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*Research Article*

# Ulam Stability and Non-Stability of Additive Functional Equation in IFN-Spaces and 2-Banach Spaces by Different Methods

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This paper introduces a new dimension of an additive functional equation and obtains its general solution. The main goal of this study is to examine the Ulam stability of this equation in IFN-spaces (intuitionistic fuzzy normed spaces) with the help of direct and fixed point approaches and 2-Banach spaces. Also, we use an appropriate counterexample to demonstrate that the stability of this equation fails in a particular case.

## 1. Introduction

The study of stability problems for functional equations is one of the essential research areas in mathematics, which originated in issues related to applied mathematics. The first question concerning the stability of homomorphisms was given by Ulam [1] as follows.

Given a group $(G, *)$, a metric group $(G', \cdot)$ with the metric $d$, and a mapping $f$ from $G$ and $G'$, does $\delta > 0$ exist such that

$$d(f(x * y), f(x) \cdot f(y)) \leq \delta, \tag{1}$$

for all $x, y \in G$. If such a mapping exists, then does a homomorphism $h: G \longrightarrow G'$ exist such that

$$d(f(x), h(x)) \leq \varepsilon, \tag{2}$$

for all $x \in G$? Ulam defined such a problem in 1940 and solved it the following year for the Cauchy functional equation

$$\psi(u + v) = \psi(u) + \psi(v), \tag{3}$$

by the way of Hyers [2]. The consequence of Hyers becomes stretched out by Aoki [3] with the aid of assuming the unbounded Cauchy contrasts. Hyers theorem for additive mapping was investigated by Rassias [4], and then Rassias results were generalized by Gavruta [5].

As of late, Nakmahachalasint [6] gave the overall answer and HUR (briefly, Hyers–Ulam–Rassias) stability of finite variable functional equation; furthermore, Khodaei and Rassias [7] examined the stability of generalized additive functions in several variables. The stability result of additive functional equations was examined by means of Najati and Moghimi [8], Shin et al. [9], and Gordji [10]. Stability problems of various functional equations have been investigated by many researchers, and there are various interesting results about this problem (see [11–14]).

Zadeh [15] established the concept of fuzzy sets, which is a tool for demonstrating weakness and ambiguity in several scientific and technological problems. The possibility of IFN-spaces, from the start, has been presented in [16]. Saadati [17] have examined the modified intuitionistic fuzzy metric spaces and proven some fixed point theorems in these spaces.

The IFN-spaces and IF2N-spaces (briefly, intuitionistic fuzzy 2-normed spaces) have been studied by a number of researchers [18–20]. Furthermore, several researchers have discussed the generalized Ulam–Hyers stability of various functional equations in IFN-spaces (see [21–24]).

In this current work, we present a new kind of additive functional equation:

$$\sum_{1\le a<b<c\le s}\phi\left(-v_a-v_b-v_c+\sum_{d=1,d\neq a\neq b\neq c}^{s}v_d\right)-\left(\frac{s^3-9s^2+20s-12}{6}\right)\sum_{a=1}^{s}\left[\frac{\phi(v_a)-\phi(-v_a)}{2}\right]=0, \tag{4}$$

where $s>4$ is a fixed integer, and obtain its general solution. The main goal of this study is to examine the Ulam–Hyers stability of this equation in IFN-spaces with the help of direct and fixed point approaches and 2-Banach spaces by using the direct approach. Also, we use an appropriate counter-example to demonstrate that the stability of equation (4) fails in a particular case.

## 2. General Solution

**Theorem 1.** *If a mapping $\phi$ between two real vector spaces W and F satisfies functional equation (4), then the function $\phi$ is additive.*

*Proof.* Setting $v_1=\cdots=v_s=0$ in (4), we have $\phi(0)=0$. Replacing $(v_1,v_2,\ldots,v_s)$ by $(v,\underbrace{0,0,\ldots,0}_{(s-1)-\text{times}})$ in (4), we get $\phi(-v)=-\phi(v)$ for all $v\in W$. Hence, $\phi$ is an odd function. Replacing $(v_1,v_2,v_3,\ldots,v_s)$ by $(v,v,\underbrace{0,0,\ldots,0}_{(s-2)-\text{times}})$ in (4), we have

$$\phi(2v)=2\phi(v), \tag{5}$$

for all $v\in W$. Replacing $v$ by $2v$ in (5), we have

$$\phi\left(2^2v\right)=2^2\phi(v), \tag{6}$$

for all $v\in W$. Again, replacing $v$ with $2v$ in (6), we get

$$\phi\left(2^3v\right)=2^3\phi(v), \tag{7}$$

for all $v\in W$. In general, for any non-negative integer $a>0$, we have

$$\phi(2^a v)=2^a\phi(v), \tag{8}$$

for all $v\in W$. Replacing $(v_1,v_2,v_3,\ldots,v_s)$ by $(s,t,\underbrace{0,0,\ldots,0}_{(s-2)-\text{times}})$ in (4), we obtain (3) for all $s,t\in W$. □

*Remark 1.* If a mapping $\phi$ between two real vector spaces $W$ and $F$ satisfies functional equation (3), then the function $\phi$ satisfies additive functional equation (4), for all $v_1,v_2,v_3,\ldots,v_s\in W$.

For our notational handiness, we define a mapping $\phi\colon W\longrightarrow F$ by

$$D\phi(v_1,v_2,\ldots,v_s)=\sum_{1\le a<b<c\le s}\phi\left(-v_a-v_b-v_c+\sum_{d=1,d\neq a\neq b\neq c}^{s}v_d\right)-\left(\frac{s^3-9s^2+20s-12}{6}\right)\sum_{a=1}^{s}\left[\frac{\phi(v_a)-\phi(-v_a)}{2}\right], \tag{9}$$

for all $v_1,v_2,\ldots,v_s\in W$.

## 3. Stability Results in IFN-Spaces

We can recall some basic notions and preliminaries from [25] and using the alternative fixed point theorem which are important results in fixed point theory [26].

*Definition 1* (see [25]). Consider a membership degree $\mu$ and non-membership degree $\nu$ of an intuitionistic fuzzy set from $W\times(0,+\infty)$ to $[0,1]$ such that $\mu_v(t)+\nu_v(t)\le 1$ for all $v\in W$ and $t>0$. The triple $(W,I_{\mu,\nu},\Upsilon)$ is called as an Intuitionistic Fuzzy Normed-space (briefly, IFN-space) if a vector space $W$, a continuous $t$-representable $\Upsilon$ and $I_{\mu,\nu}\colon W\times(0,+\infty)\longrightarrow L^*$ satisfying $v_1,v_2\in W$ and $t,s>0$,

(IFN1) $I_{\mu,\nu}(v_1,0)=0_{L^*}$.

(IFN2) $I_{\mu,\nu}(v_1,t)=1_{L^*}$ if and only if $v_1=0$.

(IFN3) $I_{\mu,\nu}(\alpha v_1,t)=I_{\mu,\nu}(v_1,(t/|\alpha|))$, for all $\alpha\neq 0$.

(IFN4) $I_{\mu,\nu}(v_1+v_2,t+s)\ge_{L^*}\Upsilon(I_{\mu,\nu}(v_1,t),I_{\mu,\nu}(v_2,s))$.

In this case, $I_{\mu,\nu}$ is called an intuitionistic fuzzy norm, where $I_{\mu,\nu}(v_1,t)=(\mu_{v_1}(t),\nu_{v_1}(t))$.

*Definition 2* (see [25]). A sequence $\{v_m\}$ in $W$ is called as a Cauchy sequence if for every $\epsilon>0$ and $t>0$, there exists $m_0$ such that

$$I_{\mu,\nu}\left(v_{m+p}-v_m,t\right)>1-\varepsilon,\quad m\ge m_0, \tag{10}$$

for all $p>0$.

*Remark 2.* In an intuitionistic fuzzy normed space, every convergent sequence is a Cauchy sequence.

If every Cauchy sequence is convergent, then the intuitionistic fuzzy normed space is called as complete.

*Definition 3* (see [25]). A mapping $\phi$ between two IFN-spaces $W$ and $F$ is continuous at $v_0$ if for every $\{v_m\}$ converging to $v_0$ in $W$, the sequence $\phi\{v_m\}$ converges to $\phi\{v_0\}$. If

$\phi$ is continuous at each point $v_0 \in W$, then the mapping $\phi$ is called as a continuous mapping on $W$.

