligned} \quad (21)$$

For $s > 3/2$, and from (21), we get

$$\lim_{j\longrightarrow\infty}\left\|y_j(.,t)\right\|_{\mathscr{L}^2(\mathscr{D})} \geq \lim_{j\longrightarrow\infty}\frac{\lambda_j^{s-3/2}}{\lambda_1^{-1}+k}\frac{T^\beta - t^\beta}{\beta} \longrightarrow +\infty. \quad (22)$$

Thus, Problem (1), in general, ill-posed in the Hadamard sense in $\mathscr{L}^2(\mathscr{D})$-norm.

*4.2. Regularization of inverse Problem (1) in $\mathscr{L}^p(\mathscr{D})$ space.* From (15), we know that the explicit formula of the mild solution

$$\begin{aligned}y(x,t) &= \sum_{m=1}^{+\infty}\exp\left(\frac{\lambda_m^s}{1+k\lambda_m}\frac{T^\beta - t^\beta}{\beta}\right)\left(\int_{\mathscr{D}} y_T(x)e_m(x)dx\right)e_m(x) \\ &\quad - \sum_{m=1}^{+\infty}\frac{1}{1+k\lambda_m}\left(\int_t^T \theta^{\beta-1}\exp\left(\frac{\lambda_m^s}{1+k\lambda_m}\frac{\theta^\beta - t^\beta}{\beta}\right)\right. \\ &\quad \cdot\left.\left(\int_{\mathscr{D}} F(x,\theta)e_m(x)dx\right)d\theta\right)e_m(x).\end{aligned} \quad (23)$$

By applying the Fourier truncation method, we have its approximation

$$\begin{aligned}y_\delta(x,t) &= \sum_{m\leq\mathscr{M}_\delta}\exp\left(\frac{\lambda_m^s}{1+k\lambda_m}\frac{T^\beta - t^\beta}{\beta}\right)\left(\int_{\mathscr{D}} y_{T,\delta}(x)e_m(x)dx\right)e_m(x) \\ &\quad - \sum_{m\leq\mathscr{M}_\delta}\frac{1}{1+k\lambda_m}\left(\int_t^T \theta^{\beta-1}\exp\left(\frac{\lambda_m^s}{1+k\lambda_m}\frac{\theta^\beta - t^\beta}{\beta}\right)\right. \\ &\quad \cdot\left.\left(\int_{\mathscr{D}} F_\delta(x,\theta)e_m(x)dx\right)d\theta\right)e_m(x).\end{aligned} \quad (24)$$

Here, $\mathscr{M}_\delta$ is parameter regularization which is defined later.

**Theorem 3.** *For $s > 1$, taking $(y_T, F) \in \mathscr{L}^p(0,T) \times \mathscr{L}^\infty(0,T;\mathscr{L}^p(\mathscr{D}))$ for any $0 \leq t \leq T$ for any $1/\beta < p < 2$, assume that $(y_T, F)$ is observed by the couple $(y_{T,\delta}, F_\delta)$ such that*

$$\left\|y_{T,\delta} - y_T\right\|_{\mathscr{L}^p(\mathscr{D})} + \|F - F_\delta\|_{\mathscr{L}^\infty(0,T;\mathscr{L}^p(\mathscr{D}))} \leq \delta, \delta > 0. \quad (25)$$

*Let us assume that $u \in \mathscr{L}^\infty(0,T;\mathscr{X}^{n+\sigma})$ for $\sigma > 0$ and $0 < n < N/4$. With $\mathscr{M}_\delta$ such that*

$$\lim_{\delta\longrightarrow 0}\mathscr{M}_\delta = +\infty, \lim_{\delta\longrightarrow 0}|\mathscr{M}_\delta|^{n+N/2p-N/4}\exp\left(\frac{T^\beta}{\beta}\mathscr{M}_\delta^{s-1}k^{-1}\right)\delta = 0. \quad (26)$$

*Then, the error estimate*

$$\|y_\delta - y\|_{\mathscr{L}^{2N/N-4n}(\mathscr{D})} \text{ is of order } \max\cdot\left\{|\mathscr{M}_\delta|^{n+N/2p-N/4}\exp\left((\mathscr{M}_\delta)^{s-1}k^{-1}\frac{T^\beta}{\beta}\right)\delta, |\mathscr{M}_\delta|^{-\sigma}\right\}. \quad (27)$$

*Remark 4.* One choice for $\mathscr{M}_\delta$ such that

$$\mathscr{M}_\delta = \left(T^{-\beta}\beta(1-\alpha)k\right)^{1/s-1}\left[\log\left(\frac{1}{\delta}\right)\right]^{1/s-1}, \text{for } 0 < \alpha < 1. \quad (28)$$

then $\|y_\delta - y\|_{\mathscr{L}^{2N/(N-4n)}(\mathscr{D})}$ is of order $\max\{\delta^\alpha [\log(1/\delta)]^{(n+N/2p-N/4)/(s-1)}, [\log(1/\delta)]^{-\sigma/(s-1)}\}$.

