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Research Article

On the Fractional Variable Order Thermostat Model: Existence Theory on Cones via Piece-Wise Constant Functions

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In the current manuscript, we intend to investigate the existence, uniqueness, and the stability of positive solution in relation to a fractional version of variable order thermostat model equipped with nonlocal boundary values in the Caputo sense. In fact, we will get help from the constant piece-wise functions for transforming our variable order model into an auxiliary standard model of thermostat. By Guo-Krasnoselskii's fixed point theorem on cones, we derive the required conditions ensuring the existence property for positive solutions. An example is illustrated to examine the validity of the observed results.

1. Introduction

One of the most significant subbranches of mathematics in other existing computational and applied disciplines is arbitrary order calculus, notably fractional order calculus. Because of its applicability in a variety of fields including applied science and engineering, as well as its flexibility to model different systems, processes, and phenomena with memory effects, an arbitrary order calculus theory is regarded as an important subject of research for most researchers, mathematicians, scientists, and engineers. Fractional derivatives are powerful tools for describing the memory and heredity qualities of a wide range of materials and processes. The experts have contributed significantly to the study of fractional differential equations in recent years, for example, see some papers about the modeling of the Sturm-Liouville-Langevin problem [1], studying the Langevin boundary value problem (BVP) [2], investigation of the combination synchronization of a Caputo-Hadamard system [3], modeling of the Caputo-conformable pantograph problem [4], modeling of the hybrid thermostat [5], impulsive fractional systems [6], and modeling COVID-19 cases in the fractional settings [7], (for more studies, we refer to [8–13]).

The discussions of BVPs have attracted the focus of many scholars, and in this direction, valuable results have been obtained. Of course, various methods have been utilized to study fractional BVP such as the Banach contraction map principle [14], fixed point theorems [15], monotone iterative method [16], variational method [17], fixed point index theory and coincidence degree theory [18, 19], and numerical methods [20, 21]. On the other hand, numerous researches and review studies have been done in exploring the existence and stability of solutions to constant order fractional problems, while the subject of existence and stability in relation to the variable order problems is discussed seldom in the newly published papers including investigation of the variable order model of alcoholism [22], multiterm variable order BVPs [23], nonsingular variable order problems [24], initial value problems with conformable variable orders [25], approximate solutions on half-axis for a variable order IVP [26], singular variable order BVPs [27], Ulam-

Hyers-Rassias stability for an implicit variable order BVP [28], and the generalized variable order Lyapunov-type inequality [29].

Infante and Webb [30] modeled a mechanical device, i.e., thermostat in terms of the following nonlocal second order boundary value problem which is insulated at $t = 0$ with a heat-dissipating controller at $t = 1$ based on the temperature that is measured by a mechanical sensor at the time $t = c$:

$$-y'' = H(t, y), t \in I, y'(0) = 0, \gamma y'(1) + y(c) = 0, I \coloneqq [0, 1]. \tag{1}$$

Keeping in view the importance and accuracy of fractional operators to study of boundary value problems, Nieto and Pimentel [31] consider the fractional analogue of thermostat control model as

$$\begin{cases} -{}^cD_0^{\alpha} y(t) = H(t, y(t)), \\ y'(0) = 0, \gamma {}^cD_0^{\alpha-1} y(1) + y(c) = 0, \end{cases} \tag{2}$$

where $\gamma > 0, c \in (0, 1), \alpha \in (1, 2)$, ${}^cD_0^{\theta}$ denotes the derivative in the form of Caputo of order $\theta \in \{\alpha, \alpha - 1\}$ and H is continuous on $[0, 1] \times [0,\infty)$. These two mathematicians discussed the existence of positive solution to the proposed problem using fixed point approach.

In 2015, Shen et al. [32] turned to the fractional model of thermostat with a parameter

$$\begin{cases} -{}^cD_0^{\alpha} y(t) = aH(t, y(t)), \\ y'(0) = 0, \gamma {}^cD_0^{\alpha-1} y(1) + y(c) = 0, \end{cases} \tag{3}$$

where $\gamma > 0, c \in [0, 1], \alpha \in (1, 2]$. They proved the existence along with the nonexistence of positive solutions for different values of parameter a.

On the other side, keeping in view the advancements in studies of fractional calculus, it was observed that the constant order fractional calculus is not the ultimate instrument for the modeling of physical problems. In this regard, Lorenze and Hartley [33], in 1998, proposed the variable order fractional operator to model complex dynamics problems and further study in this field were given by them in [34].

In consistent with the theory of variable order fractional calculus, we aim to consider a simple type of the variable order Caputo fractional thermostat model in terminal points which possesses a mathematical structure as

$$\begin{cases} -{}^cD^{u(t)} y(t) = f(t, y(t)), & t \in J = [0, S], \\ y'(0) = 0, & {}^cD^{u(t)-1} y(M) + y(0) = 0, \end{cases} \tag{4}$$

where $u(t)$: $J \longrightarrow (1, 2]$ stands for a function as the variable order, $f : J \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.

There are no research manuscripts about the application of cones on variable order systems. This gap motivates us to investigate the generalized version of thermostat model in the context of fractional variable orders with respect to constant piece-wise functions. Furthermore, the presented structure is expressed in a unique and simple case, allowing us to generalize several typical specific cases previously addressed, and also this structure makes the novelty of our work. The ultimate goal of this study is to explore some results in relation to existence theory for the solutions of variable order BVP for the Caputo fractional thermostat model (4). Note that in this model, we use the Caputo fractional derivative. In fact, the importance of the fractional derivatives is their memory property which can describe the dynamical behavior of a system during a time interval. This kind of derivatives is not local and that is why we can analyze the solutions of a system with respect to various values of fractional orders. Also, we emphasize that our model is a generalization of two constant order thermostat models given in [31, 32] to variable order version.

The structure of the manuscript is designed as follows: the following section is based on some basic notions related to the current study. In Section 3, to get help of Guo-Krasnoselskii fixed point theorem, a suitable partition of the underlying interval J is proposed. This variable order BVP for the Caputo fractional thermostat model (4) is divided into the finite number of BVPs consisting of the Caputo fractional differential equations of constant orders, and then using the concept of Green's function, the equivalent solution of the proposed problem is obtained and some properties of Green's function are discussed. Finally, the existence result for (4) is proved based on the cones in Section 4. In the next section (Section 5), an example is prepared to give the illustrations of the main findings. We end our research by conclusions in Section 6.