*Example 1. Let* $(W, \|\cdot\|)$ *be a normed space. Let* $T(a,b) = (a, b, \min(a_2 + b_2, 1))$ *for all* $a = (a_1, a_2)$; $b = (b_1, b_2) \in L^*$ *and* $\mu, \nu$ *be membership and non-membership degree of an intuitionistic fuzzy set defined by*

$$I_{\mu,\nu}(v,t) = \left(\mu_v(t), \nu_v(t)\right) = \left(\frac{t}{t+\|v\|}, \frac{\|v\|}{t+\|v\|}\right), \ t \in R^+. \quad (11)$$

Then, $(W, I_{\mu,\nu}, T)$ is an IFN-space.

**Theorem 2** (see [26]). *Let* $(W, d)$ *be a generalized complete metric space and a strictly contractive mapping* $M: W \longrightarrow W$ *with Lipschitz constant* $L < 1$. *Then, for all* $v_1 \in W$, *either*

$$d\left(M^m v_1, M^{m+1} v_1\right) = \infty, \quad m \geq m_0, \quad (12)$$

*or there exists a positive integer* $m_0$ *such that*

*(i)* $d(M^m v_1, M^{m+1} v_1) < \infty$, $m \geq m_0$.

*(ii) The sequence* $\{M^m v_1\}_{m\in\mathbb{N}}$ *converges to a fixed point* $v_1^*$ *of* $M$.

*(iii)* $v_1^*$ *is the unique fixed point of* $M$ *in* $W^* = \{v_2 \in W \mid d(M^{m_0} v_1, v_2) < \infty\}$.

*(iv)* $d(v_2, v_1^*) \leq (1/1-L) d(M v_2, v_2)$, *for all* $v_2 \in W^*$.

*3.1. Stability Results: Direct Technique.* In this section, we assume that $W$, $(Z, I'_{\mu,\nu}, Y)$, and $(F, I_{\mu,\nu}, Y)$ are linear space, IFN-space, and complete IFN-space, respectively.

**Theorem 3.** *If a mapping* $\varphi: W^s \longrightarrow Z$ *with* $0 < (\varsigma/2) < 1$,

$$I'_{\mu,\nu}(\varphi(2v, 2v, 0, \ldots, 0), \varepsilon) \geq_{L^*} I'_{\mu,\nu}(\varsigma\varphi(v, v, 0, \ldots, 0), \varepsilon), \quad (13)$$

$$\lim_{k\longrightarrow\infty} I'_{\mu,\nu}\left(\varphi\left(2^k v_1, 2^k v_2, \ldots, 2^k v_s\right), 2^k \varepsilon\right) = 1_{L^*}, \quad (14)$$

*for all* $v, v_1, v_2, \ldots, v_s \in W$ *and all* $\epsilon > 0$. *If a mapping* $\phi: W \longrightarrow F$ *satisfies*

$$I_{\mu,\nu}(D\phi(v_1, v_2, \ldots, v_s), \varepsilon) \geq_{L^*} I'_{\mu,\nu}(\varphi(v_1, v_2, \ldots, v_s), \varepsilon), \quad (15)$$

*for all* $v_1, v_2, \ldots, v_s \in W$ *and all* $\epsilon > 0$, *then the limit*

$$I_{\mu,\nu}\left(A_1(v) - \frac{\phi\left(2^k v\right)}{2^k}, \epsilon\right) \longrightarrow 1_{L^*} \text{ as } k \longrightarrow \infty, \quad (16)$$

*exists and there exists a unique additive mapping* $A_1: W \longrightarrow F$ *satisfying functional equation (4) and*

$$I_{\mu,\nu}(\phi(v) - A_1(v), \epsilon) \geq_{L^*} I'_{\mu,\nu}\left(\varphi(v, v, 0, \ldots, 0), \left(\frac{s^3 - 9s^2 + 20s - 12}{6}\right)\epsilon(2-\varsigma)\right), \quad (17)$$

*for all* $v \in W$ *and all* $\epsilon > 0$.

*Proof.* Fix $v \in W$ and all $\epsilon > 0$. Replacing $(v_1, v_2, \ldots, v_s)$ by $(v, v, 0, \ldots, 0)$ in (15), we have

$$\begin{aligned} I_{\mu,\nu}&\left(\left(\frac{s^3 - 9s^2 + 20s - 12}{6}\right)\phi(2v) - \left(\frac{2\left(s^3 - 9s^2 + 20s - 12\right)}{6}\right)\phi(v), \varepsilon\right) \\ &\geq_{L^*} I'_{\mu,\nu}(\varphi(v, v, 0, \ldots, 0), \varepsilon). \end{aligned} \quad (18)$$

Replacing $v$ by $2^k v$ in (18) and using (IFN3), we obtain

$$\begin{aligned} I_{\mu,\nu}&\left(\frac{\phi\left(2^{k+1}v\right)}{2} - \phi\left(2^k v\right), \left(\frac{6\epsilon}{2\left(s^3 - 9s^2 + 20s - 12\right)}\right)\right) \\ &\geq_{L^*} I'_{\mu,\nu}\left(\varphi\left(2^k v, 2^k v, 0, \ldots, 0\right), \epsilon\right). \end{aligned} \quad (19)$$

By the inequality (13) and (IFN3) in (19), we have

$$\begin{aligned} I_{\mu,\nu}&\left(\frac{\phi\left(2^{k+1}v\right)}{2} - \phi\left(2^k v\right), \left(\frac{6\epsilon}{2\left(s^3 - 9s^2 + 20s - 12\right)}\right)\right) \\ &\geq_{L^*} I'_{\mu,\nu}\left(\varphi(v, v, 0, \ldots, 0), \frac{\epsilon}{\varsigma^k}\right). \end{aligned} \quad (20)$$

Clearly, we can show from inequality (20) that

$$I_{\mu,\nu}\left(\frac{\phi\left(2^{k+1}\nu\right)}{2^{k+1}}-\frac{\phi\left(2^{k}\nu\right)}{2^{k}},\left(\frac{6\epsilon}{2^{k+1}\left(s^3-9s^2+20s-12\right)}\right)\right)\geq_{L^*} I'_{\mu,\nu}\left(\varphi(\nu,\nu,0,\ldots,0),\frac{\epsilon}{\varsigma^k}\right). \tag{21}$$

Replacing $\epsilon$ by $\varsigma^k\epsilon$ in (21), we get

$$I_{\mu,\nu}\left(\frac{\phi\left(2^{k+1}\nu\right)}{2^{k+1}}-\frac{\phi\left(2^{k}\nu\right)}{2^{k}},\left(\frac{6\varsigma^k\epsilon}{2^{k+1}\left(s^3-9s^2+20s-12\right)}\right)\right)\geq_{L^*} I'_{\mu,\nu}\left(\varphi(\nu,\nu,0,\ldots,0),\epsilon\right). \tag{22}$$

Clearly,

$$\frac{\phi\left(2^{k}\nu\right)}{2^{k}}-\phi(\nu)=\sum_{a=0}^{k-1}\frac{\phi\left(2^{a+1}\nu\right)}{2^{a+1}}-\frac{\phi\left(2^{a}\nu\right)}{2^{a}}. \tag{23}$$

It follows from (22) and (23) that

$$\begin{aligned}
I_{\mu,\nu}&\left(\frac{\phi\left(2^{k}\nu\right)}{2^{k}}-\phi(\nu),\sum_{a=0}^{k-1}\frac{6\varsigma^a\epsilon}{2^{a+1}\left(s^3-9s^2+20s-12\right)}\right)\\
&\geq_{L^*} \Upsilon_{a=0}^{k-1}\left\{I'_{\mu,\nu}\left(\frac{\phi\left(2^{a+1}\nu\right)}{2^{a+1}}-\frac{\phi\left(2^{a}\nu\right)}{2^{a}},\frac{6\varsigma^a\epsilon}{2^{a+1}\left(s^3-9s^2+20s-12\right)}\right)\right\}\\
&\geq_{L^*} \Upsilon_{a=0}^{k-1}\left\{I'_{\mu,\nu}\left(\varphi(\nu,\nu,0,\ldots,0),\epsilon\right)\right\}\\
&\geq_{L^*} I'_{\mu,\nu}\left(\varphi(\nu,\nu,0,\ldots,0),\epsilon\right),
\end{aligned} \tag{24}$$

for all $\nu\in W$ and $\epsilon>0$. Replacing $\nu$ by $2^t\nu$ in (24) and with the help of (13), we have

$$I_{\mu,\nu}\left(\frac{\phi\left(2^{k+t}\nu\right)}{2^{k+t}}-\frac{\phi\left(2^{t}\nu\right)}{2^{t}},\sum_{a=0}^{k-1}\frac{6\varsigma^a\epsilon}{2^{a+t}2\left(s^3-9s^2+20s-12\right)}\right)\geq_{L^*} I'_{\mu,\nu}\left(\varphi(\nu,\nu,0,\ldots,0),\frac{\epsilon}{\varsigma^t}\right), \tag{25}$$

for every $t,k\geq 0$. Replacing $\epsilon$ by $\varsigma^t\epsilon$ in (25), we have

$$I_{\mu,\nu}\left(\frac{\phi\left(2^{k+t}\nu\right)}{2^{k+t}}-\frac{\phi\left(2^{t}\nu\right)}{2^{t}},\sum_{a=t}^{k+t-1}\frac{6\varsigma^a\epsilon}{2^{a+1}\left(s^3-9s^2+20s-12\right)}\right)\geq_{L^*} I'_{\mu,\nu}\left(\varphi(\nu,\nu,0,\ldots,0),\epsilon\right). \tag{26}$$