*Proof.* Let

$$\begin{aligned}\mathscr{V}_\delta(x,t) &= \sum_{m\leq\mathscr{N}_\delta}\exp\left(\frac{\lambda_m^s}{1+k\lambda_m}\frac{T^\beta - t^\beta}{\beta}\right)\left(\int_{\mathscr{D}} y_T(x)e_m(x)dx\right)e_m(x) \\ &\quad - \sum_{m\leq\mathscr{M}_\delta}\frac{1}{1+k\lambda_m}\left(\int_t^T \theta^{\beta-1}\exp\left(\frac{\lambda_m^s}{1+k\lambda_m}\frac{\theta^\beta - t^\beta}{\beta}\right)\right. \\ &\quad \cdot\left.\left(\int_{\mathscr{D}} F(x,\theta)e_m(x)dx\right)d\theta\right)e_m(x).\end{aligned} \quad (29)$$

It is clear that

$$\begin{aligned}\|y_\delta(\cdot,t) - y(\cdot,t)\|_{\mathscr{X}^n(\mathscr{D})} &\leq \|y_\delta(\cdot,t) - \mathscr{V}_\delta(\cdot,t)\|_{\mathscr{X}^n(\mathscr{D})} \\ &\quad + \|\mathscr{V}_\delta(\cdot,t) - y(\cdot,t)\|_{\mathscr{X}^n(\mathscr{D})}.\end{aligned} \quad (30)$$

We continue to consider the two components of the right hand side.

Step 1:

$$\begin{aligned} y_\delta(x,t) - \mathscr{V}_\delta(x,t) &= \sum_{\lambda_m \le \mathscr{M}_\delta} \exp\left(\frac{\lambda_m^s}{1+k\lambda_m}\frac{T^\beta - t^\beta}{\beta}\right) \cdot \left(\int_{\mathscr{D}} \left(y_{T,\delta}(x) - y_T(x)\right) e_m(x)dx\right) e_m(x) \\ &\quad - \sum_{\lambda_m \le \mathscr{M}_\delta} \frac{1}{1+k\lambda_m}\left(\int_t^T \theta^{\beta-1} \exp\left(\frac{\lambda_m^s}{1+k\lambda_m}\frac{\theta^\beta - t^\beta}{\beta}\right) \cdot \left(\int_{\mathscr{D}} (F_\delta(x,\theta) - F(x,\theta)) e_m(x)dx\right) d\theta\right)) e_m(x) \\ &= \mathscr{H}_1(x,t) - \mathscr{H}_2(x,t). \end{aligned} \tag{31}$$

For $s > 1$, it is easy to see that $\lambda_m^s (1 + k\lambda_m)^{-1} \le \lambda_m^{s-1} (\lambda_m^{-1} + k)^{-1} \le \lambda_m^{s-1} k^{-1}$. The first term $\mathscr{H}_1(x,t)$ on $\mathscr{X}^n(\mathscr{D})$ is bounded by

$$\begin{aligned} \|\mathscr{H}_1(.,t)\|^2_{\mathscr{X}^n(\mathscr{D})} &= \sum_{\lambda_m \le \mathscr{M}_\delta} \lambda_m^{2n} \exp\left(\frac{2\lambda_m^s}{1+k\lambda_m}\frac{T^\beta - t^\beta}{\beta}\right) \cdot \left(\int_{\mathscr{D}} \left(y_{T,\delta}(x) - y_T(x)\right) e_m(x)dx\right)^2 \\ &= \sum_{\lambda_m \le \mathscr{M}_\delta} \lambda_m^{2n+N/p-N/2} \lambda_m^{Np-2N/2p} \exp \cdot \left(2\lambda_m^{s-1} k^{-1} \frac{T^\beta}{\beta}\right) \left(\int_{\mathscr{D}} \left(y_{T,\delta}(x) - y_T(x)\right) e_m(x)dx\right)^2 \\ &\le |\mathscr{M}_\delta|^{2n+N/p-N/2} \exp\left(\frac{2|\mathscr{M}_\delta|^{s-1} k^{-1} T^\beta}{\beta}\right) \sum_{\lambda_m \le \mathscr{M}_\delta} \lambda_m^{Np-2N/2p} \cdot \left(\int_{\mathscr{D}} \left(y_{T,\delta}(x) - y_T(x)\right) e_m(x)dx\right)^2 \\ &\le |\mathscr{M}_\delta|^{2n+N/p-N/2} \exp\left(\frac{2|\mathscr{M}_\delta|^{s-1} k^{-1} T^\beta}{\beta}\right) \left\|y_{T,\delta} - y_T\right\|_{\mathscr{X}^{Np-2N/4p}}. \end{aligned} \tag{32}$$