2. Preliminaries

The collection $C(J, \mathbb{R})$ of all functions $\varkappa : J \longrightarrow \mathbb{R}$ having the property of continuity will be a Banach space via $\|\varkappa\| = \sup \{|\varkappa(t)| : t \in J\}$..

Definition 1 (see [35, 36]). The left Riemann-Liouville fractional integral and left Caputo fractional derivative of variable order $u(t)$ of H are recalled as

$$I_{0^+}^{u(t)} H(t) = \frac{1}{\Gamma(u(t))} \int_0^t (t - \mathfrak{w})^{u(t)-1} H(\mathfrak{w}) d\mathfrak{w}, t \in J, \tag{5}$$

$${}^cD_{0^+}^{u(t)} H(t) = \frac{1}{\Gamma(n - u(t))} \int_0^t (t - \mathfrak{w})^{n-u(t)-1} H^{(n)}(\mathfrak{w}) d\mathfrak{w}, \tag{6}$$

with $u(t)$: $J \longrightarrow (n - 1, n]$.

Remark 2 (see [37, 38]). In Equations (5) and (6), by choosing the constant order u, the Caputo fractional derivative and the Riemann-Liouville fractional integral of variable order will be the same conventional of the Caputo fractional derivative and Riemann-Liouville fractional integral of constant order, respectively.

Lemma 3 (see [38]). *Suppose $\alpha_1, b_1 > 0$ and $n = [\alpha_1] + 1$, then*

$$I_{b_1^+}^{\alpha_1}\left({}^cD_{b_1^+}^{\alpha_1}\psi(t)\right) = h(t) + \sum_{k=0}^{n-1}\frac{\psi^{(k)}(0)}{k!}t^k. \quad (7)$$

Lemma 4 (see [39]). *Let $\alpha_1, \alpha_2 > 0$, $\psi \in L^1(b_1, b_2)$, and ${}^cD_{b_1^+}^{\alpha_1}\psi \in L^1(b_1, b_2)$. Then, the differential equation*

$${}^cD_{b_1^+}^{\alpha_1}\psi = 0 \quad (8)$$

has unique solution

$$\psi(t) = \varpi_0 + \varpi_1(t - b_1) + \varpi_2(t - b_1)^2 + \cdots + \varpi_{n-1}(t - b_1)^{n-1} \quad (9)$$

and

$$I_{b_1^+}^{\alpha_1}\left({}^cD_{b_1^+}^{\alpha_1}\right)\psi(t) = \psi(t) + \varpi_0 + \varpi_1(t - b_1) + \varpi_2(t - b_1)^2 + \cdots + \varpi_{n-1}(t - b_1)^{n-1} \quad (10)$$

under the conditions $n - 1 < \alpha_1 \le n$, $\varpi_j \in \mathbb{R}$.

Moreover,

$$\begin{aligned} {}^cD_{b_1^+}^{n}\psi(t) &= \psi^{(n)}(t), n \in \mathbb{N}, \\ {}^cD_{b_1^+}^{\alpha_1}\left(I_{b_1^+}^{\alpha_1}\right)\psi(t) &= \psi(t), \\ I_{b_1^+}^{\alpha_1}\left(I_{b_1^+}^{\alpha_2}\right)\psi(t) = I_{b_1^+}^{\alpha_2}\left(I_{b_1^+}^{\alpha_1}\right)\psi(t) &= I_{b_1^+}^{\alpha_1+\alpha_2}\psi(t). \end{aligned} \quad (11)$$

Remark 5 (see [26, 27]). It is observed that the property of semigroup is not consistent for arbitrary functions $u(t)$, $v(t)$ in the role of variable orders, i.e.,

$$I_{b_1^+}^{u(t)}I_{b_1^+}^{v(t)}\psi(t) \ne I_{b_1^+}^{u(t)+v(t)}\psi(t). \quad (12)$$

Example 1. Assume $y(t) \equiv 1$ for $t \in J = [0, 3]$ and $u(t) = t/2$ and $v(t) = \begin{cases} 1, & t \in [0, 1] \\ 2, & t \in]1, 3]. \end{cases}$ Then,

$$\begin{aligned} I_{0^+}^{u(t)}\left(I_{0^+}^{v(t)}y(t)\right) &= \int_0^t \frac{(t-s)^{u(t)-1}}{\Gamma(u(t))}\int_0^s \frac{(s-\tau)^{v(s)-1}}{\Gamma(v(s))}y(\tau)d\tau ds \\ &= \int_0^t \frac{(t-s)^{u(t)-1}}{\Gamma(u(t))}\left[\int_0^1 \frac{(s-\tau)^0}{\Gamma(1)}d\tau + \int_1^s \frac{(s-\tau)}{\Gamma(2)}d\tau\right] \\ &\cdot ds = \int_0^t \frac{(t-s)^{u(t)-1}}{\Gamma(u(t))}\left[\frac{s^2}{2} - s + \frac{3}{2}\right]ds, \\ I_{0^+}^{u(t)+v(t)}y(t) &= \frac{1}{\Gamma(u(t)+v(t))}\int_0^t (t-s)^{u(t)+v(t)-1}y(s)ds. \end{aligned} \quad (13)$$

For $t = 2$, we write

$$I_{0^+}^{u(t)}\left(I_{0^+}^{v(t)}y(t)\right)\Big|_{t=2} = \int_0^2 \frac{(2-s)^0}{\Gamma(1)}\left[\frac{s^2}{2} - s + \frac{3}{2}\right]ds = \int_0^2\left(\frac{s^2}{2} - s + \frac{3}{2}\right)ds = \frac{7}{3}, \quad (14)$$

and accordingly,

$$\begin{aligned} I_{0^+}^{u(t)+v(t)}y(t)\Big|_{t=2} &= \int_0^2 \frac{(2-s)^{u(t)+v(t)-1}}{\Gamma(u(t)+v(t))}y(s)ds = \int_0^1 \frac{(2-s)^1}{\Gamma(2)} \\ &\cdot ds + \int_1^2 \frac{(2-s)^2}{\Gamma(3)}ds = \frac{3}{2} + \frac{1}{6} = \frac{5}{3}. \end{aligned} \quad (15)$$

As a result, the semigroup property is not valid in the generalized case for the Riemann-Liouville fractional integral of variable order.

The next results will be used in the main results concerning the Riemann-Liouville fractional integral given in (5).