Using (IFN3) in (26), we obtain

$$I_{\mu,\nu}\left(\frac{\phi\left(2^{k+t}\nu\right)}{2^{k+t}}-\frac{\phi\left(2^{t}\nu\right)}{2^{t}},\epsilon\right)\geq_{L^*} I'_{\mu,\nu}\left(\varphi(\nu,\nu,0,\ldots,0),\frac{\epsilon}{\sum_{a=t}^{k+t-1}\left(6\varsigma^a/2^a2\left(s^3-9s^2+20s-12\right)\right)}\right), \tag{27}$$

for all $t,k\geq 0$. Since $0<\varsigma<2$ and $\sum_{a=0}^{k}(\varsigma/2)^a<\infty$, the Cauchy criterion for convergence in IFNS shows that $\{\phi(2^k\nu)/2^k\}$ is Cauchy sequence in $(F,I_{\mu,\nu},\Upsilon)$. Since $(F,I_{\mu,\nu},\Upsilon)$ is a complete, this sequence converges to some point $A_1(\nu)\in F$. Then, we can define the mapping $A_1\colon W\longrightarrow F$ by

$$I_{\mu,\nu}\left(A_1(\nu)-\frac{\phi\left(2^{k}\nu\right)}{2^{k}}\right)\longrightarrow 1_{L^*}\ \text{as}\ k\longrightarrow\infty. \tag{28}$$

Setting $t=0$ in inequality (29), we obtain

$$I_{\mu,\nu}\left(\frac{\phi(2^k\nu)}{2^k}-\phi(\nu),\epsilon\right)\geq_{L^*}I'_{\mu,\nu}\left(\varphi(\nu,\nu,0,\ldots,0),\frac{\epsilon}{\sum_{a=0}^{k-1}\left(6\varsigma^a/2^a2\left(s^3-9s^2+20s-12\right)\right)}\right). \tag{29}$$

Taking the limit as $k\longrightarrow\infty$ in (29), we obtain

$$I_{\mu,\nu}(\phi(\nu)-A_1(\nu),\epsilon)\geq_{L^*}I'_{\mu,\nu}\left(\varphi(\nu,\nu,0,\ldots,0),\left(\frac{s^3-9s^2+20s-12}{6}\right)\epsilon(2-\varsigma)\right). \tag{30}$$

Next, we want to prove that the function $A_1$ satisfies functional equation (4); replacing $(\nu_1,\nu_2,\ldots,\nu_s)$ by $(2^k\nu_1,2^k\nu_2,\ldots,2^k\nu_s)$ in (15), we have

$$\begin{aligned}&I_{\mu,\nu}\left(\frac{1}{2^k}D\phi\left(2^k\nu_1,\ldots,2^k\nu_s\right),\epsilon\right)\\&\quad\geq_{L^*}I'_{\mu,\nu}\left(\varphi\left(2^k\nu_1,\ldots,2^k\nu_s\right),2^k\epsilon\right),\end{aligned} \tag{31}$$

for all $\nu_1,\nu_2,\ldots,\nu_s\in W$ and all $\epsilon>0$. Since

$$\lim_{k\longrightarrow\infty}I'_{\mu,\nu}\left(\varphi\left(2^k\nu_1,2^k\nu_2,\ldots,2^k\nu_s\right),2^k\varepsilon\right)=1_{L^*}, \tag{32}$$

the function $A_1$ satisfies functional equation (4). Thus, the function $A_1$ is additive. Finally, we want to prove that the function $A_1$ is unique; consider another additive mapping $A_2$: $W\longrightarrow F$ satisfying functional equations (4) and (17). Hence,

$$\begin{aligned}I_{\mu,\nu}(A_1(\nu)-A_2(\nu),\epsilon)&=I_{\mu,\nu}\left(\frac{A_1(2^k\nu)}{2^k}-\frac{A_2(2^k\nu)}{2^k},\epsilon\right)\geq_{L^*}\\&\Upsilon\left\{I_{\mu,\nu}\left(\frac{A_1(2^k\nu)}{2^k}-\frac{\phi(2^k\nu)}{2^k},\frac{\epsilon}{2}\right),I_{\mu,\nu}\left(\frac{\phi(2^k\nu)}{2^k}-\frac{A_2(2^k\nu)}{2^k},\frac{\epsilon}{2}\right)\right\}\\&\geq_{L^*}I'_{\mu,\nu}\left(\varphi\left(2^k\nu,2^k\nu,0,\ldots,0\right),\frac{\left(s^3-9s^2+20s-12\right)2^k\epsilon(2-\varsigma)}{12}\right)\\&\geq_{L^*}I'_{\mu,\nu}\left(\varphi(\nu,\nu,0,\ldots,0),\frac{\left(s^3-9s^2+20s-12\right)2^k\epsilon(2-\varsigma)}{12\varsigma^k}\right),\end{aligned} \tag{33}$$

for all $\nu\in W$ and all $\epsilon>0$. As

$$\lim_{s\longrightarrow\infty}\frac{\left(s^3-9s^2+20s-12\right)2^k\epsilon(2-\varsigma)}{12\varsigma^k}=\infty, \tag{34}$$

we obtain

$$\lim_{k\longrightarrow\infty}I'_{\mu,\nu}\left(\varphi(\nu,\nu,0,\ldots,0),\frac{\left(s^3-9s^2+20s-12\right)2^k\epsilon(2-\varsigma)}{12\varsigma^k}\right)=1_{L^*}. \tag{35}$$

Thus, $I_{\mu,\nu}(A_1(\nu)-A_2(\nu),\epsilon)=1_{L^*}$.

Therefore, $A_1(\nu)=A_2(\nu)$. Thus, the additive function $A_1(\nu)$ is unique. This ends the proof. □

**Theorem 4.** *If a mapping $\varphi$: $W^s\longrightarrow Z$ with $0<(2/\varsigma)<1$,*

$$I'_{\mu,\nu}\left(\varphi\left(2^{-1}\nu,2^{-1}\nu,0,\ldots,0\right),\epsilon\right)\geq_{L^*}I'_{\mu,\nu}\left(\frac{1}{\varsigma}\varphi(\nu,\nu,0,\ldots,0),\epsilon\right), \tag{36}$$

$$\lim_{k\longrightarrow\infty}I'_{\mu,\nu}\left(\varphi\left(2^{-k}\nu_1,2^{-k}\nu_2,\ldots,2^{-k}\nu_s\right),2^{-k}\epsilon\right)=1_{L^*}, \tag{37}$$

*for all* $v, v_1, v_2, \ldots, v_s \in W$ *and all* $\epsilon > 0$. *If a mapping* $\phi: E \longrightarrow F$ *satisfies (15), then the limit* $I_{\mu,\nu}(A_1(v) - 2^k\phi(v/2^k), \epsilon) \longrightarrow 1_{L^*}$ *as* $k \longrightarrow \infty$ *exists and there exists a unique additive mapping* $A_1: W \longrightarrow F$ *satisfying functional equation (4) and*

$$I_{\mu,\nu}(\phi(v) - A_1(v), \epsilon) \geq_{L^*} I'_{\mu,\nu}\left(\varphi(v, v, 0, \ldots, 0), \left(\frac{s^3 - 9s^2 + 20s - 12}{6}\right)\epsilon(\varsigma - 2)\right), \quad (38)$$

*for all* $v \in W$ *and all* $\epsilon > 0$.