Since the Sobolev space embedding $\mathscr{L}^p(\mathscr{D}) \longrightarrow \mathscr{X}^{(Np-2N)/4p}(\mathscr{D})$, we have

$$\left\|y_{T,\delta} - y_T\right\|_{\mathscr{X}^{Np-2N/4p(\mathscr{D})}} \le C_1(N,p) \left\|y_{T,\delta} - y_T\right\|_{\mathscr{L}^p(\mathscr{D})}. \tag{33}$$

This follows from (32) that

$$\|\mathscr{H}_1(.,t)\|_{\mathscr{X}^n(\mathscr{D})} \le (\mathscr{M}_\delta)^{n+N/2p-N/4} \exp\left(\frac{(\mathscr{M}_\delta)^{s-1} k^{-1} T^\beta}{\beta}\right) C_1(N,p)\delta. \tag{34}$$

The second term $\mathscr{H}_2(x,t)$ is estimated as follows:

$$\|\mathscr{H}_2(.,t)\|^2_{\mathscr{X}^n(\mathscr{D})} = \sum_{\lambda_m \le \mathscr{M}_\delta} \frac{\lambda_m^{2n}}{1+k\lambda_m}\left(\int_t^T \theta^{\beta-1} \exp\left(\frac{\lambda_m^s}{1+k\lambda_m}\frac{\theta^\beta - t^\beta}{\beta}\right) \cdot \left(\int_{\mathscr{D}} (F(x,\theta) - F_\delta(x,\theta)) e_m(x)dx\right) d\theta\right))^2 \tag{35}$$

By the same arguments as above, we find that

$$\begin{aligned} \|\mathscr{H}_2(\cdot,t)\|^2_{\mathscr{X}^n(\mathscr{D})} &\le \sum_{\lambda_m \le \mathscr{M}_\delta} \lambda_m^{2n} \left(\int_t^T \theta^{\beta-1} \exp\left(\lambda_m^{s-1} k^{-1} \frac{\theta^\beta - t^\beta}{\beta}\right) \cdot \left(\int_{\mathscr{D}} (F(x,\theta) - F_\delta(x,\theta)) e_m(x)dx\right) d\theta\right)^2 \\ &\le \exp\left(2(\mathscr{M}_\delta)^{s-1} k^{-1} \frac{T^\beta}{\beta}\right) \sum_{\lambda_m \le \mathscr{M}_\delta} \lambda_m^{2n} \cdot \left(\int_t^T \theta^{\beta-1} \left(\int_{\mathscr{D}} (F(x,\theta) - F_\delta(x,\theta)) e_m(x)dx\right) d\theta\right)^2 \end{aligned} \tag{36}$$

We can see that $\lambda_m^{2n+N/p-N/2} \le |\mathscr{M}_\delta|^{2n+N/p-N/2}$ and $\int_t^T \theta^{\beta-1} d\theta \le (T^\beta - t^\beta)/\beta \le T^\beta/\beta$. From (36), using Holder's inequality, we get

$$\begin{aligned} &\sum_{\lambda_m \le \mathscr{M}_\delta} \lambda_m^{2n} \left(\int_t^T \theta^{\beta-1} \left(\int_{\mathscr{D}} (F(x,\theta) - F_\delta(x,\theta)) e_m(x)dx\right) d\theta\right)^2 \\ &\le \sum_{\lambda_m \le \mathscr{M}_\delta} \lambda_m^{2n+N/p-N/2} \lambda_m^{Np-2N/2p} \left(\int_t^T \theta^{\beta-1} d\theta\right) \cdot \left(\int_t^T \theta^{\beta-1} \left(\int_{\mathscr{D}} (F(x,\theta) - F_\delta(x,\theta)) e_m(x)dx\right)^2 d\theta\right) \\ &\le \beta^{-1} T^\beta |\mathscr{M}_\delta|^{2n+N/p-N/2} \left(\int_t^T \theta^{\beta-1} \sum_{\lambda_m \le \mathscr{M}_\delta} \lambda_m^{2n+N/p-N/2} \cdot \left(\int_{\mathscr{D}} (F(x,\theta) - F_\delta(x,\theta)) e_m(x)dx\right)^2 d\theta\right) \\ &\le \beta^{-1} T^\beta |\mathscr{M}_\delta|^{2n+N/p-N/2} \left(\int_t^T \theta^{\beta-1} \|F_\delta(\cdot,t) - F(\cdot,t)\|^2_{\mathscr{X}^{Np-2N/4p}} d\theta\right). \end{aligned} \tag{37}$$