Lemma 6 (see [26]). *Let $u \in C(J, (1, 2])$, then for*

$$h \in C_\rho(J, \mathbb{R}) = \{h(t) \in C(J, \mathbb{R}), t^\rho h(t) \in C(J, \mathbb{R}), 0 \le \rho \le 1\}, \quad (16)$$

we have

$$\begin{aligned} &\exists I_{0^+}^{u(t)}h(t), \forall t \in J; \\ &I_{0^+}^{u(t)}h(t) \in C(J, \mathbb{R}). \end{aligned} \quad (17)$$

Definition 7 (see [26, 27, 40]). Consider a subset $\mathscr{I}$ of $\mathbb{R}$:

(a) A generalized interval is either empty, singleton subset, or an interval

(b) The finite set $\mathscr{P}$ of generalized interval is a partition of $\mathscr{I}$ whenever every $s \in \mathscr{I}$ belongs to exactly one of the generalized intervals $\mathscr{E}$ in $\mathscr{P}$

(c) $h : \mathscr{I} \longrightarrow \mathbb{R}$ is piece-wise constant with respect to $\mathscr{P}$ of $\mathscr{I}$ if $\forall \mathscr{E} \in \mathscr{P}$, h is constant on $\mathscr{E}$

Theorem 8 (Guo-Krasnoselskii fixed point theorem [41]). *Assume a cone P and bounded subsets B_1, B_2 of a Banach space E with*

$$0 \in B_1 \subset \overline{B_1} \subset B_2, \quad (18)$$

and assume a completely continuous operator $T : P \cap (\overline{B_2} \setminus B_1) \longrightarrow P$ such that

(i) $\|Ty\| \ge \|y\|, y \in P \cap \partial B_1$ and $\|Ty\| \le \|y\|, y \in P \cap \partial B_2$; or

(ii) $\|Ty\| \le \|y\|, y \in P \cap \partial B_1$ and $\|Ty\| \ge \|y\|, y \in P \cap \partial B_2$.

Then, operator T possesses a fixed point in $P \cap (\overline{B_2} \setminus B_1)$.

3. Auxiliary BVP and Green's Function

Before moving to the main results, we first assume the following assertions:

Let $n \in \mathbb{N}$, and the finite sequence $\{S_j\}_{j=0}^{n}$ of points in a way that $0 = S_0 < S_j < S_n = S, j = 1, \cdots, n-1$. Designate $J_j := (S_{j-1}, S_j], j = 1, 2, \cdots, n$. Then, the partition of interval J is $\mathscr{P} = \{J_j : j = 1, 2, \cdots, n\}$.

Suppose a constant piece-wise map $u(t): J \longrightarrow (1, 2]$ with respect to $\mathscr{P}$, i.e., $u(t) = \sum_{j=1}^{n} u_j I_j(t)$, where $u_j \in (1, 2]$, and I_j stands for the interval $J_j, j = 1, 2, \cdots, n$:

$$I_j(t) = \begin{cases} 1, & \text{for } t \in J_j, \\ 0, & elsewhere. \end{cases} \quad (19)$$

For all $j \in \mathbb{N}_1^n$, $E_j = C(J_j, \mathbb{R})$, denotes Banach spaces via $\|y\|_{E_j} = \sup_{t \in J_j} | y(t) |$ ..

To achieve our main findings, let us first reduce the variable order FBVP (4) to a standard constant order system. With the aid of (6), the variable order BVP for the Caputo fractional thermostat model (4) takes the form

$$-\sum_{k=1}^{j-1} \int_{S_{k-1}}^{S_k} \frac{(t-s)^{1-u_k}}{\Gamma(2-u_k)} y''(s) ds - \int_{S_{j-1}}^{t} \frac{(t-s)^{1-u_j}}{\Gamma(2-u_j)} y''(s) ds = f(t, y(t)). \quad (20)$$

If $\tilde{y} \in C(J_j, \mathbb{R})$ is $\tilde{y} \equiv 0$ on $[0, S_{j-1}]$ and is a solution of (20), then (20) becomes

$$-{}^{c}D_{S_{j-1}^{+}}^{u_j} \tilde{y}(t) = f(t, \tilde{y}(t)), t \in J_j. \quad (21)$$

Taking the above into consideration $\forall j \in \mathbb{N}_1^n$ and with piece-wise functions u_j, we take into account the next auxiliary BVP of constant order, the Caputo fractional thermostat model is formulated as

$$\begin{cases} -{}^{c}D_{S_{j-1}^{+}}^{u_j} y(t) = f(t, y(t)), & t \in J_j \\ y'(S_{j-1}) = 0, & {}^{c}D^{u_j-1} y(S_j) + y(S_{j-1}) = 0. \end{cases} \quad (22)$$

In the following, relevant Green's function is obtained in relation to auxiliary BVP of constant order of the Caputo fractional thermostat model (22).

Lemma 9. *Let $1 < u_j \le 2, \forall j \in \mathbb{N}_1^n$. Then, $y \in E_j$ is a solution of the auxiliary BVP of constant order of the Caputo fractional thermostat model*

$$\begin{cases} -{}^{c}D_{S_{j-1}^{+}}^{u_j} y(t) = f(t), & t \in J_j \\ y'(S_{j-1}) = 0, & {}^{c}D^{u_j-1} y(S_j) + y(S_{j-1}) = 0. \end{cases} \quad (23)$$

if it fulfills

$$y(t) = \int_{S_{j-1}}^{S_j} G_j(t, s) f(s) ds, \quad (24)$$

where Green's function $G_j(t, s)$ is stated as

$$G_j(t, s) = \begin{cases} 1 - \dfrac{(t-s)^{u_j-1}}{\Gamma(u_j)}, & s \le t \\ 1, & s > t. \end{cases} \quad (25)$$

Proof. Using Lemma 3, there are $c_0, c_1 \in \mathbb{R}$ such that

$$y(t) = -I^{u_j} f(t) + c_0 + c_1 t = -\int_{S_{j-1}}^{t} \frac{(t-s)^{u_j-1}}{\Gamma(u_j)} f(s) ds + c_0 + c_1 t. \quad (26)$$

In view of Lemma 4,

$$y'(t) = -\int_{S_{j-1}}^{t} \frac{(t-s)^{u_j-2}}{\Gamma(u_j-1)} f(s) ds + c_1. \quad (27)$$