*Proof.* Fix $v \in W$ and all $\epsilon > 0$. Replacing $(v_1, v_2, \ldots, v_s)$ by $(v, v, 0, \ldots, 0)$ in (15), we have

$$I_{\mu,\nu}\left(\left(\frac{s^3 - 9s^2 + 20s - 12}{6}\right)\phi(2v) - \left(\frac{2(s^3 - 9s^2 + 20s - 12)}{6}\right)\phi(v), \epsilon\right) \geq_{L^*} I'_{\mu,\nu}(\varphi(v, v, 0, \ldots, 0), \epsilon). \quad (39)$$

From (39), we obtain that

$$I_{\mu,\nu}\left(\phi(2v) - 2\phi(v), \frac{6\epsilon}{2(s^3 - 9s^2 + 20s - 12)}\right) \geq_{L^*} I'_{\mu,\nu}(\varphi(v, v, 0, \ldots, 0), \epsilon). \quad (40)$$

Replacing $v$ by $v/2$ in (40), we get

$$I_{\mu,\nu}\left(\phi(v) - 2\phi\left(\frac{v}{2}\right), \frac{6\epsilon}{2(s^3 - 9s^2 + 20s - 12)}\right) \geq_{L^*} I'_{\mu,\nu}\left(\varphi\left(\frac{v}{2}, \frac{v}{2}, 0, \ldots, 0\right), \epsilon\right). \quad (41)$$

Replacing $v$ by $v/2^k$ in (41) and using (IFN3), we have

$$I_{\mu,\nu}\left(\phi\left(\frac{v}{2^k}\right) - 2\phi\left(\frac{v}{2^{k+1}}\right), \frac{6\epsilon}{2(s^3 - 9s^2 + 20s - 12)}\right) \geq_{L^*} I'_{\mu,\nu}\left(\varphi\left(\frac{v}{2^{k+1}}, \frac{v}{2^{k+1}}, 0, \ldots, 0\right)\right). \quad (42)$$

With the help of inequality (36) and (IFN3) in (42), we obtain that

$$I_{\mu,\nu}\left(\phi\left(\frac{v}{2^k}\right) - 2\phi\left(\frac{v}{2^{k+1}}\right), \frac{6\epsilon}{2(s^3 - 9s^2 + 20s - 12)}\right) \geq_{L^*} I'_{\mu,\nu}\left(\varphi(v, v, 0, \ldots, 0), \epsilon\varsigma^{k+1}\right). \quad (43)$$

The remaining part of the proof can be proven in the same way as Theorem 3. □

**Corollary 1.** *Let* $\theta \in \mathbb{R}^+$. *If a mapping* $\phi: W \longrightarrow F$ *such that*

$$I_{\mu,\nu}(D\phi(v_1, v_2, \ldots, v_s), \epsilon) \geq_{L^*} I'_{\mu,\nu}(\theta, \epsilon), \quad (44)$$

*for all* $v_1, v_2, \ldots, v_s \in W$ *and all* $\epsilon > 0$, *then there exists a unique additive mapping* $A_1: W \longrightarrow F$ *satisfying*

$$I_{\mu,\nu}(\phi(v) - A_1(v), \epsilon) \geq_{L^*} I'_{\mu,\nu}\left(6\theta, |2 - 1|\epsilon(s^3 - 9s^2 + 20s - 12)\right), \quad (45)$$

*for all* $v \in W$ *and all* $\epsilon > 0$.

*Proof.* The proof holds from Theorems 3 and 4 by letting $\varphi(v_1, v_2, \ldots, v_s) = \theta$ and $\varsigma = 2^0$. □

**Corollary 2.** *Let* $\theta, \xi \in \mathbb{R}^+$ *with* $\xi \in (0, 1) \cup (1, +\infty)$. *If a mapping* $\phi: W \longrightarrow F$ *such that*

$$I_{\mu,\nu}(D\phi(v_1, v_2, \ldots, v_s), \epsilon) \geq_{L^*} I'_{\mu,\nu}\left(\theta \sum_{a=1}^{s} \|v_a\|^{\xi}, \epsilon\right), \quad (46)$$

*for all* $v_1, v_2, \ldots, v_s \in W$ *and all* $\epsilon > 0$, *then there exists a unique additive mapping* $A_1: W \longrightarrow F$ *satisfying*

$$I_{\mu,\nu}(\phi(v) - A_1(v), \epsilon) \geq_{L^*} I'_{\mu,\nu}\left(12\theta\|v\|^{\xi}, \left|2 - 2^{\xi}\right|(s^3 - 9s^2 + 20s - 12)\epsilon\right), \quad (47)$$

*for all* $v \in W$ *and all* $\epsilon > 0$.

*Proof.* The proof holds from Theorems 3 and 4 by setting $\varphi(v_1, v_2, \ldots, v_s) = \theta \sum_{a=1}^{s} \|v_a\|^{\xi}$ and $\varsigma = 2^{\xi}$. □

**Corollary 3.** *Let* $\theta, \xi, \gamma, \tau \in \mathbb{R}^+$ *with* $s\xi, s\tau \in (0, 1) \cup (1, +\infty)$. *If a mapping* $\phi: W \longrightarrow F$ *such that*

$$I_{\mu,\nu}(D\phi(v_1, v_2, \ldots, v_s), \epsilon) \geq_{L^*} I'_{\mu,\nu}\left(\theta \sum_{a=1}^{s} \|v_a\|^{s\xi} + \gamma \prod_{a=1}^{s} \|v_a\|^{\tau}, \epsilon\right), \quad (48)$$

*for all* $v_1, v_2, \ldots, v_s \in W$ *and all* $\epsilon > 0$, *then there exists a unique additive mapping* $A_1: W \longrightarrow F$ *satisfying*

$$I_{\mu,\nu}(\phi(v)-A_1(v),\epsilon) \geq_{L^*} I'_{\mu,\nu}\left(12\theta\|v\|^{s\xi}, \left|2-2^{s\xi}\right|\left(s^3-9s^2+20s-12\right)\epsilon\right), \quad (49)$$

*for all* $v \in W$ *and all* $\epsilon > 0$.

*Proof.* The proof holds from Theorems 3 and 4 by setting $\varphi(v_1, v_2, \ldots, v_s) = \theta\sum_{a=1}^{s}\|v_a\|^{s\xi} + \gamma\prod_{a=1}^{s}\|v_a\|^{\tau}$ and $\varsigma = 2^{s\xi}$. □

**Corollary 4.** *Let* $\gamma, \tau \in \mathbb{R}^+$ *with* $0 < s\tau \neq 1$. *If a mapping* $\phi: W \longrightarrow F$ *such that*

$$I_{\mu,\nu}(D\phi(v_1, v_2, \ldots, v_s), \epsilon) \geq_{L^*} I'_{\mu,\nu}\left(\gamma\prod_{a=1}^{s}\|v_a\|^{\tau}, \epsilon\right), \quad (50)$$

*for all* $v_1, v_2, \ldots, v_s \in W$ *and all* $\epsilon > 0$, *then the mapping* $\phi$ *is additive.*

*Proof.* The proof is valid from Theorems 3 and 4 by setting $\varphi(v_1, v_2, \ldots, v_s) = \gamma\prod_{a=1}^{s}\|v_a\|^{\tau}$. □

*3.2. Stability Results: Fixed Point Technique.* Before we begin, let us consider a constant $\beta_a$ such that

$$\beta_a = \begin{cases} 2, & \text{if } a = 0, \\ \dfrac{1}{2}, & \text{if } a = 1, \end{cases} \quad (51)$$

and $\Psi$ is the set such that $\Psi = \{n_1 | n_1: W \longrightarrow F, n_1(0) = 0\}$.

**Theorem 5.** *Consider a mapping* $\phi: W \longrightarrow F$ *for which there is a mapping* $\varphi: W^s \longrightarrow Z$ *with*

$$\lim_{l\longrightarrow\infty} I'_{\mu,\nu}\left(\varphi\left(2^l v_1, 2^l v_2, \ldots, 2^l v_s\right), 2^l\epsilon\right) = 1_{L^*}, \quad (52)$$

*satisfying functional inequality (15). If there is* $L = L(a)$ *such that* $v \longrightarrow \eta(v) = 6/(s^3 - 9s^2 + 20s - 12)\varphi((v/2), (v/2), 0, \ldots, 0)$ *has the property*

$$I'_{\mu,\nu}\left(L\frac{1}{\beta_a}\eta(\beta_a v), \epsilon\right) = I'_{\mu,\nu}(\eta(v), \epsilon), \quad (53)$$

*then there exists a unique additive mapping* $A_1: W \longrightarrow F$ *satisfying functional equation (4) and*

$$I_{\mu,\nu}(\phi(v) - A_1(v), \epsilon) n \geq_{L^*} I'_{\mu,\nu}\left(\frac{L^{1-a}}{1-L}\eta(v), \epsilon\right), \quad (54)$$

*for all* $v \in W$ *and all* $\epsilon > 0$.