This latter inequality together with Sobolev embedding $\mathscr{L}^p(\mathscr{D}) \longrightarrow \mathscr{X}^{(Np-2N)/4p}(\mathscr{D})$ gives us

$$\begin{aligned} &\sum_{\lambda_m \le \mathscr{M}_\delta} \lambda_m^{2n} \left(\int_t^T \theta^{\beta-1} \left(\int_{\mathscr{D}} (F(x,\theta) - F_\delta(x,\theta)) e_m(x)dx\right) d\theta\right)^2 \\ &\le \beta^{-1} |C_2(N,p)|^2 T^\beta (\mathscr{M}_\delta)^{2n+N/p-N/2} \cdot \left(\int_t^T \theta^{\beta-1} \|F(x,\theta) - F_\delta(x,\theta)\|^2_{\mathscr{L}^p(\mathscr{D})} d\theta\right) \\ &\le \beta^{-2} |C_2(N,p)|^2 T^{2\beta} |\mathscr{M}_\delta|^{2n+N/p-N/2} \|F - F_\delta\|_{\mathscr{L}^\infty(0,T;\mathscr{L}^p(\mathscr{D}))} \\ &\le \beta^{-2} |C_2(N,p)|^2 T^{2\beta} |\mathscr{M}_\delta|^{2n+N/p-N/2} \delta^2. \end{aligned} \tag{38}$$

Combining (36) and (38), we get

$$\|\mathscr{H}_2(\cdot,t)\|^2_{\mathscr{X}^n(\mathscr{D})} \le \exp\left(2(\mathscr{M}_\delta)^{s-1}k^{-1}\frac{T^\beta}{\beta}\right) \cdot |C_2(N,p)|^2 T^{2\beta}|\mathscr{M}_\delta|^{2n+N/p-N/2}\beta^{-2}\delta^2. \tag{39}$$

Taking the square root on the both sides, we have

$$\|\mathscr{H}_2(\cdot,t)\|_{\mathscr{X}^n(\mathscr{D})} \le \exp\left((\mathscr{M}_\delta)^{s-1}k^{-1}\frac{T^\beta}{\beta}\right) \cdot |C_2(N,p)|T^\beta|\mathscr{M}_\delta|^{n+N/2p-N/4}\beta^{-1}\delta. \tag{40}$$

From (34) and (40), we deduce that

$$\begin{aligned}\|y_\delta(\cdot,t)-\mathscr{V}_\delta(\cdot,t)\|_{\mathscr{X}^n(\mathscr{D})} &\le \|\mathscr{H}_1(\cdot,t)\|_{\mathscr{X}^n(\mathscr{D})} + \|\mathscr{H}_2(\cdot,t)\|_{\mathscr{X}^n(\mathscr{D})}\\ &\le |\mathscr{M}_\delta|^{n+N/2p-N/4}\exp\left(\frac{(\mathscr{M}_\delta)^{s-1}k^{-1}T^\beta}{\beta}\right)C_1(N,p)\delta + \exp\\ &\quad\cdot\left((\mathscr{M}_\delta)^{s-1}k^{-1}\frac{T^\beta}{\beta}\right)|C_2(N,p)|T^\beta|\mathscr{M}_\delta|^{n+N/2p-N/4}\beta^{-1}\delta\\ &\le |\mathscr{M}_\delta|^{n+N/2p-N/4}\exp\left((\mathscr{M}_\delta)^{s-1}k^{-1}\frac{T^\beta}{\beta}\right)\delta\\ &\quad\cdot\left(C_1(N,p)+|C_2(N,p)|T^\beta\beta^{-1}\right).\end{aligned} \tag{41}$$

Step 2: Estimate of $\|u(\cdot,t)-\mathscr{V}_\delta(\cdot,t)\|_{\mathscr{X}^n(\mathscr{D})}$.
From the definition (23) and (29), we have

$$\begin{aligned}\|y(\cdot,t)-\mathscr{V}_\delta(\cdot,t)\|^2_{\mathscr{X}^n(\mathscr{D})} &= \sum_{\lambda_m>\mathscr{M}_\delta}\lambda_m^{2n}\left(\int_{\mathscr{D}}u(\cdot,t)e_m(x)dx\right)^2\\ &= \sum_{\lambda_m>\mathscr{M}_\delta}\lambda_m^{-2\sigma}\lambda_m^{2n+2\sigma}\left(\int_{\mathscr{D}}u(\cdot,t)e_m(x)dx\right)^2\\ &\le |\mathscr{M}_\delta|^{-2\sigma}\|u\|^2_{\mathscr{L}^\infty(0,T;\mathscr{X}^{n+\sigma}(\mathscr{D}))}.\end{aligned} \tag{42}$$

Therefore, we get

$$\|y(\cdot,t)-\mathscr{V}_\delta(\cdot,t)\|_{\mathscr{X}^n(\mathscr{D})} \le |\mathscr{M}_\delta|^{-\sigma}\|u\|_{L^\infty(0,T;\mathscr{X}^{n+\sigma}(\mathscr{D})}. \tag{43}$$