By $y'(0) = 0$, we get that $c_1 = 0$.. Moreover,

$${}^{C}D^{u_j-1} y(t) = -I^1 y(t). \quad (28)$$

Using the boundary condition ${}^{C}D^{\alpha-1} y(S_j) + y(S_{j-1}) = 0$, we get

$$c_0 = \int_{S_{j-1}}^{S_j} f(s) ds. \quad (29)$$

At the end, (26) takes the form

$$y(t) = \int_{S_{j-1}}^{S_j} f(s) ds - \int_{S_{j-1}}^{t} \frac{(t-s)^{u_j-1}}{\Gamma(u_j)} f(s) ds = \int_{S_{j-1}}^{S_j} G_j(t, s) f(s) ds. \quad (30)$$

The proof is completed. □

Remark10. Notice that for each fixed $s \in [S_{j-1}, S_j]$, $\partial G_j/\partial t = 0$ for $t \le s$ and $\partial G_j/\partial t < 0$ for $t > s$; this turns $G_j(t, s)$ a decreasing function. Therefore

$$\max_{t \in [S_{j-1}, S_j]} G_j(t, s) = G_j(S_{j-1}, s) = \begin{cases} 1, & s > S_{j-1}, \\ 1 + \dfrac{(S_{j-1} - s)^{u_j-1}}{\Gamma(u_j)}, & s \le S_{j-1}, \end{cases}$$

$$\min_{t \in [S_{j-1}, S_j]} G_j(t, s) = G_j(S_j, s) = \begin{cases} 1 - \dfrac{(S_j - s)^{u_j-1}}{\Gamma(u_j)} & s > S_{j-1}, \\ \dfrac{(S_{j-1} - s)^{u_j-1} - (S_j - s)^{u_j-1}}{\Gamma(u_j)} + 1, & s \le S_{j-1}. \end{cases} \quad (31)$$

As a result, due to behavior of $G_j(t,s)$ with respect to s, we have

$$\min_{t,s\in[S_{j-1},S_j]} G_j(t,s) = 1 - \frac{\left(S_j - S_{j-1}\right)^{u_j-1}}{\Gamma(u_j)},$$
$$\max_{t,s\in[S_{j-1},S_j]} G_j(t,s) = 1 + \frac{S_{j-1}^{u_j-1}}{\Gamma(u_j)}. \tag{32}$$

We shall prove that $Gj(t,s)$ meets the following property to ensure the existence of positive solution of the variable order Caputo fractional thermostat model (4) proposed by Lan and Webb in [42]:

(H2): $\exists\varphi : [S_{j-1}, S_j] \longrightarrow [0,\infty)$ as a measurable mapping, $[c,d] \subseteq [S_{j-1}, S_j]$ and $\xi \in [0,1]$ such that

$$|G_j(t,s)| \le \varphi(s), \quad \forall t,s \in \left[S_{j-1}, S_j\right],$$
$$G_j(t,s) \ge \xi\varphi(s), \quad \forall t \in [c,d], \forall s \in \left[S_{j-1}, S_j\right]. \tag{33}$$

Lemma 11. *If $\Gamma(u_j) > (S_j - S_{j-1})^{u_j-1}$, then $G_j(t,s) > 0$ for all $t,s \in [S_{j-1}, S_j]$, and $G_j(t,s)$ satisfies (H2).*

Proof. Taking $[c,d] = [S_{j-1}, S_j]$,, then

$$|G_j(t,s)| = G_j(t,s) \le 1 + \frac{S_{j-1}^{u_j-1}}{\Gamma(u_j)} \coloneqq \varphi(s),$$
$$G_j(t,s) \ge \xi\varphi(s) \quad \forall s,t \in \left[S_{j-1}, S_j\right], \tag{34}$$

where

$$\xi = \frac{\Gamma(u_j) - \left(S_j - S_{j-1}\right)^{u_j-1}}{\Gamma(u_j) + S_{j-1}^{u_j-1}}. \tag{35}$$

The proof is completed. □

Lemma 12. *If $\Gamma(u_j) = (S_j - S_{j-1})^{u_j-1}$, then $G_j(t,s) \ge 0$ for all $t,s \in [S_{j-1}, S_j]$, and $G_j(t,s)$ satisfies (H2).*

Proof. Taking $[c,d] = [Sj-1, d]$ such that $S_{j-1} \le d < S_j$, and using the preceding lemma arguments, we obtain

$$|G_j(t,s)| \le 1 + \frac{S_{j-1}^{u_j-1}}{\Gamma(u_j)} \coloneqq \varphi(s), \forall t,s \in \left[S_{j-1}, S_j\right]. \tag{36}$$

Setting

$$\xi = \frac{\Gamma(u_j) - \left(d - S_{j-1}\right)^{u_j-1}}{\Gamma(u_j) + S_{j-1}^{u_j-1}}, \tag{37}$$

we get

$$G(t,s) \ge \xi\varphi(s), \quad \forall t \in \left[S_{j-1}, d\right], \forall s \in \left[S_{j-1}, S_j\right]. \tag{38}$$

The proof is completed. □

Lemma 13. *If $\Gamma(u_j) < (S_j - S_{j-1})^{u_j-1}$, then $G(t,s)$ changes sign on $[S_{j-1}, S_j] \times [S_{j-1}, S_j]$, and $G_j(t,s)$ satisfies (H2).*

Proof. Let $[c,d] = [S_{j-1}, d]$ be such that $S_{j-1} \le d < S_j$ and $\Gamma(u_j) > (d - S_{j-1})^{u_j-1}$. We have

$$|G_j(t,s)| \le \max\left\{1 + \frac{S_{j-1}^{u_j-1}}{\Gamma(u_j)}, \frac{\left(1 - S_{j-1}\right)^{u_j-1}}{\Gamma(u_j)} - 1\right\}$$
$$\coloneqq \varphi(s), \forall t,s \in \left[S_{j-1}, S_j\right],$$
$$G(t,s) \ge \xi\varphi(s), \forall t \in \left[S_{j-1}, d\right], \forall s \in \left[S_{j-1}, S_j\right], \tag{39}$$

where

$$\xi = \min\left\{\frac{\Gamma(u_j) - \left(d - S_{j-1}\right)^{u_j-1}}{\Gamma(u_j) + S_{j-1}^{u_j-1}}, \frac{\Gamma(u_j) -- \left(d - S_{j-1}\right)^{u_j-1}}{\left(S_j - S_{j-1}\right)^{u_j-1} - \Gamma(u_j)}\right\}. \tag{40}$$