*Proof.* Let $\varsigma$ be a general metric on $\Psi$:

$$\varsigma(n_1, n_2) = \inf\left\{t \in (0, \infty) | I_{\mu,\nu}(n_1(v) - n_2(v), \epsilon) \geq_{L^*} I'_{\mu,\nu}(t\eta(v), \epsilon), v \in W, \epsilon > 0\right\}. \quad (55)$$

Clearly, $(\Psi, \varsigma)$ is complete. Define a mapping $\Upsilon: \Psi \longrightarrow \Psi$ by $\Upsilon n_1(v) = (1/\beta_a)n_1(\beta_a v)$, for all $v \in W$. For $n_1, n_2 \in \Psi$, we have

$$\begin{aligned} &\varsigma(n_1, n_2) \leq t, \\ &\Rightarrow I_{\mu,\nu}(n_1(v) - n_2(v), \varepsilon) \geq_{L^*} I'_{\mu,\nu}(t\eta(v), \varepsilon) \\ &\Rightarrow I_{\mu,\nu}\left(\frac{n_1(\beta_a v)}{\beta_a} - \frac{n_2(\beta_a v)}{\beta_a}, \varepsilon\right) \geq_{L^*} I'_{\mu,\nu}\left(\frac{t\eta(\beta_a v)}{\beta_a}, \varepsilon\right) \\ &\Rightarrow I_{\mu,\nu}(\Upsilon n_1(v) - \Upsilon n_2(v), \varepsilon) \geq_{L^*} I_{\mu,\nu}(tL\eta(v), \varepsilon) \\ &\Rightarrow \varsigma(\Upsilon n_1(v), \Upsilon n_2(v)) \leq tL \\ &\Rightarrow \varsigma(\Upsilon n_1, \Upsilon n_2) \leq L\varsigma(n_1, n_2). \end{aligned} \quad (56)$$

Thus, the function $\Upsilon$ is strictly contractive on $\Psi$ with $L$ (Lipschitz constant). Replacing $(v_1, v_2, \ldots, v_s)$ by $(v, v, 0, \ldots, 0)$ in (15), we have

$$I_{\mu,\nu}\left(\left(\frac{s^3-9s^2+20s-12}{6}\right)\phi(2\nu)-\left(\frac{2\left(s^3-9s^2+20s-12\right)}{6}\right)\phi(\nu),\epsilon\right)\geq_{L^*}I'_{\mu,\nu}(\varphi(\nu,\nu,0,\ldots,0),\epsilon). \tag{57}$$

Using (IFN3) in (57), we have

$$I_{\mu,\nu}\left(\frac{\phi(2\nu)}{2}-\phi(\nu),\epsilon\right)\geq_{L^*}I'_{\mu,\nu}\left(\left(\frac{6}{2\left(s^3-9s^2+20s-12\right)}\right)\varphi(\nu,\nu,0,\ldots,0),\epsilon\right). \tag{58}$$

Using equation (53) for the case $a=0$, we have

$$I_{\mu,\nu}\left(\frac{\phi(2\nu)}{2}-\phi(\nu),\epsilon\right)\geq_{L^*}I'_{\mu,\nu}(L\eta(\nu),\epsilon)$$
$$\Rightarrow\varsigma(\Upsilon\phi,\phi)\leq L=L^1=L^{1-a}. \tag{59}$$

Replacing $\nu$ by $(\nu/2)$ in (57), we have

$$I_{\mu,\nu}\left(\phi(\nu)-2\phi\left(\frac{\nu}{2}\right),\epsilon\right)\geq_{L^*}I'_{\mu,\nu}\left(\left(\frac{6}{\left(s^3-9s^2+20s-12\right)}\right)\varphi\left(\frac{\nu}{2},\frac{\nu}{2},0,\ldots,0\right),\epsilon\right), \tag{60}$$

for all $\nu\in W$ and all $\epsilon>0$; using (53) for the case $a=1$, we have

$$I_{\mu,\nu}\left(\phi(\nu)-2\phi\left(\frac{\nu}{2}\right),\epsilon\right)\geq_{L^*}I'_{\mu,\nu}(\eta(\nu),\epsilon)$$
$$\Rightarrow\varsigma(\phi,\Upsilon\phi)\leq 1=L^0=L^{1-a}. \tag{61}$$

We can conclude from equations (59) and (61) that

$$\varsigma(\phi,\Upsilon\phi)\leq L^{1-a}<\infty. \tag{62}$$

By the fixed point alternative in both cases, there is a fixed point $A_1$ of $\Upsilon$ in $\Psi$ such that

$$\lim_{k\longrightarrow\infty}I_{\mu,\nu}\left(\frac{\phi\left(\beta_a^k\nu\right)}{\beta_a^k}-A_1(\nu),\epsilon\right)\longrightarrow 1_{L^*},\quad \nu\in W,\ \epsilon>0. \tag{63}$$

Replacing $(\nu_1,\nu_2,\ldots,\nu_s)$ by $(\beta_a\nu_1,\beta_a\nu_2,\ldots,\beta_a\nu_s)$ in (15), we obtain

$$I_{\mu,\nu}\left(\frac{1}{\beta_a}D\phi(\beta_a\nu_1,\beta_a\nu_2,\ldots,\beta_a\nu_s),\epsilon\right)\geq_{L^*}I'_{\mu,\nu}(\varphi(\beta_a\nu_1,\beta_a\nu_2,\ldots,\beta_a\nu_s),\beta_a\epsilon), \tag{64}$$

for all $\nu_1,\nu_2,\ldots,\nu_s\in W$ and all $\epsilon>0$. By same manner of Theorem 3, we can show that the function $A_1$ satisfies functional equation (4). By Theorem 2, as $A_1$ is a unique fixed point of $\Upsilon$ in $\Delta=\{\phi\in\Psi|\varsigma(\phi,A_1)<\infty\}$, the function $A_1$ is unique such that

$$I_{\mu,\nu}(A_1(\nu)-\phi(\nu),\varepsilon)\geq_{L^*}I'_{\mu,\nu}(t\eta(\nu),\varepsilon),\quad t>0. \tag{65}$$

Using fixed point alternative, we reach

$$\varsigma(\phi,A_1)\leq\frac{1}{1-L}\varsigma(\phi,\Upsilon\phi)$$
$$\Rightarrow\varsigma(\phi,A_1)\leq\frac{L^{1-a}}{1-L}$$
$$\Rightarrow I_{\mu,\nu}(\phi(\nu)-A_1(\nu),\epsilon)\geq_{L^*}I'_{\mu,\nu}\left(\frac{L^{1-a}}{1-L}\eta(\nu),\epsilon\right), \tag{66}$$

for all $\nu\in W$ and all $\epsilon>0$. Hence, the proof of the theorem is now completed. □

**Corollary 5.** *Let $\theta,\xi\in\mathbb{R}_+$ with $\theta>0$. If a mapping $\phi: W\longrightarrow F$ such that*

$$I_{\mu,\nu}(D\phi(\nu_1,\nu_2,\ldots,\nu_s),\epsilon)\geq_{L^*}\begin{cases}I'_{\mu,\nu}(\theta,\epsilon),\\ I'_{\mu,\nu}\left(\theta\sum_{j=1}^{s}\left\|\nu_j\right\|^{\xi},\epsilon\right),\\ I'_{\mu,\nu}\left(\theta\left(\prod_{j=1}^{s}\left\|\nu_j\right\|^{\xi}+\sum_{j=1}^{s}\left\|\nu_j\right\|^{s\xi}\right),\epsilon\right),\end{cases} \tag{67}$$

*for all $\nu_1,\nu_2,\ldots,\nu_s\in W$ and $\epsilon>0$, then there exists a unique additive mapping $A_1: W\longrightarrow F$ satisfying*

$$I_{\mu,\nu}(\phi(\nu)-A_1(\nu),\epsilon)\geq_{L^*}\begin{cases}I'_{\mu,\nu}\left(|6|\theta,\left(s^3-9s^2+20s-12\right)\epsilon\right),\\ I'_{\mu,\nu}\left(12\theta\|\nu\|^s,\left(s^3-9s^2+20s-12\right)\left|2-2^{\xi}\right|\epsilon\right),\xi<1\text{ or }\xi>1,\\ I'_{\mu,\nu}\left(12\theta\|\nu\|^s,\left(s^3-9s^2+20s-12\right)\left|2-2^{s\xi}\right|\epsilon\right),\xi<\dfrac{1}{s}\text{ or }\xi>\dfrac{1}{s},\end{cases}\tag{68}$$

*for all* $\nu\in W$ *and all* $\epsilon>0$.