Combining two steps and noting that $\mathscr{X}^n(\mathscr{D})\hookrightarrow \mathscr{L}^{2N/(N-4n)}$, $(0<n<N/4)$, we deduce that

$$\begin{aligned}\|y_\delta(\cdot,t)-y(\cdot,t)\|_{\mathscr{L}(\mathscr{D})} &\le C_3(N,n)\|y_\delta(\cdot,t)-\mathscr{V}_\delta(\cdot,t)\|_{\mathscr{X}^n(\mathscr{D})}\\ &\quad + C_3(N,n)\|\mathscr{V}_\delta(\cdot,t)-y(\cdot,t)\|_{\mathscr{X}^n(\mathscr{D})}\\ &\le C_3(N,n)|\mathscr{M}_\delta|^{n+N/2p-N/4}\exp\\ &\quad\cdot\left((\mathscr{M}_\delta)^{s-1}k^{-1}\frac{T^\beta}{\beta}\right)\delta\left(C_1(N,p)+|C_2(N,p)|T^\beta\beta^{-1}\right)\\ &\quad + C_3(N,n)|\mathscr{M}_\delta|^{-\sigma}\|u\|_{\mathscr{L}^\infty(0,T;\mathscr{X}^{n+\sigma}(\mathscr{D})}.\end{aligned} \tag{44}$$

The proof of Theorem 3 is completed. In the following theorem, we give a regularization result in the case that $F$ has a split form $F(x,t)=\Phi(t)f(x)$. □

**Theorem 5.** *For $s>1$, let us assume that the input data $\Phi_\delta$, $g_\delta, f_\delta$ such that*

$$\|\Phi_\delta-\Phi\|_{\mathscr{L}^p(0,T)} + \left\|y_{T,\delta}-y_T\right\|_{\mathscr{L}^p(\mathscr{D})} + \|f_\delta-f\|_{\mathscr{L}^p(\mathscr{D})} \le \delta. \tag{45}$$

*Assume that $u\in\mathscr{L}^\infty(0,T;\mathscr{X}^{n+\sigma}(\mathscr{D}))$ for any $\sigma>0$, then we construct a regularized solution defined by*

$$\begin{aligned}\mathscr{W}_\delta(x,t) &= \sum_{m\le\mathscr{B}_\delta}\exp\left(\frac{\lambda_m^s}{1+k\lambda_m}\frac{T^\beta-t^\beta}{\beta}\right)\left(\int_{\mathscr{D}}y_{T,\delta}(x)e_m(x)dx\right)e_m(x)\\ &\quad - \sum_{m\le\mathscr{B}_\delta}\frac{1}{1+k\lambda_m}\left(\int_t^T\theta^{\beta-1}\exp\left(\frac{\lambda_m^s}{1+k\lambda_m}\frac{\theta^\beta-t^\beta}{\beta}\right)\right.\\ &\quad\left.\cdot\left(\int_{\mathscr{D}}f(x)e_m(x)dx\right)\Phi_\delta(\theta)d\theta\right)e_m(x).\end{aligned} \tag{46}$$

*Then, the error $\|\mathscr{W}_\delta(.,t)-y(.,t)\|_{\mathscr{L}^{2N/(N-4n)}(\mathscr{D})}$ is of order $\max\{\delta|\mathscr{B}_\delta|^{-\sigma}, |\mathscr{B}_\delta|^{n+(N/2p)-(N/4)}\exp((\mathscr{B}_\delta^{s-1}k^{-1}T^\beta)/\beta)\delta\}$.*

*Remark 6.* $\mathscr{B}_\delta=(T^{-\beta}\beta k)^{1/(s-1)}(1-\alpha)^{1/(s-1)}\log(1/\delta)^{1/(s-1)}$, then the error

$$\|\mathscr{W}_\delta(.,t)-y(.,t)\|_{\mathscr{L}^{2N/N-4n}(\mathscr{D})}\ \text{is of order}\ \max\cdot\left\{\delta\left|\log\left(\frac{1}{\delta}\right)\right|^{-\sigma/s-1}\left|\log\left(\frac{1}{\delta}\right)\right|^{n+N/2p-N/4/s-1}\delta^\alpha\right\}. \tag{47}$$

*Proof.* Since $F(x,t)=\Phi(t)f(x)$, we know that

$$\begin{aligned}\mathscr{Z}_\delta(x,t) &= \sum_{m\le\mathscr{B}_\delta}\exp\left(\frac{\lambda_m^s}{1+k\lambda_m}\frac{T^\beta-t^\beta}{\beta}\right)\left(\int_{\mathscr{D}}y_T(x)e_m(x)dx\right)e_m(x)\\ &\quad - \sum_{m\le\mathscr{B}_\delta}\frac{1}{1+k\lambda_m}\left(\int_t^T\theta^{\beta-1}\exp\left(\frac{\lambda_m^s}{1+k\lambda_m}\frac{\theta^\beta-t^\beta}{\beta}\right)\right.\\ &\quad\left.\cdot\left(\int_{\mathscr{D}}f(x)e_m(x)dx\right)\Phi(\theta)d\theta\right)e_m(x).\end{aligned} \tag{48}$$

The triangle inequality allows us to obtain that

$$\|\mathscr{W}_\delta(\cdot,t)-y(\cdot,t)\|_{\mathscr{X}^n(\mathscr{D})} \le \|\mathscr{W}_\delta(\cdot,t)-\mathscr{Z}_\delta(\cdot,t)\|_{\mathscr{X}^n(\mathscr{D})} + \|\mathscr{Z}_\delta(\cdot,t)-y(\cdot,t)\|_{\mathscr{X}^n(\mathscr{D})}. \tag{49}$$