The proof is completed. □

4. Existence and Uniqueness

We now derive the existence result with the aid of Guo-Krasnoselskii's result [41]. Before starting, set

$$f_0 = \lim_{y\longrightarrow 0^+} \min_{t\in[S_{j-1},S_j]} \frac{f(t,y)}{y}, \quad f_0^\star = \lim_{y\longrightarrow 0^+} \max_{t\in[S_{j-1},S_j]} \frac{f(t,y)}{y},$$
$$f_\infty = \lim_{y\longrightarrow\infty} \min_{t\in[S_{j-1},S_j]} \frac{f(t,y)}{y}, \quad f_\infty^\star = \lim_{y\longrightarrow\infty} \max_{t\in[S_{j-1},S_j]} \frac{f(t,y)}{y}. \tag{41}$$

Theorem 14. *Let $f(s, y(s)) \in C([S_{j-1}, S_j]) \times [0,\infty)$. Assuming one of the below mentioned cases:*

(1) $f_0 = \infty$ and $f_\infty = 0$ (Sublinear case)

(2) $f_0^\star = 0$ and $f_\infty^\star = \infty$ (Superlinear case),

if $\Gamma(u_j) > (S_j - S_{j-1})^{u_j-1}$, then there is at least positive solution of the auxiliary BVP of constant order Caputo fractional thermostat model (22).

Proof. Define $T : C([S_{j-1}, S_j]) \longrightarrow C([S_{j-1}, S_j])$ as

$$Ty(t) = \int_{S_{j-1}}^{S_j} G_j(t,s) f(s, y(s))\,ds. \tag{42}$$

We now define the cone

$$P=\left\{y\setminus y\in C\left[S_{j-1},S_j\right],y(t)\geq 0,\min_{t\in\left[S_{j-1},S_j\right]}y(t)\geq\xi\|y\|\right\},\quad(43)$$

where ξ is as in (35).

In the start, we prove that $T(P)\subset P$. Since the functions f and G_j are positive and continuous, it follows that if $y\in P$, then $Ty\in C([S_{j-1},S_j])$ and $Ty(t)\geq 0$ for all $t\in[S_{j-1},S_j]$. We reach to the inequality derived below for a fixed $y\in P$ for all $t\in[S_{j-1},S_j]$ and that $G_j(t,s)$ satisfies (H2)

$$\begin{aligned}Ty(t)&=\int_{S_{j-1}}^{S_j}G_j(t,s)f(s,y(s))ds\geq\xi\int_{S_{j-1}}^{S_j}\varphi(s)f(s,y(s))\\&\cdot ds\geq\xi\int_{S_{j-1}}^{S_j}\max_{t\in\left[S_{j-1},S_j\right]}G_j(t,s)f(s,y(s))\\&\cdot ds\geq\xi\max_{t\in\left[S_{j-1},S_j\right]}\int_{S_{j-1}}^{S_j}G_j(t,s)f(s,y(s))ds=\xi\|Ty\|,\end{aligned}\quad(44)$$

which leads to the first claim. In the next step, we prove the complete continuity of operator $T:P\longrightarrow P$. Because of the continuous behavior of G_j and f, continuity of $T:P\longrightarrow P$ is followed immediately. Let $B\subset P$ be bounded. Define

$$L=\max_{S_{j-1}\leq t\leq S_j,0\leq y\leq M}|f(t,y)|+1.\quad(45)$$

So for all $y\in B$, we have

$$|Ty(t)|\leq\int_{S_{j-1}}^{S_j}G_j(t,s)f(s,y(s))ds\leq L\int_{S_{j-1}}^{S_j}G_j(t,s)ds,\quad(46)$$

for all $t\in[S_{j-1},S_j]$. This defines the boundedness of $T(B)$. Now, for each $y\in B$ and $t_1,t_2\in[S_{j-1},S_j]$ such that $t_1<t_2$, we write

$$\begin{aligned}|Ty(t_2)-Ty(t_1)|&=\left|-\int_{S_{j-1}}^{t_2}\frac{(t_2-s)^{u_j-1}}{\Gamma(u_j)}f(s,y(s))\right.\\&\cdot ds+\left.\int_{S_{j-1}}^{t_1}\frac{(t_1-s)^{u_j-1}}{\Gamma(u_j)}f(s,y(s))ds\right|\leq\frac{1}{\Gamma(u_j)}\int_{S_{j-1}}^{t_1}\left((t_2-s)^{u_j-1}\right.\\&\left.-(t_1-s)^{u_j-1}\right)|f(s,y(s))|ds+\frac{1}{\Gamma(u_j)}\int_{t_1}^{t_2}(t_2-s)^{u_j-1}|f(s,y(s))\\&\cdot|ds\leq\frac{L}{\Gamma(u_j)}\left(\int_{S_{j-1}}^{t_1}\left((t_2-s)^{u_j-1}-(t_1-s)^{u_j-1}\right)ds+\int_{t_1}^{t_2}(t_2-s)^{u_j-1}ds\right)\\&=\frac{L}{u_j\Gamma(u_j)}\left(-(t_2-t_1)^{u_j}+t_2^{u_j}-t_1^{u_j}+(t_2-t_1)^{u_j}\right)=\frac{L}{\Gamma(u_j+1)}\left(t_2^{u_j}-t_1^{u_j}\right).\end{aligned}\quad(47)$$

Tending $t_1\longrightarrow t_2$ implies that the RHS of the above inequality goes to 0 and thus $T(B)$ is equicontinuous. At the end, the Arzela-Ascoli theorem confirms complete continuity of operator $T:P\longrightarrow P$.

Next, assume that (2) is true. As $f_0=\infty$, a $\rho_1>0$ exists such that $f(t,y)\geq\delta_1 y,\forall 0<y\leq\rho_1$, where δ_1 satisfies

$$\delta_1\left(1-\frac{\left(S_j-S_{j-1}\right)^{u_j-1}}{\Gamma(u_j)}\right)\geq 1.\quad(48)$$

Taking $y\in P$ with $\|y\|=\rho_1$, then

$$\begin{aligned}Ty&=\int_{S_{j-1}}^{S_j}G_j(t,s)f(s,y(s))ds\geq\delta_1\int_{S_{j-1}}^{S_j}G_j(t,s)y(s)ds\geq\delta_1\|y\|\\&\cdot\left(1-\frac{\left(S_j-S_{j-1}\right)^{u_j-1}}{\Gamma(u_j)}\right)\geq\|y\|.\end{aligned}\quad(49)$$

Let $B_1=\{y\in C([S_{j-1},S_j])\setminus\|y\|<\rho_1\}$. Hence, we have $\|Ty\|\geq\|y\|,y\in P\cap\partial B_1$.