*Proof.* Set

$$\varphi(\nu_1,\nu_2,\ldots,\nu_s)=\begin{cases}\theta,\\ \theta\sum_{j=1}^{s}\left\|\nu_j\right\|^{\xi},\\ \theta\left(\prod_{j=1}^{s}\left\|\nu_j\right\|^{\xi}+\sum_{j=1}^{s}\left\|\nu_j\right\|^{s\xi}\right).\end{cases}\tag{69}$$

Then,

$$I'_{\mu,\nu}\left(\varphi\left(\beta_a^l\nu_1,\beta_a^l\nu_2,\ldots,\beta_a^l\nu_s\right),\beta_a^l\epsilon\right)=\begin{cases}I'_{\mu,\nu}\left(\theta,\left(\beta_a\right)^l\epsilon\right),\\ I'_{\mu,\nu}\left(\theta\sum_{j=1}^{s}\left\|\nu_j\right\|^{\xi},\left(\beta_a^{1-\xi}\right)^l\epsilon\right),\\ I'_{\mu,\nu}\left(\theta\left(\prod_{j=1}^{s}\left\|\nu_j\right\|^{\xi}+\sum_{j=1}^{s}\left\|\nu_j\right\|^{s\xi}\right),\left(\beta_a^{1-s\xi}\right)^l\epsilon\right),\end{cases}$$
$$=\begin{cases}\longrightarrow 1_{L^*}\text{ as }l\longrightarrow\infty,\\ \longrightarrow 1_{L^*}\text{ as }l\longrightarrow\infty,\\ \longrightarrow 1_{L^*}\text{ as }l\longrightarrow\infty.\end{cases}\tag{70}$$

Thus, (52) holds. But we have that

$$\eta(\nu)=\frac{6}{\left(s^3-9s^2+20s-12\right)}\varphi\left(\frac{\nu}{2},\frac{\nu}{2},0,\ldots,0\right),\tag{71}$$

has the property

$$I'_{\mu,\nu}\left(L\frac{1}{\beta_a}\eta(\beta_a\nu),\epsilon\right)\geq_{L^*}I'_{\mu,\nu}(\eta(\nu),\epsilon),\quad \nu\in W,\ \epsilon>0.\tag{72}$$

Hence,

$$I'_{\mu,\nu}(\eta(\nu),\epsilon)=I'_{\mu,\nu}\left(\frac{6}{\left(s^3-9s^2+20s-12\right)}\varphi\left(\frac{\nu}{2},\frac{\nu}{2},0,\ldots,0\right),\epsilon\right)$$
$$=\begin{cases}I'_{\mu,\nu}\left(6\theta,\left(s^3-9s^2+20s-12\right)\epsilon\right),\\ I'_{\mu,\nu}\left(\dfrac{12\theta}{2^{\xi}}\|\nu\|^{\xi},\left(s^3-9s^2+20s-12\right)\epsilon\right),\\ I'_{\mu,\nu}\left(\dfrac{12\theta}{2^{s\xi}}\|\nu\|^{s\xi},\left(s^3-9s^2+20s-12\right)\epsilon\right).\end{cases}\tag{73}$$

Now,

$$I'_{\mu,\nu}\left(\frac{1}{\beta_a}\eta(\beta_a\nu),\epsilon\right)=\begin{cases} I'_{\mu,\nu}\left(\frac{6\theta}{\beta_a},\left(s^3-9s^2+20s-12\right)\epsilon\right), \\ I'_{\mu,\nu}\left(\frac{12\theta}{2^{\xi}\beta_a}\|\beta_a\nu\|^{\xi},\left(s^3-9s^2+20s-12\right)\epsilon\right), \\ I'_{\mu,\nu}\left(\frac{12\theta}{2^{s\xi}\beta_a}\|\beta_a\nu\|^{s\xi},\left(s^3-9s^2+20s-12\right)\epsilon\right), \end{cases}$$
$$=\begin{cases} I'_{\mu,\nu}\left(\beta_a^{-1}\eta(\nu),\epsilon\right), \\ I'_{\mu,\nu}\left(\beta_a^{\xi-1}\eta(\nu),\epsilon\right), \\ I'_{\mu,\nu}\left(\beta_a^{s\xi-1}\eta(\nu),\epsilon\right). \end{cases} \tag{74}$$

From inequality (53), we can verify the following cases for conditions of $\beta_a$. □

*Case 1.* $L=2^{-1}$ if $a=0$.

$$I_{\mu,\nu}\left(\phi(\nu)-A_1(\nu),\epsilon\right)\geq_{L^*}I'_{\mu,\nu}\left(\frac{2^{-1}}{1-2^{-1}}\eta(\nu),\epsilon\right) = I'_{\mu,\nu}\left(6\theta,\left(s^3-9s^2+20s-12\right)\epsilon\right). \tag{75}$$

*Case 2.* $L=2$ if $a=1$.

$$I_{\mu,\nu}\left(\phi(\nu)-A_1(\nu),\epsilon\right)\geq_{L^*}I'_{\mu,\nu}\left(\frac{1}{1-2}\eta(\nu),\epsilon\right) = I'_{\mu,\nu}\left(-6\theta,\left(s^3-9s^2+20s-12\right)\epsilon\right). \tag{76}$$

*Case 3.* $L=2^{\xi-1}$ for $\xi<1$ if $a=0$.

$$I_{\mu,\nu}\left(\phi(\nu)-A_1(\nu),\epsilon\right)\geq_{L^*}I'_{\mu,\nu}\left(\frac{2^{\xi-1}}{1-2^{\xi-1}}\eta(\nu),\epsilon\right) = I'_{\mu,\nu}\left(12\theta\|\nu\|^{\xi},\left(s^3-9s^2+20s-12\right)\cdot\left(2-2^{\xi}\right)\epsilon\right). \tag{77}$$

*Case 4.* $L=2^{1-\xi}$ for $\xi>1$ if $a=1$.

$$I_{\mu,\nu}\left(\phi(\nu)-A_1(\nu),\epsilon\right)\geq_{L^*}I'_{\mu,\nu}\left(\frac{1}{1-2^{1-\xi}}\eta(\nu),\epsilon\right) = I'_{\mu,\nu}\left(12\theta\|\nu\|^{\xi},\left(s^3-9s^2+20s-12\right)\cdot\left(2^{\xi}-2\right)\epsilon\right). \tag{78}$$

*Case 5.* $L=2^{s\xi-1}$ for $\xi<(1/s)$ if $a=0$.

$$I_{\mu,\nu}\left(\phi(\nu)-A_1(\nu),\epsilon\right)\geq_{L^*}I'_{\mu,\nu}\left(\frac{2^{s\xi-1}}{1-2^{s\xi-1}}\eta(\nu),\epsilon\right) = I'_{\mu,\nu}\left(12\theta\|\nu\|^{s\xi},\left(s^3-9s^2+20s-12\right)\cdot\left(2-2^{s\xi}\right)\epsilon\right). \tag{79}$$

*Case 6.* $L=2^{1-s\xi}$ for $\xi<(1/s)$ if $a=1$.

$$I_{\mu,\nu}\left(\phi(\nu)-A_1(\nu),\epsilon\right)\geq_{L^*}I'_{\mu,\nu}\left(\frac{1}{1-2^{1-s\xi}}\eta(\nu),\epsilon\right) = I'_{\mu,\nu}\left(12\theta\|\nu\|^{s\xi},\left(s^3-9s^2+20s-12\right)\cdot\left(2^{s\xi}-2\right)\epsilon\right). \tag{80}$$

## 4. Stability Results in 2-Banach Spaces

In 1960, Gahler [27, 28] developed the concept of linear 2-normed spaces.

*Definition 4.* Consider a linear space $W$ over $\mathbb{R}$ with dimension $W>1$ and consider a mapping $\|\cdot,\cdot\|\colon W^2 \longrightarrow \mathbb{R}$ with the following conditions:

(a) $\|r,s\| = 0$ if and only if $r$ and $s$ are linearly dependent.