Next, we will evaluate the right side of (49), by the same way as demonstrated in (42),

$$\|y(\cdot,t)-\mathscr{Z}_\delta(\cdot,t)\|^2_{\mathscr{X}^n(\mathscr{D})} \le |\mathscr{B}_\delta|^{-2\sigma}\|u\|^2_{\mathscr{L}^\infty(0,T;\mathscr{X}^{n+\sigma}(\mathscr{D}))}. \tag{50}$$

It is easy to see that

$$\mathscr{W}_\delta(x,t)-\mathscr{Z}_\delta(x,t)=\mathscr{J}_1(x,t)+\mathscr{J}_2(x,t)+\mathscr{J}_3(x,t). \tag{51}$$

whereby

$$\begin{aligned}
\mathscr{J}_1(x,t) &= \sum_{\lambda_m\le\mathscr{B}_\delta} \exp\left(\frac{\lambda_m^s}{1+k\lambda_m}\frac{T^\beta-t^\beta}{\beta}\right) \cdot \left(\int_{\mathscr{D}}\left(y_{T,\delta}(x)-y_T(x)\right)e_m(x)dx\right)e_m(x),\\
\mathscr{J}_2(x,t) &= -\sum_{\lambda_m\le\mathscr{B}_\delta}\left[\int_t^T \theta^{\beta-1}\exp\left(\frac{\lambda_m^s}{1+k\lambda_m}\frac{\theta^\beta-t^\beta}{\beta}\right)\Phi_\delta(\theta)d\theta\right] \cdot \left(\int_{\mathscr{D}}(f_\delta(x)-f(x))e_m(x)\right)e_m(x),\\
\mathscr{J}_3(x,t) &= \sum_{\lambda_m\le\mathscr{B}_\delta}\left[\int_t^T \theta^{\beta-1}\exp\left(\frac{\lambda_m^s}{1+k\lambda_m}\frac{\theta^\beta-t^\beta}{\beta}\right)(\Phi(\theta)-\Phi_\delta(\theta))d\theta\right] \cdot \left(\int_{\mathscr{D}}f(x)e_m(x)dx\right)e_m(x).
\end{aligned} \tag{52}$$

We will divide this review into several steps as follows:

Step 1: Estimate of $\|\mathscr{J}_1(\cdot,t)\|_{\mathscr{X}^n(\mathscr{D})}$, we obtain that

$$\|\mathscr{J}_1(\cdot,t)\|_{\mathscr{X}^n(\mathscr{D})} \le C_4(N,p)|\mathscr{B}_\delta|^{n+N/2p-N/4}\exp\left(\mathscr{B}_\delta^{s-1}k^{-1}\frac{T^\beta}{\beta}\right)\delta. \tag{53}$$

Step 2: Due to Parseval's equality, the term $\|\mathscr{J}_2(\cdot,t)\|_{\mathscr{X}^n(\mathscr{D})}$ can be bounded as follows:

$$\|\mathscr{J}_2(\cdot,t)\|_{\mathscr{X}^n(\mathscr{D})} = \sum_{\lambda_m\le\mathscr{B}_\delta}\lambda_m^{2n}\left[\int_t^T\theta^{\beta-1}\exp\left(\frac{\lambda_m^s}{1+k\lambda_m}\frac{\theta^\beta-t^\beta}{\beta}\right)\Phi_\delta(\theta)d\theta\right]^2 \cdot \left(\int_{\mathscr{D}}(f_\delta(x)-f(x))e_m(x)dx\right)^2. \tag{54}$$

Thank to Holder's inequality, we derive that for $p>1$ and $p^*=1+(1/(p-1))$, one has

$$\begin{aligned}
&\left|\int_t^T\theta^{\beta-1}\exp\left(\frac{\lambda_m^s}{1+k\lambda_m}\frac{\theta^\beta-t^\beta}{\beta}\right)\Phi_\delta(\theta)d\theta\right|\\
&\le\left(\int_0^T|\Phi_\delta|^p d\theta\right)^{1/p}\left(\int_t^T\theta^{p^*(\beta-1)}\exp\left(p^*\frac{\lambda_m^s}{1+k\lambda_m}\frac{\theta^\beta-t^\beta}{\beta}\right)d\theta\right)^{1/p^*}\\
&\le\exp\left(\lambda_m^{s-1}k^{-1}\frac{T^\beta}{\beta}\right)\|\Phi_\delta\|_{\mathscr{L}^{p^*}(0,T)}\left(\int_0^T\theta^{p^*(\beta-1)}d\theta\right)^{1/p^*}\\
&\le\left(\frac{p-1}{p\beta-1}\right)^{p-1/p}T^{p\beta-1/p}\exp\left(\lambda_m^{s-1}k^{-1}\frac{T^\beta}{\beta}\right)\|\Phi_\delta\|_{\mathscr{L}^p(0,T)}.
\end{aligned} \tag{55}$$