Due the continuity of $f(t,.)$ on $[0,\infty)$, a function $\tilde{f}(t,y)=\max_{z\in[0,y]}\{f(t,z)\}$ can be defined which is nondecreasing on $(0,\infty)$ by assumption and

$$\lim_{y\longrightarrow\infty}\left\{\max_{t\in\left[T_{j-1},T_j\right]}\frac{\tilde{f}(t,y)}{y}\right\}=0.\quad(50)$$

Therefore, there exists $\rho_2>\rho_1>0$ such that $\tilde{f}(t,y)\leq\delta_2 y,\forall y\geq\rho_2$, where δ_2 satisfies

$$\delta_2\left(1+\frac{S_{j-1}^{u_j-1}}{\Gamma(u_j)}\right)\leq 1.\quad(51)$$

Define $\Omega_2=\{y\in C([S_{j-1},S_j])\setminus\|y\|<\rho_2\}$ and let $y\in P$ such that $\|y\|=\rho_2$. Then,

$$Ty=\int_{S_{j-1}}^{S_j}G_j(t,s)f(s,y(s))ds\leq\int_{S_{j-1}}^{S_j}G_j(t,s)\tilde{f}(s,\|y\|)ds\leq\delta_2\|y\|\left(1+\frac{S_{j-1}^{u_j-1}}{\Gamma(u_j)}\right)\leq\|y\|.\quad(52)$$

Hence, we have $\|Ty\|\leq\|y\|,y\in P\cap\partial B_2$. In conclusion, there is at least one positive solution by (i) of Theorem 8 for the auxiliary BVP of constant order Caputo fractional thermostat model (22).

Now, assume that (3) is true. For $\delta_2>0$ and by assumption, there is $r_1>0$ with $f(t,y)\leq\delta_2 y$ for $0\leq y\leq r_1$. Let $y\in P$ such that $\|y\|=r_1$. Then,

$$Ty=\int_{S_{j-1}}^{S_j}G_j(t,s)f(s,y(s))ds\leq\delta_2\int_{S_{j-1}}^{S_j}G_j(t,s)y(s)ds\leq\delta_2\|y\|\left(1+\frac{S_{j-1}^{u_j-1}}{\Gamma(u_j)}\right)\leq\|y\|.\quad(53)$$

If we let $B_1=\{y\in C[S_{j-1},S_j]\setminus\|y\|<r_1\}$, then $\|Ty\|\leq\|y\|$ for $y\in P\cap\partial B_1$. Again by assumption, there is $r>0$ such that $f(t,y)\geq\delta_1 u,\forall y\geq r$. Define $B_2=\{y\in C([S_{j-1},S_j])\setminus\|y\|<r_2\}$, where $r_2=\max(2r_1,(r/\xi))$. Then, $y\in P$ and $\|y\|=r_2$ imply that

$$\min y(t) \geq \xi \|y\| = \xi r_2 \geq r, \quad (54)$$

and so we obtain

$$Ty = \int_{S_{j-1}}^{S_j} G_j(t,s) f(s, y(s)) ds \geq \delta_1 \int_{S_{j-1}}^{S_j} G_j(t,s) y(s) \cdot ds \geq \delta_1 \|y\| \left(1 - \frac{\left(S_j - S_{j-1}\right)^{u_j - 1}}{\Gamma(u_j)}\right) \geq \|y\|. \quad (55)$$

This shows that $\|Ty\| \geq \|y\|$ for $y \in P \cap \partial B_2$. In conclusion, it is found at least one positive solution, by (ii) of Theorem 8, for the auxiliary BVP of constant order Caputo fractional thermostat model (22) as $\widetilde{y_j} \in P \cap (\overline{B_2} \setminus B_1)$. □

Now, we show the uniqueness of solutions for the auxiliary BVP of constant order Caputo fractional thermostat model (22) based on the Banach contraction principle. We consider the following assumption:

Let $f \in C(J \times \mathbb{R}, \mathbb{R})$ and there exists a number $\delta \in (0,1)$ such that $t^\delta f \in C(J \times \mathbb{R}, \mathbb{R})$ and there exists a constant $K > 0$ such that

$$t^\delta |f(t, y_1) - f(t, y_2)| \leq K |y_1 - y_2|, \text{for } any\, y_1, y_2 \in \mathbb{R} \text{ and } t \in J. \quad (56)$$

Theorem 15. *Let the conditions (H1) and (H3) be satisfied and the inequality*

$$\frac{K\left(S_j^{1-\delta} - S_{j-1}^{1-\delta}\right)}{1-\delta} \left(1 + \frac{S_{j-1}^{u_j - 1}}{\Gamma(u_j)}\right) < 1 \quad (57)$$

holds. If $\Gamma(u_j) \geq (S_j - S_{j-1})^{u_j - 1}$, then the thermostat BVP (22) has a unique positive solution in E_j.

Proof. We shall use the Banach contraction principle to prove that T has unique fixed point. For $x(t), y(t) \in E_j$, by Lemma 11 and Lemma 12, we obtain

$$\begin{aligned} &|Tx(t) - Ty(t)| \\ &\cdot ds| \leq \int_{S_{j-1}}^{S_j} G_j(t,s) |f(s, x(s)) - f(s, y(s)) \\ &\cdot |ds \leq \left(1 + \frac{S_{j-1}^{u_j-1}}{\Gamma(u_j)}\right) \int_{S_{j-1}}^{S_j} s^{-\delta} (K \mid x(s) - y(s) \mid) \\ &\cdot ds \leq K \left(1 + \frac{S_{j-1}^{u_j-1}}{\Gamma(u_j)}\right) \|x - y\|_{E_j} \int_{S_{j-1}}^{S_j} s^{-\delta} ds \leq \frac{K\left(S_j^{1-\delta} - S_{j-1}^{1-\delta}\right)}{1-\delta} \\ &\cdot \left(1 + \frac{S_{j-1}^{u_j-1}}{\Gamma(u_j)}\right) \|x - y\|_{E_j}. \end{aligned} \quad (58)$$

Consequently by (57), the operator T is a contraction. Hence, by Banach's contraction principal, T has a unique fixed point $\widetilde{y_j} \in E_j$, which is a unique positive solution of the auxiliary BVP of the constant order Caputo fractional thermostat model (22). □

In the next result, we generalize the existence criteria for the variable order BVP of the main Caputo fractional thermostat model (4).