(b) $\|r,s\| = \|s,r\|$,

(c) $\|\lambda r,s\| = |\lambda|\|r,s\|$,

(d) $\|r,s+w\| \leq \|r,s\| + \|r,w\|$,for all $r,s,w \in W$ and $\lambda \in \mathbb{R}$.

Then, the function $\|\cdot,\cdot\|$ is called as a 2-norm on $W$ and the pair $(W,\|\cdot,\cdot\|)$ is called as a linear 2-normed space. A typical example of 2-normed space is $\mathbb{R}^2$ with 2-norm defined as $|r,s|$ = the area of the triangle with the vertices 0, $r$, and $s$ is a typical example of a 2-normed space.

As a result of (d), it follows that

$$\|r+s,w\| \leq \|r,w\| + \|s,w\| \text{ and } |\,\|r,w\| - \|s,w\|\,| \leq \|r-s,w\|. \tag{81}$$

Thus, $r \longrightarrow \|r,s\|$ are continuous mappings of $W$ into $\mathbb{R}$ for any fixed $s \in W$.

*Definition 5.* A sequence $\{r_j\}$ in a linear 2-normed space $W$ is known as a Cauchy sequence if there exist two points $s,w \in W$ such that $s$ and $w$ are linearly independent.

$$\begin{aligned} \lim_{i,j\longrightarrow\infty}\left\|r_i - r_j, s\right\| &= 0,\\ \lim_{i,j\longrightarrow\infty}\left\|r_i - r_j, w\right\| &= 0. \end{aligned} \tag{82}$$

*Definition 6.* A sequence $\{r_j\}$ in a linear 2-normed space $W$ is called as a convergent sequence if there exists an element $r \in W$ such that

$$\lim_{i,j\longrightarrow\infty}\left\|r_j - r, s\right\| = 0, \tag{83}$$

for all $s \in W$. If $\{r_j\}$ converges to $r$, then we denote $r_j \longrightarrow r$ as $j \longrightarrow \infty$ and say that $r$ is the limit point of $\{r_j\}$. We also write in this instance

$$\lim_{j\longrightarrow\infty} r_j = r. \tag{84}$$

*Definition 7.* A 2-Banach space is a linear 2-normed space in which every Cauchy sequence is convergent.

**Lemma 1** (see [29]). *Let $(W,\|\cdot,\cdot\|)$ be a linear 2-normed space. If $r \in W$ and $\|r,s\| = 0$ for every $s \in W$, then $r = 0$.*

**Lemma 2** (see [29]). *For a convergent sequence $\{r_j\}$ in a linear 2-normed space $W$,*

$$\lim_{j\longrightarrow\infty}\left\|r_j, w\right\| = \left\|\lim_{j\longrightarrow\infty} r_j, s\right\|, \tag{85}$$

*for every $s \in W$.*

Park studied approximate additive mappings, approximate Jensen mappings, and approximate quadratic mappings in 2-Banach spaces in his paper [29]. In [30], Park examined the superstability of the Cauchy functional inequality and the Cauchy–Jensen functional inequality in 2-Banach spaces under certain conditions.

In this section, we let $W$ be a normed linear space and $F$ be a 2-Banach space.

**Theorem 6.** *Let $\varphi\colon W \longrightarrow [0,+\infty)$ be a function such that*

$$\lim_{i\longrightarrow\infty}\frac{1}{2^i}\varphi\left(2^i v_1, 2^i v_2, \ldots, 2^i v_s, w\right) = 0, \tag{86}$$

*for all $v_1,v_2,\ldots,v_s,w \in W$. If a mapping $\phi\colon W \longrightarrow F$ such that $\phi(0) = 0$ and*

$$\left\|D\phi\left(v_1,v_2,\ldots,v_s\right), w\right\| \leq \varphi\left(v_1,v_2,\ldots,v_s,w\right), \tag{87}$$

$$\widehat{\varphi}(v,w) =: \sum_{j=0}^{\infty}\frac{1}{2^j}\varphi\left(2^j v, 2^j v, 0, \ldots, 0, w\right) < \infty, \tag{88}$$

*for all $v_1,v_2,\ldots,v_s,w \in W$. Then, there exists a unique additive mapping $A_1\colon W \longrightarrow F$ satisfying*

$$\left\|\phi(v) - A_1(v), w\right\| \leq \frac{6}{2\left(s^3 - 9s^2 + 20s - 12\right)}\widehat{\varphi}(v,w), \tag{89}$$

*for all $v,w \in W$.*

*Proof.* Replacing $(v_1,v_2,\ldots,v_s)$ by $(v,v,0,\ldots,0)$ in (87), we get

$$\begin{aligned} &\left\|\frac{\left(s^3 - 9s^2 + 20s - 12\right)}{6}\phi(2v) - \frac{2\left(s^3 - 9s^2 + 20s - 12\right)}{6}\phi(v), w\right\| \\ &\quad \leq \varphi(v,v,0,\ldots,0,w), \end{aligned} \tag{90}$$

for all $v,w \in W$. Replacing $v$ by $2^n v$ in (90) and dividing both sides by $2^{n-1}$, we have

$$\begin{aligned} &\left\|\frac{1}{2^{(n+1)}}\phi\left(2^{n+1}v\right) - \frac{1}{2^n}\phi\left(2^n v\right), w\right\| \\ &\quad \leq \frac{6}{2^{n+1}\left(s^3 - 9s^2 + 20s - 12\right)}\varphi\left(2^i v, 2^i v, 0, \ldots, 0, w\right), \end{aligned} \tag{91}$$

for all $v,w \in W$ and all non-negative integers $i$. Hence,

$$\begin{aligned} &\left\|\frac{1}{2^{n+1}}\phi\left(2^{n+1}v\right) - \frac{1}{2^m}\phi\left(2^m v\right), w\right\| \\ &\quad \leq \sum_{j=m}^{i}\left\|\frac{1}{2^{j+1}}\phi\left(2^{j-1}v\right) - \frac{1}{2^j}\phi\left(2^j v\right), w\right\| \\ &\quad \leq \frac{6}{2\left(s^3 - 9s^2 + 20s - 12\right)}\sum_{j=m}^{i}\frac{1}{2^j}\varphi\left(2^j v, 2^j v, 0, \ldots, 0, w\right), \end{aligned} \tag{92}$$

for all $v, w \in W$ and all non-negative integers $m$ and $i$ with $i \geq m$. Therefore, it follows from (15) and (19) that the sequence $\{(1/2^i)\phi(2^i v)\}$ is Cauchy in $F$ for every $v \in W$. Since $F$ is complete, the sequence $\{(1/2^i)\phi(2^i v)\}$ converges in $F$ for all $v \in W$. Thus, we may define a mapping $A_1: W \longrightarrow F$ by

$$A_1(v): \lim_{i\longrightarrow\infty} \frac{1}{2^i}\phi\left(2^i v\right), \tag{93}$$

for all $v \in W$. Therefore,

$$\lim_{i\longrightarrow\infty}\left\|\frac{1}{2^i}\phi\left(2^i v\right) - A_1(v), w\right\| = 0, \tag{94}$$

for all $v, w \in W$. Letting $m = 0$ and taking the limit as $i \longrightarrow \infty$ in (94), we have (89). Next, we want to prove that the function $A_1$ is additive. From inequalities (86), (87), and (94) and Lemma 2,

$$\begin{aligned}\left\|D\phi(v_1, v_2, \ldots, v_s), w\right\| &= \lim_{i\longrightarrow\infty}\left\|D\phi\left(2^i v_1, 2^i v_2, \ldots, 2^i v_s\right), w\right\| \\ &\leq \lim_{i\longrightarrow\infty}\frac{1}{2^i}\varphi\left(2^i v_1, 2^i v_2, \ldots, 2^i v_s, w\right) = 0,\end{aligned} \tag{95}$$

for all $v_1, v_2, \ldots, v_s, w \in W$. By Lemma 1,

$$DA_1(v_1, v_2, \ldots, v_s) = 0, \tag{96}$$

for all $v_1, v_2, \ldots, v_s \in W$. Hence, according to Theorem 1, the mapping $A_1: W \longrightarrow F$ is additive.