The latter inequality leads to

$$\begin{aligned}
&\|\mathscr{J}_2(.,t)\|^2_{\mathscr{X}^n(\mathscr{D})}\\
&=\sum_{\lambda_m\le\mathscr{B}_\delta}\lambda_m^{2n}\left[\int_t^T\theta^{\beta-1}\exp\left(\frac{\lambda_m^s}{1+k\lambda_m}\frac{\theta^\beta-t^\beta}{\beta}\right)\Phi_\delta(\theta)d\theta\right]^2 \cdot\left(\int_{\mathscr{D}}(f_\delta(x)-f(x))e_m(x)dx\right)^2\\
&\le\left(\frac{p-1}{p\beta-1}\right)^{2p-2/p}T^{2p\beta-2/p}\|\Phi_\delta(\theta)\|^2_{\mathscr{L}^p(0,T)} \times\sum_{\lambda_m\le\mathscr{B}_\delta}\lambda_m^{2n+N/p-N/2}\lambda_m^{Np-2N/2p}\exp \cdot\left(\frac{2T^\beta\lambda_m^{s-1}k^{-1}}{\beta}\right)\left(\int_{\mathscr{D}}(f_\delta(x)-f(x))e_m(x)dx\right)^2\\
&\le\left(\frac{p-1}{p\beta-1}\right)^{2p-2/p}T^{2p\beta-2/p}\|\Phi_\delta(\theta)\|^2_{\mathscr{L}^p(0,T)}|\mathscr{B}_\delta|^{2n+N/p-N/2}\exp \cdot\left(\frac{2T^\beta|\mathscr{B}_\delta|^{s-1}k^{-1}}{\beta}\right)\times\sum_{\lambda_m\le\mathscr{B}_\delta}\lambda_m^{2n+N/p-N/2}\left(\int_{\mathscr{D}}(f_\delta(x)-f(x))e_m(x)dx\right)^2\\
&\le\left(\frac{p-1}{p\beta-1}\right)^{2p-2/p}T^{2p\beta-2/p}\|\Phi_\delta(\theta)\|^2_{\mathscr{L}^p(0,T)}|\mathscr{B}_\delta|^{2n+N/p-N/2} \times\exp\left(\frac{2T^\beta|\mathscr{B}_\delta|^{s-1}k^{-1}}{\beta}\right)\|f_\delta-f\|_{\mathscr{X}^{Np-2N/4p}(\Omega)}.
\end{aligned} \tag{56}$$

where $s>1/\beta$. In view of Sobolev embedding $\mathscr{L}^p(\mathscr{D})^\circ\mathscr{X}^{(Np-2N)/4p}(\mathscr{D})$, we derive that the following estimate

$$\|\mathscr{J}_2(.,t)\|_{\mathscr{X}^n(\mathscr{D})}\le C_6(p,\beta,T,N)\|\Phi_\delta\|_{\mathscr{L}^p(0,T)}|\mathscr{B}_\delta|^{n+N/2p-N/4}\exp \cdot\left(\frac{T^\beta|\mathscr{B}_\delta|^{s-1}k^{-1}}{\beta}\right)\delta, \tag{57}$$

where

$$C_6(p,\beta,T,N)=\left(\frac{p-1}{p\beta-1}\right)^{p-1/p}T^{p\beta-1/p}C_5(N,p). \tag{58}$$

Step 3: Let us now to consider the term $\|\mathcal{J}_3(.,t)\|_{\mathcal{X}^n(\mathcal{D})}$. By applying Hölder's inequality, we get that for $p > 1$ and $p^* = 1/(p-1)$ ·

$$\begin{aligned} &\left|\int_t^T \theta^{\beta-1} \exp\left(\frac{\lambda_m^s}{1+k\lambda_m}\frac{\nu^\beta - t^\beta}{\beta}\right)(\Phi(\theta)-\Phi_\delta(\theta))d\theta\right| \\ &\le \left(\int_0^T |\Phi(\theta)-\Phi_\delta(\theta)|^s d\nu\right)^{1/p} \\ &\quad \cdot \left(\int_t^T \theta^{p^*(\beta-1)} \exp\left(p^* \frac{\lambda_m^s}{1+k\lambda_m}\frac{\theta^\beta - t^\beta}{\beta}\right)d\nu\right)^{1/p^*} \\ &\le \exp\left(\frac{\lambda_m^{s-1}k^{-1}T^\beta}{\beta}\right)\|\Phi_\delta - \Phi\|_{\mathcal{L}^p(0,T)}\left(\int_t^T \nu^{s^*(\beta-1)}d\nu\right)^{1/p^*} \\ &\le \left(\frac{p-1}{p\beta-1}\right)^{p-1/p} T^{p\beta-1/p} \exp\left(\frac{T^\beta \lambda_m^{s-1}k^{-1}}{\beta}\right)\|\Phi-\Phi_\delta\|_{\mathcal{L}^p(0,T)} \\ &\le \left(\frac{p-1}{p\beta-1}\right)^{p-1/p} T^{p\beta-1/p} \exp\left(\frac{T^\beta \lambda_m^{s-1}k^{-1}}{\beta}\right)\delta. \end{aligned} \tag{59}$$