Theorem 16. *Assuming (H1)-(H2), the variable order BVP of the main Caputo fractional thermostat model (4) admits at least one solution in $C(J, \mathbb{R})$.*

Proof. We know that the solution $\widetilde{y_j} \in E_j$ fulfills the auxiliary BVP of constant order Caputo fractional thermostat model (22) by Theorem 14 for any $j \in \{1, 2, \cdots, n\}$. Now, the continuous function on $[0, S_j]$ is defined as

$$y_j = \begin{cases} 0, & t \in [0, S_{j-1}], \\ \tilde{y}_j, & t \in J_j, \end{cases} \quad (59)$$

is a solution of (20) for $t \in J_j$, $j \in \{1, 2, \cdots, n\}$.

Therefore, $y(t) = y_j(t), t \in J_j, j \in \{1, 2, \cdots, n\}$ solves the variable order BVP of the main Caputo fractional thermostat model (4). The proof is completed. □

5. Ulam-Hyers Stability

In this section, we are going to investigate the Ulam-Hyers stability for solutions of the given variable model of thermostat.

Definition 17 (see [43]). The variable order Caputo fractional thermostat model (4) is Ulam-Hyers stable if there exists $c_f > 0$ such that for each $\varepsilon > 0$ and for every solution $z \in C(J, \mathbb{R})$ of the following inequality

$$|-{}^c D_{0^+}^{u(t)} z(t) - f(t, z(t))| \leq \varepsilon, t \in J, \quad (60)$$

there exists a solution $y \in C(J, \mathbb{R})$ of (4) with

$$|z(t) - y(t)| \leq c_f \varepsilon, t \in J. \quad (61)$$

Theorem 18. *Assume that the conditions (H1) and (H2) to be held. Then, the Caputo fractional thermostat model (4) is Ulam-Hyers stable.*

Proof. Let $\varepsilon > 0$ be an arbitrary number and the function $z(t)$ belonging to $C(J, \mathbb{R})$ satisfies the following inequality

$$|-{}^c D_{0^+}^{u(t)} z(t) - f(t, z(t))| \leq \varepsilon, t \in J. \quad (62)$$

For any $j \in \{1, 2, \cdots, n\}$, we define the functions $z_1(t) \equiv z(t), t \in [0, S_1]$ and for $j = 2, 3, \cdots, n$:

$$z_j(t) = \begin{cases} 0, & t \in [0, S_{j-1}], \\ z(t), & t \in J_j. \end{cases} \quad (63)$$

For any $j \in \{1, 2, \cdots, n\}$ and according to the equality (6), for $t \in J_j$, we get

$$ {}^{c}D_{0^+}^{u(t)} z_j(t) = \int_{S_{j-1}}^{t} \frac{(t-s)^{1-u_j}}{\Gamma(2-u_j)} z^{(2)}(s)ds. \tag{64}$$

Taking $I_{S_{j-1}^+}^{u_j}$ on both sides of the inequality (62), we obtain

$$ \begin{aligned} &|-z_j(t) - \int_{S_{j-1}}^{S_j} G_j(t,s) f(s, z_j(s)) ds| \le \varepsilon \int_{S_{j-1}}^{t} \frac{(t-s)^{u_j-1}}{\Gamma(u_j)} \\ &\cdot ds \le \varepsilon \frac{(S_j - S_{j-1})^{u_j}}{\Gamma(u_j+1)}. \end{aligned} \tag{65}$$

According to Theorem 16, the Caputo fractional thermostat model (4) has a positive solution $y \in C(J, \mathbb{R})$ defined by $y(t) = y_j(t)$ for $t \in J_j$, $j = 1, 2, \cdots, n$, where

$$ y_j = \begin{cases} 0, & t \in [0, S_{j-1}], \\ \tilde{y}_j, & t \in J_j, \end{cases} \tag{66}$$

and $\tilde{y}_j \in E_j$ is a positive solution of (22). According to Lemma 9, the integral equation

$$ \tilde{y}_j(t) = \int_{S_{j-1}}^{S_j} G_j(t,s) f\left(s, \tilde{y}_j(s)\right) ds, \tag{67}$$

holds. Let $t \in J_j$, $j = 1, 2, \cdots, n$. Then, by Eq (66) and (67) we get

$$ \begin{aligned} &|z(t) - y(t)| = |z(t) - y_j(t)| = |z_j(t) - \tilde{y}_j(t)| \\ &= |z_j(t) - \int_{S_{j-1}}^{S_j} G_j(t,s) f\left(s, \tilde{y}_j(s)\right) ds| \le |z_j(t) - \int_{S_{j-1}}^{S_j} G_j(t,s) \\ &\quad \cdot f(s, z_j(s)) ds| + |\int_{S_{j-1}}^{S_j} G_j(t,s) f(s, z_j(s)) \\ &\quad \cdot ds - \int_{S_{j-1}}^{t} G_j(t,s) f\left(s, \tilde{y}_j(s)\right) ds| \le |-z_j(t) \\ &\quad - \int_{S_{j-1}}^{S_j} G_j(t,s) f(s, z_j(s)) ds| + |\int_{S_{j-1}}^{S_j} G_j(t,s) f(s, z_j(s)) \\ &\quad \cdot ds - \int_{S_{j-1}}^{t} G_j(t,s) f\left(s, \tilde{y}_j(s)\right) ds| \le \varepsilon \frac{(S_j - S_{j-1})^{u_j}}{\Gamma(u_j+1)} \\ &\quad + \left(1 + \frac{S_{j-1}^{u_j-1}}{\Gamma(u_j)}\right) \int_{S_{j-1}}^{S_j} |f(s, z_j(s)) ds - f(s, \tilde{y}_i(s)) \\ &\quad \cdot |ds \le \varepsilon \frac{(S_j - S_{j-1})^{u_j}}{\Gamma(u_j+1)} + \left(1 + \frac{S_{j-1}^{u_j-1}}{\Gamma(u_j)}\right) \int_{S_{j-1}}^{S_j} s^{-\delta} \\ &\quad \cdot \left(K \,|\, z_j(s) - \tilde{y}_j(s) \,|\, ds \le \varepsilon \frac{(S_j - S_{j-1})^{u_j}}{\Gamma(u_j+1)} + \left(1 + \frac{S_{j-1}^{u_j-1}}{\Gamma(u_j)}\right)\right. \\ &\quad \cdot \left(K \|z_i - \tilde{y}_j\|_{E_j}\right) \int_{S_{j-1}}^{S_j} s^{-\delta} ds \le \varepsilon \frac{(S_j - S_{j-1})^{u_j}}{\Gamma(u_j+1)} + \frac{K\left({S_j}^{1-\delta} - {S_{j-1}}^{1-\delta}\right)}{1-\delta} \\ &\quad \cdot \left(1 + \frac{S_{j-1}^{u_j-1}}{\Gamma(u_j)}\right) \|z_j - \tilde{y}_j\|_{E_j} \le \varepsilon \frac{(S_j - S_{j-1})^{u_j}}{\Gamma(u_j+1)} + \mu \|z - y\|, \end{aligned} \tag{68}$$

where

$$ \mu = \max_{j=1,2,\cdots,n} \frac{K\left({S_j}^{1-\delta} - {S_{j-1}}^{1-\delta}\right)}{1-\delta} \left(1 + \frac{S_{j-1}^{u_j-1}}{\Gamma(u_j)}\right). \tag{69}$$