To prove that the function $A_1$ is unique, we consider another additive mapping $A_1': W \longrightarrow F$ satisfying (89). Then,

$$\begin{aligned}\left\|A_1(v) - A_1'(v), w\right\| &= \lim_{i\longrightarrow\infty}\frac{1}{2^i}\left\|A_1\left(2^i v\right) - \phi\left(2^i v\right) + \phi\left(2^i v\right) - A_1'\left(2^i v\right), w\right\| \\ &\leq \frac{6}{\left(s^3 - 9s^2 + 20s - 12\right)}\lim_{i\longrightarrow\infty}\frac{1}{2^i}\widehat{\varphi}\left(2^i v, w\right) = 0,\end{aligned} \tag{97}$$

for all $v, w \in W$. By Lemma 1, $A_1(v) - A_1'(v) = 0$ for all $v \in W$. Therefore, $A_1 = A_1'$. □

*Remark 3.* A theorem analogous to (93) can be formulated, in which the sequence

$$A_1(v) := \lim_{i\longrightarrow\infty} 2^i\phi\left(\frac{v}{2^i}\right) \tag{98}$$

is defined with appropriate assumptions for $\varphi$.

**Corollary 6.** *Let $\lambda: [0,\infty) \longrightarrow [0,\infty)$ be a mapping such that $\lambda(0) = 0$ and*

*(i) $\lambda(pq) \leq \lambda(p)\lambda(s)$.*

*(ii) $\lambda(p) < p$ for all $p > 1$.*

If a mapping $\phi: W \longrightarrow F$ with $\phi(0) = 0$ and

$$\left\|D\phi(v_1, v_2, \ldots, v_s), w\right\| \leq \sum_{i=1}^{s}\lambda\left(\left\|v_i\right\|\right) + \lambda(\|w\|), \tag{99}$$

for all $v_1, v_2, \ldots, v_s, w \in W$, then there exists a unique additive mapping $A_1: W \longrightarrow F$ satisfying

$$\left\|\phi(v) - A_1(v), w\right\| \leq \frac{6}{\left(s^3 - 9s^2 + 20s - 12\right)}\left[\frac{2\lambda(\|v\|)}{2 - \lambda(2)} + \lambda(\|w\|)\right], \tag{100}$$

for all $v, w \in W$.

*Proof.* Let

$$\varphi(v_1, v_2, \ldots, v_s, w) = \sum_{i=1}^{s}\lambda\left(\left\|v_i\right\|\right) + \lambda(\|w\|), \tag{101}$$

for all $v_1, v_2, \ldots, v_s, w \in W$. It follows from (i) that

$$\begin{aligned}\lambda\left(2^i\right) &\leq (\lambda(2))^i, \\ \varphi\left(2^i v_1, 2^i v_2, \ldots, 2^i v_s, w\right) &\leq (\lambda(2))^i\left(\sum_{i=1}^{s}\lambda\left(\left\|v_i\right\|\right)\right) + \lambda(\|w\|).\end{aligned} \tag{102}$$

By using Theorem 6, we obtain (96). □

**Corollary 7.** *Let $q$ be a positive real number such that $q < 1$ and let $H: [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ be a homogeneous mapping with degree $q$. If a mapping $\phi: W \longrightarrow F$ with $\phi(0) = 0$ and*

$$\left\|D\phi(v_1, v_2, \ldots, v_s), w\right\| \leq H\left(\left\|v_1\right\|, \left\|v_2\right\|, \ldots, \left\|v_s\right\|\right) + \|w\|, \tag{103}$$

*for all $v_1, v_2, \ldots, v_s, w \in W$, then there exists a unique additive mapping $A_1: W \longrightarrow F$ satisfying*

$$\left\|\phi(v)-A_1(v),w\right\|\leq\frac{6}{\left(s^3-9s^2+20s-12\right)}\frac{H(\|v\|,\|v\|,0,\ldots,0)+\|w\|}{2-q},\tag{104}$$

*for all* $v,w\in W$.

*Proof.* Let

$$\varphi\left(v_1,v_2,\ldots,v_s,w\right)=H\left(\left\|v_1\right\|,\left\|v_2\right\|,\ldots,\left\|v_s\right\|\right)+\|w\|,\tag{105}$$

for all $v_1,v_2,\ldots,v_s,w\in W$. By using Theorem 6, we have (104). □

**Corollary 8.** *Let* $q\in\mathbb{R}^+$ *such that* $q<1$ *and let* $H\colon[0,\infty)\times[0,\infty)\longrightarrow[0,\infty)$ *be a homogeneous mapping with degree* $q$. *If a mapping* $\phi\colon W\longrightarrow F$ *with* $\phi(0)=0$ *and*

$$\left\|D\phi\left(v_1,v_2,\ldots,v_s\right),w\right\|\leq H\left(\left\|v_1\right\|,\left\|v_2\right\|,\ldots,\left\|v_s\right\|\right)\|w\|,\tag{106}$$

*for all* $v_1,v_2,\ldots,v_s,w\in W$, *then there exists a unique additive mapping* $A_1\colon W\longrightarrow F$ *satisfying*

$$\left\|\phi(v)-A_1(v),w\right\|\leq\frac{6}{2\left(s^3-9s^2+20s-12\right)}\cdot\frac{H(\|v\|,\|v\|,0,\ldots,0)\|w\|}{2-2^q},\tag{107}$$

*for all* $v,w\in W$.

*Proof.* Let

$$\varphi\left(v_1,v_2,\ldots,v_s,w\right)=H\left(\left\|v_1\right\|,\left\|v_2\right\|,\ldots,\left\|v_s\right\|\right)\|w\|,\tag{108}$$

for all $v_1,v_2,\ldots,v_s,w\in W$. By using Theorem 6, we have (110). □

**Corollary 9.** *Let* $p\in\mathbb{R}^+$ *such that* $p<1$. *If a mapping* $\phi\colon W\longrightarrow F$ *with* $\phi(0)=0$ *and*

$$\left\|D\phi\left(v_1,v_2,\ldots,v_s\right),w\right\|\leq\sum_{i=1}^{s}\left\|v_i\right\|^p+\|w\|,\tag{109}$$

*for all* $v_1,v_2,\ldots,v_s,w\in W$, *then there exists a unique additive mapping* $A_1\colon W\longrightarrow F$ *satisfying*

$$\left\|\phi(v)-A_1(v),w\right\|\leq\frac{6}{\left(s^3-9s^2+20s-12\right)}\frac{2\|v\|^p+\|w\|}{2-p},\tag{110}$$

*for all* $v,w\in W$.

We use an appropriate example to demonstrate that the stability of the functional equation (4) fails in the singular case. We provide the following counterexample, which shows the instability in a particular condition $p=2$ in Corollary 9 of functional equation (4), inspired by Gajda's excellent example in [31].

*Remark 4.* If a mapping $\phi\colon\mathbb{R}\longrightarrow W$ satisfies (4), then the following assertions hold:

(1) $\phi(m^c v)=m^c\phi(v), v\in\mathbb{R}, m\in\mathbb{Q}$, and $c\in\mathbb{Z}$.

(2) $\phi(v)=v\phi(1), v\in\mathbb{R}$ if the function $\phi$ is continuous.

*Example 2. Let a mapping* $\phi\colon\mathbb{R}\longrightarrow\mathbb{R}$ *be defined by*

$$\phi(v)=\sum_{p=0}^{\infty}\frac{\psi\left(2^p v\right)}{2^p},\tag{111}$$

where

$$\psi(v)=\begin{cases}\lambda v, & -1<v<1,\\ \lambda, & \text{else.}\end{cases}\tag{112}$$

Then, the mapping $\phi\colon\mathbb{R}\longrightarrow\mathbb{R}$ satisfies

$$\left|D\phi\left(v_1,v_2,\ldots,v_s\right)\right|\leq\left(\frac{n^4-8n^3+5n^2+34n-32}{4}\right)\cdot\left(\frac{4}{3}\right)\lambda\left(\sum_{j=1}^{s}\left|v_j\right|\right),\tag{113}$$

for all $v_1,v_2,\ldots,v_s\in\mathbb{R}$, but there does not exist an additive mapping $A_1\colon\mathbb{R}\longrightarrow\mathbb{R}$ satisfying

$$\left|\phi(v)-A_1(v)\right|\leq\delta|v|,\quad v\in\mathbb{R},\tag{114}$$

where $\lambda$ and $\delta$ are constants.

## 5. Conclusion

In this work, a new dimensional additive functional (equation (4)) has been introduced. We primarily found its solution and examined Hyers–Ulam stability in IFN-spaces using the direct approach in Section 3.1 and the fixed point approach in Section 3.2. In Section 4, we investigated the Hyers–Ulam stability in 2-Banach space by using the direct method. Also, we provided the counterexample, which shows the instability in a particular condition $p=2$ in Corollary 9 of equation (4), by the way of Gajda.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Authors' Contributions

All authors contributed equally to this study. All authors have read and approved the final version of the manuscript.

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