This inequality together with Parseval's equality allows us to derive that

$$\begin{aligned} &\|\mathcal{J}_3(.,t)\|^2_{\mathcal{X}^n(\mathcal{D})} \\ &= \sum_{\lambda_m \le \mathcal{B}_\delta} \lambda_m^{2n}\left(\int_t^T \theta^{\beta-1}\exp\left(\frac{\lambda_m^s}{1+k\lambda_m}\frac{\theta^\beta - t^\beta}{\beta}\right)(\Phi(\theta)-\Phi_\delta(\theta))d\theta\right)^2 \\ &\quad \cdot\left(\int_{\mathcal{D}} f(x)e_m(x)dx\right)^2 \\ &\le \left(\frac{p-1}{p\beta-1}\right)^{2p-2/p} T^{2p\beta-2/p}\delta^2 \sum_{\lambda_m\le\mathcal{B}_\delta} \lambda_m^{2n+N/p-N/2}\lambda_m^{Np-2N/2p} \exp \\ &\quad \cdot\left(\frac{2\lambda_m^{s-1}k^{-1}T^\beta}{\beta}\right)\left(\int_{\mathcal{D}} f(x)e_m(x)dx\right)^2 \\ &\le \left(\frac{p-1}{p\beta-1}\right)^{2p-2/p} T^{2p\beta-2/p}\delta^2 |\mathcal{B}_\delta|^{2n+N/p-N/2} \exp \\ &\quad \cdot\left(\frac{2T^\beta|\mathcal{B}_\delta|^{s-1}k^{-1}}{\beta}\right)\|f\|_{\mathcal{X}^{Np-2N/4p}(\Omega)}. \end{aligned} \tag{60}$$

By the fact that $\mathcal{L}^p(\mathcal{D})^\circ\mathcal{X}^{(Np-2N)/4p}(\mathcal{D})$, we deduce that

$$\begin{aligned} \|\mathcal{J}_3(.,t)\|_{\mathcal{X}^n(\mathcal{D})} \le C_6(p,\beta,T,N)\|f\|_{\mathcal{L}^p(D)}|\mathcal{B}_\delta|^{n+N/2p-N/4} \exp \\ \cdot\left(\frac{T^\beta|\mathcal{B}_\delta|^{s-1}k^{-1}}{\beta}\right)\delta. \end{aligned} \tag{61}$$

Combining Step 1 to Step 3, we get

$$\begin{aligned} &\|\mathcal{J}_1(.,t)\|_{\mathcal{X}^n(\mathcal{D})}+\|\mathcal{J}_2(.,t)\|_{\mathcal{X}^n(\mathcal{D})}+\|\mathcal{J}_3(.,t)\|_{\mathcal{X}^n(\mathcal{D})} \\ &\quad \le C_4(N,p)|\mathcal{B}_\delta|^{n+N/2p-N/4}\exp\left(\frac{|\mathcal{B}_\delta|^{s-1}k^{-1}T^\beta}{\beta}\right)\delta \\ &\qquad + C_6(p,\beta,T,N)|\mathcal{B}_\delta|^{n+N/2p-N/4}\exp\left(\frac{T^\beta|\mathcal{B}_\delta|^{s-1}k^{-1}}{\beta}\right)\delta \\ &\qquad \cdot\left(\|\Phi_\delta\|_{\mathcal{L}^p(0,T)} + \|f\|_{\mathcal{L}^p(D)}\right). \end{aligned} \tag{62}$$

Finally, combining the reviews from above, we conclude that

$$\begin{aligned} &\|\mathcal{W}_\delta(\cdot,t) - y(\cdot,t)\|_{\mathcal{X}^n(\mathcal{D})} \\ &\quad \le \delta|\mathcal{B}_\delta|^{-\sigma}\|u\|_{\mathcal{L}^\infty(0,T;\mathcal{X}^{n+\sigma}(\mathcal{D}))} \\ &\qquad + C_4(N,p)|\mathcal{B}_\delta|^{n+N/2p-N/4}\exp\left(\mathcal{B}_\delta^{s-1}k^{-1}\frac{T^\beta}{\beta}\right)\delta \\ &\qquad + C_6(p,\beta,T,N)|\mathcal{B}_\delta|^{n+N/2p-N/4}\exp \\ &\qquad \cdot\left(\frac{T^\beta|\mathcal{B}_\delta|^{s-1}k^{-1}}{\beta}\right)\delta\left(\|\Phi_\delta\|_{\mathcal{L}^p(0,T)} + \|f\|_{\mathcal{L}^p(D)}\right). \end{aligned} \tag{63}$$

□

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no competing interests. The corresponding author is a full-time member of the School of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran

## Authors' Contributions

All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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