Then,

$$ \|z - y\|(1-\mu) \le \frac{(S_j - S_{j-1})^{u_j}}{\Gamma(u_j+1)} \varepsilon. \tag{70}$$

We obtain, for each $t \in J_j$

$$ |z(t) - y(t)| \le \|z - y\| \le \frac{(S_j - S_{j-1})^{u_j}}{(1-\mu)\Gamma(u_j+1)} \varepsilon \coloneqq c_f \varepsilon. \tag{71}$$

Therefore, by Definition 17, the Caputo fractional thermostat model (4) is Ulam-Hyers stable and the proof is completed. □

6. Example

We here simulate the simple form of our variable order BVP of the main Caputo fractional thermostat model (4) by giving an example numerically.

Example 2. Consider the nonlinear function

$$ f(t,y) = \frac{e^t + 1}{\sqrt{y}}, \tag{72}$$

on $(t, y) \in [0, 2] \times [0, +\infty)$ and

$$ u(t) = \begin{cases} 1.4, & t \in J_1 \coloneqq [0, 1], \\ 1.5, & t \in J_2 \coloneqq]1, 2]. \end{cases} \tag{73}$$

Then, in consistent with (22) and corresponding to the variable order BVP of fractional differential equation

$$ \begin{cases} -{}^{c}D^{u(t)} y(t) = \dfrac{e^t + 1}{\sqrt{y(t)}}, & t \in J \coloneqq [0, 2], \\ y'(0) = 0, & {}^{c}D^{u(t)-1} y(2) + y(0) = 0, \end{cases} \tag{74}$$

the constant order auxiliary BVPs are

$$ \begin{cases} -{}^{c}D_{0^+}^{1.4} y(t) = \dfrac{e^t + 1}{\sqrt{y(t)}}, & t \in J_1, \\ y'(0) = 0, & {}^{c}D_{0^+}^{0.4} y(1) + y(0) = 0, \end{cases} \tag{75}$$

$$ \begin{cases} {}^{c}D_{1^+}^{1.5} y(t) = \dfrac{e^t + 1}{\sqrt{y(t)}}, & t \in J_2, \\ y'(1) = 0, & {}^{c}D_{1^+}^{0.5} y(2) + y(1) = 0. \end{cases} \tag{76}$$

Clearly, $f_0 = \infty$ and $f_\infty = 0$.

For $j = 1$, we get $\Gamma(1.4) - 1 \approx -0.11274 < 0$, we take

$$\xi_1 = \min\left\{\frac{\Gamma(u_j) - (d - S_{j-1})^{u_j-1}}{\Gamma(u_j) + S_{j-1}^{u_j-1}}, \frac{\Gamma(u_j) - (d - S_{j-1})^{u_j-1}}{(S_j - S_{j-1})^{u_j-1} - \Gamma(u_j)}\right\} = \frac{\Gamma(1.4) - d^{u_j-1}}{\Gamma(1.4)} \tag{77}$$

and define the cone $P_1 = \{y \setminus y \in C[0, 1], \min_{t\in[0,1]} y(t) \geq \xi_1 \|y\|\}$. By Theorem 14, it is deduced that that the auxiliary BVP of constant order Caputo fractional thermostat model (75) possesses a positive solution $\tilde{y}_1 \in P_1$.

For $j = 2$, we get $\Gamma(1.5) - 1 \approx -0.11377 < 0$, we take

$$\xi_2 = \min\left\{\frac{\Gamma(u_j) - (d - S_{j-1})^{u_j-1}}{\Gamma(u_j) + S_{j-1}^{u_j-1}}, \frac{\Gamma(u_j) - (d - S_{j-1})^{u_j-1}}{(S_j - S_{j-1})^{u_j-1} - \Gamma(u_j)}\right\} = \frac{\Gamma(1.5) - (d-1)^{1.5}}{\Gamma(1.5) + 1} \tag{78}$$

and consider the cone $P_2 = \{y \setminus y \in C[1, 2], \min_{t\in[1,2]} y(t) \geq \xi_2 \|y\|\}$.

According to Theorem 14, the auxiliary BVP of constant order Caputo fractional thermostat model (76) admits a positive solution $\tilde{y}_2 \in P_2$, and by Theorem 16, the BVP for variable order Caputo fractional thermostat model (74) has a solution

$$y(t) = \begin{cases} \tilde{y}_1(t), & t \in J_1, \\ y_2(t), & t \in J_2, \end{cases} \tag{79}$$

where

$$y_2(t) = \begin{cases} 0, & t \in J_1, \\ \tilde{y}_2(t), & t \in J_2. \end{cases} \tag{80}$$

7. Conclusion

In science and technology, fractional differential equations are utilized to model and describe a variety of natural processes. In connection with standard fractional models, variable fractional models appear to be more important for complex natural phenomena. The focus of this study was to analyze the BVP of variable order Caputo fractional thermostat model (4) and to explore its solutions' existence utilizing techniques from fixed point theory on cones. To do such a method, we first noticed the invalidity of semigroup property for the Riemann-Liouville fractional integral of variable order. In order to obtain the solution of problem and solve this issue, we considered a partition of the interval J and the corresponding auxiliary constant order BVP of the Caputo fractional thermostat model (22) was derived from the variable order one. Some properties of relevant Green's function were reviewed. For the solutions' existence, utilizing fixed point attributed to Guo-Krasnoselskii's on cones, the main theorems (Theorem 14 and 16) were deduced. An example to confirm the validity of theoretical findings was provided. This technique can be used to consider various physical models with variable order fractional operators. Moreover, with the help of our results in this research paper, investigations on this open research problem can be also possible and one can extend the proposed BVP to other complicated fractional models. In the future, we want to study these boundary value problems with different conditions involving integral conditions or integroderivative conditions or nonlocal conditions along with infinite delay.

Data Availability

No data were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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